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Real-Time Identification and Control of Satellite Signal Impairments—Solution and Application of the Stratonovich Equation

Part 1. Theoretical Development

Robert M. Manning
Glenn Research Center, Cleveland, Ohio

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Glenn Research Center
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Part 1. Theoretical Development

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National Aeronautics and Space Administration
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Cleveland, Ohio 44135

Abstract

As satellite communications systems become both more complex and reliant with respect to their operating environment, it has become imperative to be able to identify, during real-time operation, the onset of one or more impairments to the quality of overall communications system integrity. One of the most important aspects to monitor of a satellite link operating within the Earth's atmosphere is the signal fading due to the occurrence of rain and/or phase scintillations. This, of course, must be done in the presence of the associated measurement uncertainty or potentially faulty measurement equipment such as in the Advanced Communication Technology Satellite (ACTS) experiment. In the present work, an approach originally suggested in 1991, and apparently still considered iconoclastic, will be significantly developed and applied to the satellite communications link on which the deleterious composite signal fade is the result of one or many component fade mechanisms. Through the measurement (with the attendant uncertainty or 'error' in the measurement) of such a composite fading satellite signal, it is desired to extract the level of each of the individual fading mechanisms so they can be appropriately mitigated before they impact the overall performance of the communications network. Rather than employing simple-minded deterministic filtering to the real-time fading, the present approach is built around all the models and/or descriptions used to describe the individual fade components, including their dynamic evolution. The latter is usually given by a first-order Langevin equation. This circumstance allows the description of the associated temporal transition probability densities of each of the component processes. By using this description, along with the real-time measurements of the composite fade (along with the measurement errors), one can obtain statistical estimates of the levels of each of the component fading mechanisms as well as their predicted values into the future. This is all accomplished by the use of the well-known Stratonovich integro-differential equation that results from the model of the measured signal fade that is also tailored to adaptively adjust the values of the parameters used in the statistical models of the individual fade mechanisms. Three examples of increasing complexity are addressed and solved for the iterative determination of fade component levels from the measured composite signal fade in the presence of measurement error and, in the last case, with uncertainty in the model parameters.

1.0 Introduction

The propagation impairments that are peculiar in and above the 30/20 GHz Ka-band are of such a dynamic and composite nature as to require fade processing methods not found in earlier satellite link attenuation assessments. In particular, not only is one faced with rain attenuation characterized with deeper fades and "faster" dynamics as compared to that in the Ku-band, but also the phenomena of cloud and/or clear air scintillation (essentially, enhancements and degradations of signal level due to constructive and destructive interference initiated by phase perturbations of the propagating wave), with

effects comparable to that of rain attenuation, becomes significant, especially at 30 GHz. One must then be able to identify and separate these components from the composite fade since each must be dealt with in a different manner, e.g., rain fade by power control and scintillation by time diversity transmission. Even if scintillation is not by itself deleterious to communication link integrity, it must still be separated from the rain fade component for efficient and optimal power control implementation.

In addition to the identification problem outlined above, one will also always encounter situations where the signal from which one is to obtain rain fade information is corrupted by measurement uncertainty or “noise” due to the measurement technique and/or equipment. What is more, if frequency scaling is applied to noisy attenuation data that has (before proper filtering and identification) such random variations, these variations can become greatly exaggerated after the scaling operation simply because they are not governed by the same mechanisms that are responsible for the rain attenuation frequency scaling law. The ever-present measurement uncertainty that prevails in the assessment of communications satellite link integrity must always be properly addressed before one can reliably capture associated performance metrics.

1.1 The Dynamics of the ACTS Rain Attenuation Prediction Model

The above problems all have had to be addressed within the Advanced Communication Technology Satellite (ACTS) Project in the design and implementation of ground terminals for the satellite. These have been collectively met by the formulation of the problem within the context of the ACTS Rain Attenuation Prediction Model (Refs. 1, 2, 3, and 4] and Nonlinear Markov Filtering Theory (Ref. 5). In particular, due to the structure adopted for the ACTS Rain Attenuation Prediction Model, the problem was couched within the context of the state variable approach of stochastic control theory, analyzed in terms of the optimal estimate with respect to the minimization of the least squares estimate, and analytically solved yielding a set of recursion relations, amenable to digital implementation, giving the optimal estimate of link attenuation in the presence of measurement noise. (The ‘Markov’ descriptor issues from the use of well-known Markov random process theory to derive the statistical estimates and the ‘non-linear’ descriptor comes from the non-linear functional relationship between the quantity of signal attenuation and the more basic parameters of the dynamic model.) Because of its close connection with the ACTS Rain Attenuation Prediction Model, which discerns climatological variations on the order of 0.5° in latitude and longitude in the continental U.S., the statistical filtering algorithm can be “programmed” with the attenuation statistics that are representative for any location at which it may be used. However, there too can be uncertainty in the calculations of the relevant statistical parameters of the rain attenuation process. The nature of the method used here is such that parameter uncertainty can also be incorporated into the filtering algorithm to make it adaptive to such incomplete knowledge.

1.2 The Essence of the Statistical Filtering Problem and Its Comparison to Well-Known Fourier-Transform Filtering

1.2.1 Fourier Transform Filtering

Consider the well-known problem signal decomposition of the Fourier Transform filter. For example, let the measured or observed signal $F_{obs}(F_1, F_2, t)$ be very simply given by

$$F_{obs}(F_1, F_2, t) = F_1(t)\cos(\omega_1 t) + F_2(t)\cos(\omega_2 t) + n(t) \quad (1)$$

where $F_1(t)$ is the first signal imbedded in the overall measurement. The value or level of this imbedded signal is so far unknown and, in fact, is to be ‘filtered’ out of the measurement $F_{obs}(F_1, F_2, t)$. The only

thing known about $F_1(t)$ is that it is characterized by the frequency component $\cos(\omega_1 t)$. Similarly, $F_2(t)$ is the second such signal with unknown value but characterized by $\cos(\omega_2 t)$. The function $n(t)$ is a random function that characterizes the measurement error or uncertainty. It is statistically defined by the specification of its mean value $\langle n(t) \rangle$ as well as its variance $\langle n(t) n(t') \rangle$,

$$\langle n(t) \rangle = 0, \quad \langle n(t) n(t') \rangle = \sigma_n^2 \delta(t - t') \quad (2)$$

where σ_n^2 is the variance of the measurement process. The decomposition of the measured signal given by Equation (1) into its separate components $F_1(t)$ and $F_2(t)$ is accomplished by the use of the Fourier transform relation

$$\langle F(\omega) \rangle = \int_{-\infty}^{\infty} \langle F_{obs}(t) \rangle \exp(i\omega t) dt \quad (3)$$

where the average $\langle F_{obs}(t) \rangle$ is taken to relegate the measurement noise contribution to zero as per Equation (2). So long as the temporal variation of the components $F_1(t)$ and $F_2(t)$ is such that they do not appreciably vary over intervals Δt given by $\Delta t \leq 1/\omega_{1,2}$, then, as depicted in Figure 1, using Equation (1) in Equation (3) gives in the usual way,

$$\langle F_1(t) \rangle = \langle F(\omega_1) \rangle, \quad \langle F_2(t) \rangle = \langle F(\omega_2) \rangle, \quad t > 1/\omega_{1,2} \quad (4)$$

Similarly, the variance of the filtered signals are

$$\langle F_1^2(t) \rangle = \langle F^2(\omega_1) \rangle + \sigma_n^2, \quad \langle F_2^2(t) \rangle = \langle F^2(\omega_2) \rangle + \sigma_n^2 \quad (5)$$

Hence, the values of the previously unknown signals $F_1(t)$ and $F_2(t)$ are now known; actually, due to the presence of the uncertainty of the measured value of the overall signal, only the statistical values of $F_1(t)$ and $F_2(t)$ (i.e., the mean and variance) can be determined by this filtering process.

In this example, the individual signals $F_1(t)$ and $F_2(t)$ are distinguished and separated by their frequency characteristics. This is the basis of Fourier transform filtering. The next example, and the subject of the present work, introduces a filtering methodology that uses other characteristics of the unknown signals; in this case, the characteristics are the overall statistical behavior of the signals over time imparted by the sources of each of the unknown signals that comprise the measurement $F_{obs}(t)$.

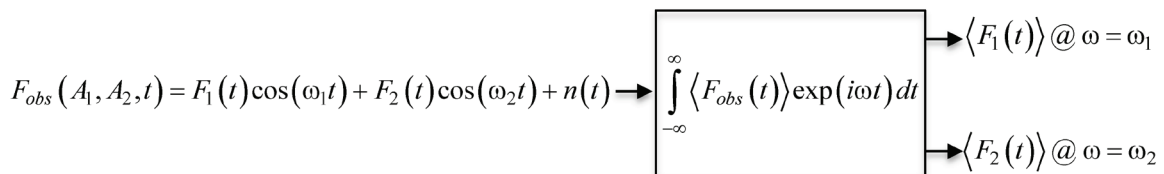


Figure 1.—A simple Fourier transform filter

1.2.2 Statistical Filtering

A similar measured signal decomposition can be based using the statistical properties of the processes $F_1(t)$ and $F_2(t)$ rather than their temporal frequency properties as above. Thus, instead of $\cos(\omega_1 t)$ characterizing the first unknown signal component, one now takes it to be characterized by some other function x_1 which takes the place of $\cos(\omega_1 t)$ in the example above. That is, instead of $A_1(t)\cos(\omega_1 t)$ giving the form of the unknown signal, it is now just given by another function $S_1(x_1, t)$ which is some given and known mathematical function of the characteristic x_1 . The characteristic function x_1 that ultimately determines the value of $S_1(x_1, t)$ can itself be defined by some mathematical representation (or model) that involves known parameters, i.e., $x_1 = f_1(a_1, b_1, \dots)$. The similar situation holds for the second signal $S_2(x_2, t)$ where $x_2 = f_2(a_2, b_2, \dots)$ is the function that characterizes S_2 just as the function $\cos(\omega_2 t)$ did in the Fourier Transform example. Thus, the measured signal, $F_{obs}(F_1, F_2, t)$ is now represented by

$$S_{obs}(S_1, S_2, t) = S_1(x_1, t) + S_2(x_2, t) + n(t), \quad x_1 = f_1(a_1, b_1, \dots), \quad x_2 = f_2(a_2, b_2, \dots) \quad (6)$$

where the function represented by $S_{obs}(S_1, S_2, t)$ is the same value as that previously represented by the function $F_{obs}(F_1, F_2, t)$; that is these are different functional models of the exact same observational data. In Equation (6), the function $f_1(a_1, b_1, \dots)$ gives the statistical quantity x_1 through which the component process $S_1(x_1, t)$ is defined. The deterministic parameters a_1, b_1, \dots that enter into the function f_1 come from some statistical model of the process $S_1(x_1, t)$. Similarly for the function $f_2(a_2, b_2, \dots)$. The measurement noise is as given by Equation (2).

Since the quantity x_1 is a statistical one, it can be given by a probability density function (PDF) $p(x_1 | S_{obs})$ conditioned on the composite observational measurement $S_{obs}(t)$. Similarly, when there are two such component processes $S_1(t)$ and $S_2(t)$ as above, one can imagine a two-dimensional conditional PDF $p(x_1, x_2 | S_{obs})$. That is, for each statistical function x_1 or x_2 that defines a particular component $S_1(x_1, t)$ or $S_2(x_2, t)$ (just as the functions $\cos(\omega_1 t)$ and $\cos(\omega_2 t)$ define the temporal frequency content of $F_1(t)\cos(\omega_1 t)$ and $F_2(t)\cos(\omega_2 t)$), there is a corresponding two-dimensional PDF $p(x_1, x_2 | S_{obs})$. As shown in Figure 2, the use of the conditional PDF immediately gives the mean values of the component processes

$$\langle S_1(t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_1(x_1, t) p(x_1, x_2 | S_{obs}) dx_1 dx_2, \quad \langle S_2(t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_2(x_2, t) p(x_1, x_2 | S_{obs}) dx_1 dx_2 \quad (7)$$

$$\langle S_1^2(t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_1(x_1, t) - \langle S_1(x_1, t) \rangle)^2 p(x_1, x_2 | S_{obs}) dx_1 dx_2 \quad (8)$$

$$\langle S_2^2(t) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S_2(x_2, t) - \langle S_2(x_2, t) \rangle)^2 p(x_1, x_2 | S_{obs}) dx_1 dx_2 \quad (9)$$

all of which are based on the observational input $S_{obs}(t)$.

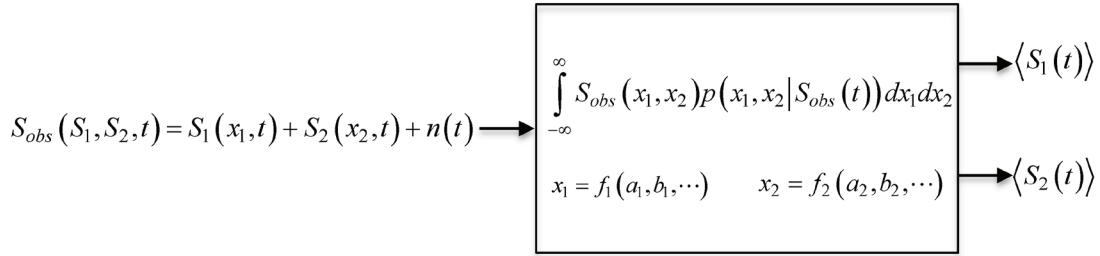


Figure 2.—A simple statistical filter.

1.3 A General Filter for Rain and Scintillation

It is the purpose of this document to obtain similar functional relationships $f_R(a_R, b_R, \dots)$ and $f_S(a_S, b_S, \dots)$ as well as the PDF $p(x_R, x_S | A_{obs})$ for the rain fade component $A_R(x_R, t)$ as well as the scintillation component $A_S(x_S, t)$ of an overall measured signal fade $A_{obs}(A_R, A_S, t)$ on a satellite link. Here, the functions $A_R(x_R, t)$, $A_S(x_S, t)$, etc., now take the place of $S_1(x_1, t)$ and $S_2(x_2, t)$ in the example above. The functions $f_R(a_R, b_R, \dots)$ and $f_S(a_S, b_S, \dots)$ will be given by mathematical models specific to the rain and scintillation processes.

A dynamic model of rain attenuation, originally developed for the dynamics portion of the ACTS Rain Attenuation Prediction Model, treats the rain attenuation process as a Dynamic Markov process which is based on a first-order differential equation, known as the Langevin equation, which gives the statistics of the dynamic rain attenuation process; this equation is represented by the function $f_R(\dots)$. A similar equation and functional relationship $f_S(\dots)$ can be written for the dynamic scintillation process. From these differential equations for the processes x_R and x_S , one can proceed to derive a similar differential equation for the related conditional PDF $p(x_R, x_S | A_{obs})$. To accomplish this, the Langevin equations for x_R and x_S are first used to obtain what is known as the Kolmogorov equation that gives the transition PDF $p(x_R(t_1), x_S(t_1) | x_R(t_0), x_S(t_0))$ for the processes from one time instant t_0 to a later time instant t_1 . It is through the use of this particular PDF that all the fade dynamics of the ACTS Rain Model were evaluated. But for the problem considered in this document, this is not enough. One must augment the description of this transition process with the uncertain (noisy) measured values of the composite fade $A_{obs}(t)$ so as to replace each of the required conditional values $x_R(t_0), x_S(t_0)$ with a single composite quantity. Such similar motivation inspired R. L. Stratonovich and his colleagues (Refs. 6 to 8) in the 1950s to develop from the Kolmogorov equation a prescription which relies on noisy measured input values to help determine the resulting PDF $p(x_R, x_S | A_{obs})$. (This approach was used by R. Kalman for the development of the simplistic linear filter applications of this theory.)

1.4 Making the Technique Adaptive

In addition to the real-time estimation of rain attenuation and scintillation levels, the approach can also be made adaptive, that is, giving it the ability to detect and correct for errors in the values of the coefficients used in the process models of the individual fade components. This is easily accomplished by taking the source of variance of the parameter uncertainty to be itself a Markov random process governed by, once again, a Langevin stochastic differential equation, etc. Adding a single adaptive adjustment to the rain and scintillation filter for one of the parameters results in a three-component Markov filter. For every parameter taken to be uncertain and requiring an adaptive adjustment, one more Markov process must be added. The use of this approach will be made in what is to follow.

1.5 Some Nomenclature and the Approach Used in the Remainder of the Work

In the development that is to follow, expressions of the form

$$A_{obs}(A_R, A_S, t) = A_R(x_R, t) + A_S(x_S, t) + n(t) \quad (10)$$

are called the **observation model**. The component functions $A_R(x_R, t)$ and $A_S(x_S, t)$ are called, respectively, **the models of rain attenuation and scintillation**. The individual equations of the form

$$x_R = f_R(a_R, b_R, \dots), \quad x_S = f_S(a_S, b_S, \dots) \quad (11)$$

are called the **process models**, i.e., the prescriptions for the statistical processes that drive the rain attenuation and scintillation. Finally, $n(t)$ along with the specifications

$$\langle n(t) \rangle = 0, \quad \langle n(t)n(t') \rangle = \sigma_n^2 \delta(t - t') \quad (12)$$

constitute the measurement **noise model**.

This work will begin with the simplest application of Nonlinear Markov filter theory which is the one originally put forward (Ref. 4). Here, only the rain attenuation component is considered; the observation model is simply

$$A_{obs}(A_R, t) = A_R(x_R, t) + n(t) \quad (13)$$

with only one process model $x_R = f_R(a_R, b_R, \dots)$. The detailed development of the Kolmogorov equation from the Langevin equation is given in Appendix A for both the single-component example as well as the more general multi-component case. Appendix B details the connection of the Kolmogorov equation to the actual measurement process by deriving the Stratonovich equation. Finally, the Stratonovich equation is solved in Appendix C and application is made of the solution to the iterative discrete-sampling real-time measurement and filtering process. The results of the appendices are then applied to this one-component case to derive an iterative filter for rain attenuation.

The next case to be considered will be the one with the two-component process of Equation (10) for both rain attenuation as well as scintillation. The analysis is the same as that of the one-component process but is rendered a bit more complicated by two components which now constitute a vector treatment of the dynamic equations rather than a scalar treatment. Also, there is an additional complication that enters due to the potential cross-correlation that may exist between the two processes of attenuation and scintillation. These nuances are thoroughly treated in the appendices. These results are then used to construct an iterative filter for both attenuation and scintillation.

Finally, the case of an adaptive filter for attenuation and scintillation is considered. Here, one of the constant parameters that are taken to characterize the rain attenuation process is now treated as one that is known only as ‘on the average’. Possible variations of its value are now incorporated into the filter by simply treating the parameter variation as driven by another Markov process. This adaptive case adds one more Markov process to the filter and becomes a three-component filter.

In all the cases considered here and in the future, only satellite signal fading due to non-modulation based variations need be considered. The ability to rid a satellite downlink signal of power fluctuations due to the attendant modulation has been dealt with elsewhere (Ref. 9).

2.0 A Simple Example to Introduce the Method—Identification of Rain Attenuation on Communications Satellite Links Using Nonlinear Markov Filtering

The problem to be addressed here is the following: Consider the measurement of the temporal behavior of attenuation due to rain fade on a satellite link. The measured or observational attenuation $A_{obs}(t)$ is taken to be composed of the actual attenuation $A_R(t)$ and a measurement uncertainty $n(t)$. For example, if the attenuation measurement is derived by a frequency scaling procedure, the measurement uncertainty is the uncertainty inherent in the frequency scaling relationship, in addition to the error imposed by the equipment used. Hence, one can write

$$A_{obs}(t) = A_R(t) + n(t) \quad (14)$$

Based on the temporal portion of the ACTS Rain Attenuation Prediction Model (Ref. 1), the actual attenuation $A_R(t)$ is given by a first order Markov Process

$$\frac{dx_R}{dt} = -\gamma_R x_R + \sqrt{2\gamma_R} \xi_R(t) \quad (15)$$

where the process variable is related to the attenuation via the non-linear prescription

$$A_R(x_R, t) = A_{mR} \exp(\sigma_{\ln A_R} x_R(t)) \quad (16)$$

which, of course, derives from the log-normal nature of the rain fade, i.e.,

$$x_R = \frac{\ln(A_R(t)) - \ln(A_{mR})}{\sigma_{\ln A_R}} \quad (17)$$

Equation (15) is the relation that is represented by $(x_R = f_R(a_R, b_R, \dots))$ where $a_R \equiv A_{mR}$ and $b_R \equiv \sigma_{\ln A_R}$. The value for the mean attenuation A_{mR} and the standard deviation of rain attenuation $\sigma_{\ln A_R}$ are not only functions of the operating frequency and propagation link geometry, they are also dependent on the geographical location of the satellite terminal. The stochastic function $\xi_R(t)$ that drives the Markov process is defined by

$$\langle \xi_R(t) \rangle = 0, \quad \langle \xi_R(t) \xi_R(t + \tau) \rangle = \delta(\tau) \quad (18)$$

i.e., a Gaussian (white noise) process characterized by an error variance of unity. The temporal decay factor $\gamma_R \approx 0.0538 \text{ min}^{-1}$ as determined by attenuation dynamics at 20 and 30 GHz (Refs. 1 and 3). The characteristic attenuation parameters A_{mR} , $\sigma_{\ln A_R}$, and γ_R are all found for a given location and link geometry from the ACTS Rain Model (Ref. 2). The problem can now be stated: Given the measurement of $A_{obs}(t)$ and the model of the actual rain attenuation given by Equations (14)-(18), what is the value of the actual attenuation $A_R(t)$ corresponding to the measurement with the defined uncertainty? The rigorous solution to this rather simplistic problem is given in Appendix A.

The statistical process described by Equation (15) has associated with it a conditional transition PDF giving the statistics connected with the evolution of the value of $x_R(t)$ at a specific time t_0 to a future time

$t, t_0 < t$. The proper and rigorous use of the information afforded by the random variable $x_R(t)$ is through the use of the governing transition PDF. As derived in Appendix A, this transition PDF is given as the solution to Equation (A28), i.e.,

$$\frac{\partial(x_R, t | x_0, t_0)}{\partial t} = D_{x_R} [(x_R, t | x_0, t_0)], \quad t > t_0 \quad (19)$$

where $D_{x_R} [\dots]$ is the Kolmogorov differential operator defined by

$$D_{x_R} [p(x_R, t | x_0, t_0)] \equiv \gamma_R \frac{\partial x_R p(x_R, t | x_0, t_0)}{\partial x_R} + \gamma_R \frac{\partial^2 p(x_R, t | x_0, t_0)}{\partial x_R^2} \quad (20)$$

Equation (19) is known as the Forward Kolmogorov Equation which is associated with the Langevin-type equation given by Equation (15). The solution to Equation (19) was discussed in detail in (Ref. 3) and is given by

$$p(x_R, t | x_0, t_0) = \sqrt{\frac{1}{2\pi\sigma^2(\Delta t)}} \exp\left[-\frac{(x_R(t) - e^{-\gamma_R \Delta t} x_0(t_0))^2}{2\sigma^2(\Delta t)}\right] \quad (21)$$

where

$$\sigma^2(\Delta t) \equiv 1 - \exp(-2\gamma_R \Delta t), \quad \Delta t \equiv t - t_0 \quad (22)$$

Using now Equation (16) to relate the process values x_R to that of the attenuation values $A_R(t)$ yields the conditional transition PDF that contains all the dynamical behavior of attenuation due to rain, i.e.,

$$p(A_R, t | A_0, t_0) = \sqrt{\frac{1}{2\pi\sigma_{\ln A}^2(\Delta t)}} \exp\left[-\frac{(\ln(A_R(t)) - \ln(A_{mR}(\Delta t)))^2}{2\sigma_{\ln A}^2(\Delta t)}\right] \quad (23)$$

where the time dependent mean attenuation and variance of attenuation are respectively given by

$$A_{mR}(\Delta t) \equiv A_{mR}^{1-e^{-\gamma_R \Delta t}} A_0 e^{-\gamma_R \Delta t}, \quad \sigma_{\ln A}^2(\Delta t) = \sigma_{\ln A}^2 (1 - e^{-2\gamma_R \Delta t}) \quad (24)$$

In the limit as $\Delta t \rightarrow \infty$, the time interval essentially represented that during a year and all the time dependent statistical parameters approach their yearly (static) values, i.e., $A_{mR}(\Delta t) \rightarrow A_{mR}$ and $\sigma_{\ln A}^2(\Delta t) \rightarrow \sigma_{\ln A}^2$, and the PDF of Equation (23) transitioned into the well-known log-normal distribution that is the basis of the ACTS attenuation model as well as other such models. The use of Equations (21)-(24) allowed the derivation of many dynamically based link performance parameters such as fade durations, etc. However, of interest here is how this dynamic description can be used to filter or identify a rain attenuation process from the noisy observed attenuation $A_{obs}(t)$. (This ability will prove itself when there are additional processes associated with that if just the rain attenuation.)

In order to apply Equation (19) to the actual measurement process, one must first admit into the analysis the fact that the sampled value of the observed attenuation $A_{obs}(t_i)$ at a particular time instant t_i

has corresponding to it, through Equation (17), a corresponding sampled value of the process $x_R(t)$. The modification of Equation (19) to the use of measured random variables characterized by a measurement uncertainty is discussed in Appendix B. There, the relation known as the Stratonovich Equation is derived, viz,

$$\frac{\partial p(x_R, t)}{\partial t} = D_{\bar{x}} [p(x_R, t)] + \left[\phi(x_R, t) - \int_{-\infty}^{\infty} \phi(x'_R, t) p(x'_R, t) dx'_R \right] p(x_R, t) \quad (25)$$

where $p(x_R, t) \equiv p(x_R, t | A_{obs}(t_i))$ (it is implicit that $t > t_i$) is the transition PDF conditioned on the observational measurement $A_{obs}(t_i)$ which enters into Equation (25) through

$$\phi(x_R, t) \equiv -\frac{1}{2\sigma_n^2} (A_{obs}(t) - A_R(x_R, t))^2 \quad (26)$$

where the functional $A_R(x_R, t)$ is given by Equation (16). Thus, the Stratonovich equation replaces the Kolmogorov equation in the case where the (noisy) measurement $A_{obs}(t)$ of a modeled function $A_R(x_R, t)$ of the process x_R is available.

One can now specify the **optimal estimate** (in the mean-square sense)

$$\langle x_R(t) \rangle \equiv \int_{-\infty}^{\infty} x_R(t) p(x_R, t) dx_R \quad (27)$$

of x_R corresponding to the measured value $A_{obs}(t)$ as well as the related variance

$$\sigma \equiv \int_{-\infty}^{\infty} (x_R - \langle x_R \rangle)^2 p(x_R, t) dx_R \quad (28)$$

which is the **error variance of the optimal estimate**. Appendix C gives a detailed solution to the Stratonovich equation for the general case of a many-component (i.e., multi-dimensional) case yielding analytical expressions for Equations (27) and (28). This general solution will be used below for the two- and three-component cases to be examined. For this particular example of only one component, the results of Appendix C can easily be adapted for the one-dimensional case as given by Equation (25). Applying the solutions for this one-dimensional discrete-time case obtained in Appendix C, i.e., employing the one-dimensional, single-component forms of Equations (C47), (C48), (C42) and (C43), one obtains the following recursive filter for rain attenuation

Optimal Estimate:

$$\langle x_R(t_{i+1}) \rangle = \exp(-\gamma_R \Delta t) \langle x_R(t_i) \rangle + \exp(-\gamma_R \Delta t) \sigma(t_i) a(t_i) \quad (29)$$

Error Covariance of Optimal Estimate:

$$\sigma(t_{i+1}) = (H^{-1}(t_i) - b(t_i))^{-1} \quad (30)$$

Error Covariance of Extrapolated Estimate:

$$H(t_i) = \exp(-2\gamma_R \Delta t) \sigma(t_{i-1}) + D(\Delta t) \quad (31)$$

Error Covariance Due to Diffusion:

$$D(\Delta t) = \exp(-2\gamma_R \Delta t) (1 - \exp(-2\gamma_R \Delta t)) \quad (32)$$

Here, the following definitions made: The fundamental transition matrix is just the scalar

$$\Phi(\Delta t) = \exp(-\gamma_R \Delta t) \quad (33)$$

Using Equation (26) in Equations (C45) and (C46) give

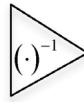
$$a(t) = \frac{\sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle) \Xi(\langle x_R \rangle) \Delta t \quad (34)$$

and

$$b(t) = \frac{\sigma_{\ln A}^2}{\sigma_n^2} A_R(\langle x_R \rangle) (\Xi(\langle x_R \rangle) - A_R(\langle x_R \rangle)) \Delta t \quad (35)$$

where $\Xi(\langle x_R \rangle) \equiv A_{obs}(t) - A_R(\langle x_R \rangle)$ is the discriminator which compares the measured value of attenuation to the predicted value.

The flow-chart of the algorithmic implementation of this iterative process is shown in Figure 3. Here, the element



is an inverter and **D** represents Equation (32). The element shown by



represents a time delay of Δt and $\Phi \equiv \exp(-\gamma_R \Delta t)$. The other designations are self-explanatory from the text.

This example demonstrates how the rain attenuation component is statistically separated from the measurement that convolves uncertainty. This identification and separation of the random processes of the actual rain attenuation from the measured data will allow, for example, the optimal use of a predictive procedure to obtain values of $A_R(\langle x_R \rangle)$ into the future so as to counter the control delay in the use of some mitigation procedure – in this case, link power control. The level of power control is driven by the filtered values $A_R(\langle x_R \rangle)$.

The utility of the method becomes apparent when two or more components enter into the composite fading process that is measured.

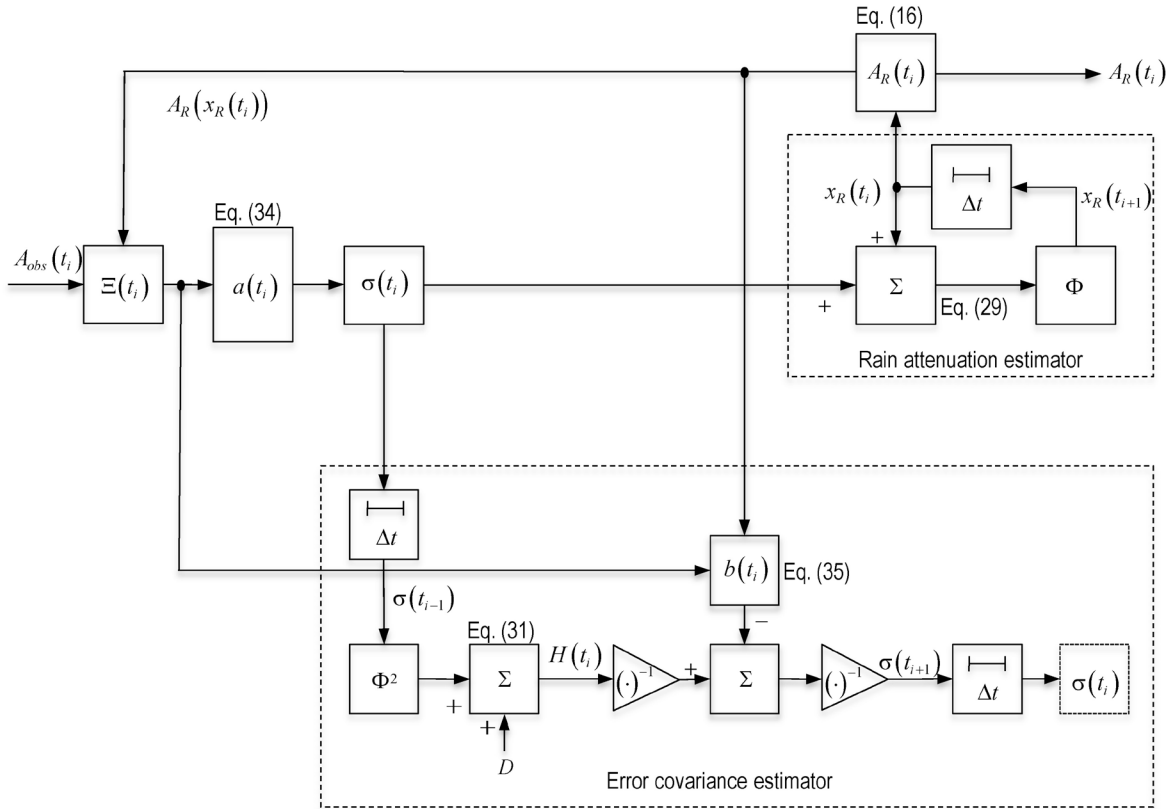


Figure 3.—Optimal estimate recursive 1-component nonlinear Markov filter.

3.0 Identification of Rain Attenuation and Phase Scintillation on Communications Satellite Links Using Nonlinear Markov Filtering

When two or more random processes combine to form an overall satellite link attenuation that is measured, the advantages of using non-linear Markov filtering become apparent. Consider the case where both 1) signal attenuation due to rain and 2) signal fades and enhancements due to atmospheric and cloud scintillation can simultaneously perturb an atmospheric propagation link of a satellite communications system as shown in Figure 4. Thus, the composite fade seen at an earth terminal can be given as

$$A_{obs}(t) = A_R(t) + A_S(t) + n(t) \quad (36)$$

where $A_{obs}(t)$ is the overall observed fade, $A_R(t)$ is that due to rain, and $A_S(t)$ is that due to phase scintillation.

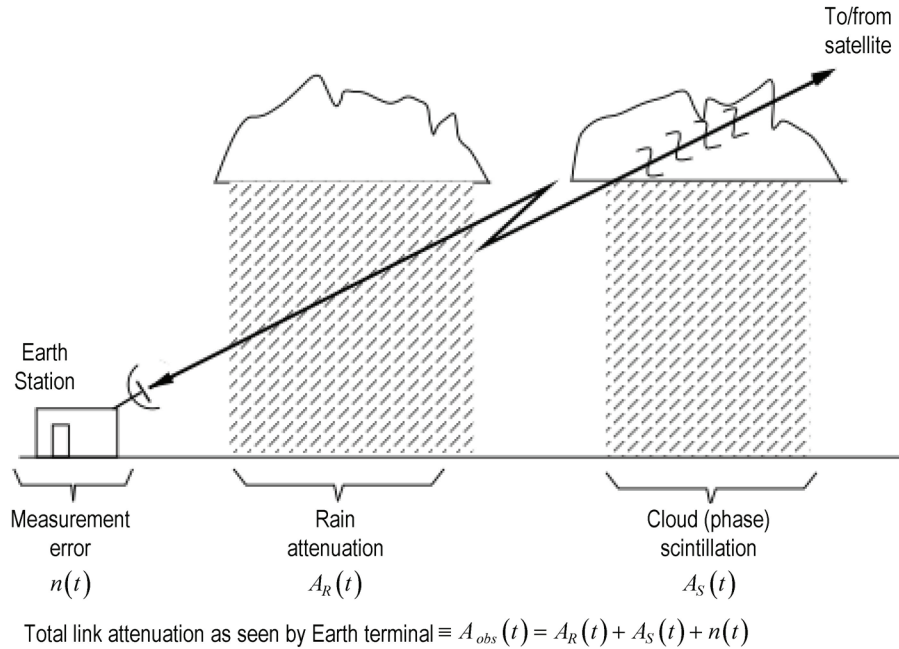


Figure 4.—Total link attenuation as seen by Earth terminal.

Scintillation is characterized by rapid (on the order of Hertz) fades and enhancements due to dynamic phase perturbations of the signal wave front propagation through the atmosphere. Scintillation can be caused by clear-air turbulence as well as scattering through clouds. Thus, unlike the rain attenuation process, one must also account for signal enhancements, precluding a statistical description using a log-normal PDF. For purposes of this work, the scintillation process will be taken to be an exponential PDF that will allow both enhancements and degradations. Thus, the statistical process to be adopted here will be of the same structure as that of rain attenuation used above, with the exception that instead of the process being related to the associated fade as given by Equation (17), the scintillation process $x_S(t)$ is connected to the signal fades and enhancements $A_S(t)$ by

$$x_S(t) = \frac{A_S(t) - A_{mS}}{\sigma_S} \quad (37)$$

where A_{mS} is the mean level of scintillation when it occurs and σ_S is the variance of fluctuations. It remains to verify this approach and obtain the governing dynamical parameters as was done for rain attenuation and this will form the subject of a future study.

Thus, the model of phase scintillation is given by

$$A_S(t) = A_{m,S} + \sigma_S x_S(t) \quad (38)$$

The associated process model (representing the function $x_S = f_S(a_S, b_S, \dots)$ used earlier in the comparison with Fourier filtering) is taken to be given by first order stochastic differential equation

$$\frac{dx_S}{dt} = -\gamma_S x_S + \sqrt{2\gamma_S} \xi_S(t) \quad (39)$$

where the corresponding random function $\xi_S(t)$ is defined by

$$\langle \xi_S(t) \rangle = 0, \langle \xi_S(t) \xi_S(t + \tau) \rangle = \delta(\tau) \quad (40)$$

A rain attenuation/phase scintillation correlation must also be allowed for; after all, since clouds can be the source of scintillation and clouds are present for rain, a correlation may prevail between the two processes sources. This is simply introduced into the model by requiring

$$\langle \xi_R(t) \xi_S(t + \tau) \rangle = B_{RS} \delta(\tau) \quad (41)$$

where B_{RS} is the correlation coefficient. The temporal coefficient γ_S can be determined by the same considerations that were used to determine that for the rain attenuation (Refs. 1 and 3); since the dynamics associated with scintillation are much faster than with rain attenuation, one can expect to have $1/\gamma_S \sim 1$ sec, i.e., $\gamma_S \sim 60 \text{ min}^{-1}$. This is vastly different from that for rain attenuation where $\gamma_R \approx 0.0538 \text{ min}^{-1}$. However, as previously mentioned, a further study will be required to determine the model parameter values A_{ms} , σ_S , γ_S , and B_{RS} . Equation (36) now becomes

$$A_{obs}(t) = A_R(x_R, t) + A_S(x_S, t) + n(t) \equiv A(\bar{x}, t) + n(t) \quad (42)$$

The governing Langevin equation is now given by the vector relationship

$$\frac{\partial \bar{x}}{\partial t} = -\bar{\Gamma} \bar{x} + \bar{G} \bar{\xi}(t), \quad \bar{x} = (x_R, x_S)^T \quad (43)$$

where

$$\bar{\Gamma} \equiv \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix}, \quad \bar{G} \equiv \begin{pmatrix} \sqrt{2\gamma_R} & 0 \\ 0 & \sqrt{2\gamma_S} \end{pmatrix}, \quad \bar{\xi}(t) \equiv \begin{pmatrix} \xi_R(t) \\ \xi_S(t) \end{pmatrix} \quad (44)$$

Hidden in the matrix $\bar{\xi}(t)$ of driving noise values, whose individual components satisfy

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_i(t') \rangle = \delta(t - t'), \quad i = R, S \quad (45)$$

is the cross-correlation of the noise components

$$\langle \xi_R(t) \xi_S(t') \rangle = B_{RS} \delta(t - t') \quad (46)$$

where $B_{RS} = B_{SR}$ is the corresponding correlation coefficient. This circumstance is introduced into the problem by defining the overall statistics governing the matrix $\bar{\xi}(t)$ by

$$\langle \bar{\xi}(t) \bar{\xi}^T(t') \rangle = \bar{B} \delta(t - t') \quad (47)$$

where \bar{B} is the process correlation coefficient which enters into the diffusion matrix given by Equation (A47), where for this example,

$$\bar{K}_2 = 2 \begin{pmatrix} \gamma_R & B_{RS} \sqrt{\gamma_R \gamma_S} \\ B_{SR} \sqrt{\gamma_R \gamma_S} & \gamma_S \end{pmatrix} \quad (48)$$

The fundamental transition matrix is

$$\bar{\Phi}(\Delta t) = \begin{pmatrix} \exp(-\gamma_R \Delta t) & 0 \\ 0 & \exp(-\gamma_S \Delta t) \end{pmatrix} \quad (49)$$

Using Equation (42) in Equation (C3b) and this intermediate result in Equation (C45) yields

$$\bar{a}(t) = \begin{pmatrix} a_R(t) \\ a_S(t) \end{pmatrix} \quad (50)$$

where

$$a_R(t) \equiv \left. \frac{\partial \phi(\bar{x}, t)}{\partial x_R} \right|_{\bar{x}=\langle \bar{x} \rangle} = \frac{\sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle) \Xi(\langle \bar{x} \rangle) \Delta t \quad (51)$$

and

$$a_S(t) \equiv \left. \frac{\partial \phi(\bar{x}, t)}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = \frac{\sigma_S}{\sigma_n^2} \Xi(\langle \bar{x} \rangle) \Delta t \quad (52)$$

where the associated discriminator which provides for the ‘correction term’ in the optimal estimation process is now given by $\Xi(\langle \bar{x} \rangle) \equiv A_{obs}(t) - A_R(\langle x_R \rangle) - A_S(\langle x_S \rangle)$. Similarly, using Equation (42) in Equation (C46), through the use of Equation (C3b), results in

$$\bar{b}(t) = \frac{\partial}{\partial \bar{x}} \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T = \begin{pmatrix} \partial / \partial x_R \\ \partial / \partial x_S \end{pmatrix} (a_R(\bar{x}, t) \quad a_S(\bar{x}, t)) \quad (53)$$

Thus,

$$b_{11}(\langle \bar{x} \rangle) = \left. \frac{\partial a_R(\bar{x})}{\partial x_R} \right|_{\bar{x}=\langle \bar{x} \rangle} = \frac{\sigma_{\ln A}^2}{\sigma_n^2} A_R(\langle x_R \rangle) (\Xi(\langle \bar{x} \rangle) - A_R(\langle x_R \rangle)) \Delta t \quad (54)$$

$$b_{12}(\langle \bar{x} \rangle) = \left. \frac{\partial a_S(\bar{x})}{\partial x_R} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle) \Delta t \quad (55)$$

$$b_{21}(\langle \bar{x} \rangle) = \left. \frac{\partial a_R(\bar{x})}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle) \Delta t \quad (56)$$

$$b_{22}(\langle \bar{x} \rangle) = \left. \frac{\partial a_S(\bar{x})}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S^2}{\sigma_n^2} \Delta t \quad (57)$$

Finally, using the two-dimensional forms of Equations (C47), (C48), (C42) and (C43) to obtain the recursive filter for rain attenuation and scintillation gives

Optimal Estimate:

$$\langle \bar{x}(t_{i+1}) \rangle = \bar{\Phi}(\Delta t) \langle \bar{x}(t_i) \rangle + \bar{\Phi}(\Delta t) \bar{\sigma}(t_i) \bar{a}(t_i) \quad (58)$$

Error Covariance of Optimal Estimate:

$$\bar{\sigma}(t_{i+1}) = \left(\bar{H}^{-1}(t_i) - \bar{b}(t_i) \right)^{-1} \quad (59)$$

Error Covariance of Extrapolated Estimate:

$$\bar{H}(t_i) = \bar{\Phi}(\Delta t) \bar{\sigma}(t_{i-1}) \bar{\Phi}^T(\Delta t) + \bar{D}(\Delta t) \quad (60)$$

Error Covariance Due to Diffusion:

$$\bar{D}(\Delta t) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \quad (61)$$

where

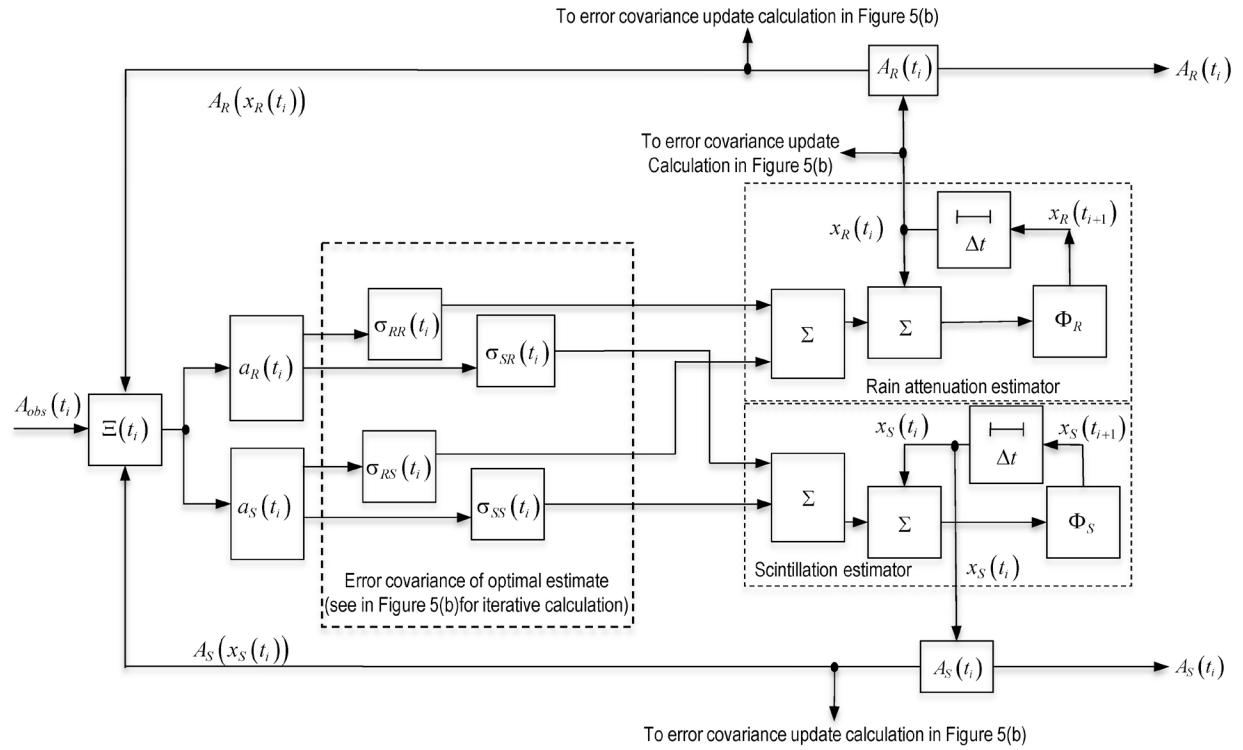
$$D_{11} \equiv \exp(-2\gamma_R \Delta t) (1 - \exp(-2\gamma_R \Delta t)) \quad (62)$$

$$D_{12} \equiv 2B_{RS} \frac{\sqrt{\gamma_R \gamma_S}}{\gamma_R + \gamma_S} \exp(-(\gamma_R + \gamma_S) \Delta t) (1 - \exp(-(\gamma_R + \gamma_S) \Delta t)) \quad (63)$$

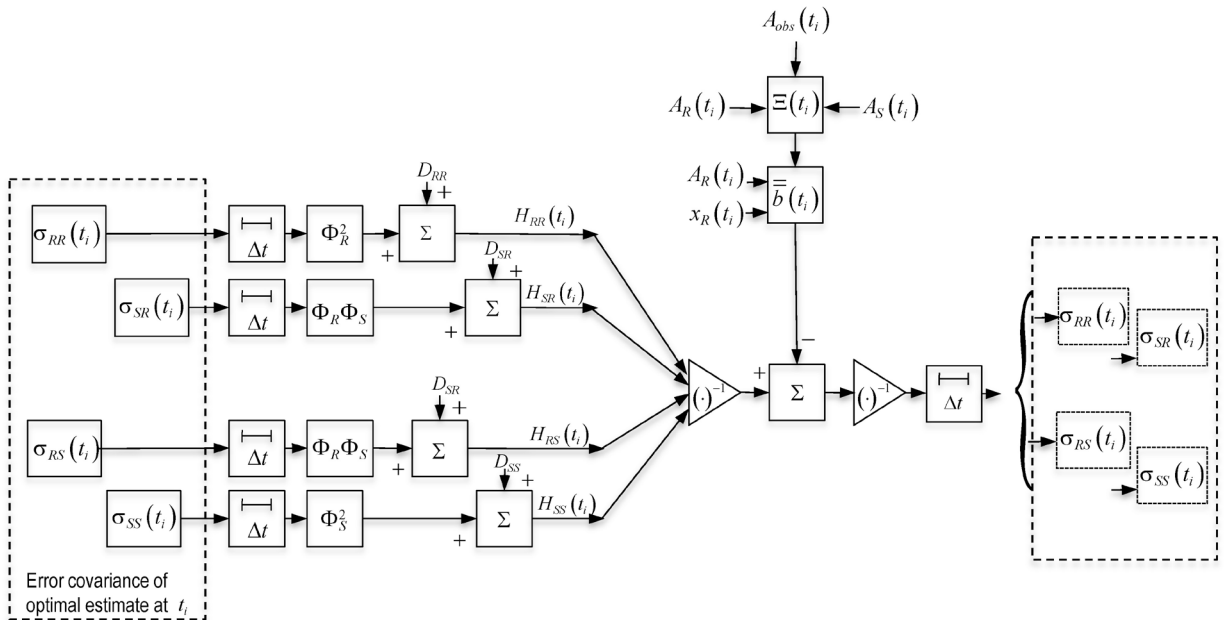
$$D_{21} \equiv 2B_{SR} \frac{\sqrt{\gamma_R \gamma_S}}{\gamma_R + \gamma_S} \exp(-(\gamma_R + \gamma_S) \Delta t) (1 - \exp(-(\gamma_R + \gamma_S) \Delta t)) \quad (64)$$

$$D_{22} \equiv \exp(-2\gamma_S \Delta t) (1 - \exp(-2\gamma_S \Delta t)) \quad (65)$$

The flow-chart of the algorithmic implementation of this iterative process is displayed in Figure 5(a) and (b). Here, $\mathbf{D}_{RR} \equiv D_{11}$, $\mathbf{D}_{RS} \equiv D_{12}$, $\mathbf{D}_{SR} \equiv D_{21}$, and $\mathbf{D}_{SS} \equiv D_{22}$. Also, $\Phi_R \equiv \exp(-\gamma_R \Delta t)$ and $\Phi_S \equiv \exp(-\gamma_S \Delta t)$. Figure 5 shows how complicated the filter implementation becomes by just adding an additional Markov component.



(a)



(b)

Figure 5.—(a) Optimal estimate recursive 2-component non-linear Markov filter. (b) Error covariance iteration for 2-component non-linear Markov filter.

4.0 The Filtering of Rain Attenuation, Scintillation and the Adaptive Determination of the Value of Average Attenuation Due to Rain

The methodology afforded by the present approach to the solution of the Stratonovich equation is also flexible enough to allow for the adaptive adjustment of the constant parameters that enter into the attenuation and scintillation models. The variation that is admitted to these parameters is essentially taken to also be governed by a Langevin-type process. In the example to follow, the case of the adaptive adjustment of the input parameter A_{mR} , i.e., the mean attenuation expected to be seen on the satellite link, will be considered. Such a potential adjustment would be required if, for example, an extreme climatological variation such as a drought that affected the rain rate were to occur.

Here, the observation model is the same as the case of Equation (42) but with an additional Markov process added to the rain attenuation model of Equation (16), i.e., the process x_{mR} that describes potential deviations from the assigned average attenuation due to rain

$$A_R(x_R) \rightarrow A_R(x_R, x_{mR}) = (A_{mR} + x_{mR}) \exp(\sigma_{\ln A} x_R(t)) \quad (66)$$

The rain attenuation model now has two statistical processes that determine its behavior, i.e., x_R and the additional process x_{mR} that adjusts the corrections to the mean value parameter A_{mR} if deemed necessary by the statistical behavior of the measurement of the composite process $A_{obs}(t)$.

Substituting Equation (66) into Equation (C3b) thus gives

$$\phi(\bar{x}, t) \equiv -\frac{1}{2\sigma_n^2} (A_{obs}(t) - A_R(x_R, x_{mR}, t) - A_S(x_S, t))^2. \quad (67)$$

The correction term x_{mR} for the parameter A_{mR} is such that the climatic process responsible for such a variation occurs over the course of several months or years. Hence, the corresponding drift coefficient γ_{mR} in the single component Langevin equation describing such a process, viz,

$$\frac{\partial x_{mR}}{\partial t} = -\gamma_{mR} x_{mR} + \sqrt{2\gamma_{mR}} \xi(t) \quad (68)$$

will be very small. This is similar to that of the rain attenuation component where, in many applications, $\gamma_R \Delta t \ll 1$ and one can neglect the temporal variation. Although the rain attenuation component can be characterized by $\gamma_R \Delta t \ll 1$, the drift coefficient γ_R is small but not zero; it is $\gamma_R \approx 0.0538 \text{ min}^{-1} = 9.0 \times 10^{-4} \text{ sec}^{-1}$. However, over the period of a climatological variation in the rainrate process that ultimately determines A_{mR} (Refs. 1 and 2) (which is typically over a period of a year), one will have $\gamma_R \approx 1.9 \times 10^{-6} \text{ min}^{-1} = 3.1 \times 10^{-8} \text{ sec}^{-1}$. Thus, the process x_{mR} is expected to operate very slowly relative to the others that prevail. Hence, letting $\gamma_{mR} \rightarrow 0$, Equation (68) becomes

$$\frac{\partial x_{mR}}{\partial t} = 0 \quad (69)$$

The corresponding Langevin equation for the entire composite process $\bar{x} = (x_R, x_S, x_{mR})^T$ is now given by

$$\frac{\partial \bar{x}}{\partial t} = -\bar{\Gamma} \bar{x} + \bar{G} \bar{\xi}(t) \quad (70)$$

with

$$\bar{\Gamma} \equiv \begin{pmatrix} \gamma_R & 0 & 0 \\ 0 & \gamma_S & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{G} \equiv \begin{pmatrix} \sqrt{2\gamma_R} & 0 & 0 \\ 0 & \sqrt{2\gamma_S} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{\xi}(t) \equiv \begin{pmatrix} \xi_R(t) \\ \xi_S(t) \\ 0 \end{pmatrix} \quad (71a)$$

Also, one now has a different form for the correlation matrix \bar{B} ,

$$\langle \bar{\xi}(t) \bar{\xi}^T(t') \rangle = \bar{B} \delta(t-t'), \quad \bar{B} = \begin{pmatrix} 1 & B_{RS} & 0 \\ B_{SR} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (71b)$$

thus giving

$$\bar{K}_2 = \bar{G} \bar{B} \bar{G}^T = 2 \begin{pmatrix} \gamma_R & B_{RS} \sqrt{\gamma_R \gamma_S} & 0 \\ B_{SR} \sqrt{\gamma_R \gamma_S} & \gamma_S & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (72)$$

The fundamental transition matrix is given in this case by

$$\bar{\Phi}(\Delta t) = \begin{pmatrix} \exp(-\gamma_R \Delta t) & 0 & 0 \\ 0 & \exp(-\gamma_S \Delta t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (73)$$

It is now required to obtain expressions for the now three-dimensional column matrix $\bar{a}(t)$ given by (see Eq. (C45))

$$\bar{a}(t) = \begin{pmatrix} a_R(t) \\ a_S(t) \\ a_{m,R}(t) \end{pmatrix} \quad (74)$$

$$a_R(t) = \frac{\partial \phi(\bar{x}, t)}{\partial x_R}, \quad a_S(t) = \frac{\partial \phi(\bar{x}, t)}{\partial x_S}, \quad a_{m,R}(t) = \frac{\partial \phi(\bar{x}, t)}{\partial x_{m,R}} \quad (75)$$

Using Equation (67) in Equation (75) yields for these matrix components

$$a_R(t) = \frac{\sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle, \langle x_{mR} \rangle) \Xi(\langle \bar{x} \rangle) \Delta t \quad (76)$$

$$a_S(t) = \frac{\sigma_S}{\sigma_n^2} \Xi(\langle \bar{x} \rangle) \Delta t \quad (77)$$

$$a_{m,R}(t) = \frac{\sigma_S}{\sigma_n^2} \exp(\sigma_{\ln A} \langle x_R \rangle) \Xi(\langle \bar{x} \rangle) \Delta t \quad (78)$$

In this example, the discriminator that drives the ‘corrector’ portion of the overall Markov filter is now given by

$$\Xi(\langle \bar{x} \rangle) \equiv A_{obs}(t) - A_R(\langle x_R \rangle, \langle x_{mR} \rangle) - A_S(\langle x_S \rangle) \quad (79)$$

Finally, using Equation (67) in Equation (C46) gives

$$\bar{b}(t) = \frac{\partial}{\partial \bar{x}} \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T = \begin{pmatrix} \frac{\partial}{\partial x_R} \\ \frac{\partial}{\partial x_S} \\ \frac{\partial}{\partial x_{m,R}} \end{pmatrix} (a_R(\bar{x}, t) \quad a_S(\bar{x}, t) \quad a_{m,R}(\bar{x}, t)) \quad (80)$$

Thus,

$$b_{11}(\langle \bar{x} \rangle) = \left. \frac{\partial a_R(\bar{x})}{\partial x_R} \right|_{\bar{x}=\langle \bar{x} \rangle} = \frac{\sigma_{\ln A}^2}{\sigma_n^2} A_R(\langle x_R \rangle, \langle x_{mR} \rangle) (\Xi(\langle \bar{x} \rangle) - A_R(\langle x_R \rangle, \langle x_{mR} \rangle)) \Delta t \quad (81)$$

$$b_{12}(\langle \bar{x} \rangle) = \left. \frac{\partial a_S(\bar{x})}{\partial x_R} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle, \langle x_{mR} \rangle) \Delta t \quad (82)$$

$$b_{13}(\langle \bar{x} \rangle) = \left. \frac{\partial a_{m,R}(\bar{x})}{\partial x_R} \right|_{\bar{x}=\langle \bar{x} \rangle} = \frac{\sigma_{\ln A}}{\sigma_n^2} \exp(\sigma_{\ln A} \langle x_R \rangle) (\Xi(\langle \bar{x} \rangle) - A_R(\langle x_R \rangle, \langle x_{mR} \rangle)) \Delta t \quad (83)$$

$$b_{21}(\langle \bar{x} \rangle) = \left. \frac{\partial a_R(\bar{x})}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle, \langle x_{mR} \rangle) \Delta t \quad (84)$$

$$b_{22}(\langle \bar{x} \rangle) = \left. \frac{\partial a_S(\bar{x})}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S^2}{\sigma_n^2} \Delta t \quad (85)$$

$$b_{23}(\langle \bar{x} \rangle) = \left. \frac{\partial a_{m,R}(\bar{x})}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S}{\sigma_n^2} \exp(\sigma_{\ln A} \langle x_R \rangle) \Delta t \quad (86)$$

$$b_{31}(\langle \bar{x} \rangle) = \left. \frac{\partial a_R(\bar{x})}{\partial x_{mR}} \right|_{\bar{x}=\langle \bar{x} \rangle} = \frac{\sigma_{\ln A}}{\sigma_n^2} \exp(\sigma_{\ln A} \langle x_R \rangle) (\Xi(\langle \bar{x} \rangle) - A_R(\langle x_R \rangle, \langle x_{mR} \rangle)) \Delta t \quad (87)$$

$$b_{21}(\langle \bar{x} \rangle) = \left. \frac{\partial a_R(\bar{x})}{\partial x_S} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle, \langle x_{mR} \rangle) \Delta t \quad (88)$$

$$b_{33}(\langle \bar{x} \rangle) = \left. \frac{\partial a_{m,R}(\bar{x})}{\partial x_{mR}} \right|_{\bar{x}=\langle \bar{x} \rangle} = -\frac{1}{\sigma_n^2} \exp(2\sigma_{\ln A} \langle x_R \rangle) \Delta t \quad (89)$$

The error covariance due to diffusion $\bar{D}(\Delta t)$ is different than the general form given by Equations (C49) due to the fact that there is no diffusion term, nor noise correlation, associated with the process x_{mR} , i.e., $B_{m,Rj} = 0$ for $j = R, S, m, R$. Thus, going back to the general Equation (C43) gives

$$\bar{D}(\Delta t) = \begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (90)$$

where the non-zero elements of this matrix are given by Equations (62)-(65).

Application of the iterative Markov filtering equations now yields the form that is isomorphic to that of Equations (58)-(61). However, for the present example, some of the filtering equations will be written out in individual component form so their overall morphology can be seen. The expressions for the coefficients of the matrix \bar{a} will also be rewritten to highlight the role of the correction discriminator.

Optimal Estimate:

$$\langle x_R(t_{i+1}) \rangle = \langle x_R(t_i) \rangle \exp(-\gamma_R \Delta t) + \exp(-\gamma_R \Delta t) \sum_{j=R,S,mR} \sigma_{Rj} c_j \Xi(\langle x_R(t_i) \rangle, \langle x_S(t_i) \rangle, \langle x_{mR}(t_i) \rangle) \quad (91)$$

$$\langle x_S(t_{i+1}) \rangle = \langle x_S(t_i) \rangle \exp(-\gamma_S \Delta t) + \exp(-\gamma_S \Delta t) \sum_{j=R,S,mR} \sigma_{Sj} c_j \Xi(\langle x_R(t_i) \rangle, \langle x_S(t_i) \rangle, \langle x_{mR}(t_i) \rangle) \quad (92)$$

$$\langle x_{mR}(t_{i+1}) \rangle = \langle x_{mR}(t_i) \rangle + \sum_{j=R,S,mR} \sigma_{mRj} c_j \Xi(\langle x_R(t_i) \rangle, \langle x_S(t_i) \rangle, \langle x_{mR}(t_i) \rangle) \quad (93)$$

with

$$\Xi(\langle \bar{x} \rangle) \equiv A_{obs}(t) - (A_{mR} + \langle x_{mR} \rangle) \exp(\sigma_{\ln A} \langle x_R(t) \rangle) - A_{mS} - \sigma_S \langle x_S \rangle \quad (94)$$

where, from Equations (76)-(78), the coefficients c_j multiplying the discriminator corrector factor are given by

$$c_R \equiv \frac{\sigma_{\ln A}}{\sigma_n^2} A_R(\langle x_R \rangle, \langle x_{mR} \rangle) \Delta t, \quad c_S \equiv \frac{\sigma_S}{\sigma_n^2} \Delta t, \quad c_{mR} \equiv \frac{\sigma_S}{\sigma_n^2} \exp(\sigma_{\ln A} \langle x_R \rangle) \Delta t \quad (95)$$

Error Covariance of Optimal Estimate:

$$\bar{\sigma}(t_{i+1}) = \left(\bar{H}^{-1}(t_i) - \bar{b}(t_i) \right)^{-1} \quad (96)$$

Here, the correction discriminator $\Xi(\langle \bar{x}(t_i) \rangle)$ also comes in through the values of the components of $\bar{b}(t_i)$.

Error Covariance of Extrapolated Estimate:

$$\bar{H}(t_i) = \bar{\Phi}(\Delta t) \bar{\sigma}(t_{i-1}) \bar{\Phi}^T(\Delta t) + \bar{D}(\Delta t) = \begin{pmatrix} H_{RR}(t_i) & H_{RS}(t_i) & H_{RmR}(t_i) \\ H_{SR}(t_i) & H_{SS}(t_i) & H_{SmR}(t_i) \\ H_{mRR}(t_i) & H_{mRS}(t_i) & H_{mRmR}(t_i) \end{pmatrix} \quad (97)$$

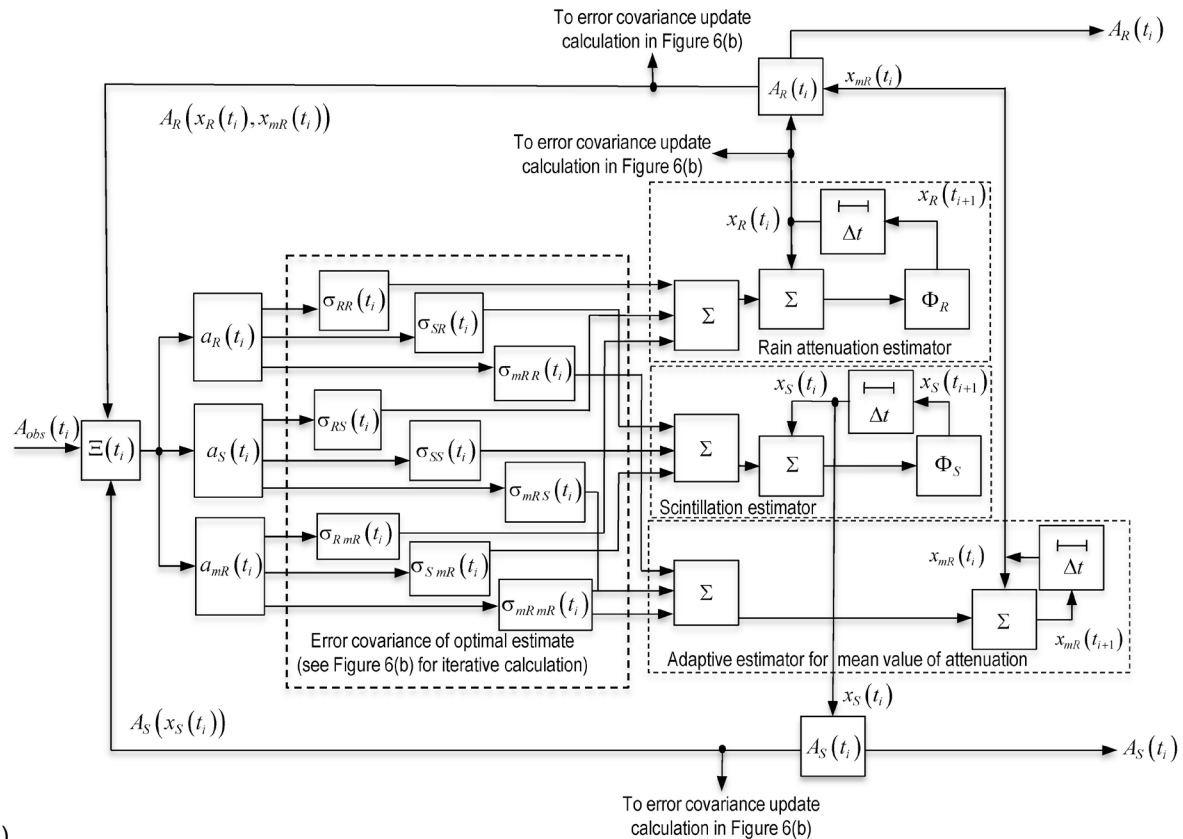
where

$$\begin{aligned} H_{RR}(t_i) &= \sigma_{RR}(t_{i-1}) \exp(-2\gamma_R \Delta t) + \exp(-2\gamma_R \Delta t) (1 - \exp(-2\gamma_R \Delta t)) \\ H_{RS}(t_i) &= \sigma_{RS}(t_{i-1}) \exp(-(\gamma_R + \gamma_S) \Delta t) \\ &\quad + 2B_{RS} \frac{\sqrt{\gamma_R \gamma_S}}{\gamma_R + \gamma_S} \exp(-(\gamma_R + \gamma_S) \Delta t) (1 - \exp(-(\gamma_R + \gamma_S) \Delta t)) \\ H_{RmR}(t_i) &= \sigma_{RmR}(t_{i-1}) \exp(-\gamma_R \Delta t) \\ H_{SR}(t_i) &= \sigma_{SR}(t_{i-1}) \exp(-(\gamma_R + \gamma_S) \Delta t) \\ &\quad + 2B_{SR} \frac{\sqrt{\gamma_R \gamma_S}}{\gamma_R + \gamma_S} \exp(-(\gamma_R + \gamma_S) \Delta t) (1 - \exp(-(\gamma_R + \gamma_S) \Delta t)) \\ H_{SS}(t_i) &= \sigma_{SS}(t_{i-1}) \exp(-2\gamma_S \Delta t) + \exp(-2\gamma_S \Delta t) (1 - \exp(-2\gamma_S \Delta t)) \\ H_{SmR}(t_i) &= \sigma_{SmR}(t_{i-1}) \exp(-\gamma_S \Delta t) \\ H_{mRR}(t_i) &= \sigma_{mRR}(t_{i-1}) \exp(-\gamma_R \Delta t) \\ H_{mRS}(t_i) &= \sigma_{mRS}(t_{i-1}) \exp(-\gamma_S \Delta t) \\ H_{mRmR}(t_i) &= \sigma_{mRmR}(t_{i-1}) \end{aligned} \quad (98)$$

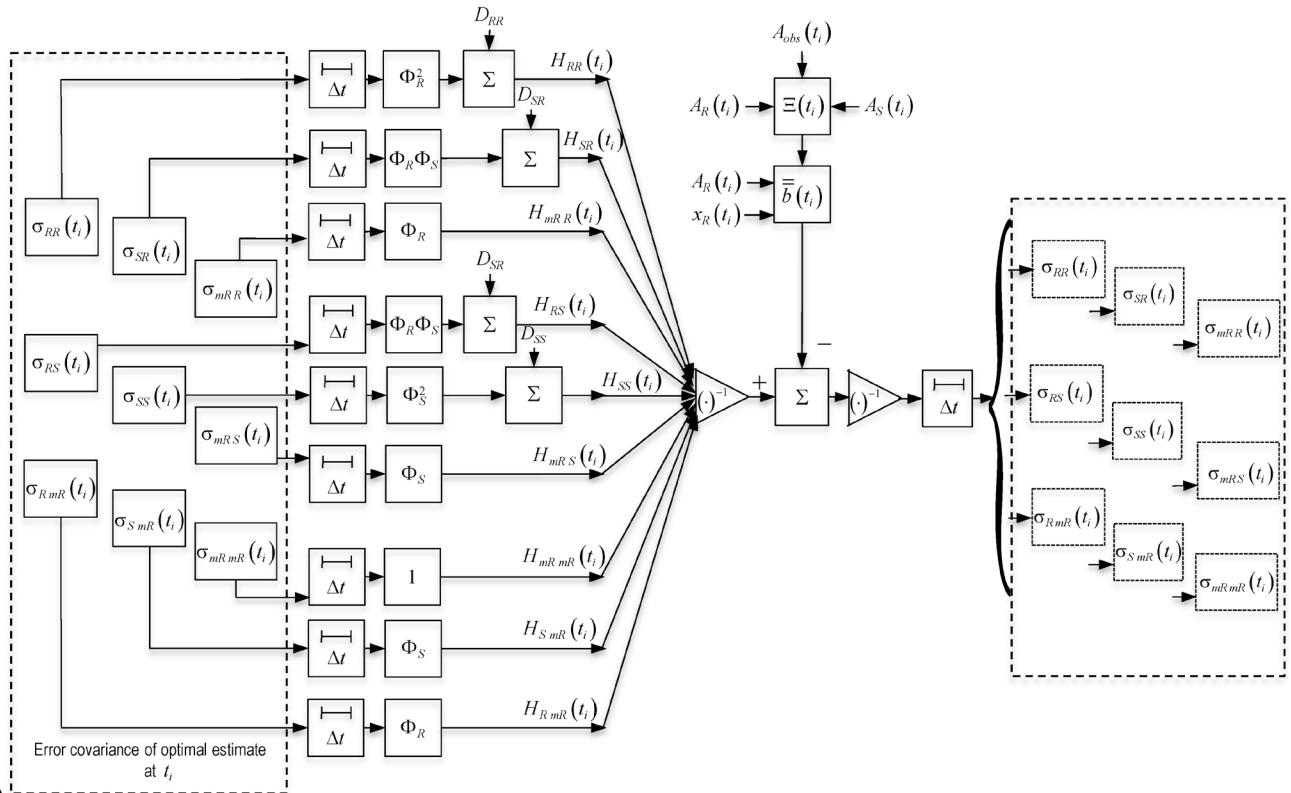
Error Covariance Due to Diffusion:

$$\bar{D}(\Delta t) = \begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (99)$$

The flowchart of the implementation of these adaptive filter equations is shown in Figure 6(a) and (b).



(a)



(b)

Figure 6.—(a) Optimal estimate recursive 3-component non-linear Markov filter. (b) Error covariance iteration for 3-component nonlinear Markov filter. (b) Error covariance iteration for 3-component nonlinear Markov filter.

5.0 Software Implementation of the Adaptive Filter Equations (91) to (99) and Their Application to Measured Satellite Link Fade

The 3-component Markov filter described in Section 4.0 will be implemented and tested with observational signal fades that include both rain attenuation and scintillation, as well as intentionally introduced uncertainty in the specification of the expected mean rain attenuation. This will form the subject to Part 2 of this work. However, time will be taken here to briefly describe the considerations that prevail for this software implementation. (Both the 1-component and 2-component Markov filters discussed above are subsets of the 3-component filter and can be easily emulated by setting appropriate coefficients to zero in the 3-component case.)

To this end, the various matrices that must be dealt with will be first introduced. With the exception of the calculation of the components comprising the observation-based quantities that enter into the $\bar{a}(t)$ and $\bar{b}(t)$ matrices, all other calculations that are required can be easily implemented using, e.g., MATLAB (The MathWorks, Inc.). This calculation procedure is recursively applied at each of the sampling times t_i separated by the sampling interval $\Delta t \equiv t_i - t_{i-1}$.

5.1 Constant Matrices

a) One first defines the Temporal Transfer Matrix $\bar{\Phi}(\Delta t)$, a 3 x 3 matrix of constants giving the temporal evolution of the Markov Processes across a fixed time interval $\Delta t \equiv t_i - t_{i-1}$ from one clock cycle t_{i-1} to the next t_i . As per Equation (73),

$$\bar{\Phi}(\Delta t) = \exp(-\bar{\Gamma} \Delta t)$$

where $\bar{\Gamma}$ is a 3 x 3 matrix of constant drift coefficients,

$$\bar{\Gamma} = \begin{pmatrix} \gamma_R & 0 & 0 \\ 0 & \gamma_S & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Typical numerical values are $\gamma_R \approx 9.0 \times 10^{-4} \text{ sec}^{-1}$ and $\gamma_S \approx 1.0 \text{ sec}^{-1}$.

b) The Matrix of Error Covariance due to Diffusion $\bar{D}(\Delta t)$, a 3 x 3 matrix of constants giving the errors defined in Equation (90), that are incurred within the filtering process due to the diffusion term within the original Langevin equation over the constant time interval Δt

$$\bar{D}(\Delta t) = \begin{pmatrix} D_{11}(\Delta t) & D_{12}(\Delta t) & 0 \\ D_{21}(\Delta t) & D_{22}(\Delta t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where, from Equations (62)-(65),

$$D_{11}(\Delta t) = \exp(-2\gamma_R \Delta t) (1 - \exp(-2\gamma_R \Delta t))$$

$$D_{12}(\Delta t) = D_{21}(\Delta t) = 2B \frac{\sqrt{\gamma_R \gamma_S}}{\gamma_R + \gamma_S} \exp(-(\gamma_R + \gamma_S)\Delta t) (1 - \exp(-(\gamma_R + \gamma_S)\Delta t))$$

$$D_{22}(\Delta t) = \exp(-2\gamma_S \Delta t) (1 - \exp(-2\gamma_S \Delta t))$$

Here, $B < 1$ is the correlation coefficient between the rain and scintillation processes. Until otherwise determined from observational data, one can take $B \approx 0.75$.

5.2 Matrices Whose Elements are Defined By and Given By Individual Equations; The Observation Model and the Filter Discriminator

a) The Matrix of the Discriminator Gradient $\bar{a}(t_i)$, a 3 x 1 matrix whose calculated elements give the gradient of the filter discriminator characteristic with respect to the process variables

$$\bar{a}(t_i) = \begin{pmatrix} a_R(t_i) \\ a_S(t_i) \\ a_{mR}(t_i) \end{pmatrix}$$

b) The Matrix of the Discriminator Second-Order Cross-Gradient $\bar{\bar{b}}(t_i)$, a 3 x 3 matrix whose calculated elements give the second-order gradient of the discriminator characteristic with respect to all possible cross-correlated process variables

$$\bar{\bar{b}}(t_i) = \begin{pmatrix} b_{11}(t_i) & b_{12}(t_i) & b_{13}(t_i) \\ b_{21}(t_i) & b_{22}(t_i) & b_{23}(t_i) \\ b_{31}(t_i) & b_{32}(t_i) & b_{33}(t_i) \end{pmatrix}$$

Before the iterative process of the Markov filter can be implemented, it is required to connect the matrix elements of $\bar{a}(t_i)$ and $\bar{\bar{b}}(t_i)$ to those of the optimal estimates given by Equations (76)-(89). This is where the discriminator exists, i.e., the comparison between 1) the predicted levels of the relevant component processes that yield a predicted observation and 2) the actual observation. ***The resulting value of this discrimination determines the required adjustments to be made by the filter during the iterative procedure at each clock sampling cycle and is the key to the entire process.*** This is where the use of the measured signal attenuation is employed.

In what is to follow, the 3 x 1 matrix of optimal estimates, i.e.,

$$\langle \bar{x}(t_i) \rangle = \begin{pmatrix} \langle x_R(t_i) \rangle \\ \langle x_S(t_i) \rangle \\ \langle x_{mR}(t_i) \rangle \end{pmatrix}$$

the components of which enter into and are iteratively calculated by the filter Equations (91)-(93), will simply be denoted by

$$\bar{x}(t_i) = \begin{pmatrix} x_1(t_i) \\ x_2(t_i) \\ x_3(t_i) \end{pmatrix}$$

for notational convenience. Thus, from Equation (66), the two optimal estimate components $x_1(t_i)$ and $x_3(t_i)$ go into determining the observation model for rain attenuation

$$A_R(t_i) = (A_{mR} + x_3(t_i)) \exp(\sigma_{\ln A} x_1(t_i))$$

where A_{mR} and $\sigma_{\ln A}$ are attenuation constants, due to the rain process, that are specific to the geographical location as well as the frequency of operation of the satellite link. Similarly for scintillation, it is determined by the component $x_2(t_i)$ and the associated attenuation is modeled by Equation (38), viz,

$$A_S(t_i) = A_{mS} + \sigma_S x_2(t_i)$$

where A_{mS} and σ_S are the constants of the scintillation model. (The numerical values of these constants are, at this point, unknown but can easily be estimated and will be done so in Part 2.) These two process models finally enter into the discriminator characteristic defined by Equation (67)

$$\phi(t) = -\frac{1}{2\sigma_n^2} \Xi^2(t), \quad \Xi(t) = A_{obs}(t) - A_R(t) - A_S(t)$$

where $\Xi(t)$ is the discriminator and σ_n^2 is the variance of the measurement uncertainty. Following the prescription given by Equations (74)-(78),

$$a_1(t_i) = \frac{\sigma_{\ln A}}{\sigma_n^2} A_R(x_1(t_i), x_3(t_i)) \Xi(t_i) \Delta t$$

$$a_2(t_i) = \frac{\sigma_S}{\sigma_n^2} \Xi(t_i) \Delta t$$

$$a_3(t_i) = \frac{\sigma_S}{\sigma_n^2} \exp(\sigma_{\ln A} x_1(t_i)) \Xi(t_i) \Delta t$$

Similarly, from Equations (80)-(89),

$$b_{11}(t_i) = \frac{\sigma_{\ln A}^2}{\sigma_n^2} A_R(x_1(t_i), x_3(t_i)) (\Xi(t_i) - A_R(x_1(t_i), x_3(t_i))) \Delta t$$

$$b_{12}(t_i) = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(x_1(t_i), x_3(t_i)) \Delta t$$

$$b_{13}(t_i) = \frac{\sigma_{\ln A}}{\sigma_n^2} \exp(\sigma_{\ln A} x_1(t_i)) (\Xi(t_i) - A_R(x_1(t_i), x_3(t_i))) \Delta t$$

$$b_{21}(t_i) = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(x_1(t_i), x_3(t_i)) \Delta t$$

$$b_{22}(t_i) = -\frac{\sigma_S^2}{\sigma_n^2} \Delta t$$

$$b_{23}(t_i) = -\frac{\sigma_S}{\sigma_n^2} \exp(\sigma_{\ln A} x_1(t_i)) \Delta t$$

$$b_{31}(t_i) = \frac{\sigma_{\ln A}}{\sigma_n^2} \exp(\sigma_{\ln A} x_1(t_i)) (\Xi(t_i) - A_R(x_1(t_i), x_3(t_i))) \Delta t$$

$$b_{32}(t_i) = -\frac{\sigma_S \sigma_{\ln A}}{\sigma_n^2} A_R(x_1(t_i), x_3(t_i)) \Delta t$$

$$b_{33}(t_i) = -\frac{1}{\sigma_n^2} \exp(2\sigma_{\ln A} x_1(t_i)) \Delta t$$

Remember, the constants A_{mR} , A_{mS} , $\sigma_{\ln A}$, σ_S , and σ_n^2 and, of course, the actual measured data $A_{obs}(t)$ are all given inputs to the calculation procedure. These matrix element expressions must be specifically coded and assigned to the appropriate matrix. The corresponding flow-chart for the calculations of these matrix elements at the sample time t_i is shown below.

5.3 Matrices Whose Elements are Directly Calculated by the Recursive Filter Equations

a) The Matrix of Optimal Estimates $\bar{x}(t_i)$, a 3 x 1 matrix whose calculated elements are the optimal estimates at the clock sample time t_i from the filter of the rain and scintillation processes as well as the adaptively estimated mean value of the rain attenuation process has already been introduced above, i.e.,

$$\bar{x}(t_i) = \begin{pmatrix} x_1(t_i) \\ x_2(t_i) \\ x_3(t_i) \end{pmatrix}$$

b) The Matrix of Error Covariance of the Optimal Estimate $\bar{\sigma}(t_i)$, a 3 x 3 matrix whose calculated elements are the errors associated with the optimal estimates at the i -th clock cycle. Its elements are defined by

$$\bar{\sigma}(t_i) = \begin{pmatrix} \sigma_{11}(t_i) & \sigma_{12}(t_i) & \sigma_{13}(t_i) \\ \sigma_{21}(t_i) & \sigma_{22}(t_i) & \sigma_{23}(t_i) \\ \sigma_{31}(t_i) & \sigma_{32}(t_i) & \sigma_{33}(t_i) \end{pmatrix}$$

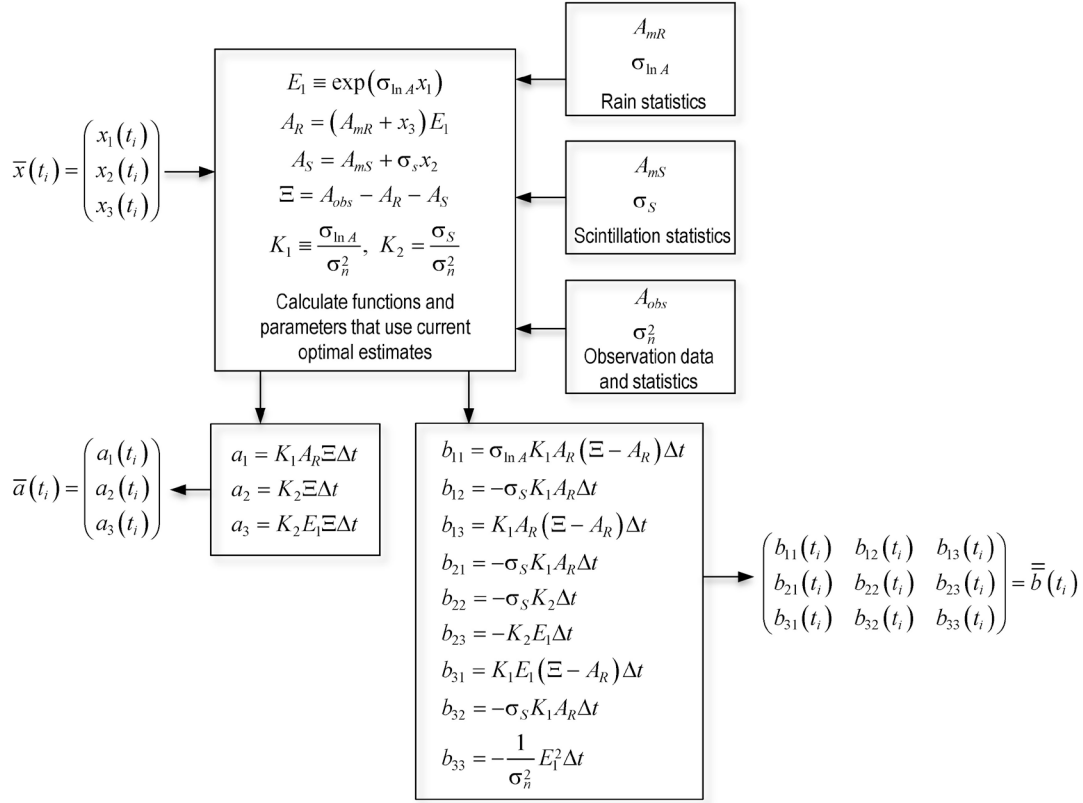


Figure 7.—Determination of the matrix components for $\bar{a}(t_i)$ and $\bar{\bar{b}}(t_i)$ using the optimal estimates for $\bar{x}(t_i)$ and the observational measurement $A_{obs}(t_i)$.

c) The Matrix of Error Covariance of Extrapolated Estimate $\bar{\bar{H}}(t_i)$, a 3 x 3 matrix whose calculated elements are the errors associated with the extrapolated estimates at the i -th clock cycle. Its elements are defined by

$$\bar{\bar{H}}(t_i) = \begin{pmatrix} H_{11}(t_i) & H_{12}(t_i) & H_{13}(t_i) \\ H_{21}(t_i) & H_{22}(t_i) & H_{23}(t_i) \\ H_{31}(t_i) & H_{32}(t_i) & H_{33}(t_i) \end{pmatrix}$$

The values of the components for the matrices $\bar{\bar{\sigma}}(t_{i+1})$ and $\bar{\bar{H}}(t_{i+1})$ are iteratively calculated from previous values $\bar{\bar{\sigma}}(t_i)$ and $\bar{\bar{H}}(t_i)$, as well as those of $\bar{x}(t_{i+1})$ and $\bar{x}(t_i)$, by application of the filter equations given by Equations (91)-(99), i.e.,

$$\bar{x}(t_{i+1}) = \bar{\Phi}(\Delta t) \bar{x}(t_i) + \bar{\Phi}(\Delta t) \bar{\bar{\sigma}}(t_i) \bar{a}(t_i)$$

$$\bar{\bar{\sigma}}(t_{i+1}) = \left(\bar{\bar{H}}^{-1}(t_i) - \bar{\bar{b}}(t_i) \right)^{-1}$$

$$\bar{\bar{H}}(t_{i+1}) = \bar{\bar{\Phi}}(\Delta t) \bar{\bar{\sigma}}(t_i) \bar{\bar{\Phi}}^T(\Delta t) + \bar{\bar{D}}(\Delta t)$$

which are easily implemented in MATLAB.

Part 2 of this work will use this procedure to evaluate the performance of the various filters that emanate from the general 3-component case using a database of the temporal evolution of signal fades on a satellite downlink to effectively separate the various fade components.

6.0 Conclusion

The integro-differential Stratonovich equation giving the observation driven temporal PDF of a multi-component Markov process was solved to obtain optimal estimates and associated error covariance's of the individual Markov components. The iterative discrete-time formulation of this general solution was then obtained. This was applied to a fading satellite communications link in which the overall fade is due to two separate fade mechanisms that are assumed to prevail, i.e., attenuation due to rain and phase scintillation. Using physics-based observation models, iterative algorithms are derived to isolate these two fade processes. It was then shown how the two-component Markov filter can be modified to be made adaptive, i.e., allowed to adjust for variations about the mean values originally assigned to the filter coefficients.

The specific filters presented here, as well as others with some variations of their observation and/or process models, will be the subject of a further investigation to follow in Part 2. Here, the various filters that can be derived will be applied to and tested with noisy measured time-series of signal levels of observed on a fading satellite link. The 'various filters' will comprise filter constructions other than those presented here, by using many variations of the theme presented here. For example, the scintillation model required to construct the scintillation portion of the filter may turn out to be of a different form than given by Equation (38). Or the model used for the adaptive adjustment of A_{mR} may be much more complicated than that given by Equation (66); it may be better represented by a Weiner process based approach. Or the statistics assumed to prevail for the random driving functions of the various Langevin equation components, i.e.,

$$\langle \xi_i(t) \xi_i(t') \rangle = \delta(t - t')$$

may indeed have variances that are not equal to unity, i.e.,

$$\langle \xi_i(t) \xi_i(t') \rangle = \sigma_i^2 \delta(t - t')$$

or may not even be given by a 'white-noise' δ -correlated process, i.e.,

$$\langle \xi_i(t) \xi_i(t') \rangle = W(t - t')$$

where $W(t - t')$ can be defined by any number of statistical characterizations. The numerical analysis afforded in Part 2 will help direct these different possibilities. What has been established here, however, is the fundamental methodology to be employed in the analytical derivation of Markov filters.

Appendix A.—From the Langevin Stochastic Differential Equation for a Random Process to the Kolmogorov Equation for the Transition Probability Density Function for the Process

The purpose of this appendix is to establish the connection between the Langevin stochastic differential equation for the random drift/diffusion process vector $\bar{x}(t)$

$$\frac{\partial \bar{x}}{\partial t} = -\bar{\Gamma} \bar{x} + \bar{G} \bar{\xi}(t)$$

where

$$\bar{\Gamma} \equiv \begin{pmatrix} \gamma_R & & & \\ & \gamma_S & & \\ & & \ddots & \\ & & & \gamma_N \end{pmatrix} \quad \text{and} \quad \bar{G} \equiv \begin{pmatrix} \sqrt{2\gamma_1} & 0 & \dots & 0 \\ 0 & \sqrt{2\gamma_2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \sqrt{2\gamma_N} \end{pmatrix}$$

and the related transition PDF $p(\bar{x}, t | \bar{x}_0, t_0)$, from which the statistics of $\bar{x}(t)$ can be obtained, given by the Kolmogorov equation,

$$\frac{\partial p(\bar{x}, t | \bar{x}_0, t_0)}{\partial t} = -\sum_i^N \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}, t | \bar{x}_0, t_0) \right) + \frac{1}{2} \sum_i^N \sum_j^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}, t | \bar{x}_0, t_0) \right)$$

where

$$\bar{K}_1(\bar{x}, t) = -\bar{\Gamma} \langle \bar{x}(t) \rangle \quad \text{and} \quad (\bar{K}_2(\bar{x}, t))_{ij} = 2B_{ij} \sqrt{\gamma_i \gamma_j}$$

in which \bar{B} is the matrix whose elements are correlation coefficients connecting all the elements of the stochastic noise vector $\bar{\xi}(t)$

$$\langle \xi_i(t) \xi_j(t') \rangle = B_{ij} \delta(t - t'), \quad B_{ii} = 1$$

The derivation of the Kolmogorov equation, a differential equation specifying the PDF for a random process given by (or defined by) a Langevin equation will be given in this appendix. For simplicity, a one-dimensional process will be used in the derivation. The extension to a multi-dimensions process will then be discussed as there exist a few subtleties in considering more than one process. The material in this appendix augments and replaces the development given in Appendix A of Reference 3.

A.1 The Kolmogorov Equation Corresponding to a One-Dimensional Langevin Equation

Consider the Langevin equation for single random function (process) $x_R(t)$, such as that for the rain attenuation process, given by

$$\frac{\partial x_R}{\partial t} = \psi(x_R, t) + g(x_R, t)\xi_R(t) \quad (\text{A1})$$

where $\psi(x_R, t)$ and $g(x_R, t)$ are both deterministic functions, in general, of both $x_R(t)$ as well as the time. (In the rain attenuation case, one simply has $\psi(x_R, t) = -\gamma_R x_R$ and $g(x_R, t) = \sqrt{2\gamma_R}$ where γ_R is the temporal constant associated with the rain dynamics.) The statistics of the process given by Equation (A1) are determined by the prescription

$$\langle \xi_R(t) \rangle = 0 \text{ and } \langle \xi_R(t)\xi_R(t') \rangle = \delta(t-t') \quad (\text{A2})$$

Equation (A2) defines the process as a ‘white noise’ process.

A description of the temporal evolution of the process $x_R(t)$ as a function of time t must necessarily be done via a statistical approach due to the random function $\xi_R(t)$ driving the process through Equation (A1). Thus, the following problem can be formulated: Given that $x_R(t)$ has some given value $x_0 = x_R(t_0)$ at time $t = t_0$, what is the probability (governed by the statistics of $\xi_R(t)$) that $x_1 = x_R(t_1)$ at a later time $t = t_1$? Let $p(x_1, t_1 | x_0, t_0)$ be the conditional probability density function (PDF), which must be determined, that gives the probability $p(x_1, t_1 | x_0, t_0)dx_1$ that, given x_0 at t_0 , one finds x_R in the range $x_1 \leq x_R \leq x_1 + dx_1$ at a later time $t = t_1$. Now all the information that can be gleaned from what is known about the random process $x_R(t)$ is given by Equations (A1) and (A2). In order to extract such information, one must complicate the formulation a bit in order to form a mathematically sound problem statement. In addition to the process value $x_1 = x_R(t_1)$ at time t_1 , conditioned on $x_0 = x_R(t_0)$, one must additionally consider the similar problem of the process value $x_2 = x_R(t_2)$ at a time $t_2 = t_1 + \Delta t$ given that $x_1 = x_R(t_1)$. Here, Δt is a general time interval which will be used in the derivation to follow. Thus, one can form the product of two such conditional PDF factors, $p(x_2, t_1 + \Delta t | x_1, t_1) p(x_1, t_1 | x_0, t_0)dx_1$. This is the PDF of $x_2 = x_R(t_2) = x_R(t_1 + \Delta t)$ given that $x_1 + dx_1 = x_R(t_1)$ which itself is conditioned on $x_0 = x_R(t_0)$. Implicit in this formulation is the fact that the process values $x_R(t)$ are uncorrelated at the times t_0 , t_1 , or t_2 . That is, the process $x_R(t)$ is taken to be a Markov random process. Considering all possible values x_1 that $x_R(t)$ can assume at $t = t_1$, one can write

$$p(x_2, t_1 + \Delta t | x_0, t_0) = \int_{-\infty}^{\infty} p(x_2, t_1 + \Delta t | x_1, t_1) p(x_1, t_1 | x_0, t_0) dx_1 \quad (\text{A3})$$

which is known as the Chapman-Kolmogorov equation for the transition PDF $p(x, t | x_0, t_0)$ of the continuous Markov process $x_R(t)$.

Given the behavior of the PDF for $x_R(t)$ as described by the Chapman-Kolmogorov equation, Equation (A3), as well as the stochastic differential equation that defines the process, Equation (A1), an equation connecting $p(x, t | x_0, t_0)$ to the parameters of Equation (A1) can be obtained. In what is to follow, one will simply write x in place of x_R . To this end, consider the integral relation defined by

$$I[x_0] \equiv \int_{-\infty}^{\infty} \frac{\partial p(x, t | x_0, t_0)}{\partial t} R(x) dx \quad (\text{A4})$$

where $R(x)$ is an arbitrary function that goes to zero as $x \rightarrow +\infty$. Using the definition of the difference quotient

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = \lim_{\Delta t \rightarrow 0} \frac{p(x,t+\Delta t|x_0,t_0) - p(x,t|x_0,t_0)}{\Delta t} \quad (\text{A5})$$

as well as Equation (A3) in Equation (A4) gives

$$\frac{\partial p(x,t|x_0,t_0)}{\partial t} = \lim_{\Delta t \rightarrow 0} \frac{\int_{-\infty}^{\infty} p(x,t+\Delta t|x',t)p(x',t|x_0,t_0)dx' - p(x,t|x_0,t_0)}{\Delta t} \quad (\text{A6})$$

Using this expression in Equation (A4) yields

$$I[x_0] = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} R(x)p(x,t+\Delta t|x',t)dx \right) p(x',t|x_0,t_0)dx' - \int_{-\infty}^{\infty} p(x,t|x_0,t_0)R(x)dx \quad (\text{A7})$$

Expanding the arbitrary function into a Taylor series about x' ,

$$R(x) \approx R(x') + \frac{\partial R(x')}{\partial x'}(x-x') + \frac{\partial^2 R(x')}{\partial x'^2}(x-x')^2 + \dots \quad (\text{A8})$$

and putting this into Equation (A7) results in

$$I[x_0] = \int_{-\infty}^{\infty} \left(\frac{\partial R(x')}{\partial x'} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (x-x')p(x,t+\Delta t|x',t)dx \right. \\ \left. + \frac{1}{2} \frac{\partial^2 R(x')}{\partial x'^2} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_{-\infty}^{\infty} (x-x')^2 p(x,t+\Delta t|x',t)dx \right) p(x',t|x_0,t_0)dx' \quad (\text{A9})$$

Finally, defining the conditional mean and variance, respectively, as

$$k_1(x',t+\Delta t) \equiv \int_{-\infty}^{\infty} (x-x')p(x,t+\Delta t|x',t)dx \quad (\text{A10})$$

and

$$k_2(x',t+\Delta t) \equiv \int_{-\infty}^{\infty} (x-x')^2 p(x,t+\Delta t|x',t)dx \quad (\text{A11})$$

and letting their temporal limits be given by

$$K_1(x',t) \equiv \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} k_1(x',t+\Delta t) \quad (\text{A12})$$

and

$$K_2(x',t) \equiv \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} k_2(x',t+\Delta t) \quad (\text{A13})$$

gives

$$I[x_0] = \int_{-\infty}^{\infty} \frac{\partial R(x')}{\partial x'} K_1(x', t) p(x', t | x_0, t_0) dx' + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial R^2(x')}{\partial x'^2} K_2(x', t) p(x', t | x_0, t_0) dx' \quad (\text{A14})$$

It is now a simple matter to integrate Equation (A14) using the assumed asymptotic properties of $R(x)$. The first term on the right side of Equation (A14) can be integrated by parts once and the second term integrated by parts twice, noting in both cases all the terms at $x \rightarrow \pm\infty$ vanish. Thus,

$$\int_{-\infty}^{\infty} \frac{\partial R(x')}{\partial x'} K_1(x', t) p(x', t | x_0, t_0) dx' = - \int_{-\infty}^{\infty} R(x') \frac{\partial (K_1(x', t) p(x', t | x_0, t_0))}{\partial x'} dx' \quad (\text{A15})$$

and

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial R^2(x')}{\partial x'^2} K_2(x', t) p(x', t | x_0, t_0) dx' = \frac{1}{2} \int_{-\infty}^{\infty} R(x') \frac{\partial^2 (K_2(x', t) p(x', t | x_0, t_0))}{\partial x'^2} dx' \quad (\text{A16})$$

allowing Equation (A14) to become

$$I[x_0] = \int_{-\infty}^{\infty} R(x') \left(- \frac{\partial K_1(x', t) p(x', t | x_0, t_0)}{\partial x'} + \frac{1}{2} \frac{\partial^2 K_2(x', t) p(x', t | x_0, t_0)}{\partial x'^2} \right) dx' \quad (\text{A17})$$

Using this relation back in Equation (A4), one obtains

$$\int_{-\infty}^{\infty} R(x') \left(\frac{\partial p(x', t | x_0, t_0)}{\partial t} + \frac{\partial K_1(x', t) p(x', t | x_0, t_0)}{\partial x'} - \frac{1}{2} \frac{\partial^2 K_2(x', t) p(x', t | x_0, t_0)}{\partial x'^2} \right) dx' = 0 \quad (\text{A18})$$

Since at the outset the function $R(x)$ was taken to be arbitrary, the only way Equation (A18) can vanish is if the quantity within the parenthesis of the integrand vanishes. Thus, reinstating the notation $x = x_R$, one has that

$$\frac{\partial p(x_R, t | x_0, t_0)}{\partial t} = - \frac{\partial K_1(x_R, t) p(x_R, t | x_0, t_0)}{\partial x_R} + \frac{1}{2} \frac{\partial^2 K_2(x_R, t) p(x_R, t | x_0, t_0)}{\partial x_R^2} \quad (\text{A19})$$

giving the sought-after relationship between the transition PDF $p(x_R, t | x_0, t_0)$ of the random process and the parameters of the stochastic differential equation specifying the dynamics of the process. This relation is the Kolmogorov equation (sometimes termed the forward Kolmogorov equation) for the PDF that describes the transition through time of the Markov random process $x_R(t)$. It now remains to finish that connection by applying Equation (A1) to the assignments given by Equations (A10)-(A13).

Equation (A10) is a specification (actually, a definition) for the ensemble average of the difference $\Delta x = x - x'$ that occurs over the time interval Δt . Thus, $k_1(x', t + \Delta t) = \langle \Delta x \rangle$ and Equation (A12) gives

$$K_1(x'_R, t) \equiv \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta x_R \rangle}{\Delta t} \quad (\text{A20})$$

Similarly, $k_2(x', t + \Delta t) = \langle (\Delta x)^2 \rangle$ and by Equation (A13)

$$K_2(x'_R, t) \equiv \lim_{\Delta t \rightarrow 0} \frac{\langle (\Delta x_R)^2 \rangle}{\Delta t} \quad (\text{A21})$$

The incorporation of the Langevin parameters into the Kolmogorov equation commences with simply integrating Equation (A1) over the small time interval Δt and averaging:

$$\langle \Delta x_R \rangle = \int_t^{t+\Delta t} \langle \psi(x_R, t') \rangle dt' + \int_t^{t+\Delta t} \langle g(x_R, t') \xi_R(t') \rangle dt' = \int_t^{t+\Delta t} \psi(x_R, t') dt' \quad (\text{A22})$$

since $\psi(x_R, t')$ and $g(x_R, t')$ are deterministic functions and $\langle \xi_R(t') \rangle = 0$ by Equation (A2). But if Δt is small enough such that the function $\psi(x_R, t')$ does not appreciably change over this interval, one has

$$\langle \Delta x_R \rangle = \int_t^{t+\Delta t} \psi(x_R, t') dt' \approx \psi(x_R, t) \Delta t \quad (\text{A23})$$

yielding

$$K_1(x'_R, t) \equiv \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta x_R \rangle}{\Delta t} = \psi(x_R, t) \quad (\text{A24})$$

Applying the same procedure for $\langle (\Delta x_R)^2 \rangle$ as required by Equation (A21) gives, upon using $\langle \xi_R(t) \xi_R(t') \rangle = \delta(t - t')$ of Equation (A2),

$$\langle (\Delta x_R)^2 \rangle = \psi^2(x_R, t') (\Delta t)^2 + g^2(x_R, t) \Delta t \quad (\text{A25})$$

Hence,

$$K_2(x'_R, t) \equiv \lim_{\Delta t \rightarrow 0} \frac{\langle (\Delta x_R)^2 \rangle}{\Delta t} = g^2(x_R, t) \quad (\text{A26})$$

Finally, the Kolmogorov equation for the one-dimensional process $x_R(t)$ defined by Equation (A1) is, by Equations (A19), (A24), and (A26),

$$\frac{\partial p(x_R, t | x_0, t_0)}{\partial t} = - \frac{\partial \psi(x_R, t) p(x_R, t | x_0, t_0)}{\partial x_R} + \frac{1}{2} \frac{\partial^2 g^2(x_R, t) p(x_R, t | x_0, t_0)}{\partial x_R^2} \quad (\text{A27})$$

As described in the text, one specifically has for the case of rain attenuation, $\psi(x_R, t) = -\gamma_R x_R$ and $g(x_R, t) = \sqrt{2\gamma_R}$ from which the conditional PDF for rain attenuation is given by

$$\frac{\partial p(x_R, t | x_0, t_0)}{\partial t} = \gamma_R \frac{\partial x_R p(x_R, t | x_0, t_0)}{\partial x_R} + \gamma_R \frac{\partial^2 p(x_R, t | x_0, t_0)}{\partial x_R^2} \quad (\text{A28})$$

the solution of which provides the PDF describing all the temporal dynamics of the ACTS Rain Attenuation Model. The form of this drift/diffusion process simplifies the structure of the theory, especially in the case of multi-dimensional processes operating to give a single composite process as will be shown in what is to follow.

A.2 Extension to the Treatment of Several Processes - The Kolmogorov Equation Corresponding to a Multi-Dimensional Langevin Equation

The procedure used above to obtain the Kolmogorov equation for a one-dimensional, single-component Markov process can be easily repeated for the general case of a several component process given by the vector Langevin equation

$$\frac{\partial \bar{x}}{\partial t} = \bar{\psi}(\bar{x}, t) + \bar{g}(\bar{x}, t) \bar{\xi}(t) \quad (\text{A29})$$

where \bar{x} is the column vector (i.e., a matrix)

$$\bar{x} = \bar{x}(t) = \begin{pmatrix} x_R(t) \\ x_S(t) \\ \vdots \\ x_Z(t) \end{pmatrix} \quad (\text{A30})$$

and similarly for

$$\bar{\psi}(\bar{x}, t) = \begin{pmatrix} \psi_R(x_R, t) \\ \psi_S(x_S, t) \\ \vdots \\ \psi_Z(x_Z, t) \end{pmatrix} \quad (\text{A31})$$

For most cases in which Markov processes apply, the matrix $\bar{g}(\bar{x}, t)$ is of the form

$$\bar{g}(\bar{x}, t) = \begin{pmatrix} g_R(x_R, t) & 0 & \cdots & 0 \\ 0 & g_S(x_S, t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & g_Z(x_Z, t) \end{pmatrix} \quad (\text{A32})$$

The nuance of this multi-dimensional case that enters is the fact that the stochastic noise terms that enter into the matrix

$$\bar{\xi}(t) = \begin{pmatrix} \xi_R(t) \\ \xi_S(t) \\ \vdots \\ \xi_Z(t) \end{pmatrix} \quad (\text{A33})$$

comes with more complicated statistical properties. As before, one still has $\langle \xi_i(t) \rangle = 0$ for each element but must now allow for possible cross-correlation to exist between different components $\xi_i(t)$. This cross-correlation is captured by the requirement that

$$\langle \xi_i(t) \xi_j(t') \rangle = B_{ij} \delta(t-t'), \text{ i.e., } \langle \bar{\xi}(t) \bar{\xi}^T(t') \rangle = \bar{B} \delta(t-t') \quad (\text{A34})$$

where B_{ij} are the correlation coefficients and the subscript T denotes the transpose of the column matrix. The only proviso on the values of B_{ij} is, of course, $B_{ii} = 1$.

Following the steps that lead to Equation (A19) gives for the multi-dimensional Kolmogorov equation specifying the corresponding multi-dimensional conditional transition PDF $(\bar{x}, t | \bar{x}_0, t_0)$

$$\frac{\partial p(\bar{x}, t | \bar{x}_0, t_0)}{\partial t} = - \sum_i \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}, t | \bar{x}_0, t_0) \right) + \frac{1}{2} \sum_i \sum_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}, t | \bar{x}_0, t_0) \right) \quad (\text{A35})$$

where, as before,

$$\bar{K}_1(\bar{x}, t) = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta \bar{x} \rangle}{\Delta t} \text{ and } \bar{K}_2(\bar{x}, t) = \lim_{\Delta t \rightarrow 0} \frac{\langle (\Delta \bar{x})^2 \rangle}{\Delta t}. \quad (\text{A36})$$

The evaluation of $\bar{K}_1(\bar{x}, t)$ yields, as before,

$$\bar{K}_1(\bar{x}, t) = \langle \bar{\psi}(\bar{x}, t) \rangle \quad (\text{A37})$$

but the similar evaluation of $\bar{K}_2(\bar{x}, t)$ involves the cross-correlations of the $\xi_i(t)$ components. In particular,

$$\begin{aligned} \langle (\Delta \bar{x})^2 \rangle &= \langle (\Delta \bar{x})(\Delta \bar{x})^T \rangle = \\ &= \int_t^{t+\Delta t} \int_t^{t+\Delta t} \langle \bar{\psi}(\bar{x}, t') \bar{\psi}^T(\bar{x}, t'') \rangle dt' dt'' + \int_t^{t+\Delta t} \int_t^{t+\Delta t} \langle \bar{g}(\bar{x}, t') \bar{\xi}(t') \bar{\xi}^T(t'') \bar{g}^T(\bar{x}, t'') \rangle dt' dt'' \\ &\approx \bar{\psi}(\bar{x}, t') \bar{\psi}^T(\bar{x}, t'') (\Delta t)^2 + \bar{g}(\bar{x}, t') \bar{B} \bar{g}^T(\bar{x}, t'') \Delta t \end{aligned} \quad (\text{A38})$$

giving

$$\bar{K}_2(\bar{x}, t) = \bar{g}(\bar{x}, t') \bar{B} \bar{g}^T(\bar{x}, t'') \quad (\text{A39})$$

To simplify matters, the function $\bar{\psi}(\bar{x}, t)$ that enters the Langevin equation, known as the drift term, is of the multiplicative form

$$\bar{\psi}(\bar{x}, t) = \begin{pmatrix} -\gamma_1 x_1 \\ -\gamma_2 x_2 \\ \vdots \\ -\gamma_N x_N \end{pmatrix} \equiv -\bar{\Gamma} \bar{x} \quad (\text{A40})$$

which allows for the introduction of the diagonal matrix for the drift coefficients

$$\bar{\Gamma} \equiv \begin{pmatrix} \gamma_R & & & \\ & \gamma_S & & \\ & & \ddots & \\ & & & \gamma_N \end{pmatrix} \quad (\text{A41})$$

The corresponding matrix of diffusion coefficients $\bar{g}(\bar{x}, t)$ can also be specialized by requiring that they result in solutions for the transition PDFs from the Kolmogorov equation that are of exponential form (Ref. 3):

$$\bar{g}(\bar{x}, t) = \bar{G} \equiv \begin{pmatrix} \sqrt{2\gamma_1} & 0 & \cdots & 0 \\ 0 & \sqrt{2\gamma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{2\gamma_N} \end{pmatrix} \quad (\text{A42})$$

Equations (A40)-(A42) apply to the multi-dimensional drift/diffusion processes that are of the same class as those for rain attenuation and phase scintillation. Thus, in this case, which will be taken to prevail for the purposes of analysis in this document, the Langevin equation can be written

$$\frac{\partial \bar{x}}{\partial t} = -\bar{\Gamma} \bar{x} + \bar{G} \bar{\xi}(t) \quad (\text{A43})$$

where, as before,

$$\bar{\xi}(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_N(t) \end{pmatrix}, \quad \langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = B_{ij} \delta(t-t'), \quad B_{ii} = 1 \quad (\text{A44})$$

Additionally, Equation (A37) now becomes

$$\bar{K}_1(\bar{x}, t) = -\bar{\Gamma} \langle \bar{x}(t) \rangle \quad (\text{A45})$$

Similarly, explicitly writing the correlation matrix

$$\bar{B} = \begin{pmatrix} 1 & B_{RS} & \cdots & B_{RZ} \\ B_{SR} & 1 & \cdots & B_{SZ} \\ \vdots & \vdots & \ddots & \vdots \\ B_{ZR} & B_{ZS} & \cdots & 1 \end{pmatrix} \quad (\text{A46})$$

Equation (39) gives

$$\left(\bar{K}_2(\bar{x}, t) \right)_{ij} = 2B_{ij} \sqrt{\gamma_i \gamma_j} \quad (\text{A47})$$

Appendix B.—Solution of the Kolmogorov Equation for Transition Probabilities of a Process Using Measurements of the Results of That Process—The Stratonovich Equation

The purpose of this appendix is to present the derivation of how the Kolmogorov equation,

$$\frac{\partial p(\bar{x}, t | \bar{x}_0, t_0)}{\partial t} = - \sum_i^N \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}, t | \bar{x}_0, t_0) \right) + \frac{1}{2} \sum_i^N \sum_j^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}, t | \bar{x}_0, t_0) \right)$$

giving the conditional transition PDF $p(\bar{x}, t | \bar{x}_0, t_0)$ of the random process $\bar{x}(t)$, is augmented with actual observational measurements of the results of that process to aid in the realistic determination of $p(\bar{x}, t | \bar{x}_0, t_0)$ in specific applications. This derivation will then give rise to the Stratonovich equation for the PDF $p(\bar{x}, t)$

$$\frac{\partial p(\bar{x}, t)}{\partial t} = D_{\bar{x}} [p(\bar{x}, t)] + \left[\phi(\bar{x}, t) - \int_{-\infty}^{\infty} \phi(\bar{x}', t) p(\bar{x}', t) d\bar{x}' \right] p(\bar{x}, t)$$

where

$$D_{\bar{x}} [p(\bar{x}, t)] \equiv - \sum_i^N \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}, t) \right) + \frac{1}{2} \sum_i^N \sum_j^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}, t) \right)$$

is the Kolmogorov operator and

$$\phi(\bar{x}, t) \equiv - \frac{1}{2\sigma_n^2} (A_{obs}(t) - A(\bar{x}, t))^2$$

provides the connection to the observed (measured) result $A_{obs}(t)$ of the composite process \bar{x} through the comparison to the observation model $A(\bar{x}, t)$ evaluated by previously determined values of \bar{x} . The solution and application of this prescription to satellite signal fade identification is given in Appendix C.

Note how the conditional PDF $p(\bar{x}, t | \bar{x}_0, t_0)$ of the Kolmogorov equation becomes the PDF $p(\bar{x}, t)$ in the Stratonovich equation since the results giving $p(\bar{x}, t | \bar{x}_0, t_0)$ are now conditioned on the observational inputs. This appendix will provide the fundamental derivation of the Stratonovich equation from the Kolmogorov equation for the probability density function for the components of a multi-dimensional Markov process.

B.1 The Problem to be Solved

Consider the observation of measurement of the attenuation $A_{obs}(t_i)$ at a time instant t_i . This measured attenuation can be the result of one or several atmospheric components that enter into the actual value that is measured. In an effort to describe such an attenuation process, a mathematical model will be used that attempts to describe the convolution of all the processes that go into the determination of the actual

attenuation. For example, when the two processes of attenuation due to rain and due to scintillation enter, one attempts to model the attenuation seen at t_i as

$$A(t_i) = A_R(t_i) + A_S(t_i) \quad (\text{B1})$$

where $A_R(t_i)$ is that portion of the observation model that attempts to describe the level of rain attenuation and $A_S(t_i)$ is that portion that attempts to give attenuation due to scintillation. Each of the models $A_R(t_i)$ and $A_S(t_i)$ are functions of corresponding underlying Markov processes $x_R(t_i)$ and $x_S(t_i)$ given by the Langevin equation of Equation (A43) and defined by Equations (A41), (A42) and (A44). Hence, $A_R(t_i) \equiv A_R(x_R(t_i))$ and $A_S(t_i) \equiv A_S(x_S(t_i))$ and Equation (B1) becomes

$$A(\bar{x}(t_i)) = A_R(x_R(t_i)) + A_S(x_S(t_i)) \quad (\text{B2})$$

where $\bar{x} = (x_R, x_S)^T$ is the matrix of the composite Markov process and T indicates the transpose. Thus, if one has faith that the model processes are reasonably correct, than one can relate the observed attenuation $A_{obs}(t_i)$ to that modeled $A(\bar{x}(t_i))$ by the use of the **observation model**

$$A_{obs}(t_i) = A(\bar{x}(t_i)) + n(t_i) = A_R(x_R(t_i)) + A_S(x_S(t_i)) + n(t_i) \quad (\text{B3})$$

where $n(t_i)$ is the uncertainty in the attenuation measurement as compared to that modeled; it constitute the measurement ‘noise’ which, for purposes in the present work, is statistically defined by

$$\langle n(t) \rangle = 0, \langle n(t)n(t') \rangle = \sigma_n^2 \delta(t - t') \quad (\text{B4})$$

where σ_n^2 is the variance of the uncertainty of the measurement process. The Langevin equation of Equations (A29) or (A43) along with Equations (A41), (A42) and (A44) are defined as the **process model**. The problem required to be addressed and solved can now be stated.

The need for all the formalism created for the process model and observation model is made apparent by the need to be able to properly separate one individual attenuation component, e.g., $A_R(x_R(t_i))$ from another, e.g., $A_S(x_S(t_i))$ given *only* the measured noisy composite observational attenuation $A_{obs}(t_i)$ at a series of time instants t_j from the past $j < i$ to the present i .

B.2 The Statistical Formulation of the Solution

From the problem statement given above and the fact that the entire process model Appendix A is available for the subsequent statistical evolution of the component processes, one reduces the problem to that of finding the particular Markov components $x_R(t_i)$ and $x_S(t_i)$ at time t_i from the set

$$\{A_{obs}\}_i \equiv \{A_{obs}(t_i), A_{obs}(t_{i-1}), \dots, A_{obs}(t_0)\} \quad (\text{B5})$$

of consecutive instantaneous observations $A_{obs}(t)$ up to the time instant t_i . Now, for no other reason than being advantageous to have, consider the conditional probability density function $p(\bar{x}(t_i) | \{A_{obs}\}_i)$ which gives the optimal estimate

$$\langle \bar{x}(t_i) \rangle = \int_{-\infty}^{\infty} p(\bar{x}(t_i) | \{A_{obs}\}_i) d\bar{x} \quad (B6)$$

derived from the minimization of the mean squared error. Here the infinite limits of the integral represent the entire range of values of the components of the matrix \bar{x} . One can then define and write the compound PDF

$$\begin{aligned} p(\bar{x}(t_i), A_{obs}(t_i) | \{A_{obs}\}_{i-1}) &= p(\bar{x}(t_i) | \{A_{obs}\}_{i-1}) p(A_{obs}(t_i) | \{A_{obs}\}_{i-1}, \bar{x}(t_i)) \\ &= p(A_{obs}(t_i) | \{A_{obs}\}_{i-1}) p(\bar{x}(t_i) | \{A_{obs}\}_{i-1}, A_{obs}(t_i)) \end{aligned} \quad (B7)$$

in which the probability of compound events is employed and the intervening PDF's have obvious meaning. Since the noise $n(t_i)$ is a sequence of random uncorrelated events, the quantity $A_{obs}(t_i)$ will not be dependent on previous $\{A_{obs}(t)\}_{i-1}$ values so

$$\begin{aligned} p(A_{obs}(t_i) | \{A_{obs}\}_{i-1}, \bar{x}(t_i)) &= p(A_{obs}(t_i) | \bar{x}(t_i)) \\ p(\bar{x}(t_i) | \{A_{obs}\}_{i-1}, A_{obs}(t_i)) &= p(\bar{x}(t_i) | \{A_{obs}\}_i) \end{aligned} \quad (B8)$$

Thus, from Equations (B7) and (B8),

$$p(\bar{x}(t_i) | \{A_{obs}\}_i) = \frac{p(\bar{x}(t_i) | \{A_{obs}\}_{i-1}) p(A_{obs}(t_i) | \bar{x}(t_i))}{p(A_{obs}(t_i) | \{A_{obs}\}_{i-1})} \quad (B9)$$

The PDF $p(A_{obs}(t_i) | \{A_{obs}\}_{i-1})$ is easily dealt with by noting that is, by hypothesis, independent to the process matrix $\bar{x}(t)$ at any time instant. It is, with respect to the evolution of $\bar{x}(t)$, a constant. Hence, $p(A_{obs}(t_i) | \{A_{obs}\}_{i-1}) = C_1^{-1}$ where C_1 is a constant allowing Equation (9) to be written

$$p(\bar{x}(t_i) | \{A_{obs}\}_i) = C_1 p(A_{obs}(t_i) | \bar{x}(t_i)) p(\bar{x}(t_i) | \{A_{obs}\}_{i-1}) \quad (B10)$$

As for the remaining PDF $p(A_{obs}(t_i) | \bar{x}(t_i))$, it is noted that, by definition, it is the PDF governing the measured attenuation $A_{obs}(t_i)$ at the time interval t_i conditioned on the prevailing value of the associated process $\bar{x}(t_i)$ at time t_i . By the observation model of Equation (B3), one has $A_{obs}(t_i) - A(\bar{x}(t_i)) = n(t_i)$. Hence, the statistics of the PDF governing a given value of $\bar{x}(t_i)$ that results in the value of $A_{obs}(t_i)$ is the same that those governing the measurement noise $n(t_i)$, i.e., is given by a Gaussian PDF. Hence, it is of the form

$$p(A_{obs}(t_i) | \bar{x}(t_i)) = C_2 \exp\left(-\frac{(A_{obs}(t_i) - A(\bar{x}(t_i)))^2}{2\sigma_{A_{obs}}^2}\right) \quad (B11)$$

The associated variance $\sigma_{A_{obs}}^2$ within the Gaussian PDF, connected to the noise statistics given by Equation (B4), can only be related, in this discrete case, to the interval $\Delta t = t_i - t_{i-1}$, i.e.,

$$\sigma_{A_{obs}}^2 = \langle n(t)n(t + \Delta t) \rangle \frac{\sigma_n^2}{\Delta t} \quad (\text{B12})$$

Hence, one can write

$$p(A_{obs}(t_i) | \bar{x}(t_i)) = C_2 \exp \left(- \frac{(A_{obs}(t_i) - A(\bar{x}(t_i)))^2}{2\sigma_n^2} \Delta t \right) = C_2 \exp(\phi(t_i) \Delta t) \quad (\text{B13})$$

where $\phi(t_i) \equiv - \frac{(A_{obs}(t_i) - A(\bar{x}(t_i)))^2}{2\sigma_n^2}$. Finally, Equation (B10) becomes

$$p(\bar{x}(t_i) | \{A_{obs}\}_i) = C_3 \exp(\phi(t_i) \Delta t) p(\bar{x}(t_i) | \{A_{obs}\}_{i-1}) \quad (\text{B14})$$

where C_3 is another constant. The PDF on the right side of Equation (B14) will now be further reduced. Given the fact that this PDF is conditioned on events that occur up to the time instant t_{i-1} , one can write $\bar{x}(t_i) = \bar{x}(t_{i-1} + \Delta t)$ occurring on both sides of Equation (B14). Hence,

$$p(\bar{x}(t_{i-1} + \Delta t) | \{A_{obs}\}_i) = C_3 \exp(\phi(t_i) \Delta t) p(\bar{x}(t_{i-1}) | \{A_{obs}\}_{i-1}) \quad (\text{B15})$$

The relationship written in this form yields the key to connecting the present development to the process model of Appendix A that will yield a differential equation for the conditional PDF from which, using Equation (B6), the equations for optimal estimates of the particular Markov components $\bar{x} = (x_R, x_S)^T$ can be obtained, thus separating the individual contributions that enter into the composite measured quantity A_{obs} .

B.3 Connecting the Foregoing to the Process Model and Obtaining the Stratonovich Equation for the Conditional Transition Probability Density Functions

To this end, note the PDF on the right side of Equation (B15); it gives the statistics of \bar{x} at time $t_{i-1} + \Delta t$ based on the set of observational inputs $\{A_{obs}\}_{i-1}$ at time t_{i-1} . Thus, in this instance, $\bar{x}(t_{i-1} + \Delta t)$ is an extrapolated estimate of the process \bar{x} based on previous observations. Now the only information concerning the evolution of the process \bar{x} from t_{i-1} to $t_{i-1} + \Delta t$ comes from the Langevin equation, Equation (A29) or Equation (A43) of Appendix A. Therefore, for times t within the interval $t_{i-1} \leq t \leq t_{i-1} + \Delta t$, the PDF $p(\bar{x}(t) | \{A_{obs}\}_{i-1})$ is determined by the Kolmogorov equation that corresponds to the Langevin equation. That is, using Equation (A35),

$$\frac{\partial p(\bar{x}(t) | \{A_{obs}\}_{i-1})}{\partial t} = - \sum_i^N \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}(t) | \{A_{obs}\}_{i-1}) \right) + \frac{1}{2} \sum_i^N \sum_j^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}(t) | \{A_{obs}\}_{i-1}) \right) \quad (\text{B16})$$

for $t_{i-1} \leq t \leq t_{i-1} + \Delta t$ and the initial condition $p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1})$. One now expands about the time interval Δt as follows:

$$\begin{aligned} p(\bar{x}(t_{i-1} + \Delta t)|\{A_{obs}\}_{i-1}) &\approx p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) + \left. \frac{\partial p(\bar{x}(t)|\{A_{obs}\}_{i-1})}{\partial t} \right|_{t=t_{i-1}} \Delta t + \dots \\ &\approx p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) + D_{\bar{x}} \left[p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) \right] \Delta t + \dots \end{aligned} \quad (\text{B17})$$

where the Kolmogorov operator $D_{\bar{x}}[\dots]$ is defined by

$$\begin{aligned} D_{\bar{x}} \left[p(\bar{x}(t)|\{A_{obs}\}_{i-1}) \right] &\equiv - \sum_i^N \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}(t)|\{A_{obs}\}_{i-1}) \right) \\ &\quad + \frac{1}{2} \sum_i^N \sum_j^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}(t)|\{A_{obs}\}_{i-1}) \right) \end{aligned} \quad (\text{B18})$$

Substituting Equation (B17) into Equation (B15) and similarly using the expansion $\exp(\phi(t_i)\Delta t) \approx 1 + \phi(t_i)\Delta t + \dots$ yields

$$\begin{aligned} p(\bar{x}(t_{i-1} + \Delta t)|\{A_{obs}\}_i) &\approx C_3 \left(p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) + D_{\bar{x}} \left[p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) \right] \Delta t \right. \\ &\quad \left. + p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) \phi(t_i) \Delta t \right) \end{aligned} \quad (\text{B19})$$

With the promise that the end of the analysis is near, one now needs to evaluate the constant C_3 . Using the fact that

$$\frac{\partial p(\bar{x}(t)|\{A_{obs}\}_{i-1})}{\partial t} = D_{\bar{x}} \left[p(\bar{x}(t)|\{A_{obs}\}_{i-1}) \right] \quad (\text{B20})$$

and, by normalization of the PDF's,

$$\int_{-\infty}^{\infty} p(\bar{x}(t)|\{A_{obs}\}_{i-1}) d\bar{x} = 1 \quad (\text{B21})$$

gives

$$\int_{-\infty}^{\infty} D_{\bar{x}} \left[p(\bar{x}(t)|\{A_{obs}\}_{i-1}) \right] d\bar{x} = 0 \quad (\text{B22})$$

Using these circumstances and integrating Equation (B19) by \bar{x} yields

$$C_3 = \left(1 + \Delta t \int_{-\infty}^{\infty} \phi(t_{i-1}) p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) d\bar{x} \right)^{-1} \approx 1 - \Delta t \int_{-\infty}^{\infty} \phi(t_{i-1}) p(\bar{x}(t_{i-1})|\{A_{obs}\}_{i-1}) d\bar{x} \quad (\text{B23})$$

Substituting this expression into Equation (B19) and rearranging terms results in

$$p(\bar{x}(t_{i-1} + \Delta t) | \{A_{obs}\}_i) - p(\bar{x}(t_{i-1}) | \{A_{obs}\}_{i-1}) \approx \Delta t \left(D_{\bar{x}} \left[p(\bar{x}(t_{i-1}) | \{A_{obs}\}_{i-1}) \right] + \left(\phi(t_{i-1}) - \int_{-\infty}^{\infty} \phi(t_{i-1}) p(\bar{x}(t_{i-1}) | \{A_{obs}\}_{i-1}) d\bar{x} \right) \right) p(\bar{x}(t_{i-1}) | \{A_{obs}\}_{i-1}) \quad (B24)$$

Finally dividing through by Δt and taking the limit $\Delta t \rightarrow 0$ (implicitly ridding of the need to specify the discrete nature of the time indicator i) gives the differential equation specifying the conditional PDF of the process \bar{x} ,

$$\frac{\partial p(\bar{x}, t)}{\partial t} = D_{\bar{x}} \left[p(\bar{x}, t) \right] + \left[\phi(\bar{x}, t) - \int_{-\infty}^{\infty} \phi(\bar{x}', t) p(\bar{x}', t) d\bar{x}' \right] p(\bar{x}, t) \quad (B25)$$

where the implicit conditional PDF $p(\bar{x}, t)$ is defined by

$$\lim_{\Delta t \rightarrow 0} \left\{ p(\bar{x}(t_{i-1}) | \{A_{obs}\}_{i-1}) \right\} = p(\bar{x}(t) | \{A_{obs}\}) \equiv p(\bar{x}, t) \quad (B26)$$

Equation (B25) is the Stratonovich equation whose solution yields the PDF $p(\bar{x}, t)$ for the vector process $\bar{x}(t)$ whose effect and determination on the measured composite process $A_{obs}(t)$ is given through the discriminator function

$$\phi(\bar{x}, t) \equiv -\frac{1}{2\sigma_n^2} (A_{obs}(t) - A(\bar{x}, t))^2 \quad (B27)$$

which prominently enters Equation (B25).

The development given above essentially followed the construction of Stratonovich in his original work in 1959. The appropriate references are the collected English translations of the papers published in Reference 6 as well as, of course, the Russian original References 7 and 8.

Appendix C.—The Stratonovich Equation for the Optimal Estimates and Related Error Covariances of the Individual Component Process Levels of a Measured Fade on a Satellite Link—Its Solution and the Related Discrete-Time Recursive Filter

This appendix will concern itself with the solution for the PDF $p(\bar{x}, t)$ given by Stratonovich integro-differential equation, derived in Appendix B, and its application to the derivation of the optimal estimate matrix $\langle \bar{x}(t) \rangle = (\langle x_R(t) \rangle, \langle x_S(t) \rangle)^T$ as well as the covariance matrix $\bar{\bar{\sigma}}(t)$ for a two-component Markov process, using the basic observational measurements of the composite quantity $A_{obs}(t)$ with a measurement uncertainty σ_n^2 . In discrete-time form for digital application for the combined noisy measurement of rain attenuation and phase scintillation, the equations for the resulting recursive filter are

$$\langle \bar{x}(t_{i+1}) \rangle = \bar{\bar{\Phi}}(t_{i+1}, t_i) \langle \bar{x}(t_i) \rangle + \bar{\bar{\Phi}}(t_{i+1}, t_i) \bar{\bar{\sigma}}(t_i) \bar{a}(t_i)$$

$$\bar{\bar{\sigma}}^{-1}(t_{i+1}) = \bar{\bar{H}}^{-1}(t_i) - \bar{\bar{b}}(t_i)$$

$$\bar{\bar{H}}(t_i) = \bar{\bar{\Phi}}(t_i, t_{i-1}) \bar{\bar{\sigma}}(t_{i-1}) \bar{\bar{\Phi}}^T(t_i, t_{i-1}) + \bar{\bar{D}}$$

$$\left(\bar{\bar{D}}(\Delta t) \right)_{jk} = 2B_{ij} \frac{\sqrt{\gamma_j \gamma_k}}{\gamma_j + \gamma_k} \exp(-(\gamma_j + \gamma_k)\Delta t) \left(1 - \exp(-(\gamma_j + \gamma_k)\Delta t) \right), \quad j, k = R, S$$

$$\bar{a}(t_i) = \left. \frac{\partial \phi(\bar{x}, t_i)}{\partial \bar{x}} \Delta t \right|_{\bar{x}=\langle \bar{x} \rangle}, \quad \bar{\bar{b}}(t_i) = \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t_i)}{\partial \bar{x}} \right)^T \Delta t \Big|_{\bar{x}=\langle \bar{x} \rangle}$$

$$\phi(\bar{x}, t) \equiv -\frac{1}{2\sigma_n^2} (A_{obs}(t) - A(\bar{x}, t))^2$$

$$A(\bar{x}, t) = A_{mR} \exp(\sigma_{\ln A_R} x_R(t)) + A_{mS} + \sigma_S x_S(t)$$

$$\bar{\bar{\Phi}}(t, t_0) = \exp(-\bar{\bar{\Gamma}}(\Delta t)), \quad \Delta t \equiv t - t_0$$

$$\bar{\bar{\Gamma}} = \begin{pmatrix} \gamma_R & 0 \\ 0 & \gamma_S \end{pmatrix}$$

The Stratonovich equation derived in Appendix B will now be solved for the PDF $p(\bar{x}, t)$ of the process \bar{x} with the goal of finding the corresponding optimal estimate

$$\langle (\bar{x}(t))_j \rangle \equiv \int_{-\infty}^{\infty} x_j(t) p(\bar{x}, t) dx_j \tag{C1}$$

as well as the associated error covariance

$$(\bar{\sigma})_{jk} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) p(\bar{x}, t) dx_j dx_k \quad (C2)$$

using the measured values of the composite quantity $A_{obs}(t)$. This will culminate in the specification and algorithm for a recursive discrete-time filter that can be applied in real-time to filter a composite fading signal and act on the level of the components in an appropriate manner. From the results of Appendix B, one has for the Stratonovich equation

$$\frac{\partial p(\bar{x}, t)}{\partial t} = -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left((\bar{K}_1)_i p(\bar{x}, t) \right) + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \left((\bar{K}_2)_{ij} p(\bar{x}, t) \right) + [\phi(\bar{x}, t) - \langle \phi(t) \rangle] p(\bar{x}, t) \quad (C3a)$$

$$\phi(\bar{x}, t) \equiv -\frac{1}{2\sigma_n^2} (A_{obs}(t) - A(\bar{x}, t))^2 \quad (C3b)$$

$$\langle \phi(t) \rangle \equiv \int_{-\infty}^{\infty} \phi(\bar{x}, t) p(\bar{x}, t) d\bar{x} \quad (C3c)$$

Although in what is to follow, an attempt is made to keep the arguments general, the final goal is always for the two-component Markov process for both rain attenuation and phase scintillation. It is now desired to obtain from the statistical description of the Markov process afforded by Equation (C3) the corresponding optimal estimates of the mean of the process as well as the associated covariance. The transition probability density determined by Equation (C3) contains all the necessary information with which to accomplish this task. Of course, there is no known analytical solution to the integro-differential equation of Equation (C3) so use of the quantity $p(\bar{x}, t)$ must be done using Equation (C3) directly.

To this end, define the characteristic functional $\Theta(\bar{u}, t)$ for an N -component process given by

$$\Theta(\bar{u}, t) \equiv \int_{-\infty}^{\infty} \exp\left(i \sum_{j=1}^N u_j x_j\right) p(\bar{x}, t) d\bar{x} \quad (C4)$$

Multiplying Equation (C3) by $\exp\left(i \sum_{j=1}^N u_j x_j\right)$ and integrating with respect to x_j (i.e., over each of the components x_j) simply gives for the left hand side

$$\int_{-\infty}^{\infty} \frac{\partial p(\bar{x}, t)}{\partial t} \exp\left(i \sum_{j=1}^N u_j x_j\right) d\bar{x} = \frac{\partial \Theta}{\partial t} \quad (C5)$$

Similarly, the first term on the right side of Equation (C3) becomes, after an integration by parts,

$$-\int_{-\infty}^{\infty} \sum_{j=1}^N \frac{\partial}{\partial x_j} \left((\bar{K}_1)_j p(\bar{x}, t) \right) \exp\left(i \sum_{k=1}^N u_k x_k\right) d\bar{x} = i \sum_{j=1}^N u_j \left\langle (\bar{K}_1)_j \exp\left(i \sum_{k=1}^N u_k x_k\right) \right\rangle \quad (C6)$$

The second term is manipulated in the same way but this time with two integrations by parts to obtain

$$\frac{1}{2} \int_{-\infty}^{\infty} \sum_{j=1}^N \sum_{k=1}^N \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \left(\left(\bar{\bar{K}}_2 \right)_{jk} p(\bar{x}, t) \right) \exp \left(i \sum_{l=1}^N u_l x_l \right) d\bar{x} = -\frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N u_j u_k \left\langle \left(\bar{\bar{K}}_2 \right)_{jk} \exp \left(i \sum_{l=1}^N u_l x_l \right) \right\rangle \quad (C7)$$

Finally, the last term in Equation (C3) becomes

$$\int_{-\infty}^{\infty} [\phi(\bar{x}, t) - \langle \phi(t) \rangle] p(\bar{x}, t) \exp \left(i \sum_{l=1}^N u_l x_l \right) d\bar{x} = \left\langle \phi(\bar{x}, t) \exp \left(i \sum_{l=1}^N u_l x_l \right) \right\rangle - \langle \phi(t) \rangle \Theta(\bar{u}, t) \quad (C8)$$

Hence, Equation (C3) becomes in characteristic functional form

$$\begin{aligned} \frac{\partial \Theta}{\partial t} = & i \sum_{j=1}^N u_j \left\langle \left(\bar{K}_1 \right)_j \exp \left(i \sum_{k=1}^N u_k x_k \right) \right\rangle - \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N u_j u_k \left\langle \left(\bar{\bar{K}}_2 \right)_{jk} \exp \left(i \sum_{l=1}^N u_l x_l \right) \right\rangle \\ & + \left\langle \phi(\bar{x}, t) \exp \left(i \sum_{l=1}^N u_l x_l \right) \right\rangle - \langle \phi(t) \rangle \Theta(\bar{u}, t) \end{aligned} \quad (C9)$$

However, it proves expedient to not use the function $\Theta(\bar{u}, t)$ but instead the function $\ln \Theta(\bar{u}, t)$. Thus, multiplying both sides of Equation (C9) by $(\Theta(\bar{u}, t))^{-1}$ gives for the transformed characteristic functional

$$\begin{aligned} \frac{\partial \ln \Theta(\bar{u}, t)}{\partial t} = & \frac{1}{\Theta(\bar{u}, t)} \frac{\partial \Theta(\bar{u}, t)}{\partial t} \\ = & \frac{1}{\exp \left(i \sum_{k=1}^N u_k x_k \right)} \left(i \sum_{j=1}^N u_j \left\langle \left(\bar{K}_1 \right)_j \exp \left(i \sum_{k=1}^N u_k x_k \right) \right\rangle \right. \\ & \left. - \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N u_j u_k \left\langle \left(\bar{\bar{K}}_2 \right)_{jk} \exp \left(i \sum_{l=1}^N u_l x_l \right) \right\rangle + \left\langle \phi(\bar{x}, t) \exp \left(i \sum_{l=1}^N u_l x_l \right) \right\rangle - \langle \phi(t) \rangle \Theta(\bar{u}, t) \right) \end{aligned} \quad (C10)$$

It is now a simple (although tedious) matter to obtain the sought after statistical quantities $\langle \bar{x}(t) \rangle_j$ and $(\bar{\sigma})_{jk} \equiv \langle (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) \rangle$, defined by the conditional probability density $p(\bar{x}, t) \equiv p(\bar{x}, t | \bar{x}_0, t_0)$, where

$$\langle \bar{x}(t) \rangle_j \equiv \int_{-\infty}^{\infty} x_j(t) p(\bar{x}, t) dx_j, \quad (\bar{\sigma})_{jk} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) p(\bar{x}, t) dx_j dx_k \quad (C11)$$

as well as $\langle \phi(t) \rangle$ given by Equation (C3). (The form of these expressions come directly from the requirement that the mean values are optimal in the mean-square sense. This is what is meant as ‘optimal’ estimates in what is to follow.) In terms of the characteristic functional Θ ,

$$\left\langle \langle \bar{x}(t) \rangle_j \right\rangle = \frac{1}{i} \frac{\partial \ln \Theta}{\partial u_j} \Big|_{u_j=0}, \quad (\bar{\sigma})_{jk} = \left(\frac{1}{i} \right)^2 \frac{\partial^2 \ln \Theta}{\partial u_j \partial u_k} \Big|_{u_j=0, u_k=0} \quad (C12)$$

Intermediate steps will not be included at this point; again, these are straightforward manipulations but they can be a bit tedious. Using Equation (C10) in the first relation of Equation (C12), one has the general result

$$\frac{\partial \langle \langle \bar{x}(t) \rangle_j \rangle}{\partial t} = \langle \langle \bar{K}_1 \rangle_j \rangle + \langle \langle x_j - \langle x_j \rangle \rangle \phi(\bar{x}, t) \rangle \quad (C13)$$

or in full matrix form

$$\frac{\partial \langle \bar{x}(t) \rangle}{\partial t} = \langle \bar{K}_1 \rangle + \langle \langle \bar{x} - \langle \bar{x} \rangle \rangle \phi(\bar{x}, t) \rangle \quad (C14)$$

Similarly, for the covariance given by the second expression of Equation (C12), using Equation (C10) yields

$$\begin{aligned} \frac{\partial (\bar{\sigma})_{jk}}{\partial t} = & \langle \langle x_j - \langle x_j \rangle \rangle \langle \bar{K}_1 \rangle_k \rangle + \langle \langle x_k - \langle x_k \rangle \rangle \langle \bar{K}_1 \rangle_j \rangle + \langle \bar{K}_2 \rangle_{jk} \\ & + \langle \langle x_j - \langle x_j \rangle \rangle \langle x_k - \langle x_k \rangle \rangle \phi(\bar{x}, t) \rangle - (\bar{\sigma})_{jk} \langle \phi(t) \rangle \end{aligned} \quad (C15)$$

Remember that in both Equations (C13) and (C15), in general, $\bar{K}_1 = \bar{K}_1(\bar{x})$ and $\bar{K}_2 = \bar{K}_2(\bar{x})$. This situation severely complicates the solution of the problem. The necessary approximation that must be made in order to make any further analytical progress is to assume the following: Given that the equations that will be derived for the mean and covariance of the process are to be recursively solved over observational sampling time intervals that are small such that the processes do not significantly change, one will have estimates of the initial values of $\langle \bar{x} \rangle$ from a previous time sample. Thus, the values $\bar{K}_1 = \bar{K}_1(\langle \bar{x} \rangle)$ and $\bar{K}_2 = \bar{K}_2(\langle \bar{x} \rangle)$ will be known, as well as that of $\phi(\langle \bar{x} \rangle, t)$ from previous (*a priori*) observations. Given the assumption that the processes do not evolve too quickly over intervals between observations, one can expand about the value $\langle \bar{x} \rangle$ these functionals in terms of Taylor series, i.e.,

$$\begin{aligned} (\bar{K}_1(\bar{x}))_j = & (\bar{K}_1(\langle \bar{x} \rangle))_j + \sum_{k=1}^N (x_k - \langle x_k \rangle) \frac{\partial (\bar{K}_1(\bar{x}))_j}{\partial x_k} \Big|_{\bar{x}=\langle \bar{x} \rangle} \\ & + \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N (x_k - \langle x_k \rangle) (x_l - \langle x_l \rangle) \frac{\partial^2 (\bar{K}_1(\bar{x}))_j}{\partial x_k \partial x_l} \Big|_{\bar{x}=\langle \bar{x} \rangle} + \dots \end{aligned} \quad (C16)$$

$$\left(\bar{\bar{K}}_2(\bar{x})\right)_{jk} = \left(\bar{\bar{K}}_2(\langle\bar{x}\rangle)\right)_{jk} + \sum_{l=1}^N (x_l - \langle x_l \rangle) \frac{\partial \left(\bar{\bar{K}}_2(\bar{x})\right)_{jk}}{\partial x_l} \Big|_{\bar{x}=\langle\bar{x}\rangle} + \dots \quad (C17)$$

$$\begin{aligned} \phi(\bar{x}, t) = & \phi(\langle\bar{x}\rangle, t) + \sum_{j=1}^N (x_j - \langle x_j \rangle) \frac{\partial \phi(\bar{x}, t)}{\partial x_j} \Big|_{\bar{x}=\langle\bar{x}\rangle} \\ & + \frac{1}{2} \sum_{j=1}^N \sum_{k=1}^N (x_j - \langle x_j \rangle)(x_k - \langle x_k \rangle) \frac{\partial^2 \phi(\bar{x}, t)}{\partial x_j \partial x_k} \Big|_{\bar{x}=\langle\bar{x}\rangle} + \dots \end{aligned} \quad (C18)$$

Using these relations in Equation (C13) and evaluating the resulting averages gives

$$\begin{aligned} \frac{\partial \langle \bar{x}(t) \rangle_j}{\partial t} = & \left(\bar{K}_1(\langle\bar{x}\rangle)\right)_j + \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N (\bar{\sigma})_{kl} \frac{\partial \left(\bar{K}_1(\bar{x})\right)_j}{\partial x_k \partial x_l} \Big|_{\bar{x}=\langle\bar{x}\rangle} \\ & + \sum_{k=1}^N (\bar{\sigma})_{jk} \frac{\partial \phi(\bar{x}, t)}{\partial x_k} \Big|_{\bar{x}=\langle\bar{x}\rangle} \approx \left(\bar{K}_1(\langle\bar{x}\rangle)\right)_j + \sum_{k=1}^N (\bar{\sigma})_{jk} \frac{\partial \phi(\bar{x}, t)}{\partial x_k} \Big|_{\bar{x}=\langle\bar{x}\rangle} \end{aligned} \quad (C19)$$

In full matrix form, Equation (C19) is

$$\boxed{\frac{\partial \langle \bar{x}(t) \rangle}{\partial t} = \bar{K}_1(\langle\bar{x}\rangle) + \bar{\sigma} \frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \Big|_{\bar{x}=\langle\bar{x}\rangle}} \quad (C20)$$

Solution of this equation yields the **optimal estimate** of the process. Similarly, using Equations (C16)–(C18) in Equation (C15) gives, after some tedious manipulations

$$\begin{aligned} \frac{\partial (\bar{\sigma})_{jk}}{\partial t} = & \sum_{l=1}^N (\bar{\sigma})_{jl} \frac{\partial \left(\bar{K}_1(\bar{x})\right)_k}{\partial x_l} \Big|_{\bar{x}=\langle\bar{x}\rangle} + \sum_{l=1}^N (\bar{\sigma})_{kl} \frac{\partial \left(\bar{K}_1(\bar{x})\right)_j}{\partial x_l} \Big|_{\bar{x}=\langle\bar{x}\rangle} \\ & + \left(\bar{\bar{K}}_2(\langle\bar{x}\rangle)\right)_{jk} + \sum_{l=1}^N \sum_{m=1}^N (\bar{\sigma})_{jl} \frac{\partial^2 \phi(\bar{x}, t)}{\partial x_l \partial x_m} \Big|_{\bar{x}=\langle\bar{x}\rangle} (\bar{\sigma})_{mk} \end{aligned} \quad (C21)$$

where the fact was used that the Markov components x_j are described by Gaussian statistics thus causing averages involving odd number of components x_j to vanish and those involving four components are reduced to three products of second order averages. The matrix form of Equation (C21) is given by

$$\boxed{\frac{\partial \bar{\sigma}}{\partial t} = \bar{\sigma} \frac{\partial \bar{K}_1(\bar{x})}{\partial \bar{x}} \Big|_{\bar{x}=\langle\bar{x}\rangle} + \frac{\partial \bar{K}_1(\bar{x})}{\partial \bar{x}} \Big|_{\bar{x}=\langle\bar{x}\rangle} \bar{\sigma} + \bar{\bar{K}}_2(\langle\bar{x}\rangle) + \bar{\sigma} \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T \Big|_{\bar{x}=\langle\bar{x}\rangle} \bar{\sigma}} \quad (C22)$$

Solution of this differential equation gives the **error covariance of the optimal estimate**. Equations (C20) and (C22), in conjunction with Equation (C3), constitute the foundation of the non-linear Markov

estimation (filtering) technique that will be applied to rain attenuation, scintillation, and the additional scenario of the adaptive adjustment of the parameter values that enter into the observation model for $A_{obs}(t)$.

Before this program can be implemented, the form of the non-linear Riccati differential equation of Equation (C22) must be simplified. To this end, consider the multiplication of Equation (C22) by the inverse matrix $\bar{\bar{\sigma}}^{-1}$ from both the left and right hand sides,

$$\bar{\bar{\sigma}}^{-1} \frac{\partial \bar{\bar{\sigma}}}{\partial t} \bar{\bar{\sigma}}^{-1} = \bar{\bar{\sigma}}^{-1} \left(\bar{\bar{\sigma}} \frac{\partial \bar{K}_1(\bar{x})}{\partial \bar{x}} \bigg|_{\bar{x}=\langle \bar{x} \rangle} + \frac{\partial \bar{K}_1(\bar{x})}{\partial \bar{x}} \bigg|_{\bar{x}=\langle \bar{x} \rangle} \bar{\bar{\sigma}} + \bar{K}_2(\langle \bar{x} \rangle) \right) \bar{\bar{\sigma}}^{-1} + \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T \bigg|_{\bar{x}=\langle \bar{x} \rangle} \quad (C23)$$

Using the matrix identity $-\frac{\partial \bar{\bar{\sigma}}^{-1}}{\partial t} = \bar{\bar{\sigma}}^{-1} \frac{\partial \bar{\bar{\sigma}}}{\partial t} \bar{\bar{\sigma}}^{-1}$ on the left side gives

$$\frac{\partial \bar{\bar{\sigma}}^{-1}}{\partial t} = -\bar{\bar{\sigma}}^{-1} \left(\bar{\bar{\sigma}} \frac{\partial \bar{K}_1(\langle \bar{x} \rangle)}{\partial \bar{x}} \bigg|_{\bar{x}=\langle \bar{x} \rangle} + \frac{\partial \bar{K}_1(\langle \bar{x} \rangle)}{\partial \bar{x}} \bigg|_{\bar{x}=\langle \bar{x} \rangle} \bar{\bar{\sigma}} + \bar{K}_2(\langle \bar{x} \rangle) \right) \bar{\bar{\sigma}}^{-1} - \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T \bigg|_{\bar{x}=\langle \bar{x} \rangle} \quad (C24)$$

The expression within the parenthesis of the first term of Equation (C24) is just the linear portion of Equation (C22). Writing this linear portion as a partial time derivative of some new matrix $\bar{\bar{H}}$, i.e.,

$$\boxed{\frac{\partial \bar{\bar{H}}}{\partial t} \equiv \bar{\bar{\sigma}} \frac{\partial \bar{K}_1(\langle x \rangle)}{\partial \bar{x}} + \frac{\partial \bar{K}_1(\langle x \rangle)}{\partial \bar{x}} \bar{\bar{\sigma}} + \bar{K}_2(\langle \bar{x} \rangle)} \quad (C25)$$

Equation (C24) can be written

$$\frac{\partial \bar{\bar{\sigma}}^{-1}}{\partial t} = -\bar{\bar{\sigma}}^{-1} \frac{\partial \bar{\bar{H}}}{\partial t} \bar{\bar{\sigma}}^{-1} - \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T \bigg|_{\bar{x}=\langle \bar{x} \rangle} \quad (C26)$$

Applying the approximation, only in Equation (C26), that $\bar{\bar{\sigma}}^{-1} \approx \bar{\bar{H}}^{-1}$ during the time interval at which $\partial \bar{\bar{H}} / \partial t$ is evaluated, one has, using the same matrix identity as above,

$$\boxed{\frac{\partial \bar{\bar{\sigma}}^{-1}}{\partial t} = \frac{\partial \bar{\bar{H}}^{-1}}{\partial t} - \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \right)^T \bigg|_{\bar{x}=\langle \bar{x} \rangle}} \quad (C27)$$

Equations (C25) and (C27) replace the Riccati equation of Equation (C22) for the solution for the error covariance matrix $\bar{\bar{\sigma}}$ of the optimal estimate. The matrix $\bar{\bar{H}}$ that was introduced above is, by its definition, also an error covariance not of the optimal estimate but of an extrapolated estimate because, unlike that of $\bar{\bar{\sigma}}$, it is devoid of any observational support within the interval $t_i \leq t < t_{i+1}$ (i.e., the last term of Equation (C22) is missing). It is very important to remember that this circumstance forms the boundary condition that will prevail for the solution of Equations (C25) and (C27). The matrix $\bar{\bar{H}}$ is the **error**

covariance of the extrapolated estimate. Only after it is corrected with the observational input in Equation (C28) does it return to that of the optimal estimate.

C.1 The Discrete Solutions for the Filter Equations (C20), (C25), and (C27)

For application purposes, the differential equations derived above that give the optimal estimates of the mean value $\langle \bar{x} \rangle$ as well as the covariance $\bar{\sigma}$ of the Markov process as defined by the expressions of Equation (C17) of the Langevin note set must now be solved as discrete equations in time. For the purpose of specializing the foregoing to the Markov process of the diffusion type considered for satellite signal attenuation discussed in Appendix A, one recalls Equations (A41)

$$\bar{\Gamma} \equiv \begin{pmatrix} \gamma_R & & & \\ & \gamma_S & & \\ & & \ddots & \\ & & & \gamma_N \end{pmatrix} \quad (C28)$$

and correspondingly, Equation (A42)

$$\bar{G} \equiv \begin{pmatrix} \sqrt{2\gamma_1} & 0 & \dots & 0 \\ 0 & \sqrt{2\gamma_2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \sqrt{2\gamma_N} \end{pmatrix} \quad (C29)$$

thus allowing the designation for the problem considered here

$$\bar{K}_1(\bar{x}, t) = -\bar{\Gamma} \bar{x} \quad (C30)$$

However, what are required are the average values $\langle \bar{x} \rangle$,

$$\bar{K}_1(\langle \bar{x} \rangle, t) = -\bar{\Gamma} \langle \bar{x} \rangle \quad (C31)$$

Finally, using the diffusion correlation coefficient matrix \bar{B} which involves the potential correlation of the various component processes, one has

$$\left(\bar{K}_2(\bar{x}, t) \right)_{ij} = 2B_{ij} \sqrt{\gamma_i \gamma_j} \quad (C32)$$

With these assignments now made, Equation (C20) becomes

$$\frac{\partial \langle \bar{x} \rangle}{\partial t} = -\bar{\Gamma} \langle \bar{x} \rangle + \bar{\sigma} \frac{\partial \phi(\bar{x}, t)}{\partial \bar{x}} \Big|_{\bar{x}=\langle \bar{x} \rangle} \quad (C33)$$

Discrete solutions of these type of differential matrix equations commence with the homogeneous form

$$\frac{\partial \langle \bar{x} \rangle}{\partial t} = -\bar{\Gamma} \langle \bar{x} \rangle \quad (\text{C34})$$

for which the solution is taken to be of the form

$$\langle \bar{x} \rangle = \langle \bar{x}(t) \rangle = \bar{\Phi}(t, t_0) \langle \bar{x}(t_0) \rangle, \quad t \geq t_0 \quad (\text{C35})$$

Using this in Equation (C34) results in

$$\left(\frac{\partial \bar{\Phi}(t, t_0)}{\partial t} + \bar{\Gamma} \bar{\Phi}(t, t_0) \right) \langle \bar{x}(t) \rangle = 0 \quad (\text{C36})$$

Since $\langle \bar{x}(t) \rangle$ is arbitrary in this particular construction, one has for the fundamental transition matrix $\bar{\Phi}(t, t_0)$ for the problem,

$$\frac{\partial \bar{\Phi}(t, t_0)}{\partial t} = -\bar{\Gamma} \bar{\Phi}(t, t_0) \quad (\text{C37})$$

Furthermore, since $\langle \bar{x}(t_0) \rangle = \bar{\Phi}(t_0, t_0) \langle \bar{x}(t_0) \rangle$, one has $(\bar{I} - \bar{\Phi}(t_0, t_0)) \langle \bar{x}(t_0) \rangle = 0$ where \bar{I} is the identity matrix. Hence, the initial condition is established,

$$\bar{I} = \bar{\Phi}(t_0, t_0) \quad (\text{C38})$$

In the case of the form of the matrix $\bar{\Gamma}$ given by Equation (C29), one has that

$$\boxed{\bar{\Phi}(t, t_0) = \exp(-\bar{\Gamma}(t - t_0))} \quad (\text{C39})$$

The discrete time solution of Equation (C33) is now straightforward. Using the transition matrix development given above for the homogeneous portion of Equation (C33), the complete solution becomes

$$\boxed{\langle \bar{x}(t_{i+1}) \rangle = \bar{\Phi}(t_{i+1}, t_i) \langle \bar{x}(t_i) \rangle + \bar{\Phi}(t_{i+1}, t_i) \bar{\sigma}(t_i) \int_{t_i}^{t_{i+1}} \frac{\partial \phi(\bar{x}, t')}{\partial \bar{x}} \Big|_{\bar{x}=\langle \bar{x} \rangle} dt'} \quad (\text{C40})$$

giving the optimal estimate of the process. Substituting Equations (C31) and (C32) into Equation (C25) yields

$$\frac{\partial \bar{H}}{\partial t} = -\bar{\sigma} \bar{\Gamma} - \bar{\Gamma} \bar{\sigma} + \bar{K}_2 \quad (\text{C41})$$

The discrete solution to this relation for the sampling time t_i , as per the boundary condition considerations mentioned earlier, is found to be

$$\boxed{\bar{H}(t_i) = \bar{\Phi}(t_i, t_{i-1}) \bar{\sigma}(t_{i-1}) \bar{\Phi}^T(t_i, t_{i-1}) + \bar{D}} \quad (C42)$$

where

$$\boxed{\bar{D} = \bar{\Phi}(t_{i+1}, t_i) \left(\int_{t_i}^{t_{i+1}} \bar{\Phi}(t', t_i) \bar{K}_2 \bar{\Phi}^T(t', t_i) dt' \right) \bar{\Phi}^T(t_{i+1}, t_i)} \quad (C43)$$

is a function of the interval $\Delta t = t_{i+1} - t_i$ and is thus a constant throughout the iterative process so long as the interval between samples remains constant. Here, the matrix \bar{D} is the error covariance of the process components related to the variations imparted by the diffusion coefficient $\bar{K}_2(\bar{x}, t)$; it is thusly termed *the error covariance due to diffusion* of the process. Finally, the simple structure of Equation (C27) gives, again noting the boundary conditions that prevail at the sampling times as discussed earlier,

$$\boxed{\bar{\sigma}^{-1}(t_{i+1}) = \bar{H}^{-1}(t_i) - \int_{t_i}^{t_{i+1}} \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t')}{\partial \bar{x}} \right)^T \Bigg|_{\bar{x}=\langle \bar{x} \rangle} dt'} \quad (C44)$$

which, as discussed above, is the error covariance of the optimal estimate.

A few more points are in order before these relationships are used to reproduce previous results. First, the observational input discriminator function $\phi(\langle \bar{x} \rangle, t)$ is only sampled at discrete times t_i separated by a constant time interval Δt , i.e., $\phi(\langle \bar{x} \rangle, t) = \delta(t - t_i) \phi(\langle \bar{x} \rangle, t) \Delta t$ for $t_i \leq t < t_{i+1}$. Hence, for the last term of Equation (C40), one has

$$\int_{t_i}^{t_{i+1}} \frac{\partial \phi(\bar{x}, t')}{\partial \bar{x}} \Bigg|_{\bar{x}=\langle \bar{x} \rangle} dt' = \frac{\partial \phi(\bar{x}, t_i)}{\partial \bar{x}} \Delta t \Bigg|_{\bar{x}=\langle \bar{x} \rangle} \equiv \bar{a}(t_i) \quad (C45)$$

Similarly, for the last term of Equation (C44),

$$\int_{t_i}^{t_{i+1}} \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t')}{\partial \bar{x}} \right)^T \Bigg|_{\bar{x}=\langle \bar{x} \rangle} dt' = \left(\frac{\partial}{\partial \bar{x}} \right) \left(\frac{\partial \phi(\bar{x}, t_i)}{\partial \bar{x}} \right)^T \Delta t \Bigg|_{\bar{x}=\langle \bar{x} \rangle} \equiv \bar{b}(t_i) \quad (C46)$$

Thus, Equations (C40) and (C44) can be rewritten in the form (Ref. 5)

$$\boxed{\langle \bar{x}(t_{i+1}) \rangle = \bar{\Phi}(t_{i+1}, t_i) \langle \bar{x}(t_i) \rangle + \bar{\Phi}(t_{i+1}, t_i) \bar{\sigma}(t_i) \bar{a}(t_i)} \quad (C47)$$

$$\boxed{\bar{\sigma}^{-1}(t_{i+1}) = \bar{H}^{-1}(t_i) - \bar{b}(t_i)} \quad (C48)$$

Second, the expression given by Equation (C43) can be simplified based on the simple structure of the matrix $\bar{\bar{R}}$ given by Equation (C32). Using additionally Equation (C39), the error covariance due to diffusion is, by its very nature, only a function of the observational sampling interval $\Delta t = t_{i+1} - t_i$

$$\boxed{\left(\bar{\bar{D}}(\Delta t)\right)_{ij} = 2B_{ij} \frac{\sqrt{\gamma_i \gamma_j}}{\gamma_i + \gamma_j} \exp(-(\gamma_i + \gamma_j)\Delta t) \left(1 - \exp(-(\gamma_i + \gamma_j)\Delta t)\right)} \quad (C49)$$

Remembering Equation (C3), i.e.,

$$\boxed{\phi(\bar{x}, t) \equiv -\frac{1}{2\sigma_n^2} (A_{obs}(t) - A(\bar{x}, t))^2} \quad (C50)$$

one now has all the necessary ingredients to apply this non-linear Markov filtering procedure to both the filtering of a multi-component diffusion process as well as the adaptive determination of the parameters that enter into the modeled observation function $A(\bar{x}, t)$.

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