

Low-Thrust Many-Revolution Trajectory Optimization via Differential Dynamic Programming and a Sundman Transformation

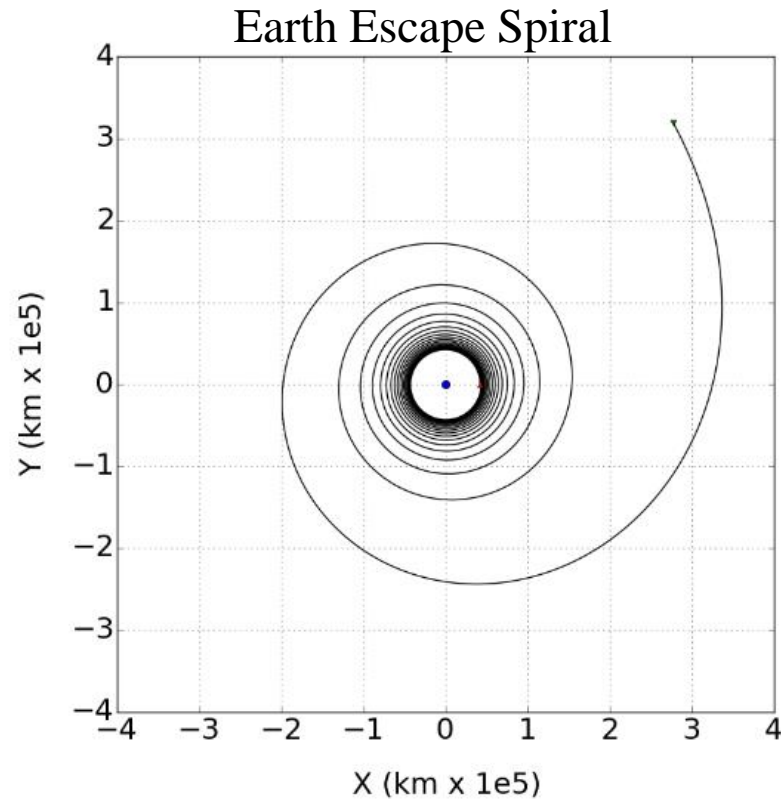
Jonathan Aziz, Jeffrey Parker, Daniel Scheeres
University of Colorado Boulder

Jacob Englander
NASA Goddard Space Flight Center

How many revolutions?

Planetary

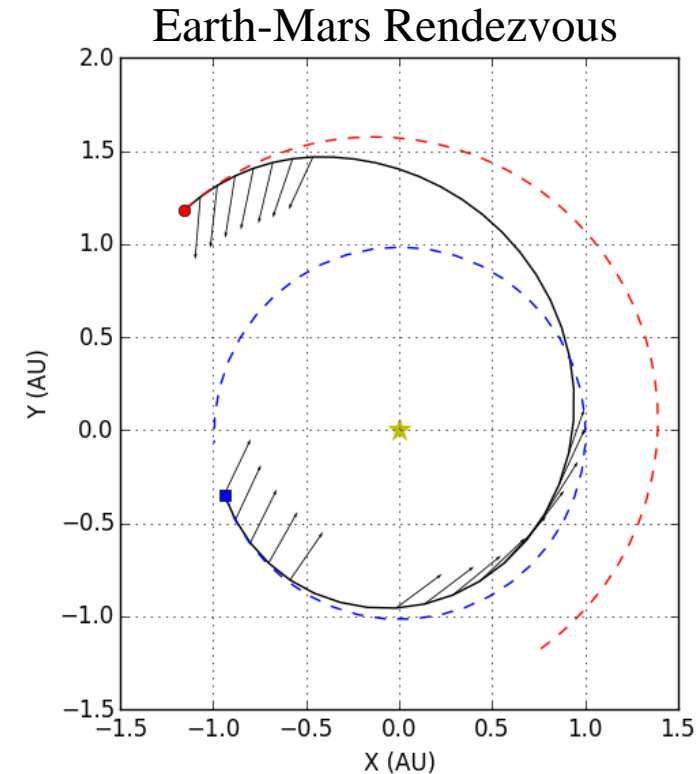
- Long transfer durations with short orbital periods span many revolutions



Number of 'revs': 10s, 100s, 1000s

Interplanetary

- Slow dynamics compared to control schedule

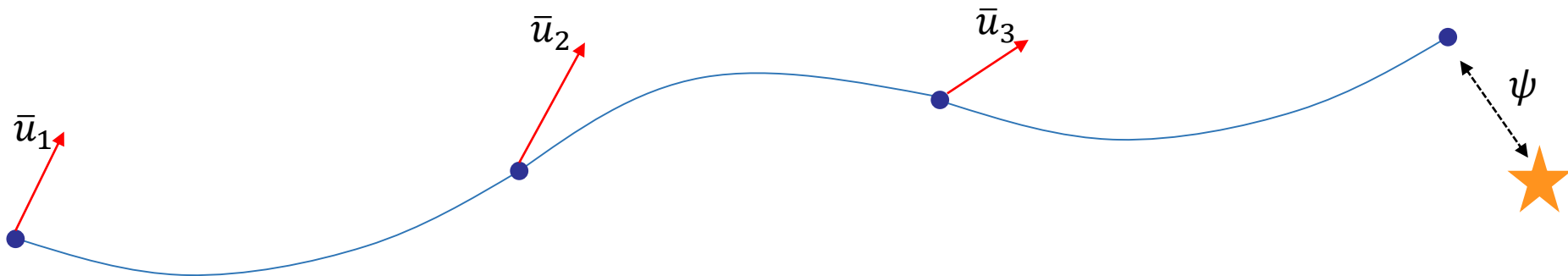


< 1, 1-10, 10s

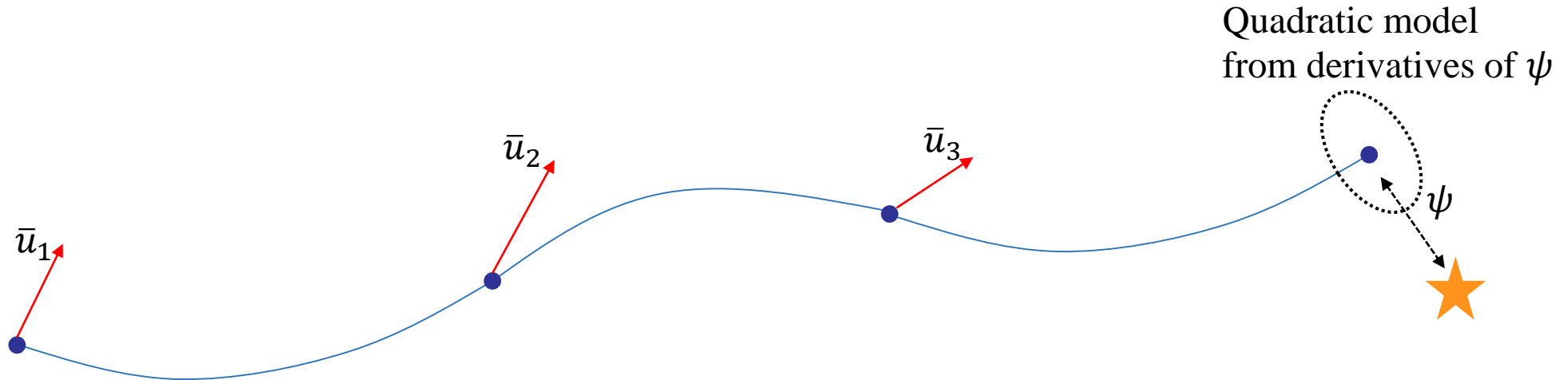
Historical Approaches

Indirect	Control Laws	Direct
<p>optimal control theory, apply Euler-Lagrange theorem and solve two point boundary value problem (TPBVP)</p>	<p>set a rule for spacecraft steering – a suboptimal policy that is acceptable by the mission designer</p>	<p>transcribe the trajectory optimization into a parameter optimization problem</p>
<p>Edelbaum Alfano Kéchichian</p>	<p>Kluever Chang Petropolous</p>	<p>Betts Whiffen Lantoine</p>

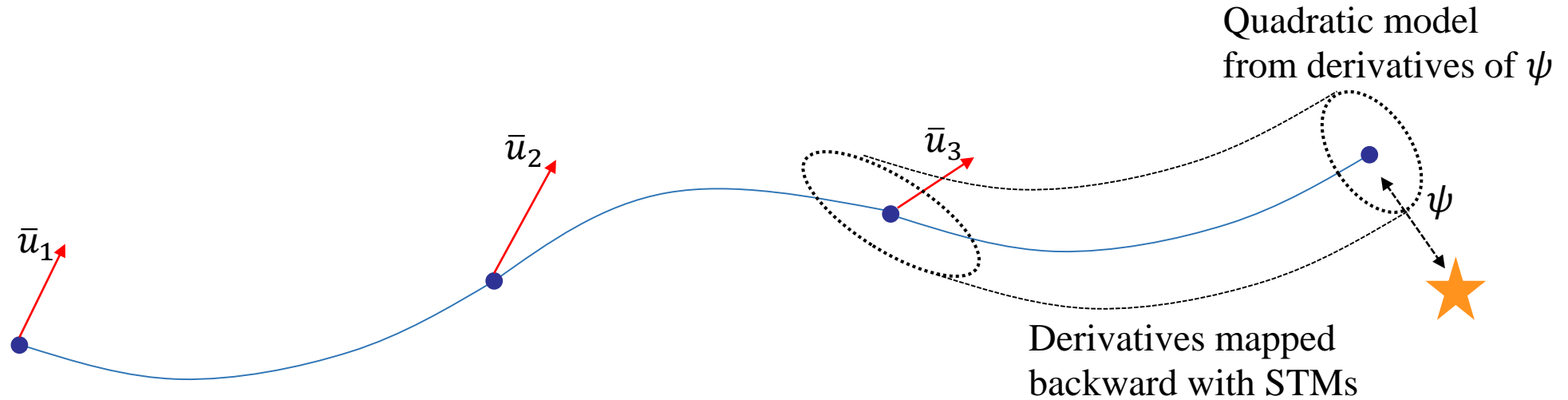
- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor



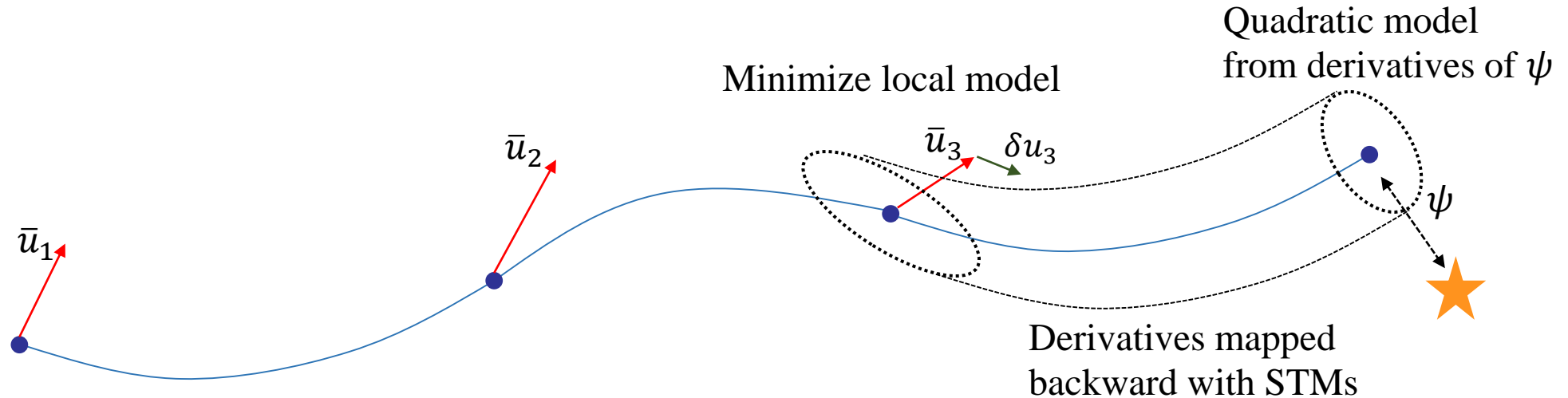
- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor



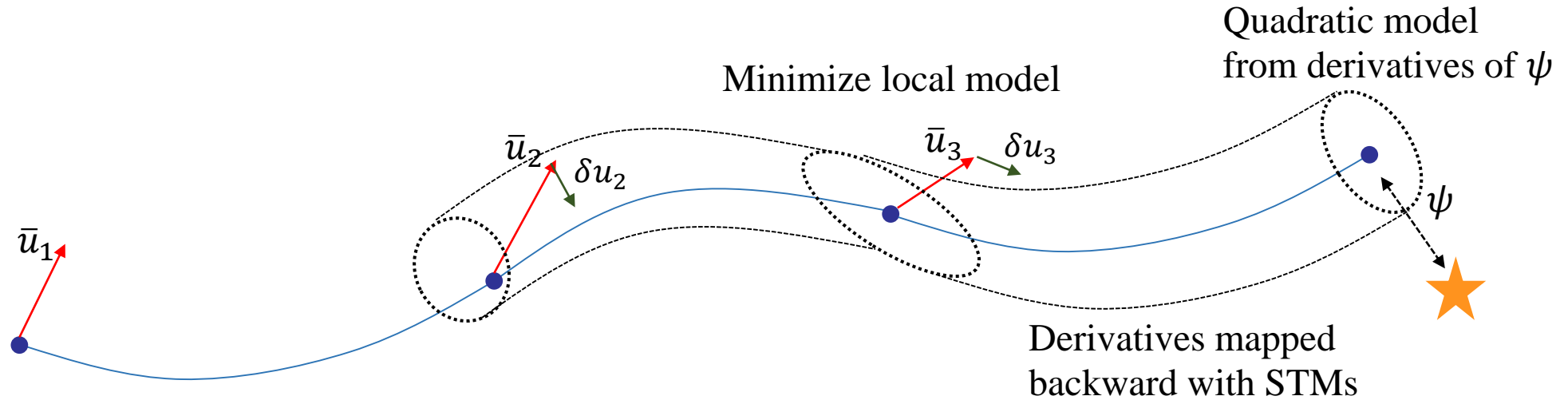
- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor



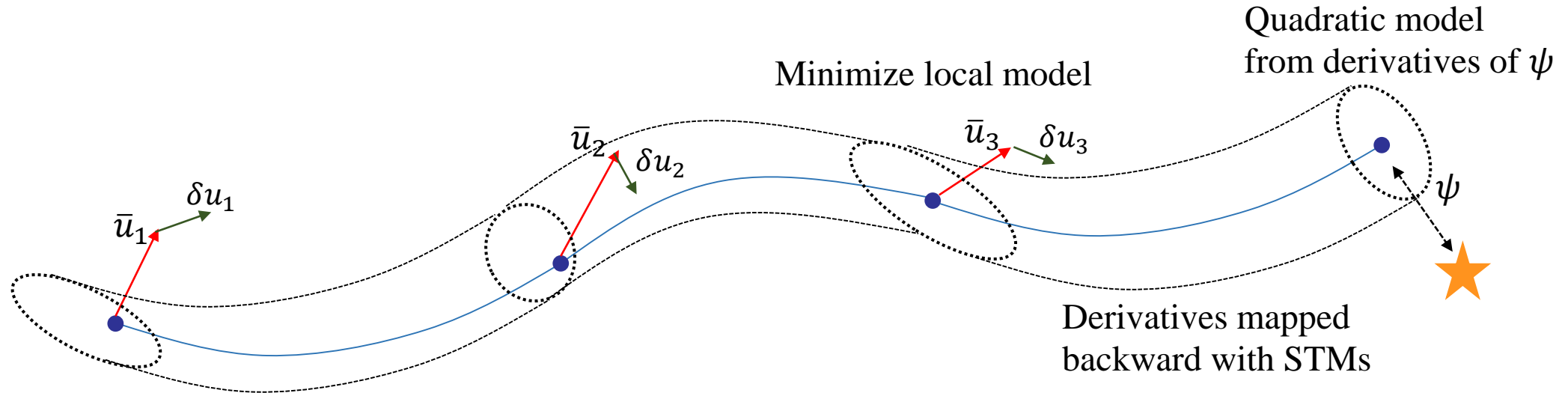
- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor



- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor

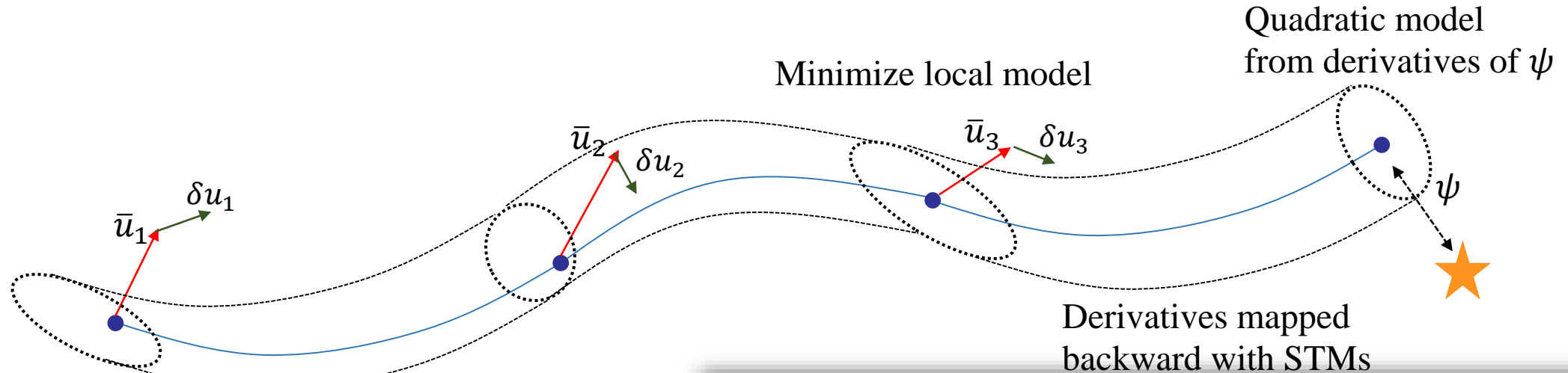


- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor



- *Forward pass*: evaluate $\bar{u} + \delta u$ in equations of motion
- *Backward sweep*: compute each δu_k

- Hybrid Differential Dynamic Programming
 - introduced by Lantoine and Russell
 - sequence of control updates that minimize quadratic model of *cost-to-go*
 - map derivatives along trajectory with state transition matrix and tensor



- *Forward pass*: evaluate $\bar{u} + \delta u$ i
- *Backward sweep*: compute each

See:

Gregory Lantoine and Ryan P. Russell. A hybrid differential dynamic programming algorithm for constrained optimal control problems. part 1: Theory. Journal of Optimization Theory and Applications, 154(2):382-417, 2012.

The Sundman Transformation

- Change independent variable from time to a function of orbital radius

$$dt = c_n r^n d\tau$$

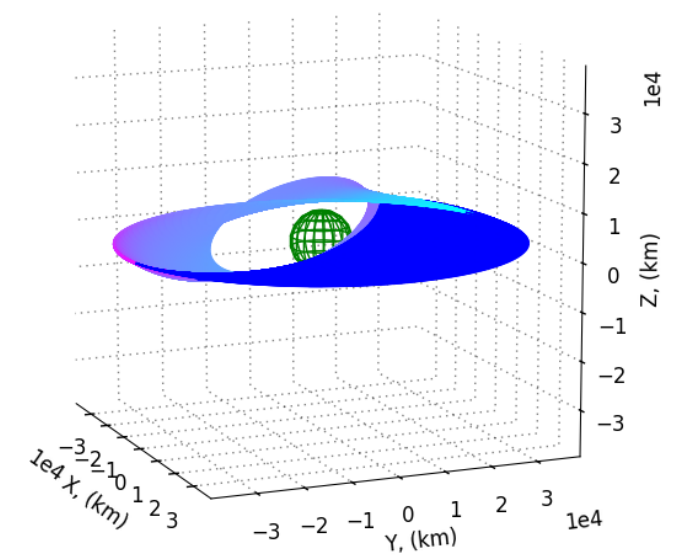
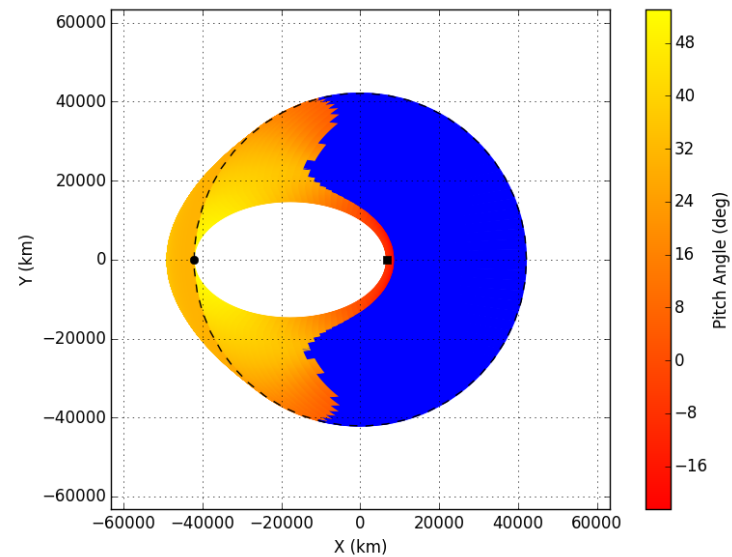
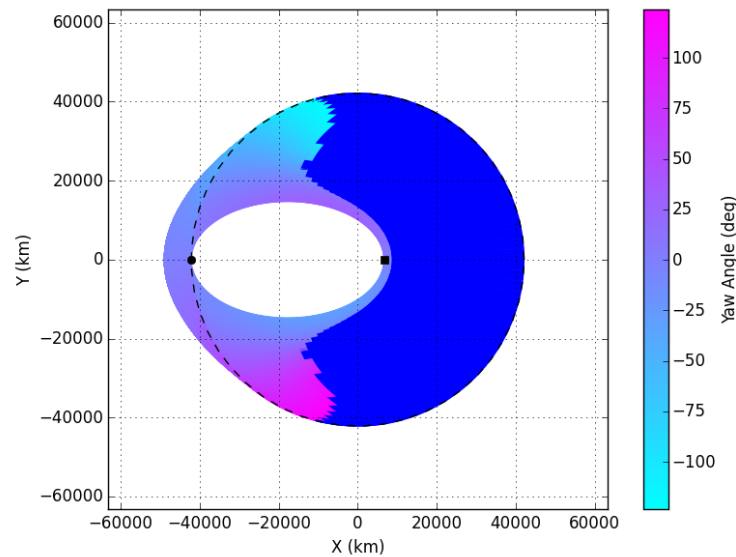
- Can choose n, c_n , so that τ is an orbit angle

Eccentric Anomaly	Mean Anomaly	True Anomaly
$dt = \sqrt{\frac{a}{\mu}} r dE$	$dt = \sqrt{\frac{a^3}{\mu}} dM$	$dt = \frac{r^2}{h} dv$

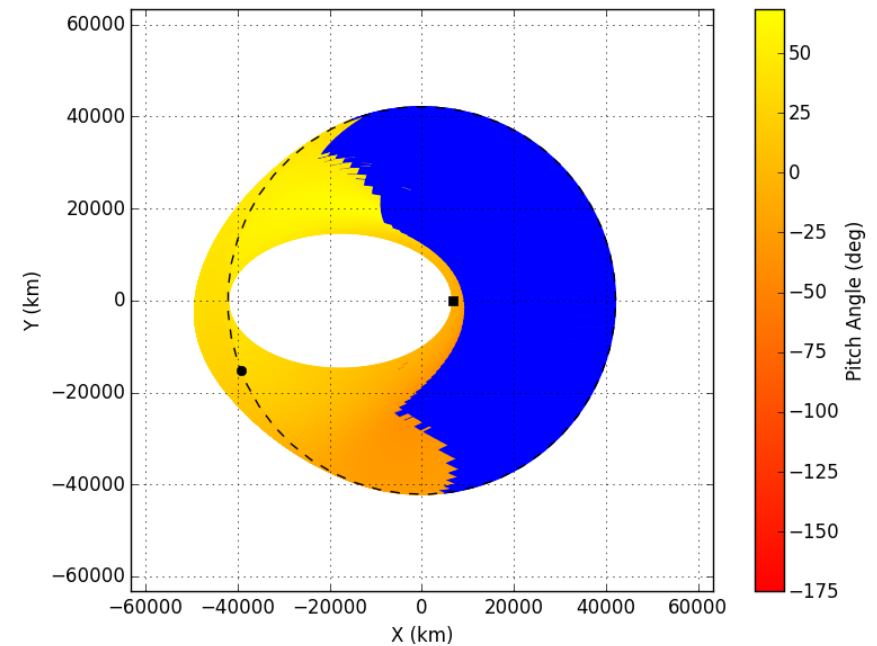
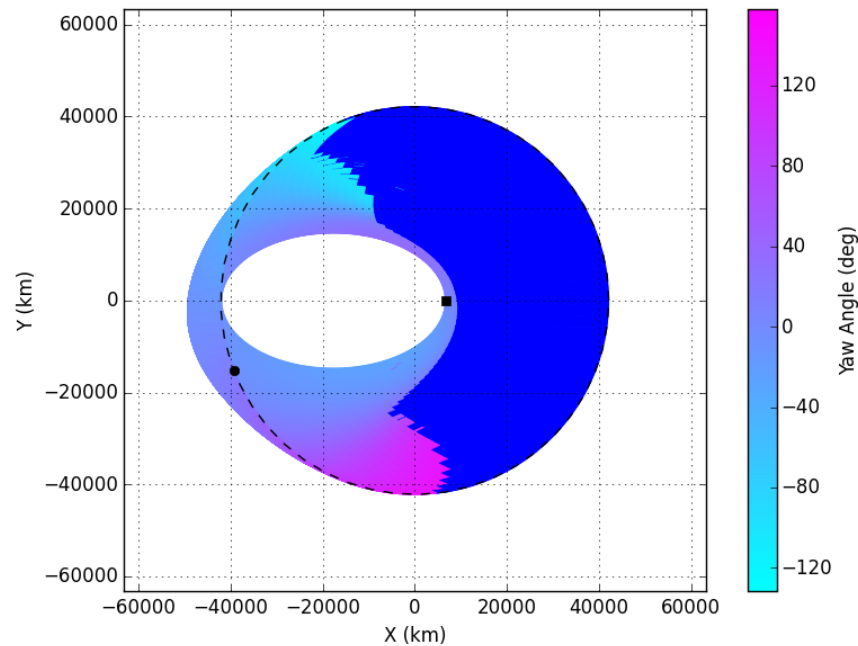
- Equations of motion become $x' = \dot{x} c_n r^n$
- Discretize in τ
- Specify τ_0, τ_f , rather than t_0, t_f
 - i.e. specify number of revolutions
- Track time in the state vector

$$x = \begin{bmatrix} t \\ x \\ y \\ z \\ \vdots \end{bmatrix}, \quad x' = \begin{bmatrix} 1 \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \vdots \end{bmatrix} c_n r^n$$

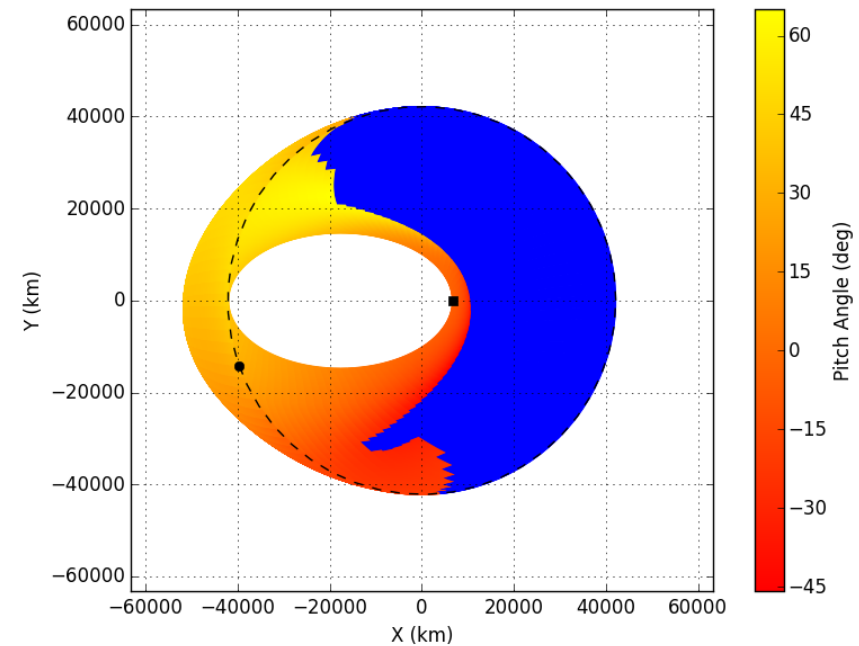
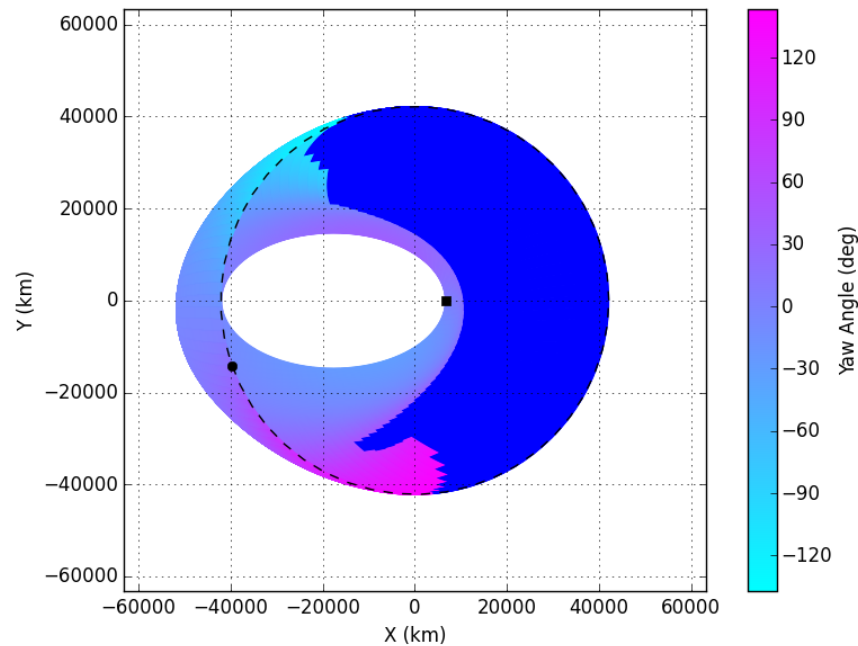
- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 450.5 revs
- 2-body dynamics
- 135,150 variables
- 54 minutes



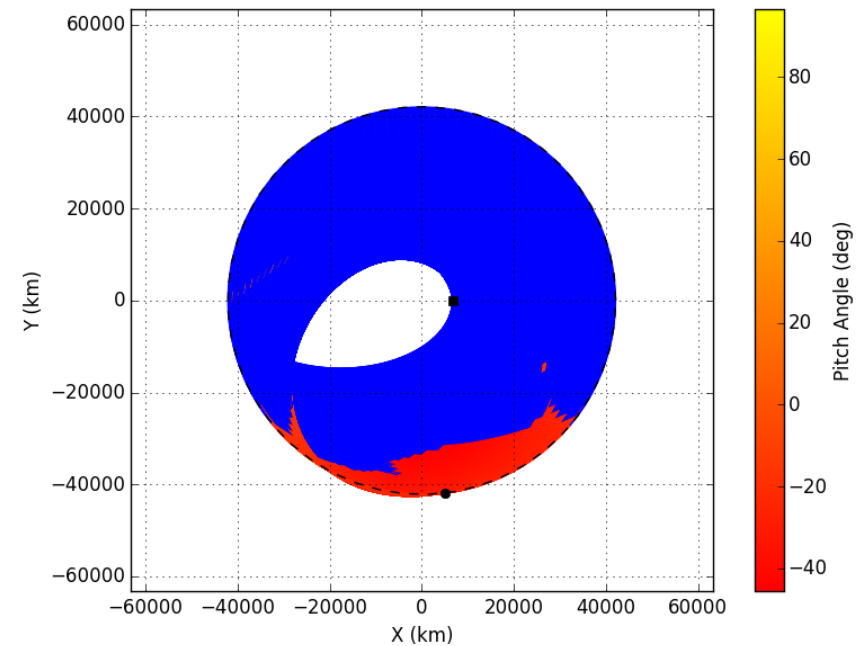
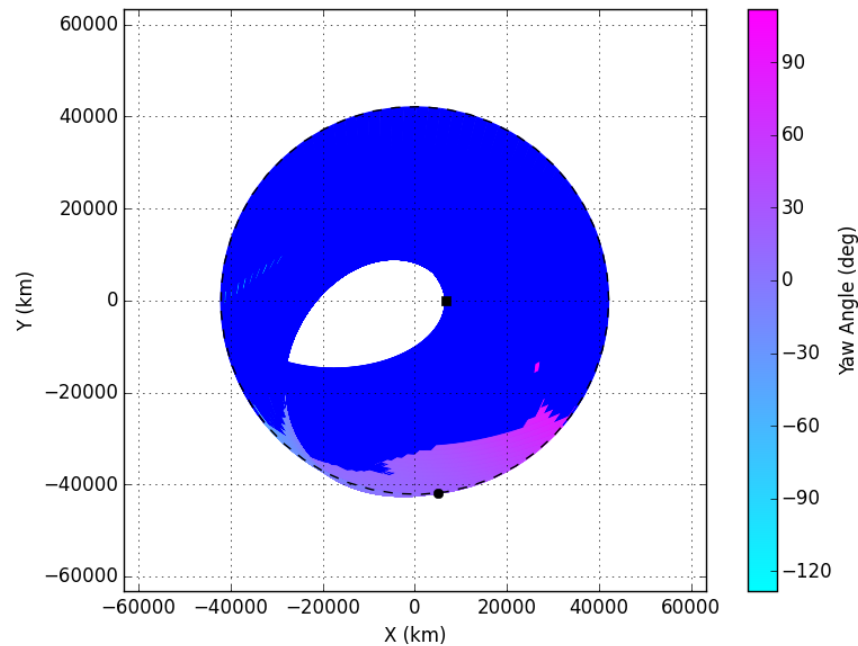
- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 450.5 revs
- 2-body dynamics + J_2
- 135,150 variables
- 61 minutes



- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 450.5 revs
- 2-body dynamics + J_2 + lunar gravity
- 135,150 variables
- 107 minutes



- Choose $\tau = E$, the eccentric anomaly
- Minimum fuel GTO to GEO in 1000.5 revs
- 2-body dynamics + J_2 + lunar gravity
- 300,150 variables
- 1359 minutes

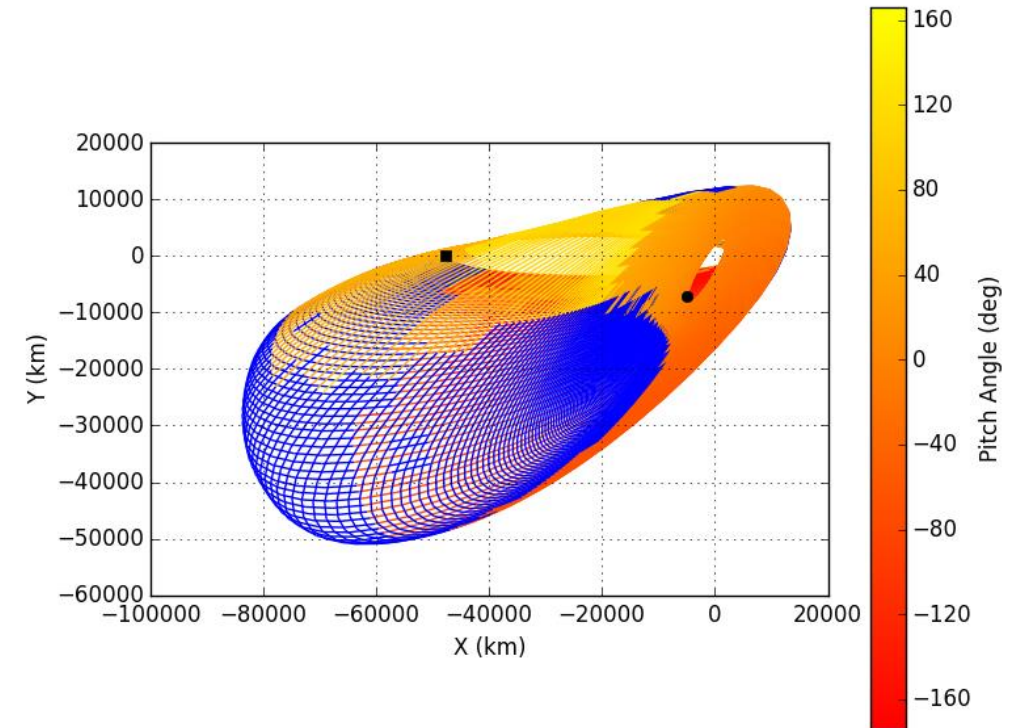
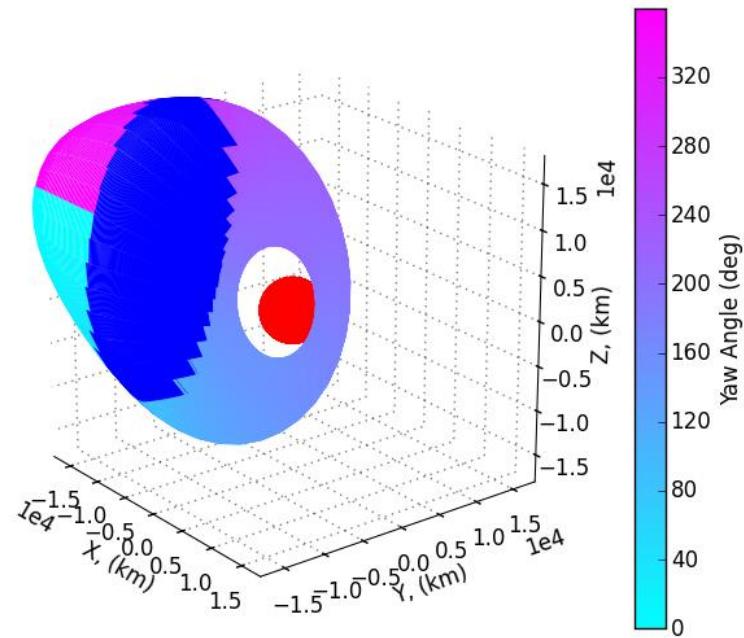


Backup Slides

Table 2.1: Summary of GTO to GEO Results.

Perturbations	N_{rev}	Iterations	Runtime (minutes)	m_f (kg)	t_f (days)	θ (deg)
None	450.5	86	54	1759.1754	315.75	180.0
J_2	450.5	111	61	1737.1949	342.13	199.8728
J_2 and Lunar Gravity	450.5	136	107	1745.3012	322.63	201.0805
J_2 and Lunar Gravity	1000.5	913	1359	1784.3632	558.86	276.7209
None (Eclipse Model)	450.5	93	70	1751.4223	325.48	180.0
None (MEE)	450.5	59	15	1758.7230	318.96	180.0

500 rev orbit lowering at Mars



and with $\Delta\Omega = 60^\circ$

Trust-Region Quadratic Subproblem

- Feedback control laws for δu , $\delta \lambda$, δw are unconstrained
- Likely to step beyond validity of quadratic expansion
- Require invertible, positive definite Hessians (negative definite for $J_{\lambda\lambda}$)
- Trust-region quadratic subproblem (TRQP):

$$\min_{\delta \mathbf{u}_k} \left[J_{u,k}^T \delta \mathbf{u}_k + \frac{1}{2} \delta \mathbf{u}_k^T J_{uu,k} \delta \mathbf{u}_k \right]$$

$$\text{s.t. } \|D\delta \mathbf{u}_k\| \leq \Delta$$

- Acceptance of an iterate:

$$\rho = \frac{\delta J}{ER_{0,0}} \quad \Delta_{p+1} = \begin{cases} \min((1 + \kappa)\Delta_p, \Delta_{\max}), & \text{if } \rho \in [1 - \epsilon_1, 1 + \epsilon_1] \\ \max((1 - \kappa)\Delta_p, \Delta_{\min}), & \text{otherwise} \end{cases}$$

$$\mathbf{X} = [t \ x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z} \ m \ T \ \alpha \ \beta]^T \quad (2)$$

$$\begin{bmatrix} \hat{r} & \hat{s} & \hat{w} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{r}}{r} & \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}}{\|(\mathbf{r} \times \mathbf{v}) \times \mathbf{r}\|} & \frac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r} \times \mathbf{v}\|} \end{bmatrix}. \quad (3)$$

$$\begin{bmatrix} T_r \\ T_s \\ T_w \end{bmatrix} = \begin{bmatrix} T \sin \alpha \cos \beta \\ T \cos \alpha \cos \beta \\ T \sin \beta \end{bmatrix}, \quad \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \hat{r} & \hat{s} & \hat{w} \end{bmatrix} \begin{bmatrix} T_r \\ T_s \\ T_w \end{bmatrix}, \quad (4)$$

$$\dot{\mathbf{X}} = \dot{\mathbf{X}}_{\oplus} + \dot{\mathbf{X}}_T + \dot{\mathbf{X}}_{J_2} + \dot{\mathbf{X}}_{\alpha}, \quad (5)$$

$$\dot{\mathbf{X}}_{\oplus} = \left[1 \ \dot{x} \ \dot{y} \ \dot{z} \ -\frac{\mu_{\oplus}}{r^3}x \ -\frac{\mu_{\oplus}}{r^3}y \ -\frac{\mu_{\oplus}}{r^3}z \ 0 \ 0 \ 0 \ 0 \right]^T, \quad (6a)$$

$$\dot{\mathbf{X}}_T = \left[0 \ 0 \ 0 \ 0 \ \frac{T_x}{m} \ \frac{T_y}{m} \ \frac{T_z}{m} \ -\frac{T}{I_{sp}g_0} \ 0 \ 0 \ 0 \right]^T, \quad (6b)$$

$$\dot{\mathbf{X}}_{J_2} = -\frac{3J_2\mu_{\oplus}R_{\oplus}^2}{2r^5} \left[0 \ 0 \ 0 \ 0 \ x(1 - 5\frac{z^2}{r^2}) \ y(1 - 5\frac{z^2}{r^2}) \ z(3 - 5\frac{z^2}{r^2}) \ 0 \ 0 \ 0 \ 0 \right]^T, \quad (6c)$$

$$\dot{\mathbf{X}}_{\alpha} = -\mu_{\alpha} \left[0 \ 0 \ 0 \ 0 \ \frac{x - x_{\alpha}}{\|\mathbf{r} - \mathbf{r}_{\alpha}\|^3} + \frac{x_{\alpha}}{r_{\alpha}^3} \ \frac{y - y_{\alpha}}{\|\mathbf{r} - \mathbf{r}_{\alpha}\|^3} + \frac{y_{\alpha}}{r_{\alpha}^3} \ \frac{z - z_{\alpha}}{\|\mathbf{r} - \mathbf{r}_{\alpha}\|^3} + \frac{z_{\alpha}}{r_{\alpha}^3} \ 0 \ 0 \ 0 \ 0 \right]^T. \quad (6d)$$

$$\dot{\mathbf{X}} = (\dot{\mathbf{X}}_{\oplus} + \dot{\mathbf{X}}_T + \dot{\mathbf{X}}_{J_2} + \dot{\mathbf{X}}_{\epsilon})\sqrt{a/\mu_{\oplus}} r \quad (15)$$

$$A^{i,j} = \frac{\partial \dot{\mathbf{X}}^i}{\partial \mathbf{X}^j}, \quad (16a)$$

$$A^{i,jk} = \frac{\partial^2 \dot{\mathbf{X}}^i}{\partial \mathbf{X}^j \partial \mathbf{X}^k}, \quad (16b)$$

$$\Lambda^{i,j} = \frac{\partial \dot{\mathbf{X}}^i}{\partial \mathbf{X}^j} \quad (17a)$$

$$\Lambda^{i,jk} = \frac{\partial^2 \dot{\mathbf{X}}^i}{\partial \mathbf{X}^j \partial \mathbf{X}^k} \quad (17b)$$

$$\eta = dt/d\tau = c_n r^n \quad (18)$$

$$\eta_{x^i} = \frac{\partial \eta}{\partial \mathbf{X}^i} \quad (19)$$

$$\eta_{xx^{i,j}} = \frac{\partial^2 \eta}{\partial \mathbf{X}^i \partial \mathbf{X}^j} \quad (20)$$

$$\Lambda^{i,j} = A^{i,j} \eta + \dot{\mathbf{X}}^i \eta_{x^j} \quad (21a)$$

$$\Lambda^{i,jk} = A^{i,jk} \eta + A^{i,j} \eta_{x^k} + A^{i,k} \eta_{x^j} + \dot{\mathbf{X}}^i \eta_{xx^{j,k}} \quad (21b)$$

$$\Phi^{i,a}(t_k, t_{k+1}) = \frac{\partial \mathbf{X}_{k+1}^i}{\partial \mathbf{X}_k^a} = \mathbf{F}_{X,k}^{i,a}$$

$$\Phi^{i,ab}(t_k, t_{k+1}) = \frac{\partial^2 \mathbf{X}_{k+1}^i}{\partial \mathbf{X}_k^a \partial \mathbf{X}_k^b} = \mathbf{F}_{XX,k}^{i,ab}$$

$$\dot{\Phi}^{i,a} = A^{i,\gamma_1} \Phi^{\gamma_1,a}$$

$$\dot{\Phi}^{i,ab} = A^{i,\gamma_1} \Phi^{\gamma_1,ab} + A^{i,\gamma_1\gamma_2} \Phi^{\gamma_1,a} \Phi^{\gamma_2,b}$$

$$A^{i,a} = \frac{\partial \dot{\mathbf{X}}^i}{\partial \mathbf{X}^a}$$

$$A^{i,ab} = \frac{\partial^2 \dot{\mathbf{X}}^i}{\partial \mathbf{X}^a \partial \mathbf{X}^b}$$

$$\dot{\Phi}^{i,j} = A^{i,a} \Phi^{a,j} \quad (27a)$$

$$\dot{\Phi}^{i,jk} = A^{i,a} \Phi^{a,jk} + A^{i,ab} \Phi^{a,j} \Phi^{b,k} \quad (27b)$$