Trajectory Design Employing Convex Optimization for Landing on Irregularly Shaped Asteroids

Robin Pinson (presenter)
NASA Marshall Space Flight Center
Ping Lu
San Diego State University

September 14, 2016
SPACE 2016 Long Beach, CA
Goal

Goal: Design an optimal powered descent trajectory on-board the spacecraft in order to softly land on an irregularly shaped asteroid.

- Algorithm needs to be autonomous, reliable, robust, and efficient.
- Designing on-board facilitates an easy change of parameters.

Convex optimization is efficient and reliable.

- Guarantees global minimum in a finite number of steps, if the problem is feasible.
- Subclasses include Second Order Cone Programming (SOCP).

Can convex optimization be used to design the asteroid powered descent trajectory?
Original Problem Formulation

- Asteroid powered descent propellant optimal problem is nonlinear and nonconvex.

\[
\begin{align*}
\min & \quad -m(t_f) \\
\text{subject to} & \quad \dot{r} = \ddot{v}, \quad \ddot{v} = \frac{T}{m} - 2\hat{\omega} \times \dot{v} - \hat{\omega} \times \dot{r} - \hat{\omega} \times (\hat{\omega} \times \dot{r}) + \nabla U(r), \quad \dot{m} = -\frac{1}{v_{ex}} \left\| \ddot{T} \right\| \\
& \quad T_{min} \leq \left\| \ddot{T} \right\| \leq T_{max}, \quad \left\| \dot{r} - \dot{r}_f \right\| \cos \theta - (\dot{r} - \dot{r}_f)^T \hat{n} \leq 0, \quad m \geq m_{dry} \\
& \quad \dot{r}(0) = \dot{r}_0, \quad \ddot{v}(0) = \ddot{v}_0, \quad m(0) = m_{wet}, \quad \dot{r}(t_f) = \dot{r}_f, \quad \ddot{v}(t_f) = \ddot{v}_f, \quad t_f \text{ given}
\end{align*}
\]

- Fixed final time two point value boundary problem
- State: $\dot{r}$, $\ddot{v}$, $m$
- Control: $\ddot{T}$
- Highlighted terms are not permissible for a convex optimization problem.
Problem Relaxation

Relax the problem by introducing a slack variable, \( T_m \).

**Original Problem**

\[
\begin{align*}
\text{min } & -m(t_f) \\
\text{s.t. } & \dot{\vec{r}} = \vec{v} \\
& \dot{\vec{v}} = \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v} - \vec{\omega} \times \dot{\vec{r}} \\
& \quad - \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \nabla U(\vec{r}) \\
& \dot{m} = -\frac{1}{v_{ex}} \left\| \vec{T} \right\|
\end{align*}
\]

**Relaxed Problem**

\[
\begin{align*}
\text{min } & -m(t_f) \\
\text{s.t. } & \dot{\vec{r}} = \vec{v} \\
& \dot{\vec{v}} = \frac{\vec{T}}{m} - 2\vec{\omega} \times \vec{v} - \vec{\omega} \times \dot{\vec{r}} \\
& \quad - \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \nabla U(\vec{r}) \\
& \dot{m} = -\frac{1}{v_{ex}} \left[ T_m \right] \\
\left\| \vec{T} \right\| & \leq T_m \\
\left\| \vec{r} - \vec{r}_f \right\| \cos \theta - (\vec{r} - \vec{r}_f)^T \hat{n} & \leq 0 \\
\left\| \vec{r} - \vec{r}_f \right\| \cos \theta - (\vec{r} - \vec{r}_f)^T \hat{n} & \leq 0 \\
m & \geq m_{dry} \\
\vec{r}(0) = \vec{r}_0, \quad \vec{v}(0) = \vec{v}_0, \quad m(0) = m_{wet} \\
\vec{r}(t_f) = \vec{r}_f, \quad \vec{v}(t_f) = \vec{v}_f, \quad t_f \text{ given}
\end{align*}
\]

Proved the optimal solution of the relaxed problem is the optimal solution of the original.
Irregularly Shaped Asteroid Gravity Models

- 4x4 Spherical Harmonics Model
  - Maximum Order and Degree 4
  - No symmetry nor coordinate system location and alignment assumptions.
  - High accuracy outside the Brillouin sphere.
  - Not valid inside the Brillouin sphere.

- Interior spherical Bessel gravity model
  - Valid inside the entire Brillouin sphere.
  - Error less than 10% for the binary asteroid Castalia.
  - Published in 2014 by Takahashi and Scheeres.
4x4 Bessel

- 4x4 spherical harmonics gravity model outside the Brillouin sphere.
- Interior spherical Bessel gravity model inside the Brillouin sphere.
- Both models are summation series.
- Highly nonlinear in terms of spacecraft position vector.
- Computational similarities between the models allows for easy transition between the models.
Successive Solution Method

- Solve a series of convex optimization problems.
- Equations of motion can be arranged as:
  \[ \dot{\mathbf{x}}^{(k)} = A \left( \mathbf{r}^{(k-1)} \right) \mathbf{x}^{(k)} + B \mathbf{u}^{(k)} + c \left( \mathbf{r}^{(k-1)} \right) \]
- \( A \) and \( c \) are evaluated using the previous solution \((k-1)\).
- In the \((k)\)th iteration, dynamics are linear and time varying.
- Iterations continue until two successive trajectories are within a set tolerance.
- This is not the same as conventional linearization, as there are no approximations in the final iteration.
- Dominant gravity term is placed in \( A \), with the higher order gravity terms in \( c \).
Successive Solution Method: A, B, c

\[ A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \omega^2 + \text{dom} & 0 & 0 & 0 & 2\omega & 0 & 0 \\ 0 & \omega^2 + \text{dom} & 0 & -2\omega & 0 & 0 & 0 \\ 0 & 0 & \text{dom} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{(k-1)} \]

\[ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{v_{ex}} \end{bmatrix} \]

\[ c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\partial U}{\partial r_x} - \text{dom} \\ \frac{\partial U}{\partial r_y} - \text{dom} \\ \frac{\partial U}{\partial r_z} - \text{dom} \\ 0 \end{bmatrix}^{(k-1)} \]

Formulation assumes rotation vector is along the +Z axis, \( \vec{\omega} = \omega \hat{z} \).

\[ \text{dom} = \begin{cases} -\frac{\mu}{r^3} \\ \alpha_{0,0} j_1 \left( \frac{\alpha_{0,0} r}{R_{\text{b}}} \right) \bar{A}_{0,0,0} + \alpha_{1,0} j_1 \left( \frac{\alpha_{1,0} r}{R_{\text{b}}} \right) \bar{A}_{1,0,0} + \alpha_{2,0} j_1 \left( \frac{\alpha_{2,0} r}{R_{\text{b}}} \right) \bar{A}_{2,0,0} \end{cases} \text{Bessel} \]
Additional Techniques

- Change of Variables
- Discretization
  - Continuous equation of motion turned into discrete equality constraints.
- Scaling

Final optimization problem is convex.
  - Linear equality constraints
  - Convex inequality constraints
  - Inequality constraints are second order cone.
  - Actually a SOCP
Optimal Flight Time

- Desire to find the optimal flight time corresponding to smallest propellant usage.
- Propellant usage is unimodal with respect to flight time.
- Create an outer optimization loop using Brent’s method to optimize the flight time.
- Use $dt = 10.0$ sec to find the optimal flight time. Design the final trajectory with $dt = 2.0$ sec.

![Optimization Flowchart]

Brent’s method searches for minimum

Optimize trajectory via cvx, $dt = 10.0$

Brent’s end criteria satisfied, rel+abs tolerance.

Yes $\Rightarrow$ yields $tf^*$

Use $tf^*$, Optimize trajectory via cvx, $dt = 2.0$

No $\Rightarrow$ continue searching
Simulation Parameters

- Asteroid Castalia
  - Period 4.07 hr along +Z axis
  - Three Landing Sites

- Spacecraft:
  - Mass 1400 kg
    - 400 kg propellant
  - Thrust 80 N – 20 N

- Initial Conditions
  - 1000 m altitude
  - Out of plane and uprange position and velocity components.

- Using CVX, a publically available Matlab based convex optimization program.
Flight Time Parameter Sweep

- Typically 3 iterations required in the successive solution method.
  - Range 3 – 7
- Low number of iterations demonstrates stability in the successive solution methodology.
Inner Loop Trajectory Design

- 400 Second Flight Time landing at LS3
- Thrust profile follows the traditional bang-bang.
Combined outer and inner loop executions took 2.2 - 2.5 minutes.

<table>
<thead>
<tr>
<th></th>
<th>Optimal Flight Time, s</th>
<th>Propellant Used, kg</th>
<th>Number of Inner Loop Executions</th>
</tr>
</thead>
<tbody>
<tr>
<td>LS1</td>
<td>512.86</td>
<td>5.31</td>
<td>7</td>
</tr>
<tr>
<td>LS2</td>
<td>512.27</td>
<td>5.34</td>
<td>7</td>
</tr>
<tr>
<td>LS3</td>
<td>513.35</td>
<td>5.34</td>
<td>7</td>
</tr>
</tbody>
</table>
Optimal Flight Time Optimal Propellant Trajectory Parameters

Aviation diagram with graphs indicating velocity magnitude, thrust magnitude, and slack variable check for LS1, LS2, and LS3.
Glide Slope Constraint

- Glide slope constraint: Constrains the vehicle to fly inside a cone around the landing site.
  - \[ ||\vec{r} - \vec{r}_f|| \cos \theta - (\vec{r} - \vec{r}_f)^T \hat{n} \leq 0 \]

- Near the landing site the vehicle must match the landing site velocity to rotate with the landing site.

- Low thrust of the vehicle (80 N) prohibits this.

- Alternate solutions for a 10 deg cone:
  - Increase max thrust to 320 N
  - Enforce the constraint for all, but the last 6 seconds.
    - Flies slightly outside the cone near the surface.
Glide Slope Results

- LS2 500 second flight time.
- 10 degree cone enforced.
Conclusions

- Asteroid powered descent trajectory design can be formulated as a convex optimization problem.
- Successive solution methodology is the key to handling a nonlinear gravity model.
- Formulated algorithm handles a wide range of parameters successfully.
- Flight time optimization is completed in an outer loop with Brent’s method.
- Inclusion of additional trajectory constraints in the algorithm is feasible.
- Viable algorithm for rapidly designing asteroid powered descent trajectories autonomously on-board the spacecraft for use in a variety of guidance algorithms.
BACK-UP
Convex Optimization and SOCP Formulation

- **Optimization problem formulation**
  \[
  \min g(x) \\
  \text{s.t. } f_i(x) \leq 0 \quad i = 1, \ldots, m \\
  h_j(x) = 0 \quad j = 1, \ldots, p
  \]

- **Convex Optimization**
  - \(g(x)\) and \(f_i(x)\) are convex functions.
  - \(h_j(x)\) is linear.
  - Convex Function: \(f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2)\)

- **Second Order Cone Program (SOCP)**
  - Subset of convex optimization
  - \(g(x)\) and \(h_j(x)\) are linear functions.
  - \(f_i(x)\) is second order cone.
  - Second order Cone: \(\|Mx + d\|_2 \leq c\)
Spherical Harmonics Gravity Model

- Fidelity determined by the coefficients and the number of terms in the summation series.

\[ N = \begin{cases} 2 & 2 \times 2 \\ 4 & 4 \times 4 \end{cases} \]

- \( \nabla U(\vec{r}) \) with respect to the Cartesian coordinate system:

\[
\frac{\partial U}{\partial r_x} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial U}{\partial \delta} \frac{\partial \delta}{\partial x} + \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial x} \\
\frac{\partial U}{\partial r_y} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial U}{\partial \delta} \frac{\partial \delta}{\partial y} + \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial y} \\
\frac{\partial U}{\partial r_z} = \frac{\partial U}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial U}{\partial \delta} \frac{\partial \delta}{\partial z} + \frac{\partial U}{\partial \lambda} \frac{\partial \lambda}{\partial z}
\]

- Partial of the gravitational potential with respect to the position vector in spherical coordinates:

\[
\frac{\partial U}{\partial r} = \sum_{l=0}^{N} \sum_{m=0}^{1} (l + 1) \frac{\mu}{r^2} \left( \frac{r_0}{r} \right)^l P_{l,m} \sin \delta \left[ C_{l,m} \cos (m\lambda) + S_{l,m} \sin (m\lambda) \right]
\]

\[
\frac{\partial U}{\partial \delta} = \sum_{l=0}^{N} \sum_{m=0}^{1} \frac{\mu}{r} \left( \frac{r_0}{r} \right)^l \left[ C_{l,m} \cos (m\lambda) + S_{l,m} \sin (m\lambda) \right] \frac{\partial P_{l,m}}{\partial \delta} \sin \delta
\]

\[
\frac{\partial U}{\partial \lambda} = \sum_{l=0}^{N} \sum_{m=0}^{1} \frac{\mu}{r} \left( \frac{r_0}{r} \right)^l mP_{l,m} \sin \delta \left[ -C_{l,m} \sin (m\lambda) + S_{l,m} \cos (m\lambda) \right]
\]

- Partial of the position vector in spherical coordinate system with respect to the Cartesian:

\[
\frac{\partial r}{\partial x} = \frac{r_x}{r}, \quad \frac{\partial r}{\partial y} = \frac{r_y}{r}, \quad \frac{\partial r}{\partial z} = \frac{r_z}{r}
\]

\[
\frac{\partial \delta}{\partial x} = -\frac{r_x r_z}{r^2 \sqrt{r_x^2 + r_y^2}}, \quad \frac{\partial \delta}{\partial y} = -\frac{r_y r_z}{r^2 \sqrt{r_x^2 + r_y^2}}, \quad \frac{\partial \delta}{\partial z} = \frac{r_x r_y}{\sqrt{r_x^2 + r_y^2}} \left( 1 - \frac{r_z^2}{r^2} \right)
\]

\[
\frac{\partial \lambda}{\partial x} = -\frac{r_y}{r^2 \sqrt{r_x^2 + r_y^2}}, \quad \frac{\partial \lambda}{\partial y} = \frac{r_x}{r^2 \sqrt{r_x^2 + r_y^2}}, \quad \frac{\partial \lambda}{\partial z} = 0
\]

- \( P_{l,m} \) associated Legendre function
- \( l, m \) order, degree
- \( r, \delta, \lambda \) radius, latitude, longitude
Interior spherical Bessel Gravity Model

- **Summation Series:** \( l_{max} = 2, n_{max} = 5, m_{max} = 5 \)

\[
\frac{\partial U}{\partial r_x} = \frac{\mu}{R_b} \sum_{l=0}^{l_{max}} \sum_{n=0}^{n_{max}} \sum_{m=0}^{m_{max}} \text{Re} \left[ \frac{\partial}{\partial x} (\tilde{\beta}_{n,m}(\alpha_{l,n})) \right] \tilde{A}_{l,n,m} + \text{Im} \left[ \frac{\partial}{\partial x} (\tilde{\beta}_{n,m}(\alpha_{l,n})) \right] \tilde{B}_{l,n,m}
\]

\[
\frac{\partial U}{\partial r_y} = \frac{\mu}{R_b} \sum_{l=0}^{l_{max}} \sum_{n=0}^{n_{max}} \sum_{m=0}^{m_{max}} \text{Re} \left[ \frac{\partial}{\partial y} (\tilde{\beta}_{n,m}(\alpha_{l,n})) \right] \tilde{A}_{l,n,m} + \text{Im} \left[ \frac{\partial}{\partial y} (\tilde{\beta}_{n,m}(\alpha_{l,n})) \right] \tilde{B}_{l,n,m}
\]

\[
\frac{\partial U}{\partial r_z} = \frac{\mu}{R_b} \sum_{l=0}^{l_{max}} \sum_{n=0}^{n_{max}} \sum_{m=0}^{m_{max}} \text{Re} \left[ \frac{\partial}{\partial z} (\tilde{\beta}_{n,m}(\alpha_{l,n})) \right] \tilde{A}_{l,n,m} + \text{Im} \left[ \frac{\partial}{\partial z} (\tilde{\beta}_{n,m}(\alpha_{l,n})) \right] \tilde{B}_{l,n,m}
\]

- **Basis Functions:** \( \tilde{\beta}_{n,m}(\alpha_{l,n}) = j_n \left[ \frac{\alpha_{l,n}}{R_b} \right] \tilde{H}_{n,m} \)

\[
\tilde{H}_{n,m} = \begin{cases} 
N_{n,m} P_{n,m} \sin(\phi) e^{im\lambda} & n \geq m \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

- **Partials of the Basis Functions:**
Thrust Profile

- Three classes of thrust profiles

a) 300 s,

b) 550 s,

c) 650 s