Position Error Covariance Matrix Validation and Correction

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Abstract

In order to calculate operationally accurate collision probabilities, the position error covariance matrices predicted at times of closest approach must be sufficiently accurate representations of the position uncertainties. This presentation will discuss why the Gaussian distribution is a reasonable expectation for the position uncertainty and how this assumed distribution type is used in the validation and correction of position error covariance matrices.

Orbit Estimation

- Secondary orbit determination (not initial O.D.)
- Two techniques come to mind:
  - Batch estimation (to which most of this presentation refers)
  - Sequential estimation (some comments at the end)
- The orbit, or state, estimate is the mean expected value of the state for the given set of observations.
- As part of the estimation process, a state error covariance matrix representing the error in the estimate is also produced.
- Historically the overall accuracy of the state error covariance has not had a high importance.

The State Estimate

- The state estimate, since it is the mean expected value of the orbit, should conform to the usual statistical expectations of mean values.
- From the "Law of Large Numbers" we infer that the state estimate is in the neighborhood of the true but unknown state.
- From the "Central Limit Theorem" we infer that the distribution of possible true mean states is approximately Gaussian (normal) in the neighborhood of the state estimate.
- This is the basis of error covariance matrix validation/scaling.

The State Error Covariance Matrix

- Produced as part of the estimation process.
- Square and symmetric.
- Should be positive definite:
  - The determinant of each leading principal minor must be positive.
  - A necessary but not sufficient condition is that all correlation coefficients be on the interval (-1,1).
- Positive semi-definite implies zero variance(s) or perfect correlation(s).
- Covariance matrices should never be negative definite.
  (Rounding, truncating or computational errors may result in negative definiteness if one or more correlations are very near 1 or -1.)

The State Error Covariance Matrix (cont.)

- The state error covariance matrix should represent the inertia matrix of the probability density function of the state errors about the mean state.
- State error covariance matrices, provided as a product of orbit estimation, should not automatically be assumed to correctly represent multivariate Gaussian distributions describing the error distributions of states.
- Assuming the distribution of state errors is Gaussian, does the calculated error covariance matrix actually represent the true variations of the state errors about the mean?

The Reality of the Error Covariance Matrix

- The state error covariance matrix is conditioned on the type and expected uncertainty of observations used in the estimation process – theoretical error covariance matrix.
- State error covariance matrices may not account for unmodeled effects such as dynamic model errors, unmodeled observation errors and/or other simulation errors.
- The net effect is that state error covariance matrices usually underestimate the uncertainty of the error in the state estimate. (This may not be true if “process noise” is added.)

Position Error Covariance Matrix Scaling

There are two sets of implications which follow from the normal distribution assumption. These implications should apply across all solutions for all objects. One set of implications deals with the expectations for any single component of the position error vector. The second set of implications deals with the behavior of the complete position vector error.

Position Error Covariance Matrix Scaling

- Position component implications:
  - Each component should have a Gaussian probability distribution.
  - Normalized components should have zero mean error.
  - Normalized components should have a variance of 1.
- Position vector implications:
  - The probability distribution of the squared, normalized vector error should be that of a chi-square variable with three degrees of freedom.
  - The expected value of the chi-square variable should be three.

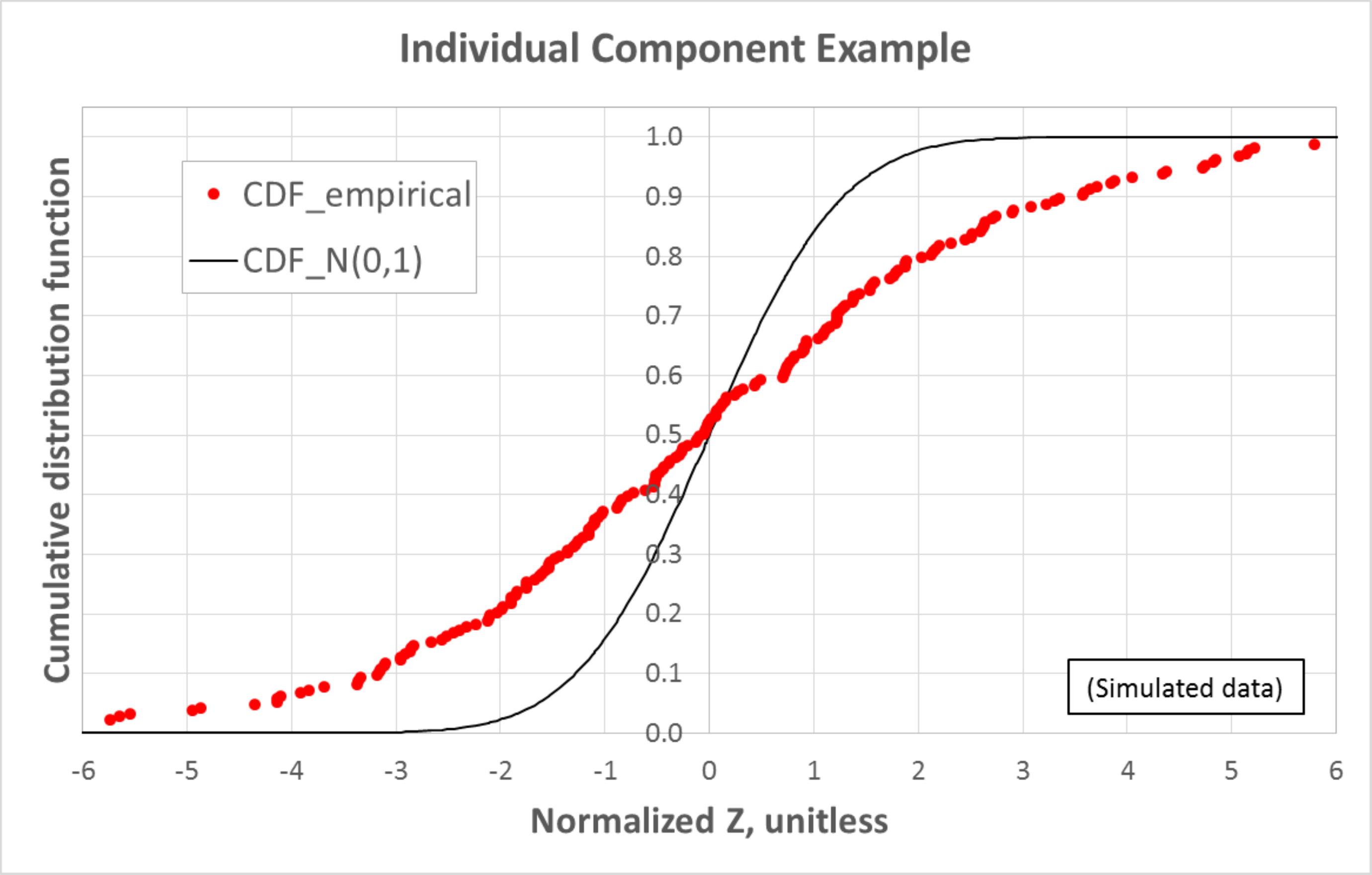
Data Description

- Two independent populations of solutions are need
  - Error covariance matrices are required for each.
  - Pairs of solutions are propagated to the same time.
  - Best if one solution represents the epoch or is within the fit interval.
- Relative errors are computed and normalized by relative variances at comparison times of interest.
  - The sum (difference) of independent Gaussian random variables is Gaussian.
  - The variance of the sum (difference) of Gaussian random variables is the sum of the variances of the independent random variables.

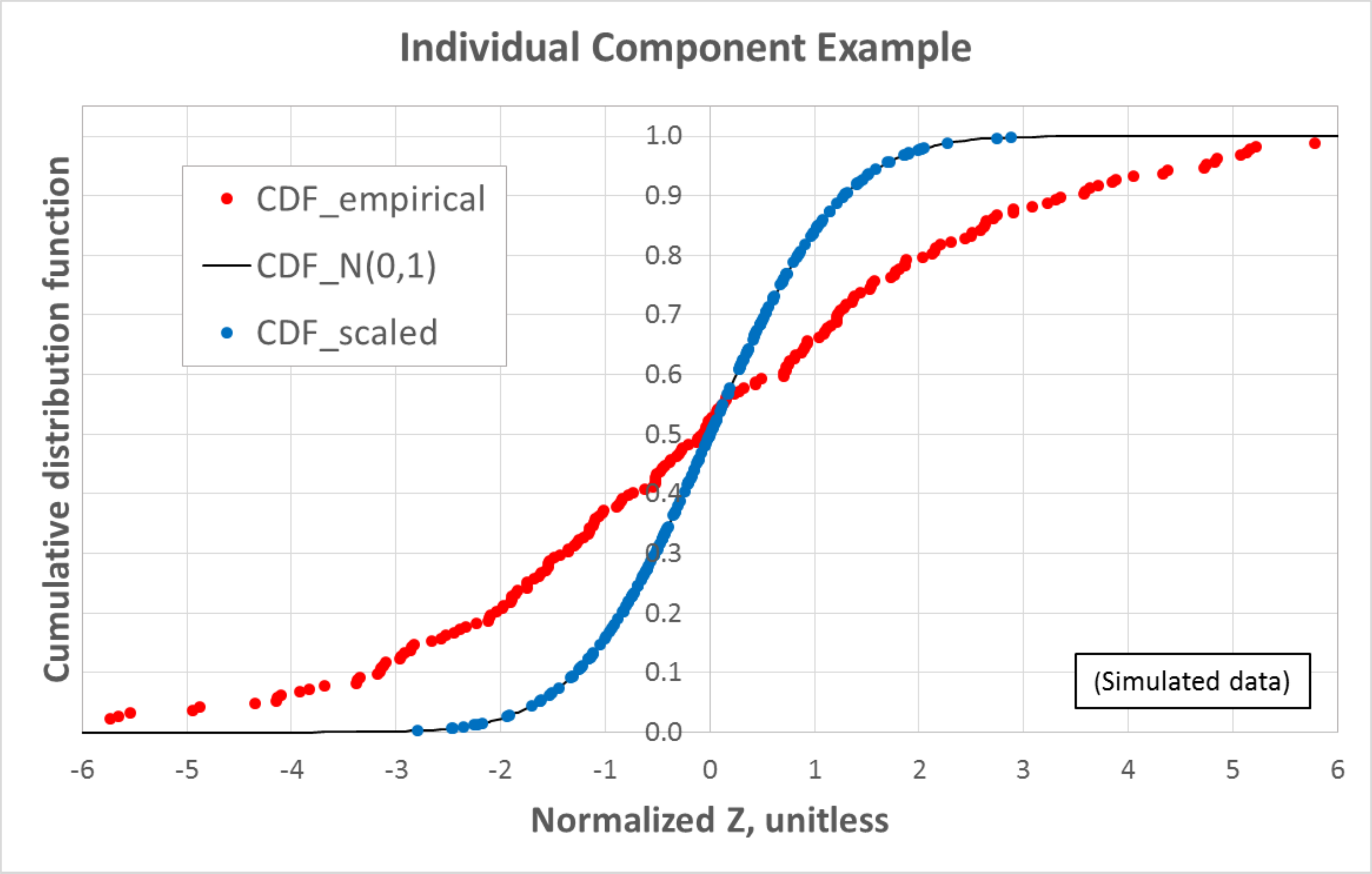
Position Error Covariance Matrix Scaling

- Individual relationships for any relative error component i and solution j:
  - In theory $z_{ij} = x_{ij} / \sigma_{ij}$ should be unit normal, N(0,1).
  - In practice: $z_{ij} = x_{ij} / (\alpha_i s_{ij})$, for proper selection of α_i, should be N(0,1).
- Vector relationships (of principal interest) for any solution j:
  - In theory $X_j^2 = r_j^T C_j^{-1} r_j$ where X_j^2 has a chi-square distribution with 3 degrees of freedom.
  - In practice (single factor) $X_j^2 = r_j^T (\alpha^2 C_j)^{-1} r_j$ should be 3-dof chi-square with proper selection of α.
  - In practice (matrix) $X_j^2 = r_j^T (A^T C_j A)^{-1} r_j$ should be 3-dof chi-square with proper selection of A. (A is usually diagonal and the components are about the same as the individual component scale factors, α_i.)

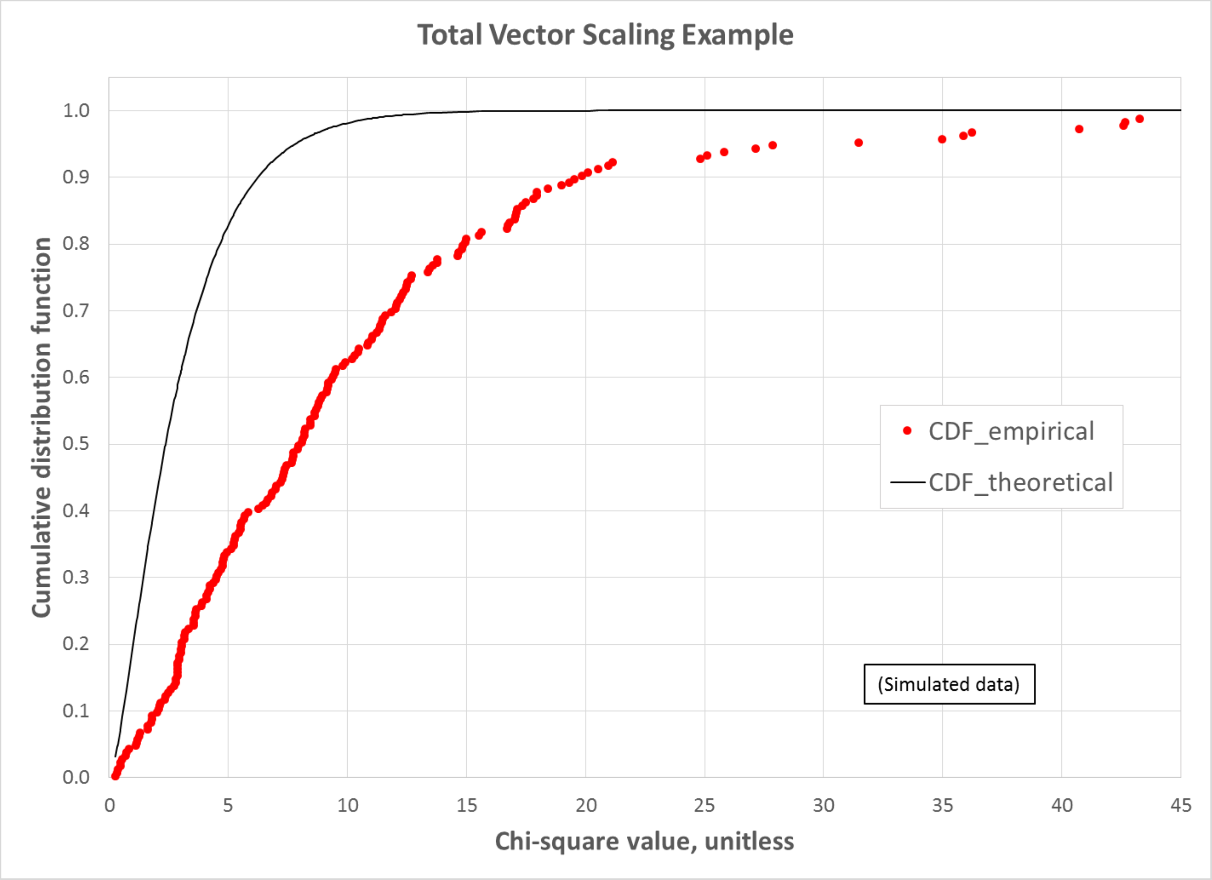
Single Component Example



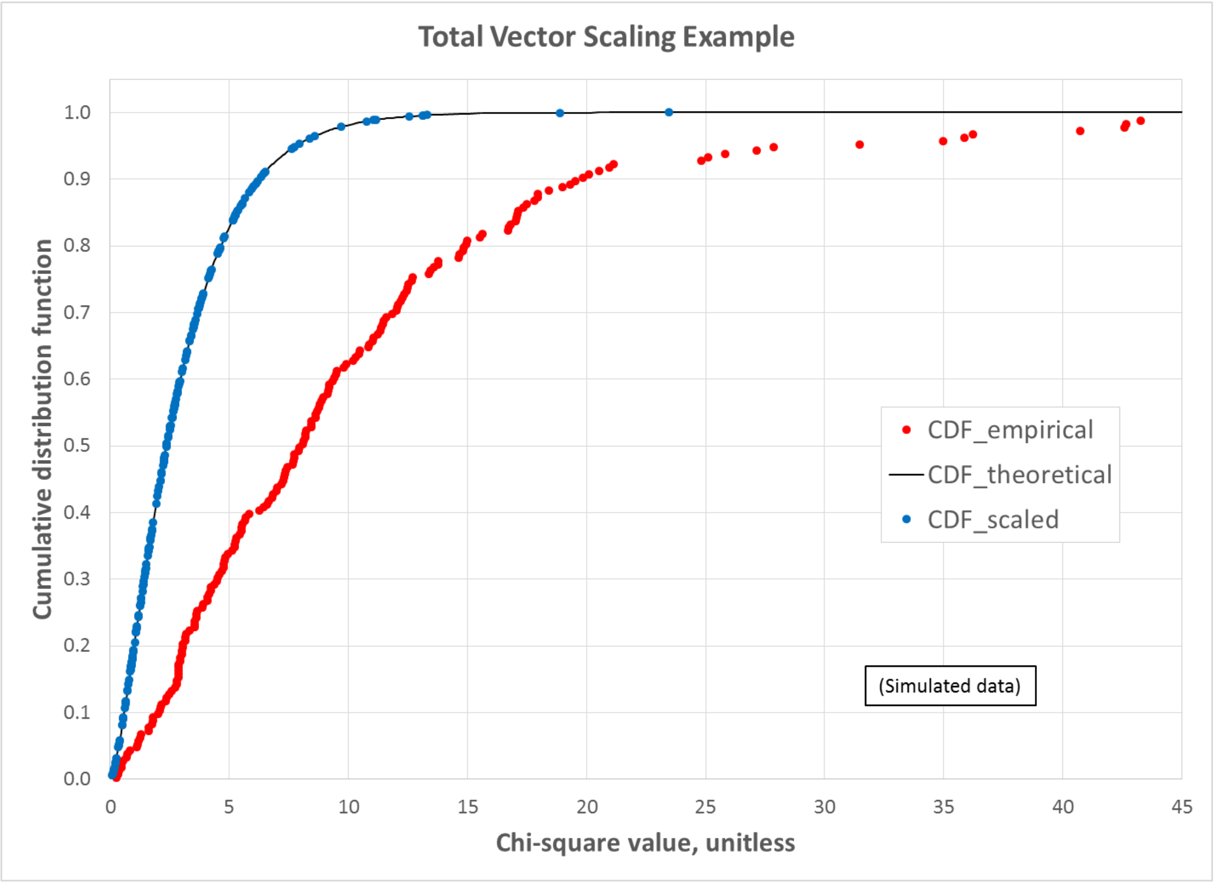
Single Component Example (continued)



Total Vector Example



Total Vector Example (continued)



Comments on the Correction Process

- This is an optimization process.
- Component based scale factors are the best starting points for total vector scaling.
- Scale factors may be propagation time and/or drag related.
- Outlier rejection is reasonable and probably to be expected.
- Types and amounts of trajectory data affect the process.
- The process should be repeated periodically.
- Depending upon what data is available and how it is used, the process may at times be unstable, i.e. converging on unrealistic values, but can be managed by the analyst.

Final Comments

- To date, virtually all experience is with batch based data.
  - In order, the best performance has been with out of plane, radial and horizontal/along track components.
  - Nonzero means in the data are usually ignorable in radial and out of plane while along track may indicate poor modeling.
  - Scale factor analyses can be an integral part of process improvement.
- Very limited experience with sequential estimation.
  - Poor performance for both components and total vector using EKF.
  - (Opinion) Sequential estimation methodology may corrupt things: process noise, observation processing, and ???.
  - very interested in any work that has been done on sequential error covariance matrix accuracy
- This is intended to promote awareness as much as give instruction.

Questions?

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