Trajectory Design Employing Convex Optimization for Landing on Irregularly Shaped Asteroids

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Goal

- Goal: Design an optimal powered descent trajectory on-board the spacecraft in order to softly land on an irregularly shaped asteroid.
  - Algorithm needs to be autonomous, reliable, robust, and efficient.
  - Designing on-board facilitates an easy change of parameters.
- Convex optimization is efficient and reliable.
  - Guarantees global minimum in a finite number of steps, if the problem is feasible.
  - Subclasses include Second Order Cone Programming (SOCP).
- ***Can convex optimization be used to design the asteroid powered descent trajectory?***

Original Problem Formulation

- Asteroid powered descent propellant optimal problem is nonlinear and nonconvex.

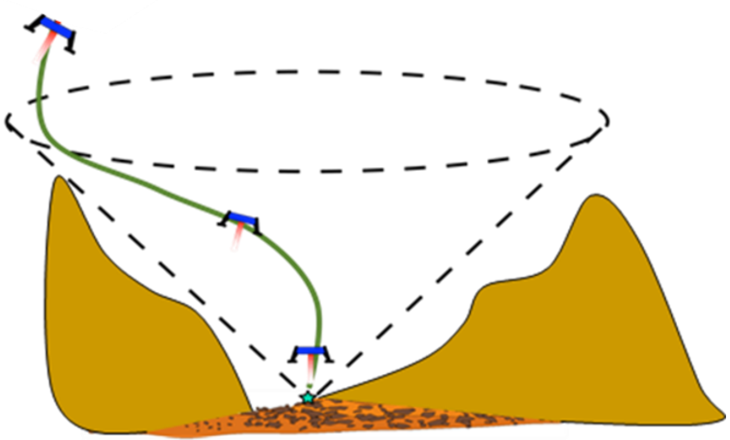
$$\min -m(t_f)$$

$$s.t.\ \dot{\vec{r}} = \vec{v},\ \dot{\vec{v}} = \frac{\vec{T}}{m} - 2\vec{\omega}\times\vec{v} - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \nabla U(\vec{r}),\ \dot{m} = -\frac{1}{v_{ex}}\left\|\vec{T}\right\|$$

$$T_{min} \le \left\|\vec{T}\right\| \le T_{max},\ \|\vec{r}-\vec{r}_f\|\,cos\theta - (\vec{r}-\vec{r}_f)^T\,\hat{n} \le 0,\ m \ge m_{dry}$$

$$\vec{r}(0) = \vec{r}_0,\ \vec{v}(0) = \vec{v}_0,\ m(0) = m_{wet},\ \vec{r}(t_f) = \vec{r}_f,\ \vec{v}(t_f) = \vec{v}_f,\ t_f\ \text{given}$$

- Fixed final time two point value boundary problem
- State: $\vec{r},\ \vec{v},\ m$
- Control: $\vec{T}$
- Highlighted terms are not permissible for a convex optimization problem.



Problem Relaxation

Relax the problem by introducing a slack variable, T_m.

Original Problem

$$\min -m(t_f)$$

$$s.t.\ \dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v} - \dot{\vec{\omega}} \times \vec{r} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \nabla U(\vec{r})$$

$$\dot{m} = -\frac{1}{v_{ex}} \left\| \vec{T} \right\|$$

$$T_{min} \leq \left\| \vec{T} \right\| \leq T_{max}$$

$$\|\vec{r} - \vec{r}_f\| \cos\theta - (\vec{r} - \vec{r}_f)^T \hat{n} \leq 0$$

$$m \geq m_{dry}$$

$$\vec{r}(0) = \vec{r}_0,\ \vec{v}(0) = \vec{v}_0,\ m(0) = m_{wet}$$

$$\vec{r}(t_f) = \vec{r}_f,\ \vec{v}(t_f) = \vec{v}_f,\ t_f \text{ given}$$

Relaxed Problem

$$\min -m(t_f)$$

$$s.t.\ \dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v} - \dot{\vec{\omega}} \times \vec{r} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \nabla U(\vec{r})$$

$$\dot{m} = -\frac{1}{v_{ex}} T_m$$ Slack Variable

$$\left\| \vec{T} \right\| \leq T_m$$ New Constraint

$$T_{min} \leq T_m \leq T_{max}$$

$$\|\vec{r} - \vec{r}_f\| \cos\theta - (\vec{r} - \vec{r}_f)^T \hat{n} \leq 0$$

$$m \geq m_{dry}$$

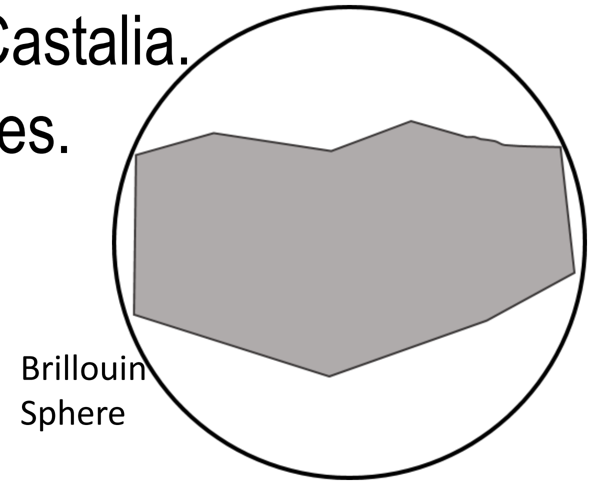
$$\vec{r}(0) = \vec{r}_0,\ \vec{v}(0) = \vec{v}_0,\ m(0) = m_{wet}$$

$$\vec{r}(t_f) = \vec{r}_f,\ \vec{v}(t_f) = \vec{v}_f,\ t_f \text{ given}$$

Proved the optimal solution of the relaxed problem is the optimal solution of the original.

Irregularly Shaped Asteroid Gravity Models

- 4x4 Spherical Harmonics Model
  - Maximum Order and Degree 4
  - No symmetry nor coordinate system location and alignment assumptions.
  - High accuracy outside the Brillouin sphere.
  - Not valid inside the Brillouin sphere.
- Interior spherical Bessel gravity model
  - Valid inside the entire Brillouin sphere.
  - Error less than 10% for the binary asteroid Castalia.
  - Published in 2014 by Takahashi and Scheeres.



4x4 Bessel

- 4x4 spherical harmonics gravity model outside the Brillouin sphere.
- Interior spherical Bessel gravity model inside the Brillouin sphere.
- Both models are summation series.
- Highly nonlinear in terms of spacecraft position vector.
- Computational similarities between the models allows for easy transition between the models.

Successive Solution Method

- Solve a series of convex optimization problems.
- Equations of motion can be arranged as:

$$\dot{\vec{x}}^{(k)} = A\left(\vec{r}^{(k-1)}\right)\vec{x}^{(k)} + B\vec{u}^{(k)} + c\left(\vec{r}^{(k-1)}\right)$$

- A and c are evaluated using the previous solution (k-1).
- In the (k)th iteration, dynamics are linear and time varying.
- Iterations continue until two successive trajectories are within a set tolerance.
- This is not the same as conventional linearization, as there are no approximations in the final iteration.
- Dominant gravity term is placed in A, with the higher order gravity terms in c.

Successive Solution Method: A, B, c

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \omega^2 + dom & 0 & 0 & 0 & 2\omega & 0 & 0 \\ 0 & \omega^2 + dom & 0 & -2\omega & 0 & 0 & 0 \\ 0 & 0 & dom & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{(k-1)} \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{v_{ex}} \end{bmatrix}$$

$$c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial U}{\partial r_x} - dom \\ \frac{\partial U}{\partial r_y} - dom \\ \frac{\partial U}{\partial r_z} - dom \\ 0 \end{bmatrix}^{(k-1)}$$

Formulation assumes rotation vector is along the +Z axis, $\vec{\omega} = \omega \hat{z}$.

$$dom = \begin{cases} -\frac{\mu}{r^3} & 4 \times 4 \\ \alpha_{0,0} j_1 \left(\frac{\alpha_{0,0} r}{R_b}\right) \bar{A}_{0,0,0} + \alpha_{1,0} j_1 \left(\frac{\alpha_{1,0} r}{R_b}\right) \bar{A}_{1,0,0} + \alpha_{2,0} j_1 \left(\frac{\alpha_{2,0} r}{R_b}\right) \bar{A}_{2,0,0} & Bessel \end{cases}$$

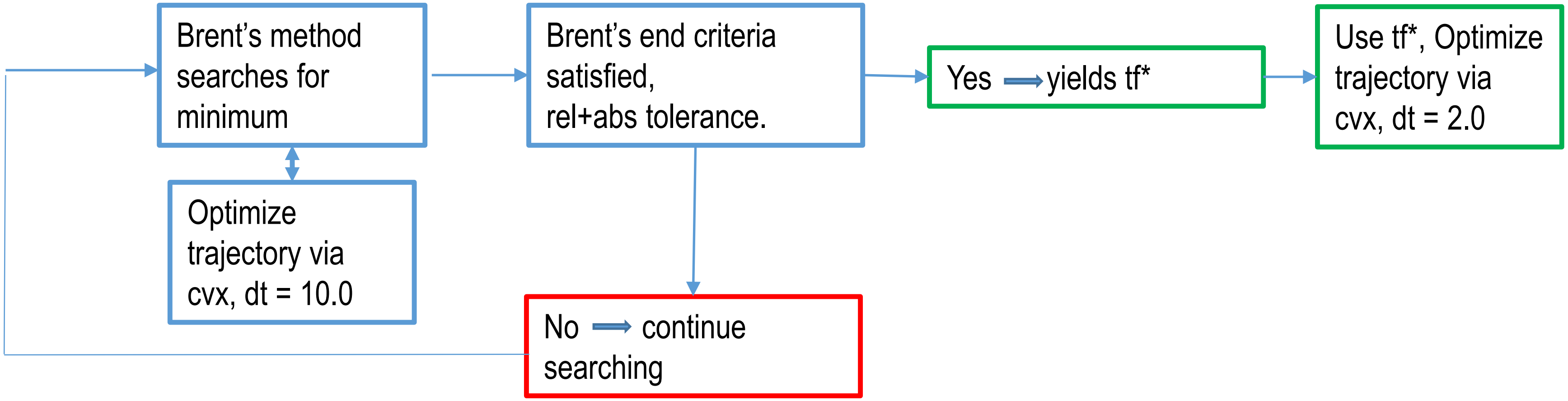
Additional Techniques

- Change of Variables
- Discretization
  - Continuous equation of motion turned into discrete equality constraints.
- Scaling

- Final optimization problem is convex.
  - Linear equality constraints
  - Convex inequality constraints
  - Inequality constraints are second order cone.
  - Actually a SOCP

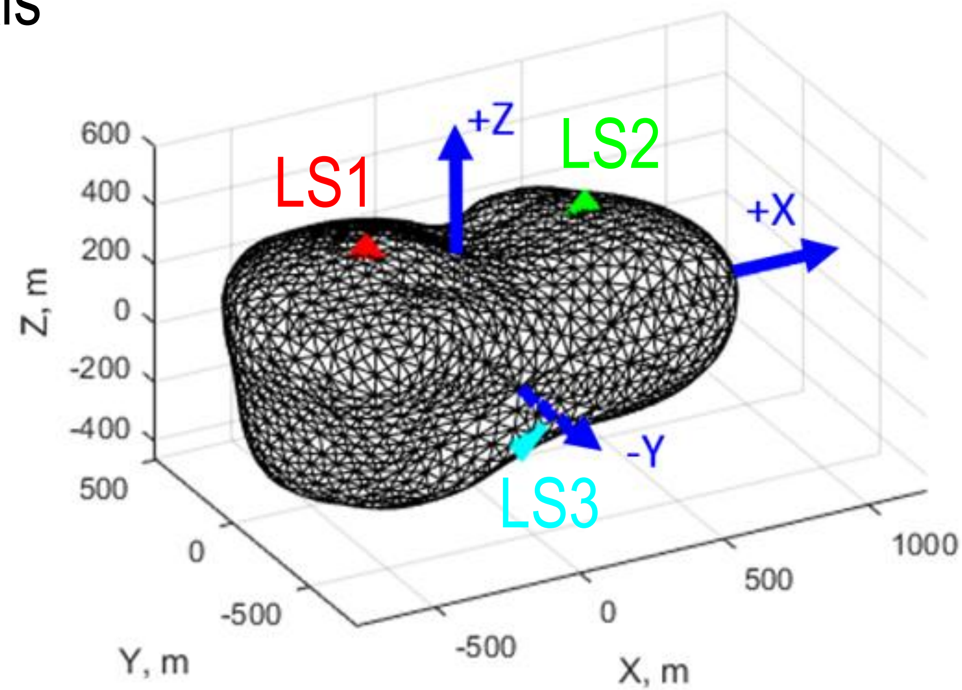
Optimal Flight Time

- Desire to find the optimal flight time corresponding to smallest propellant usage.
- Propellant usage is unimodal with respect to flight time.
- Create an outer optimization loop using Brent's method to optimize the flight time.
- Use *dt* = 10.0 sec to find the optimal flight time. Design the final trajectory with *dt* = 2.0 sec.



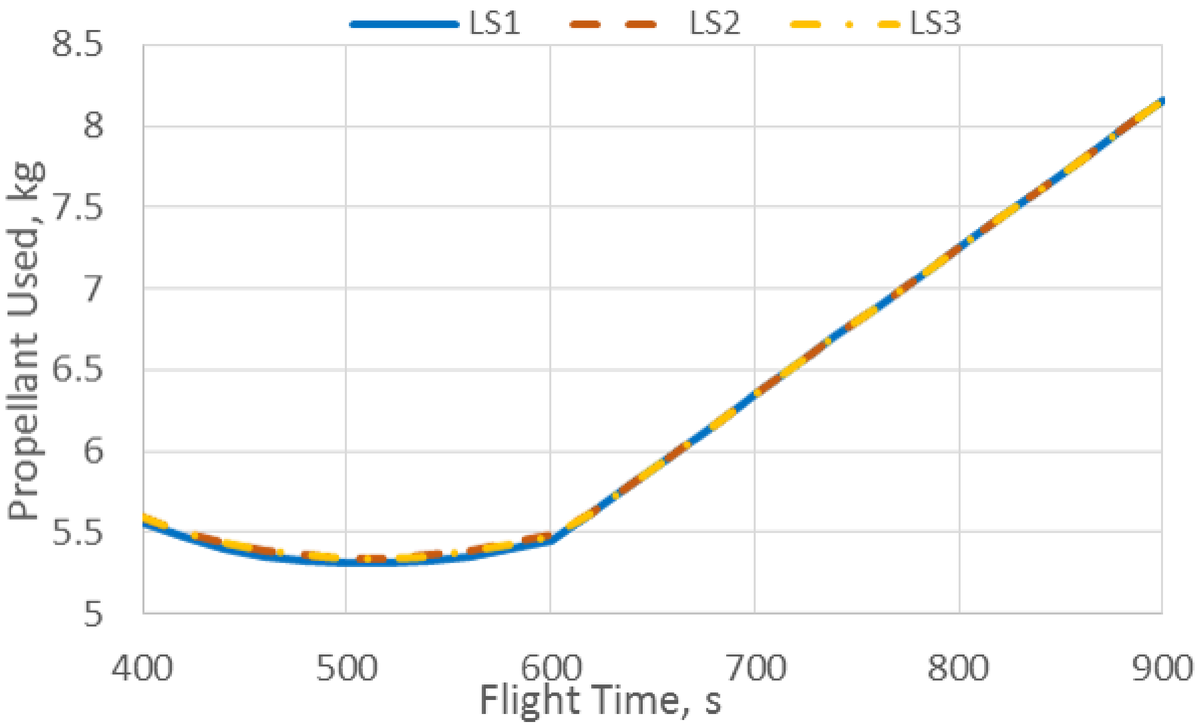
Simulation Parameters

- Asteroid Castalia
  - Period 4.07 hr along +Z axis
  - Three Landing Sites
- Spacecraft:
  - Mass 1400 kg
    - ◊ 400 kg propellant
  - Thrust 80 N – 20 N



- Initial Conditions
  - 1000 m altitude
  - Out of plane and uprange position and velocity components.
- Using CVX, a publically available Matlab based convex optimization program.

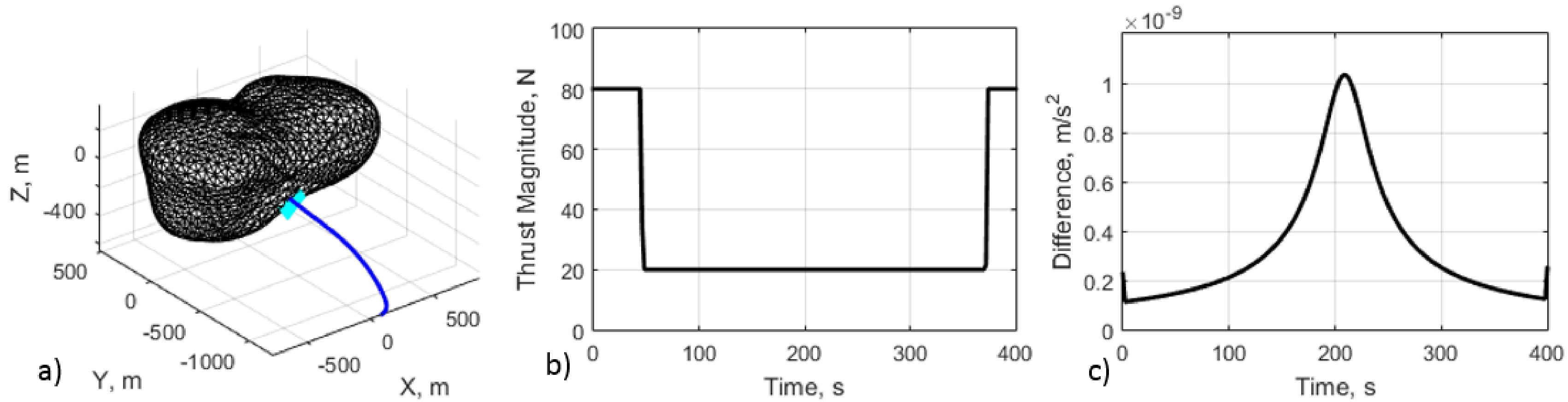
Flight Time Parameter Sweep



- Typically 3 iterations required in the successive solution method.
  - Range 3 – 7
- Low number of iterations demonstrates stability in the successive solution methodology.

Inner Loop Trajectory Design

- 400 Second Flight Time landing at LS3
- Thrust profile follows the traditional bang-bang.

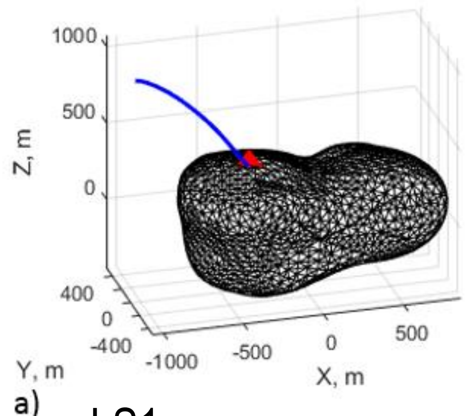


Optimal Flight Time Optimal Propellant Trajectory

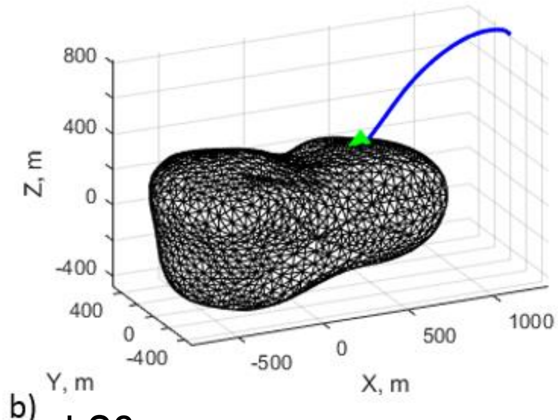
- Combined outer and inner loop executions took 2.2 - 2.5 minutes.

	Optimal Flight Time, s	Propellant Used, kg	Number of Inner Loop Executions
LS1	512.86	5.31	7
LS2	512.27	5.34	7
LS3	513.35	5.34	7

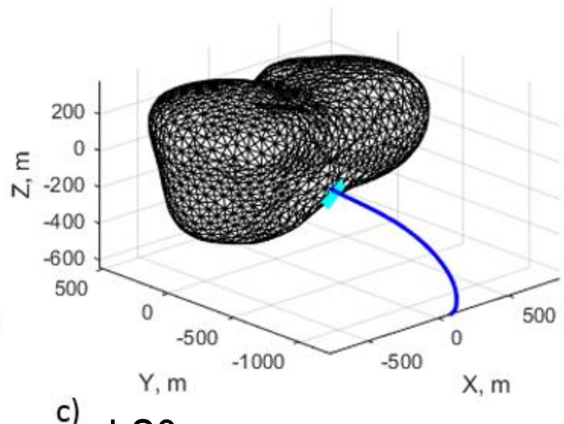
Optimal Flight Time Optimal Propellant Trajectory Parameters



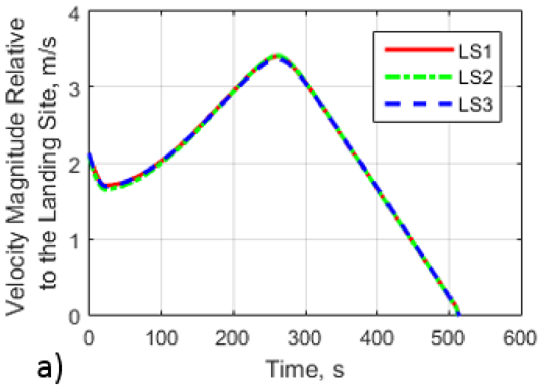
LS1



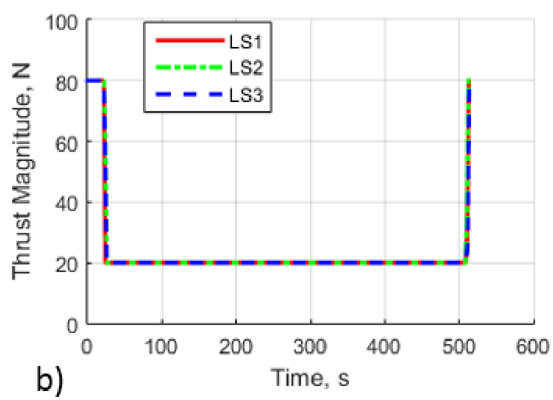
LS2



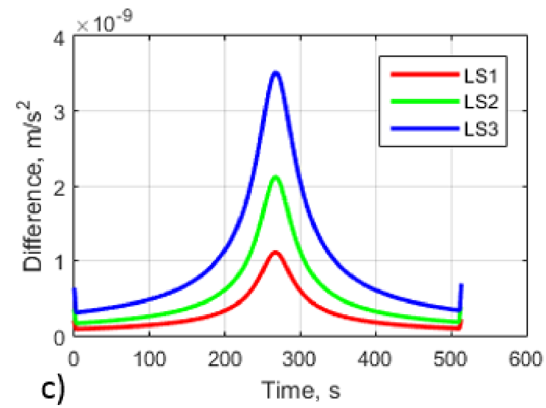
LS3



Velocity Magnitude



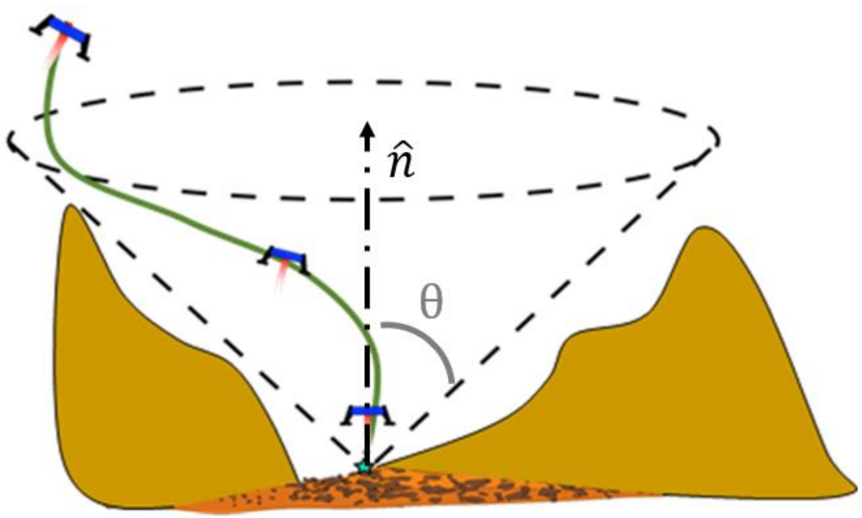
Thrust Magnitude



Slack Variable Check

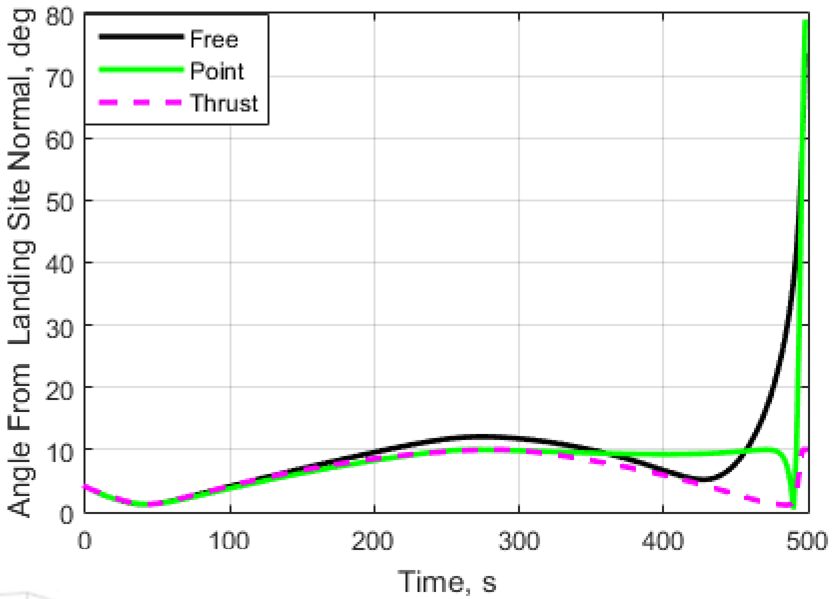
Glide Slope Constraint

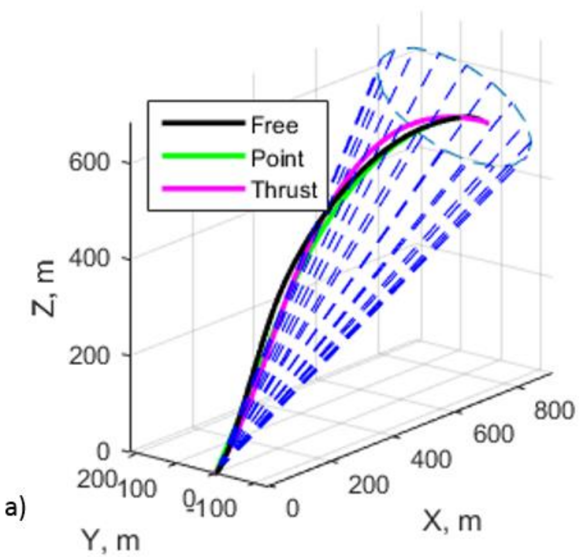
- Glide slope constraint: Constrains the vehicle to fly inside a cone around the landing site.
  - $\|\vec{r} - \vec{r}_f\| \cos\theta - (\vec{r} - \vec{r}_f)^T \hat{n} \leq 0$
- Near the landing site the vehicle must match the landing site velocity to rotate with the landing site.
- Low thrust of the vehicle (80 N) prohibits this.
- Alternate solutions for a 10 deg cone:
  - Increase max thrust to 320 N
  - Enforce the constraint for all, but the last 6 seconds.
    - ◊ Flies slightly outside the cone near the surface.

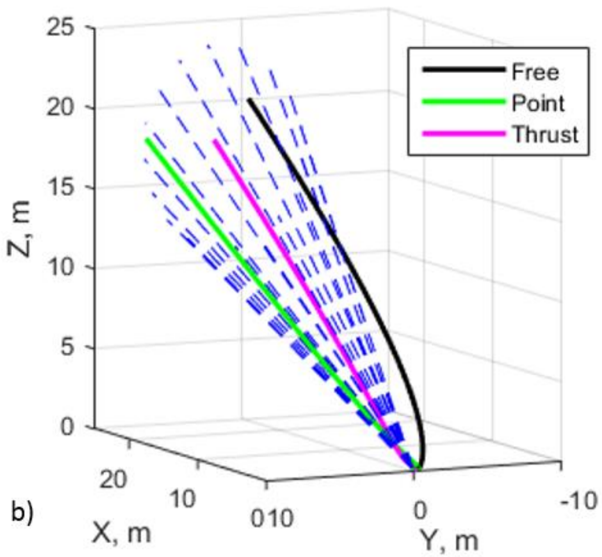


Glide Slope Results

- LS2 500 second flight time.
- 10 degree cone enforced.







Conclusions

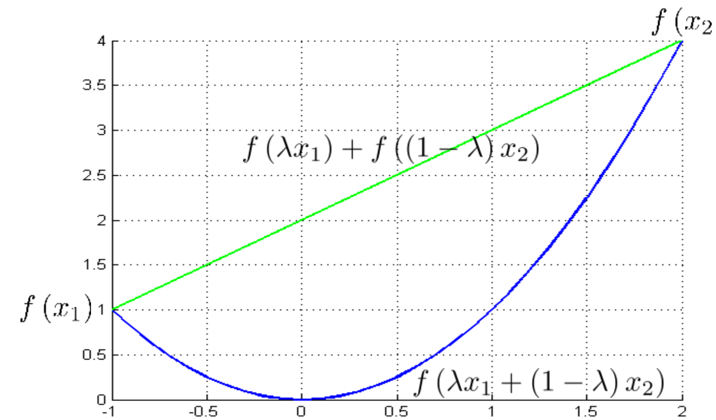
- Asteroid powered descent trajectory design can be formulated as a convex optimization problem.
- Successive solution methodology is the key to handling a nonlinear gravity model.
- Formulated algorithm handles a wide range of parameters successfully.
- Flight time optimization is completed in an outer loop with Brent's method.
- Inclusion of additional trajectory constraints in the algorithm is feasible.
- Viable algorithm for rapidly designing asteroid powered descent trajectories autonomously on-board the spacecraft for use in a variety of guidance algorithms.

BACK-UP

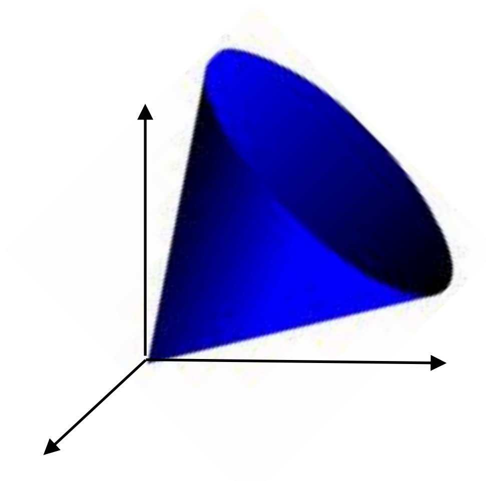
Convex Optimization and SOCP Formulation

- Optimization problem formulation

$$\min\ g(x)$$
$$s.t.\ f_i(x) \leq 0 \quad i = 1, \ldots, m$$
$$h_j(x) = 0 \quad j = 1, \ldots, p$$



- Convex Optimization
  - $g(x)$ and $f_i(x)$ are convex functions.
  - $h_j(x)$ is linear.
  - Convex Function: $f(\lambda x_1 + (1-\lambda) x_2) \leq f(\lambda x_1) + f((1-\lambda) x_2)$
- Second Order Cone Program (SOCP)
  - Subset of convex optimization
  - $g(x)$ and $h_j(x)$ are linear functions.
  - $f_i(x)$ is second order cone.
  - Second order Cone: $\|Mx + d\|_2 \leq c$



Spherical Harmonics Gravity Model

- Fidelity determined by the coefficients and the number of terms in the summation series.

$$N = \begin{cases} 2 & 2 \times 2 \\ 4 & 4 \times 4 \end{cases}$$

- $\nabla U(\vec{r})$ with respect to the Cartesian coordinate system:

$$\frac{\partial U}{\partial r_x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \delta}\frac{\partial \delta}{\partial x} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial x}$$
$$\frac{\partial U}{\partial r_y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \delta}\frac{\partial \delta}{\partial y} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial y}$$
$$\frac{\partial U}{\partial r_z} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U}{\partial \delta}\frac{\partial \delta}{\partial z} + \frac{\partial U}{\partial \lambda}\frac{\partial \lambda}{\partial z}$$

$P_{l,m}$ associated Legendre function
$l,\ m$ order, degree
$r,\ \delta,\ \lambda$ radius, latitude, longitude

- Partial of the gravitational potential with respect to the position vector in spherical coordinates:

$$\frac{\partial U}{\partial r} = \sum_{l=0}^{N}\sum_{m=0}^{l} -(l+1)\frac{\mu}{r^2}\left(\frac{r_0}{r}\right)^l P_{l,m}[\sin\delta]\left\{C_{l,m}\cos(m\lambda) + S_{l,m}\sin(m\lambda)\right\}$$
$$\frac{\partial U}{\partial \delta} = \sum_{l=0}^{N}\sum_{m=0}^{l} \frac{\mu}{r}\left(\frac{r_0}{r}\right)^l \left\{C_{l,m}\cos(m\lambda) + S_{l,m}\sin(m\lambda)\right\}\frac{\partial P_{l,m}[\sin\delta]}{\partial \delta}$$
$$\frac{\partial U}{\partial \lambda} = \sum_{l=0}^{N}\sum_{m=0}^{l} \frac{\mu}{r}\left(\frac{r_0}{r}\right)^l mP_{l,m}[\sin\delta]\left\{-C_{l,m}\sin(m\lambda) + S_{l,m}\cos(m\lambda)\right\}$$

- Partial of the position vector in spherical coordinate system with respect to the Cartesian:

$$\frac{\partial r}{\partial x} = \frac{r_x}{r},\ \frac{\partial r}{\partial y} = \frac{r_y}{r},\ \frac{\partial r}{\partial z} = \frac{r_z}{r}$$
$$\frac{\partial \delta}{\partial x} = \frac{-r_x r_z}{r^2\sqrt{r_x^2+r_y^2}},\ \frac{\partial \delta}{\partial y} = \frac{-r_y r_z}{r^2\sqrt{r_x^2+r_y^2}},\ \frac{\partial \delta}{\partial z} = \frac{1}{\sqrt{r_x^2+r_y^2}}\left(1 - \frac{r_z^2}{r^2}\right)$$
$$\frac{\partial \lambda}{\partial x} = \frac{-r_y}{r_x^2+r_y^2},\ \frac{\partial \lambda}{\partial x} = \frac{r_x}{r_x^2+r_y^2},\ \frac{\partial \lambda}{\partial z} = 0$$

Interior spherical Bessel Gravity Model

- Summation Series: $l_{max} = 2,\ n_{max} = 5,\ m_{max} = 5$

$$\frac{\partial U}{\partial r_x} = \frac{\mu}{R_b} \sum_{l=0}^{l_{max}} \sum_{n=0}^{n_{max}} \sum_{m=0}^{n} Re\left[\frac{\partial}{\partial x}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right)\right] \bar{A}_{l,n,m} + Im\left[\frac{\partial}{\partial x}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right)\right] \bar{B}_{l,n,m}$$

$$\frac{\partial U}{\partial r_y} = \frac{\mu}{R_b} \sum_{l=0}^{l_{max}} \sum_{n=0}^{n_{max}} \sum_{m=0}^{n} Re\left[\frac{\partial}{\partial y}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right)\right] \bar{A}_{l,n,m} + Im\left[\frac{\partial}{\partial y}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right)\right] \bar{B}_{l,n,m}$$

$$\frac{\partial U}{\partial r_z} = \frac{\mu}{R_b} \sum_{l=0}^{l_{max}} \sum_{n=0}^{n_{max}} \sum_{m=0}^{n} Re\left[\frac{\partial}{\partial z}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right)\right] \bar{A}_{l,n,m} + Im\left[\frac{\partial}{\partial z}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right)\right] \bar{B}_{l,n,m}$$

- Basis Functions: $\bar{\beta}_{n,m}\left(\alpha_{l,n}\right) = j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n,m}$

$$\bar{H}_{n,m} = \begin{cases} N_{n,m} P_{n,m}\left[sin(\phi)\right] e^{im\lambda} & n \geq m \geq 0 \\ 0 & otherwise \end{cases}$$

- Partials of the Basis Functions:

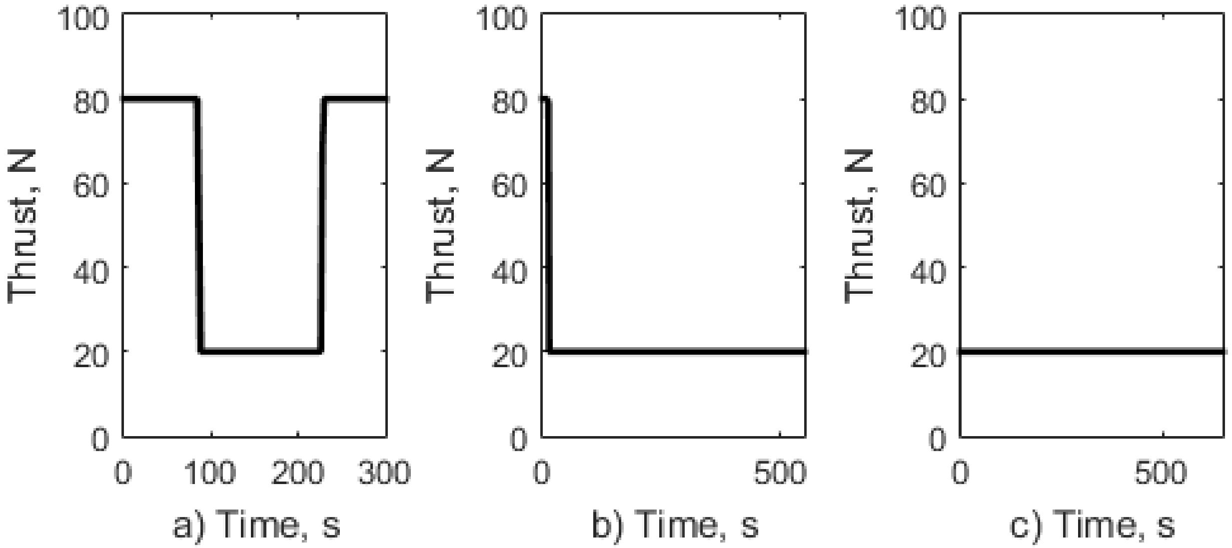
$$\frac{\partial}{\partial x}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right) = \begin{cases} -\frac{\alpha_{l,n} r_x}{R_b r} j_{n+1}\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n,0} - 2\mathscr{F}_1(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] N_{n-1,1} P_{n-1,1}\left[sin(\phi)\right] cos\lambda & m = 0 \\ -\frac{\alpha_{l,n} r_x}{R_b r} j_{n+1}\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n,m} - \mathscr{F}_1(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n-1,m+1} + \mathscr{F}_2(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n-1,m-1} & m > 0 \end{cases}$$

$$\frac{\partial}{\partial y}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right) = \begin{cases} -\frac{\alpha_{l,n} r_y}{R_b r} j_{n+1}\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n,0} - 2\mathscr{F}_1(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] N_{n-1,1} P_{n-1,1}\left[sin(\phi)\right] sin\lambda & m = 0 \\ -\frac{\alpha_{l,n} r_y}{R_b r} j_{n+1}\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n,m} + \mathscr{F}_1(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] i\,\bar{H}_{n-1,m+1} + \mathscr{F}_2(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] i\,\bar{H}_{n-1,m-1} & m > 0 \end{cases}$$

$$\frac{\partial}{\partial z}\left(\bar{\beta}_{n,m}\left(\alpha_{l,n}\right)\right) = -\frac{\alpha_{l,n} r_z}{R_b r} j_{n+1}\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n,m} + \mathscr{F}_3(n,m)\frac{1}{r} j_n\left[\frac{\alpha_{l,n} r}{R_b}\right] \bar{H}_{n-1,m}$$

Thrust Profile

- Three classes of thrust profiles



a) 300 s, b) 550 s, c) 650 s