Time Dependence of Collision Probabilities During Satellite Conjunctions

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Outline

- Motivation and objectives
- Overview of collision probability theory
- Analysis of well-studied conjunctions
- Analysis of archived conjunctions
- Conclusions

Motivation and Objectives

- **Motivation:** **The probability of collision, P_c, between two Earth-orbiting satellites can often *but not always* be approximated adequately using the "2D P_c" formulation**

- **Objective:** **Implement an improved method to estimate collision probabilities**
  - **Use Coppola's analytical "3D P_c" formulation***
  - **Validate using well-studied test cases and Monte Carlo methods**
  - **Compare 2D and 3D P_c for archived conjunctions**

*V. T. Coppola (2012) "Including Velocity Uncertainty in the Probability of Collision Between Space Objects", AAS 12-247.

Outline

- Motivation and objectives
- **Overview of collision probability theory**

  Monte Carlo methods, 3D P_c theory, 2D approximations
- Analysis of well-studied conjunctions
- Analysis of archived conjunctions
- Conclusions

Monte Carlo P_c Estimation

- **Collision probabilities can be estimated using Monte Carlo simulations**
  - **Computationally intensive, especially for low probability events**

- **Alfano* analyzes twelve conjunctions in detail using Monte Carlo simulations**
  - **Benchmark test cases that can be used for validation of the 3D P_c software**
  - **Includes cases where the 2D P_c method both succeeds and fails**

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*S. Alfano (2009) “Satellite Conjunction Monte Carlo Analysis”, AAS 09-233.

Coppola's 3D *Pc* Formulation

- **Coppola* provides an analytically-derived formulation to calculate P_c and its time derivative**

$$P_c = P_0 + \int_{t_0}^{t_0+T} \left(\frac{dP_c}{dt} \right) dt$$

1D time integral

$$\frac{dP_c}{dt} = \oint_{4\pi} I(\hat{\mathbf{r}}, t) \, d^2\hat{\mathbf{r}}$$

2D unit - sphere integral

- **These integrals must be calculated numerically**

*V. T. Coppola (2012a) "Including Velocity Uncertainty in the Probability of Collision Between Space Objects", AAS 12-247.

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2D unit - sphere integral

Analyzing the probability rate* provides new insight into the time dependence of conjunction risks

*K.J.DeMars, Y.Cheng, and M.K.Jah (2014) "Collision Probability with Gaussian Mixture Orbit Uncertainty", *J. Guidance Control and Dynamics*, **37**(3) 979-985, 2014.

Coppola's Conjunction Time Bounds

- Coppola* also provides estimates for the bounding times of a conjunction
  - These often bracket the nominal time of closest approach (TCA), ***but not always***
- These bounds only provide a first-cut approximation for the limits of the numerical integration over time
  - These limits sometimes need to be expanded to bracket sufficiently the time(s) when dP_c/dt peaks

* V. T. Coppola (2012) "Evaluating the Short Encounter Assumption of the Probability of Collision Formula", AAS 12-248.

CARA's Current 3D *Pc* Implementation

- **The 3D P_c method is general enough to use**
  - **Gaussian Mixture Model state distributions***
  - **Complex dynamical motion models**
  - **Full 6x6 time-dependent state covariances**
- **CARA's current implementation uses**
  - **Single Gaussian ECI state distributions**
  - **The Keplerian two-body motion model**
  - **Full 6x6 ECI-state covariances, propagated using an analytically-derived state transition matrix#**
- **Future plans include more advanced approaches**

*J. T. Horwood, *et al*. (2011) "Gaussian Sum Filters for Space Surveillance: Theory and Simulations," *J. of Guidance, Control, and Dynamics*, vol.34, p.1839–1851.
S.W. Shepperd (1985) "Universal Keplerian State Tranisition Matrix," *Celestial Mechanics*, vol.35, p.129-144.

Schematic Illustration of the Encounter Region

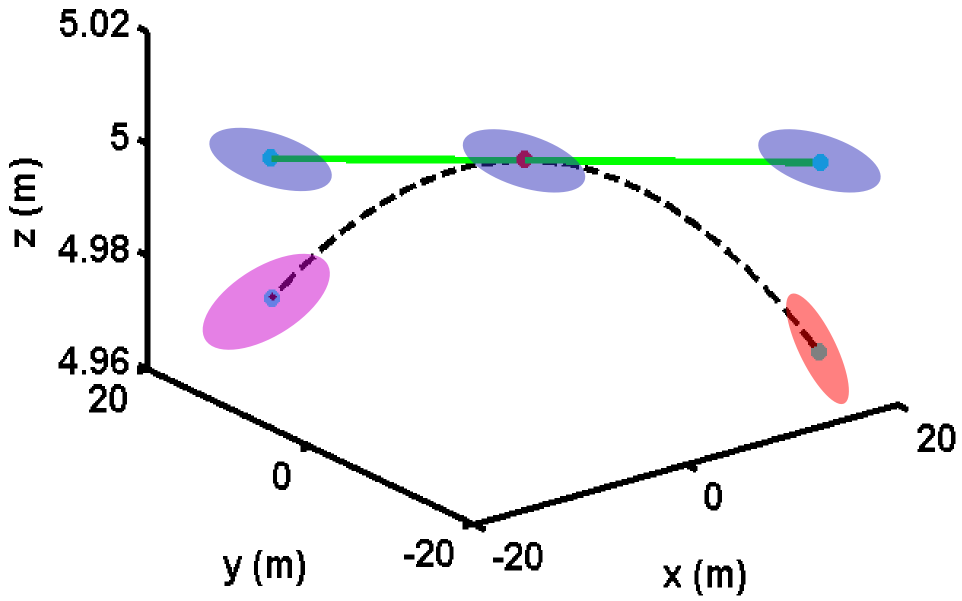

Illustration based on Alfano's test case #2*

NOTE: Actual 1σ surfaces are much larger and thinner

2D P_C assumptions:

- **Presumes straight trajectory (green)**
- **Presumes static covariances (blue)**

3D P_C assumptions:

- **Trajectories are curvilinear (black)**
- **Covariances vary throughout the encounter (pink, orange)**

*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

Approximations Used for 2D P_c Estimation

- In terms of the relative position/velocity state vector and the associated 6x6 covariance matrix:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix} \approx \begin{bmatrix} \mathbf{r}(t_{ca}) + (t - t_{ca})\mathbf{v}(t_{ca}) \\ \mathbf{v}(t_{ca}) \end{bmatrix}$$

$$\mathbf{P}(t) = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t)^T \\ \mathbf{B}(t) & \mathbf{C}(t) \end{bmatrix} \approx \begin{bmatrix} \mathbf{A}(t_{ca}) & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}$$

- Here t_{ca} = TCA = the time of closest approach

Step-by-Step Relaxation of the 2D *Pc* Approximations

Using CARA's current 3D P_c software, the three approximations used in the 2D P_c method can be relaxed in a step-by-step manner:

- Coppola 1: Linear motion, A=A(TCA), B=C=0
- Coppola 2: Kep2Body, A=A(TCA), B=C=0
- Coppola 3: Kep2Body, A=A(t), B=C=0
- Coppola 4: Kep2Body, P=P(t)

Step "Coppola 1" employs all of the 2D *Pc* assumptions

Step "Coppola 2" introduces Keplerian 2-body motion

Step "Coppola 3" introduces time-varying position covariances

Step "Coppola 4" introduces position+velocity covariances

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- Motivation and objectives

- Overview of collision probability theory

- **Analysis of well-studied conjunctions**

  Benchmark test cases analyzed by S. Alfano*

- Analysis of archived conjunctions

- Conclusions

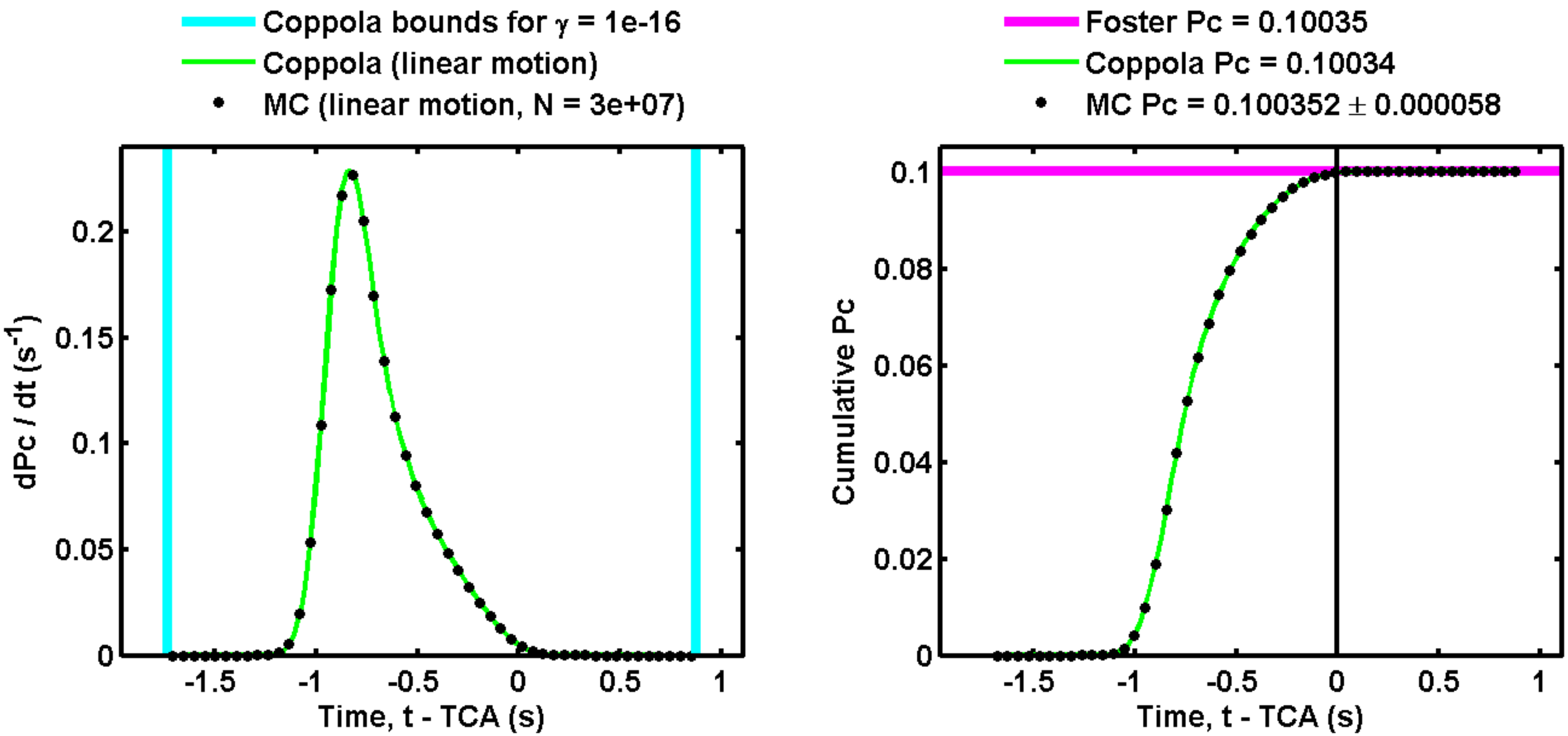
*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

Validation using Alfano's* Benchmarks

- **Benchmark case #3:**
  - "Linear" case where 2D P_c is known to be accurate
  - The 3D P_c software correctly reproduces the 2D P_c approximation, and Alfano's benchmark P_c value
- **Benchmark case #10:**
  - "Nonlinear" case where 2D P_c is known to be inaccurate
  - The 3D P_c software correctly reproduces Alfano's benchmark P_c value
- **Other benchmark cases also analyzed (but not shown)**

*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

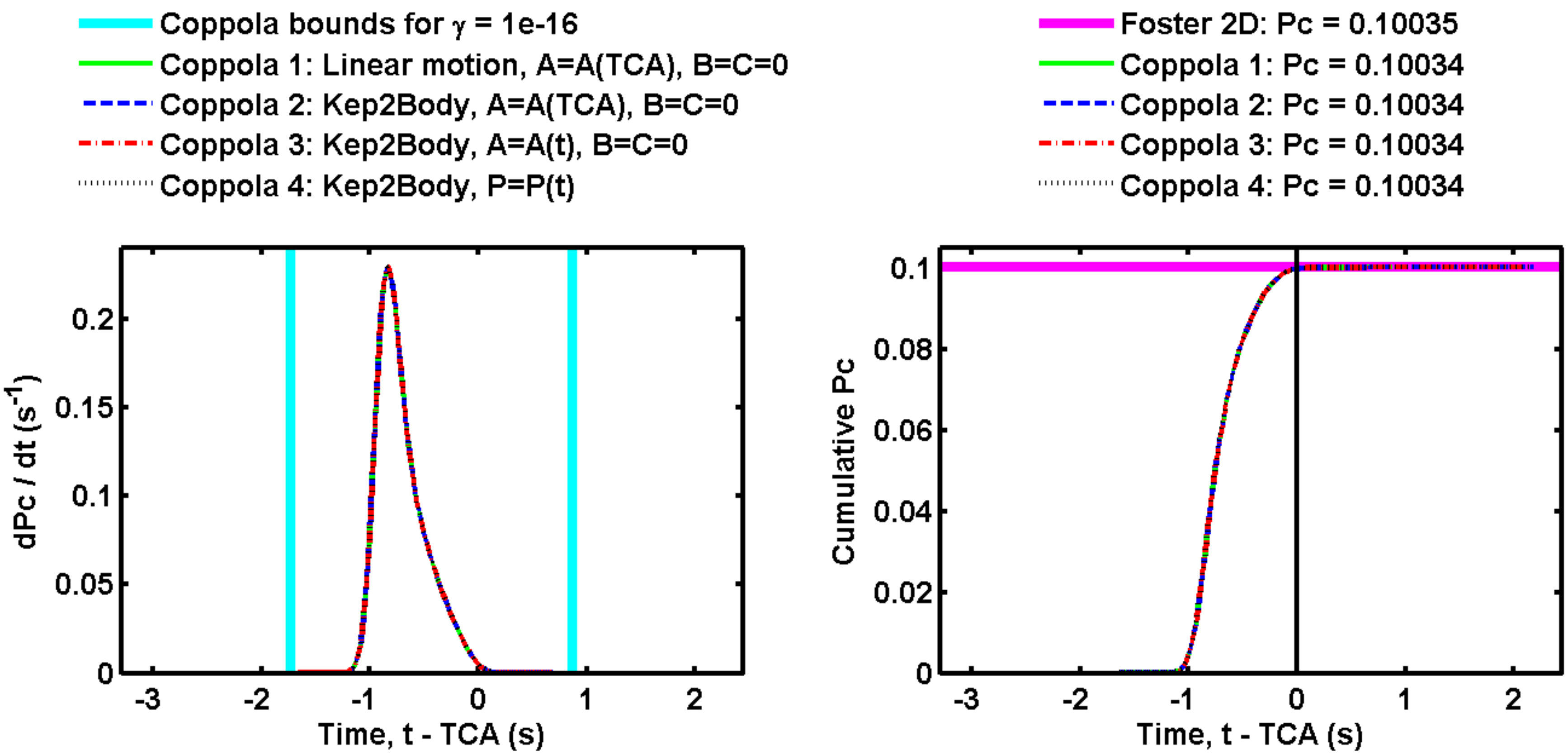
Alfano's* "Linear" Test Case #3



These plots validate that the 3D P_c software correctly reproduces both the Monte Carlo and 2D P_c estimates, when using the 2D P_c approximations

*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

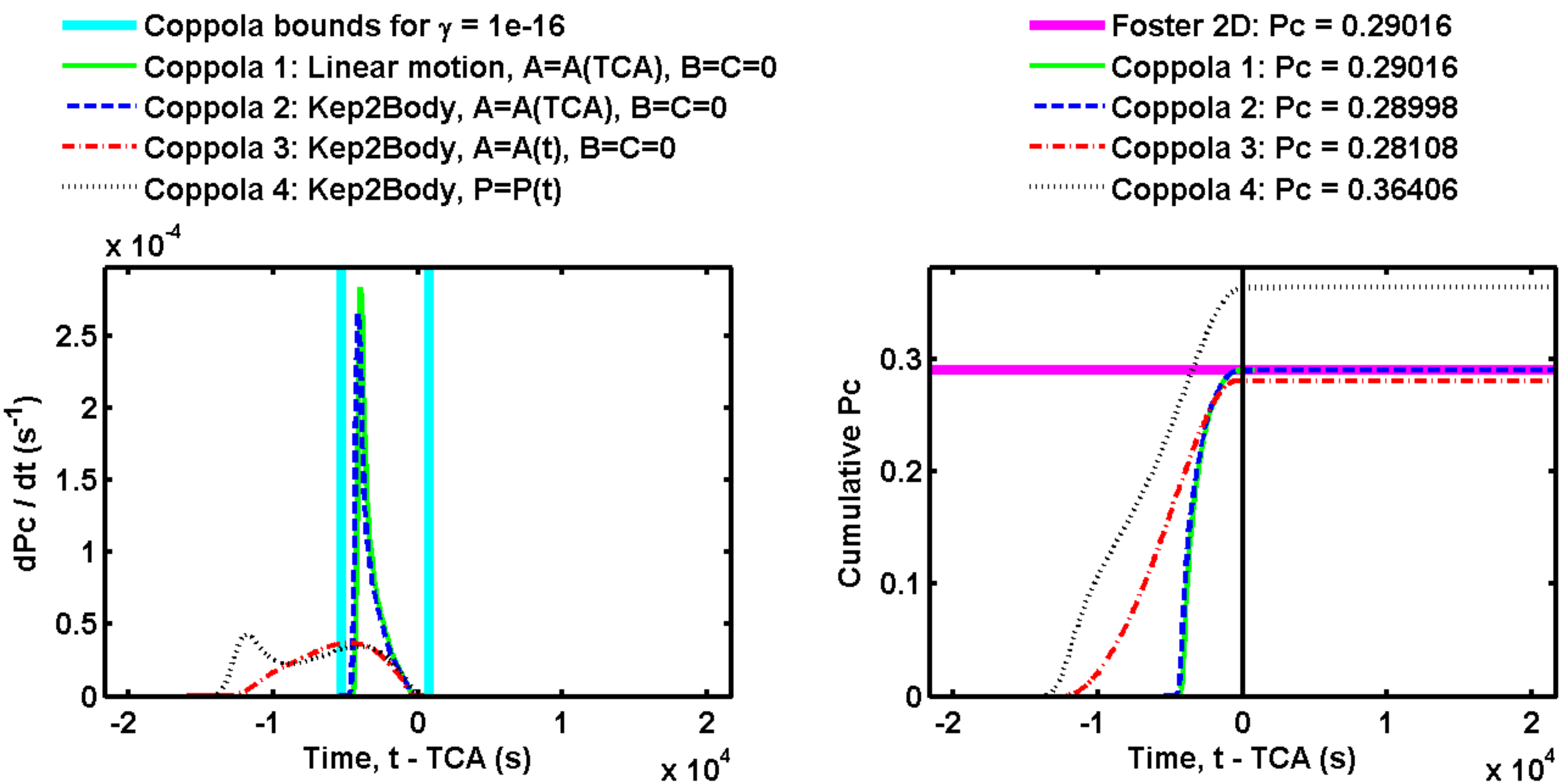
Alfano's* "Linear" Test Case #3



These plots validate that the 3D P_c software correctly reproduces the 2D P_c estimate, even when the 2D P_c approximations are fully relaxed

*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

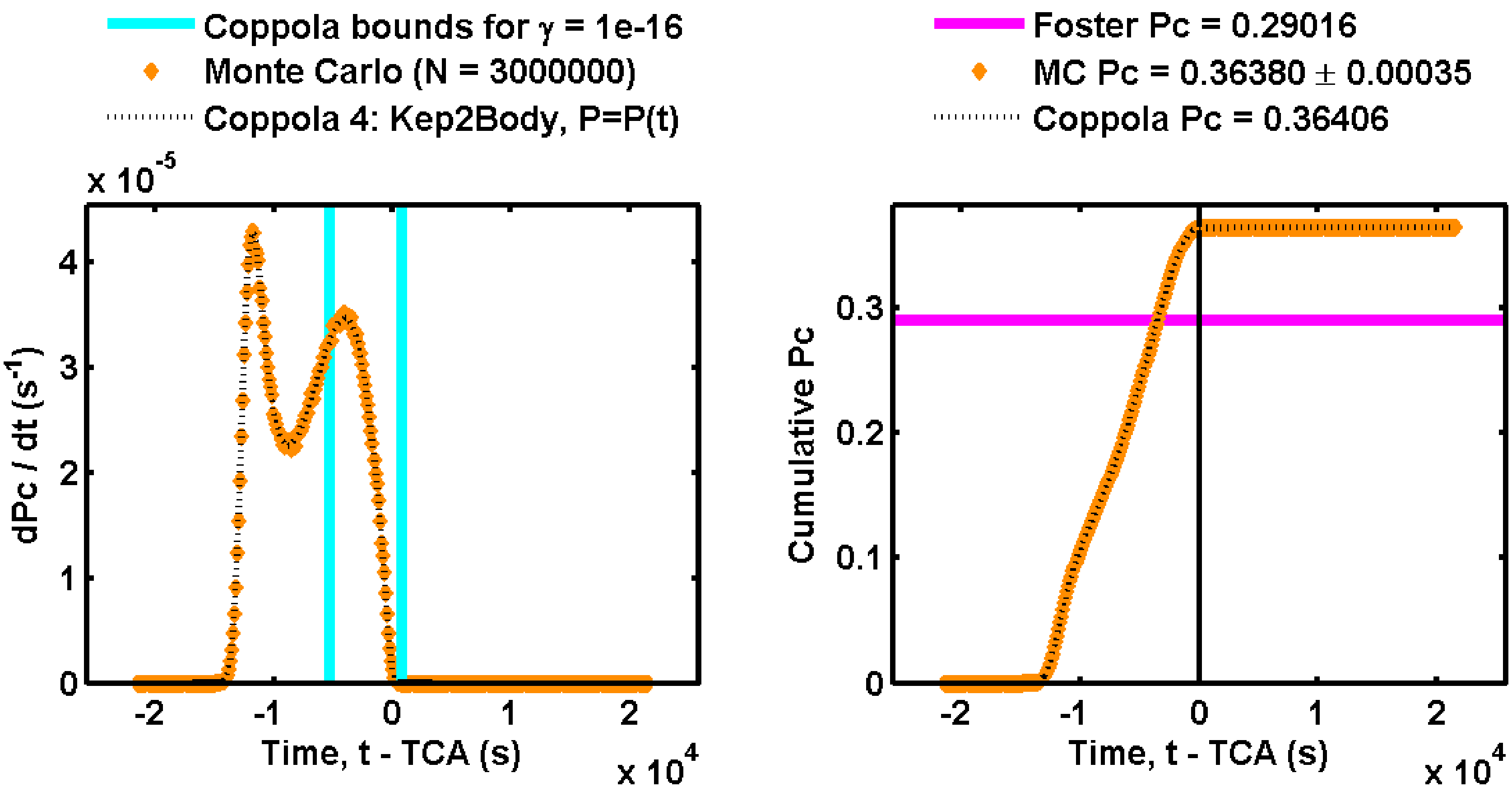
Alfano's* "Nonlinear" Test Case #10



These plots validate that the 3D P_c software correctly yields different results as the 2D P_c approximations are relaxed in a step-by-step fashion

*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

Alfano's* "Nonlinear" Test Case #10

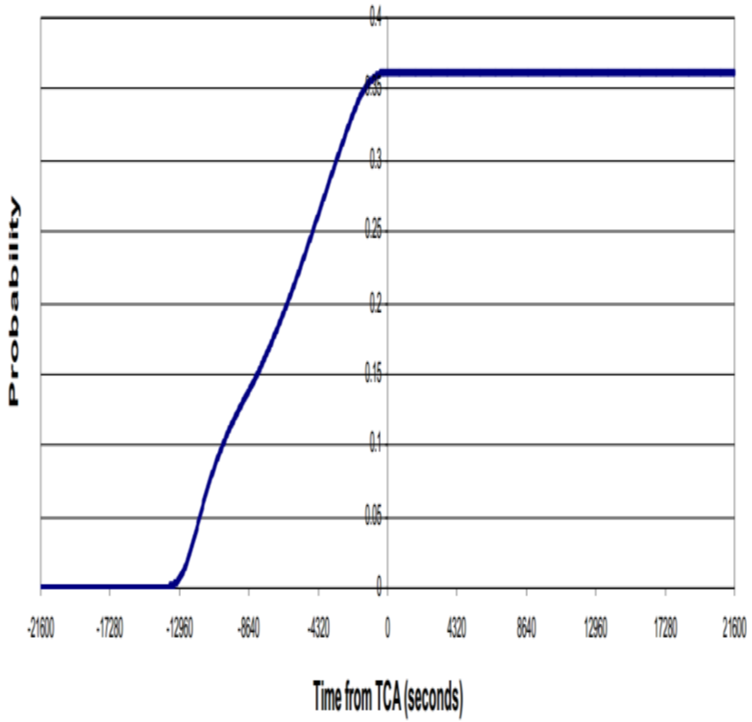


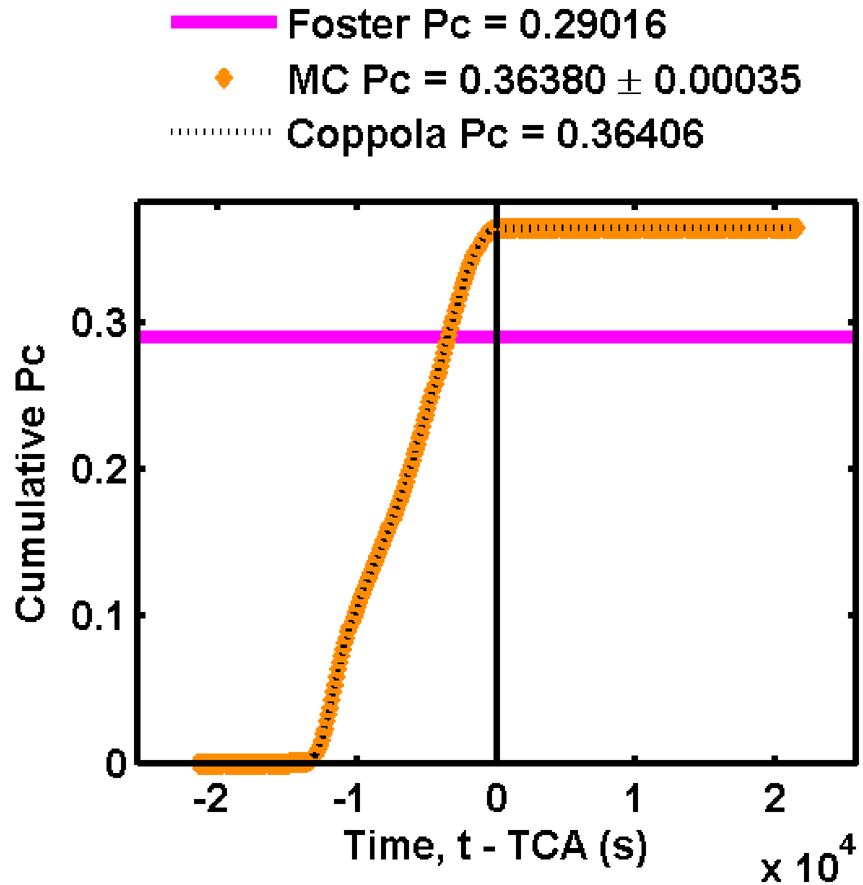
These plots validate that the 3D P_c software correctly reproduces the Monte Carlo simulation, and that the dP_c/dt profile has two blended peaks

*S. Alfano (2009) "Satellite Conjunction Monte Carlo Analysis", AAS 09-233.

Alfano's* "Nonlinear" Test Case #10

Alfano's* original plot for cumulative probability





These plots validate that the 3D P_c software correctly reproduces Alfano's benchmark Monte Carlo results

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- **Analysis of archived conjunctions**
  - 2D vs. 3D results, repeating events, small-P_c screening
- Conclusions

Analysis of Archived Conjunctions

- The 3D P_c method has been applied to 80,453 archived conjunctions
  - Actual events that occurred between 2016 April 1 and 2016 June 1

- Relatively few have appreciable 3D P_c values
  - Only 11,211 (14%) have $P_c \geq 10^{-15}$
  - Only 5,761 (7.2%) have $P_c \geq 10^{-7}$
  - Only 2,674 (3.3%) have $P_c \geq 10^{-5}$

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This is the most important set for the CARA team

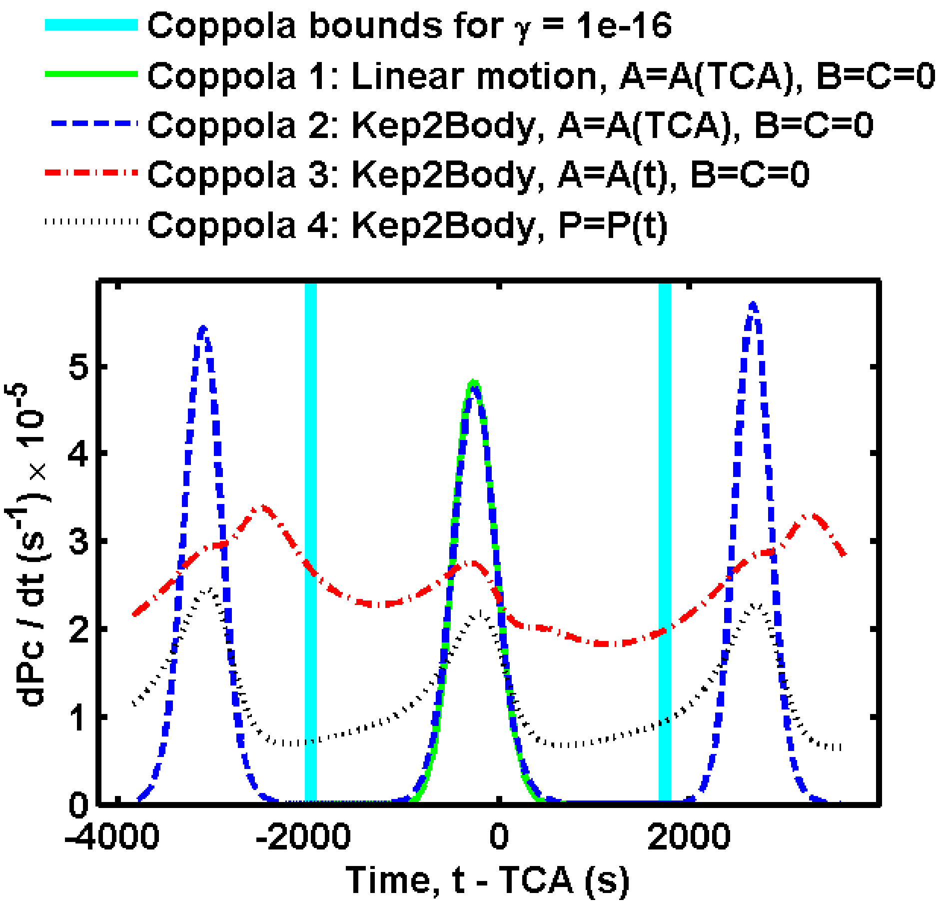
Archived Conjunctions with 3D $P_c \geq 10^{-5}$

- For these, most 2D and 3D estimates were found to be relatively close to one another
  - 71% have 2D P_c and 3D P_c within 10% of one another
  - 85% have 2D P_c and 3D P_c within 30% of one another
- But smaller subsets were found to differ significantly
  - 5.6% have 2D P_c and 3D P_c separated by a factor of 3 or more
  - 2.4% have 2D P_c and 3D P_c separated by a factor of 10 or more
- The cases where 3D P_c >> 2D P_c are of significant concern to the CARA team
  - Threatening conjunctions could be overlooked when using the 2D P_c approximation

Repeating Events for Close Proximity Satellites

- Objects persistently orbiting in close proximity can make repeated close approaches to one another
  - Satellites within formations or clusters
  - These conjunctions can often be identified by their long durations
  - This can create multiple, blended peaks in dP_c/dt
- These types of conjunctions explain some ***but not all*** of the archived cases that have 3D P_c >> 2D P_c

Archived conjunction involving two satellites flying in close proximity



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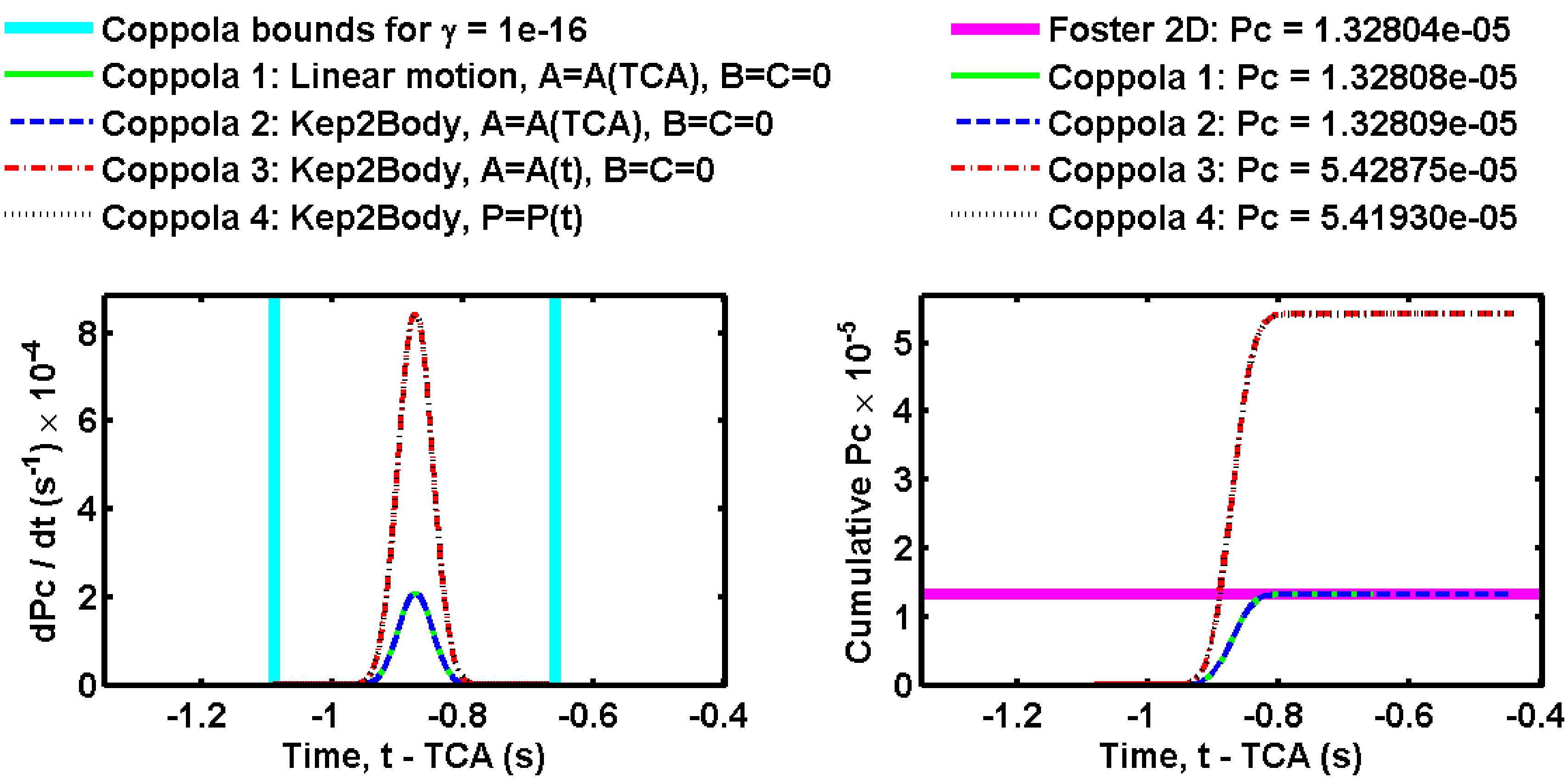
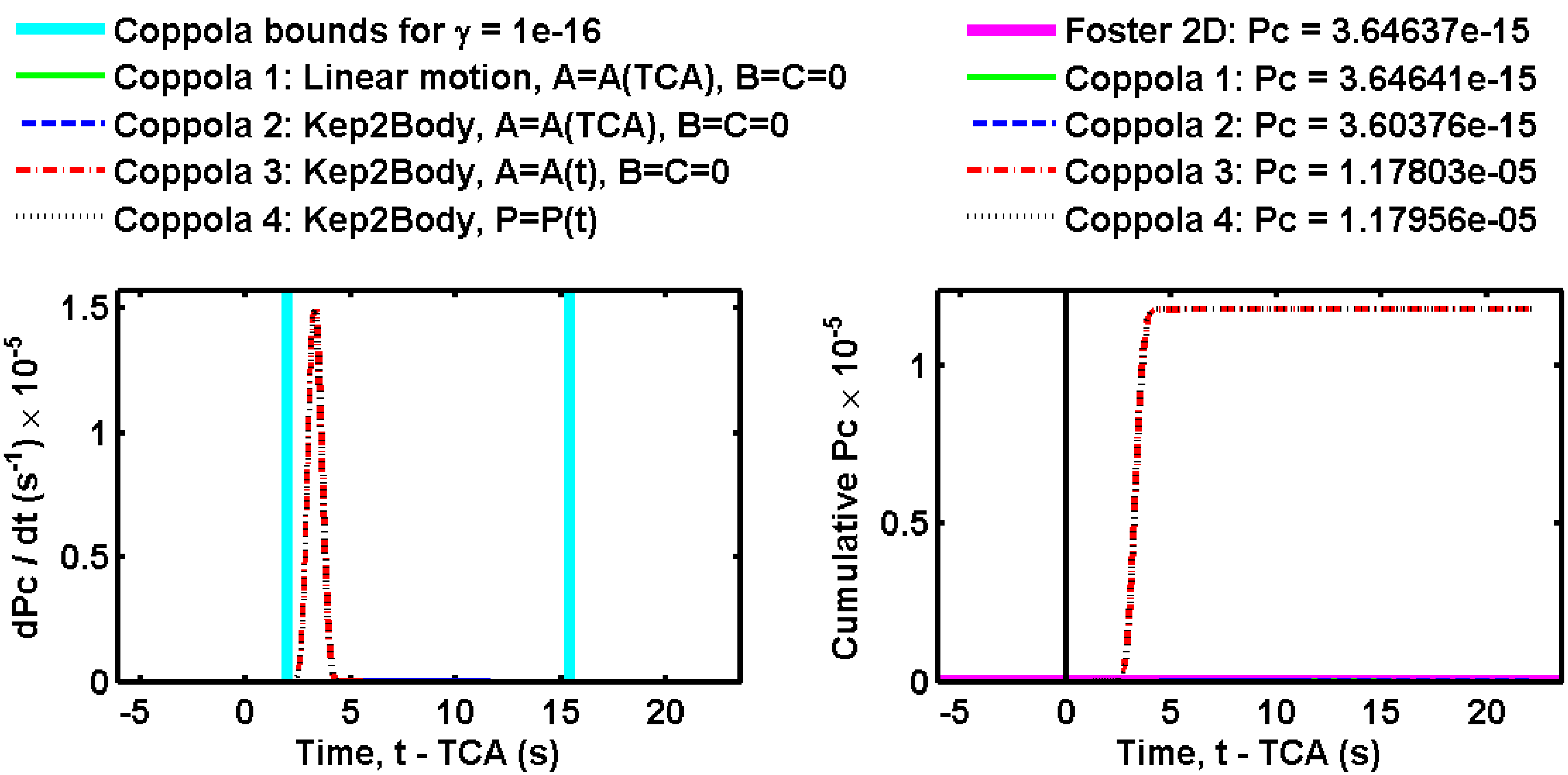
Isolated Conjunctions with 3D P_c >> 2D P_c



Archived conjunction where the 3D P_c estimate exceeds the 2D P_c estimate by a factor of about four.

Isolated Conjunctions with 3D P_c >> 2D P_c

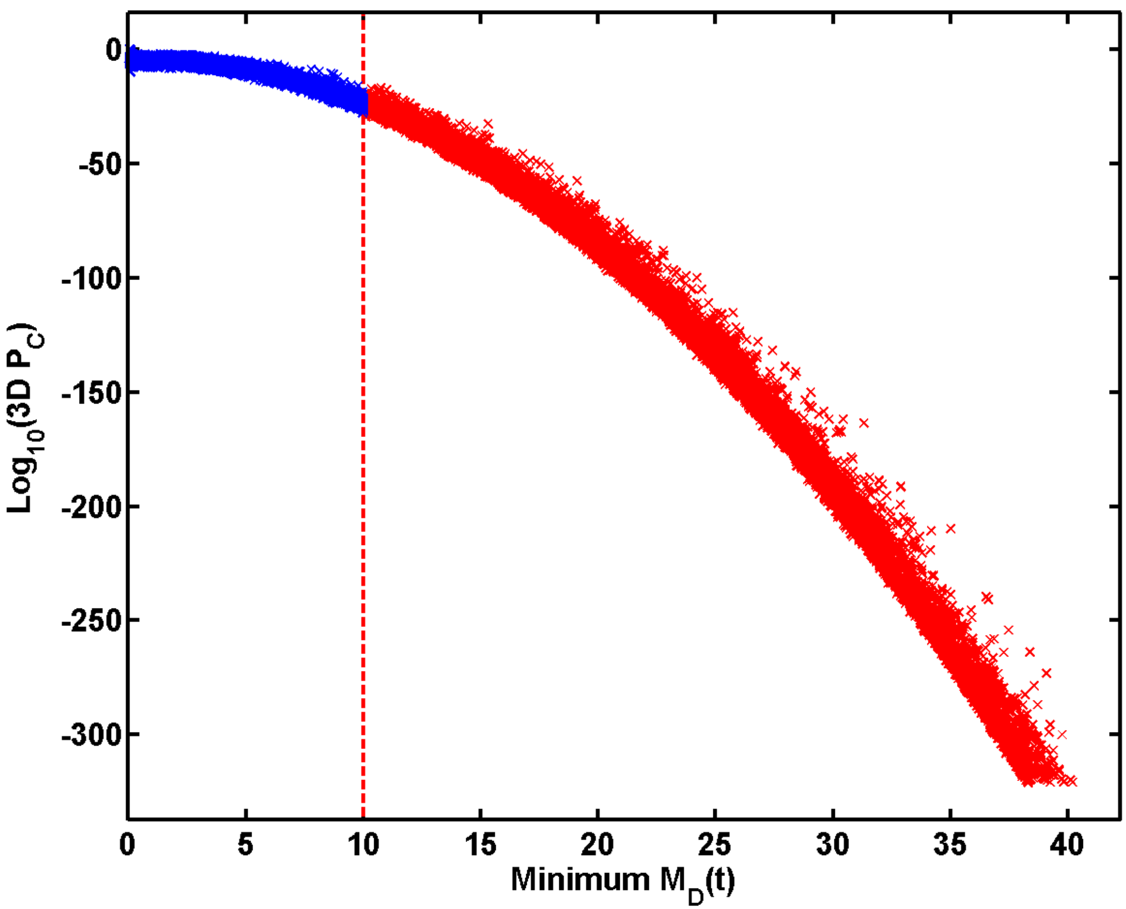


Archived conjunction where the 3D P_c estimate exceeds the 2D P_c estimate by several orders of magnitude.

A Screening Test for Small-P_c Values

- Conjunctions with large relative-position Mahalanobis distances have small 3D *Pc* values
- This correlation provides the basis for an efficient small-P_c screening test
- Applying this screening test eliminates the need to calculate 3D P_c for ≈80% of all conjunctions

About 80% of the archived conjunctions have $(M_D)_{min}$ > 10 and 3D P_c < $3{\times}10^{-17}$



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Conclusions

- **The CARA team has implemented Coppola's 3D P_c formulation into software**
  - Validated using Alfano's benchmark test cases
  - Provides estimates for both P_c and dP_c/dt
  - Provides insight into the time dependence of risk
- **Archived conjunction analysis indicates that**
  - Occasionally the 2D P_c approximation can be very inaccurate
  - An efficient small-P_c screening test can be used to speed processing for large numbers of conjunctions

Backup Slides

Schematic Illustration of 2D P_c Assumptions

Illustration of relative position trajectories for Alfano's (2009) "nonlinear" example #2

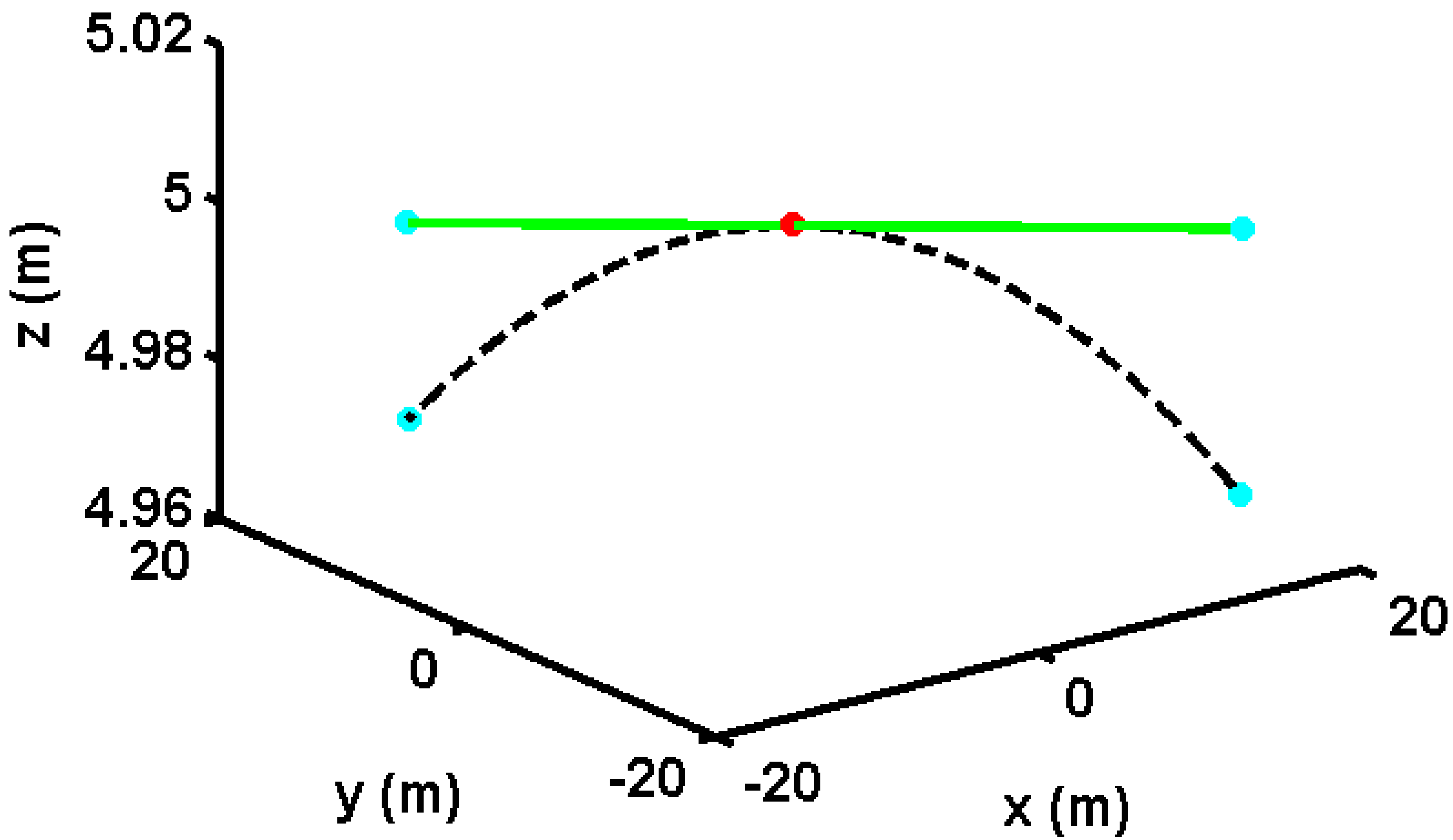


Illustration based on Alfano's test case #2

Schematic Illustration of 2D P_c Assumptions

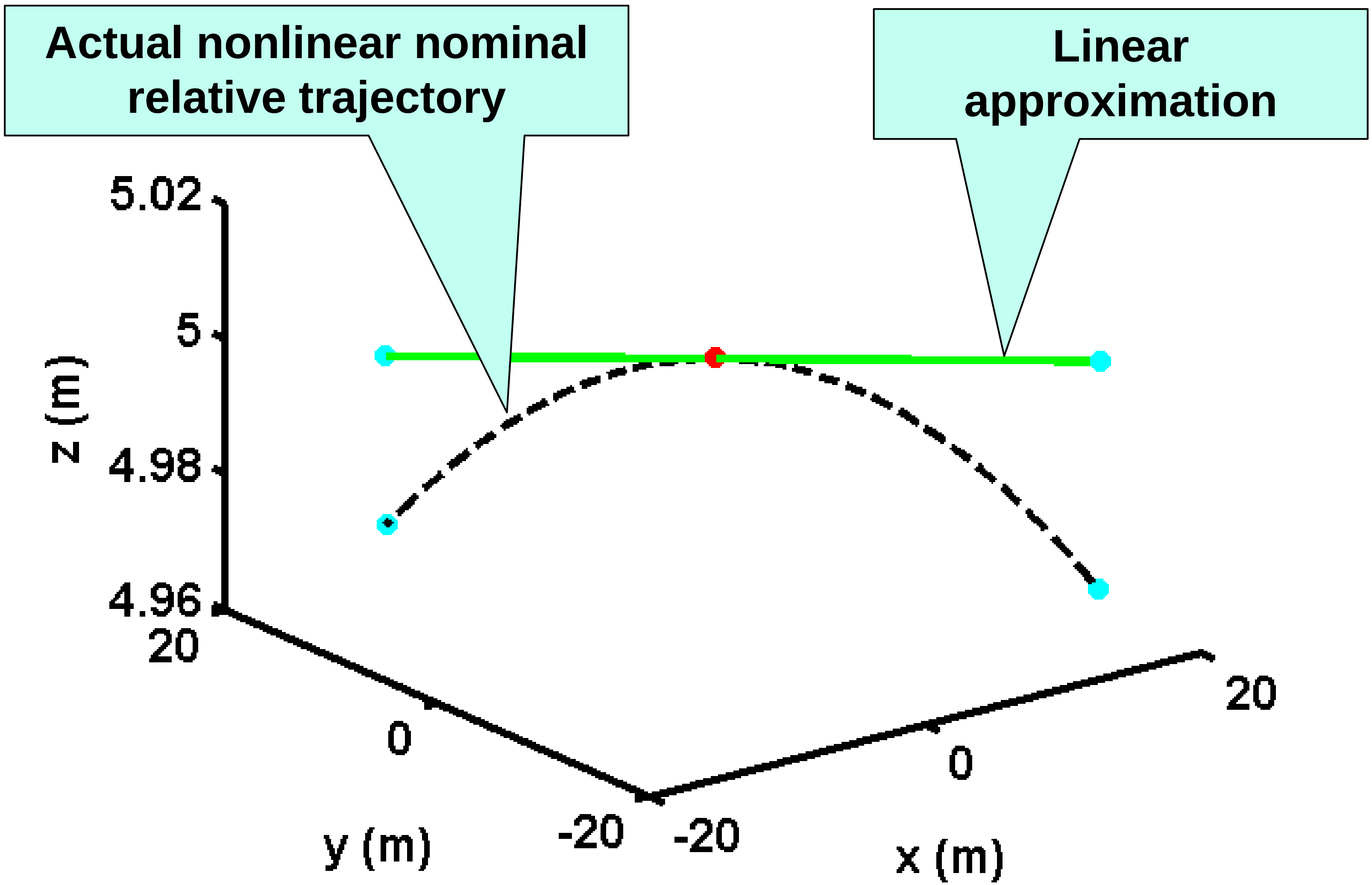


Illustration based on Alfano's test case #2

Schematic Illustration of 2D P_c Assumptions

Light blue dots show the nominal relative positions at Coppola's conjunction time bounds

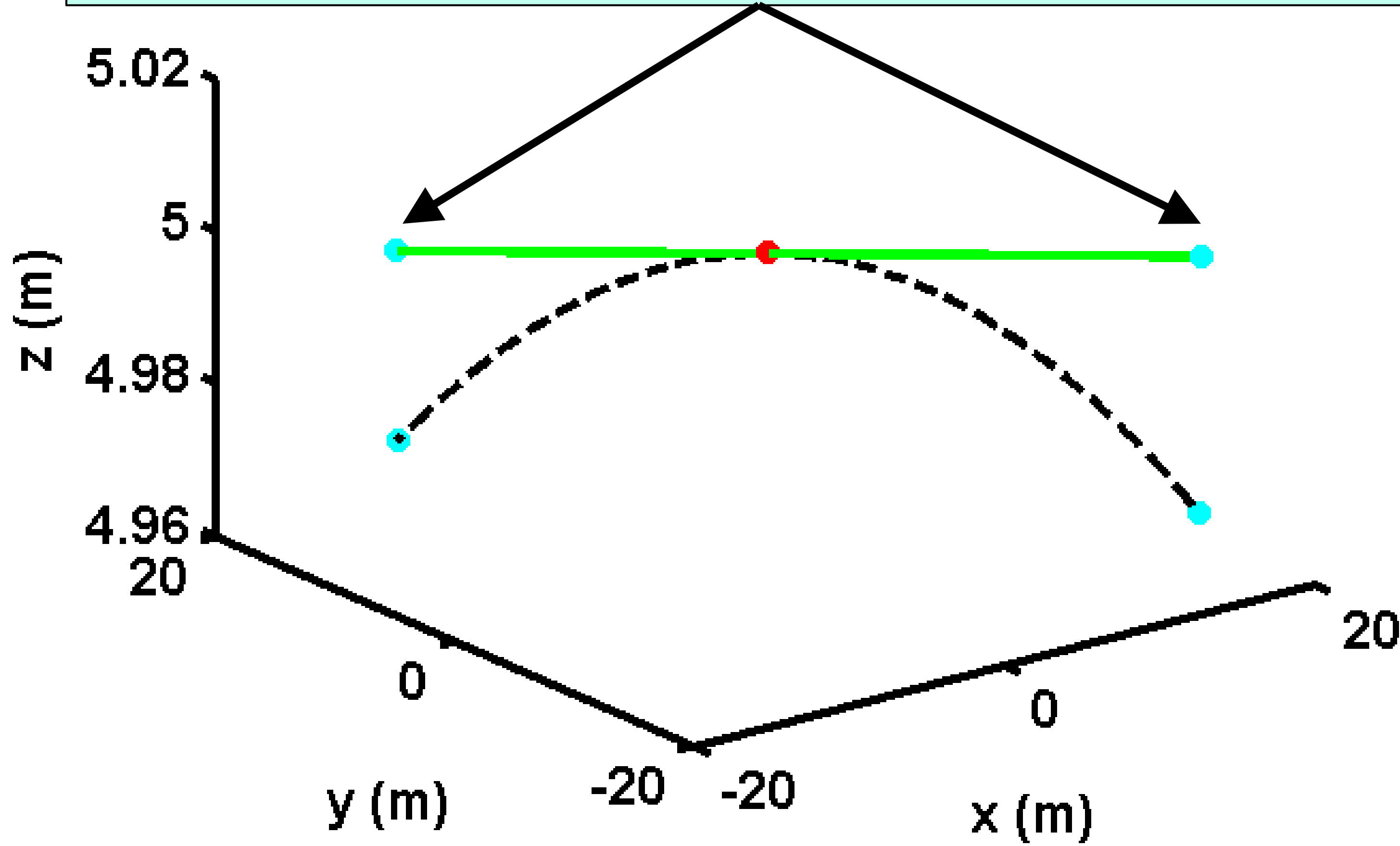


Illustration based on Alfano's test case #2

Schematic Illustration of 2D P_c Assumptions

Relative position PDFs evolve in time

NOTE: Actual 1σ surfaces are *much* larger and thinner

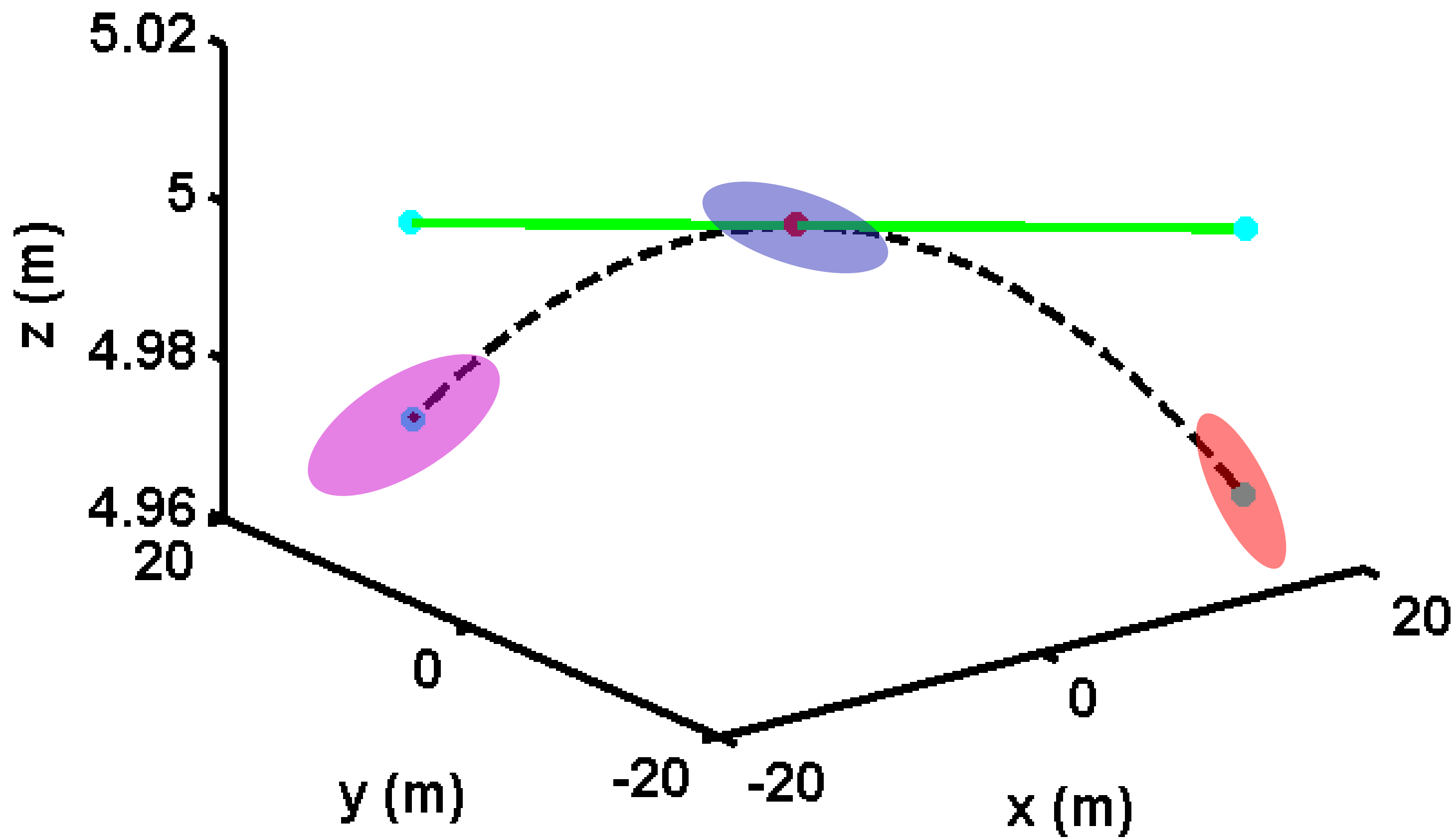


Illustration based on Alfano's test case #2

Schematic Illustration of 2D P_c Assumptions

2D *Pc* approximates the PDFs as constant, and places them along the linearized trajectory

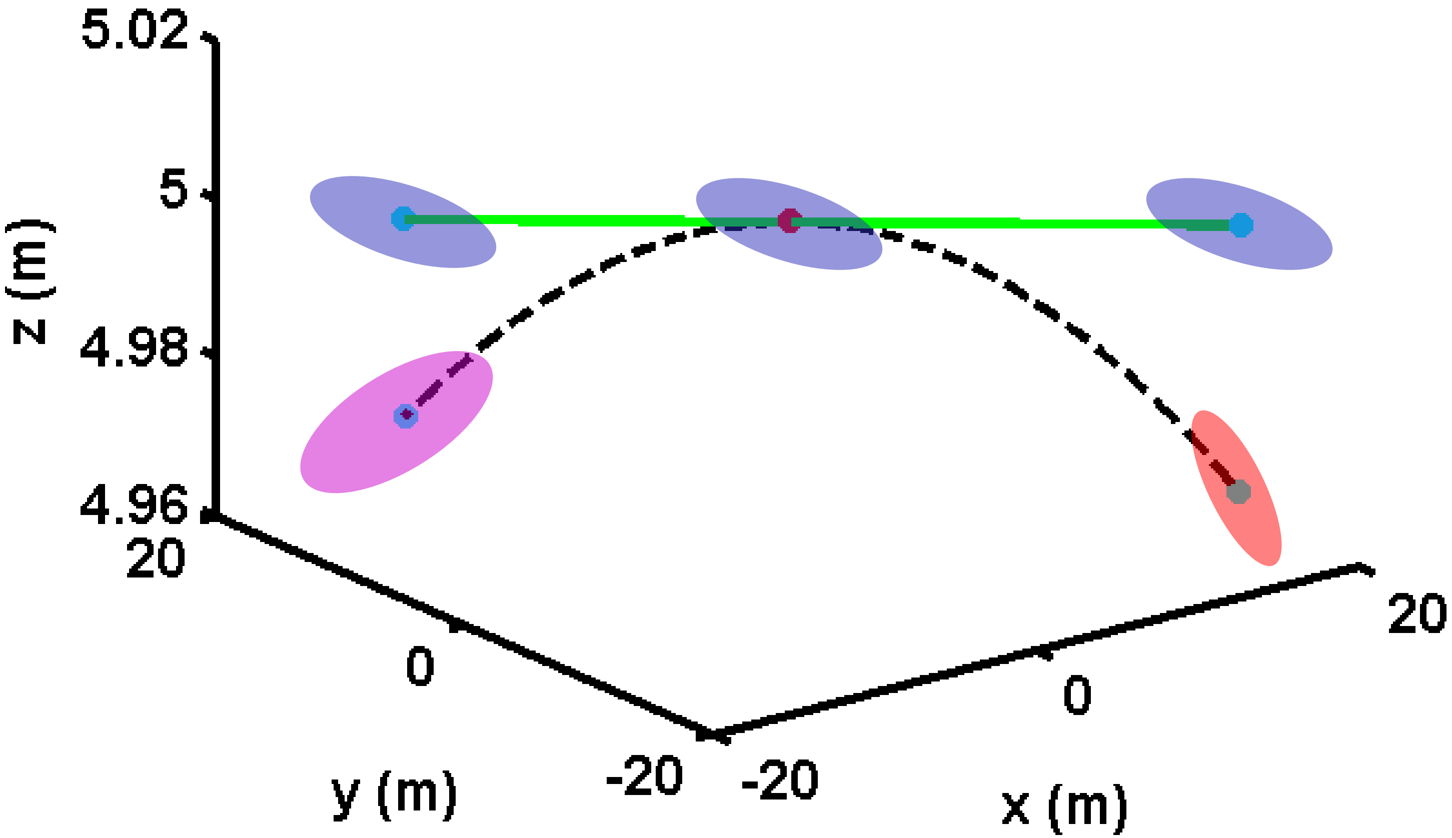


Illustration based on Alfano's test case #2

Schematic Illustration of 2D P_c Assumptions

The 2D *Pc* approximation will be inaccurate if these PDF differences become too large during the conjunction

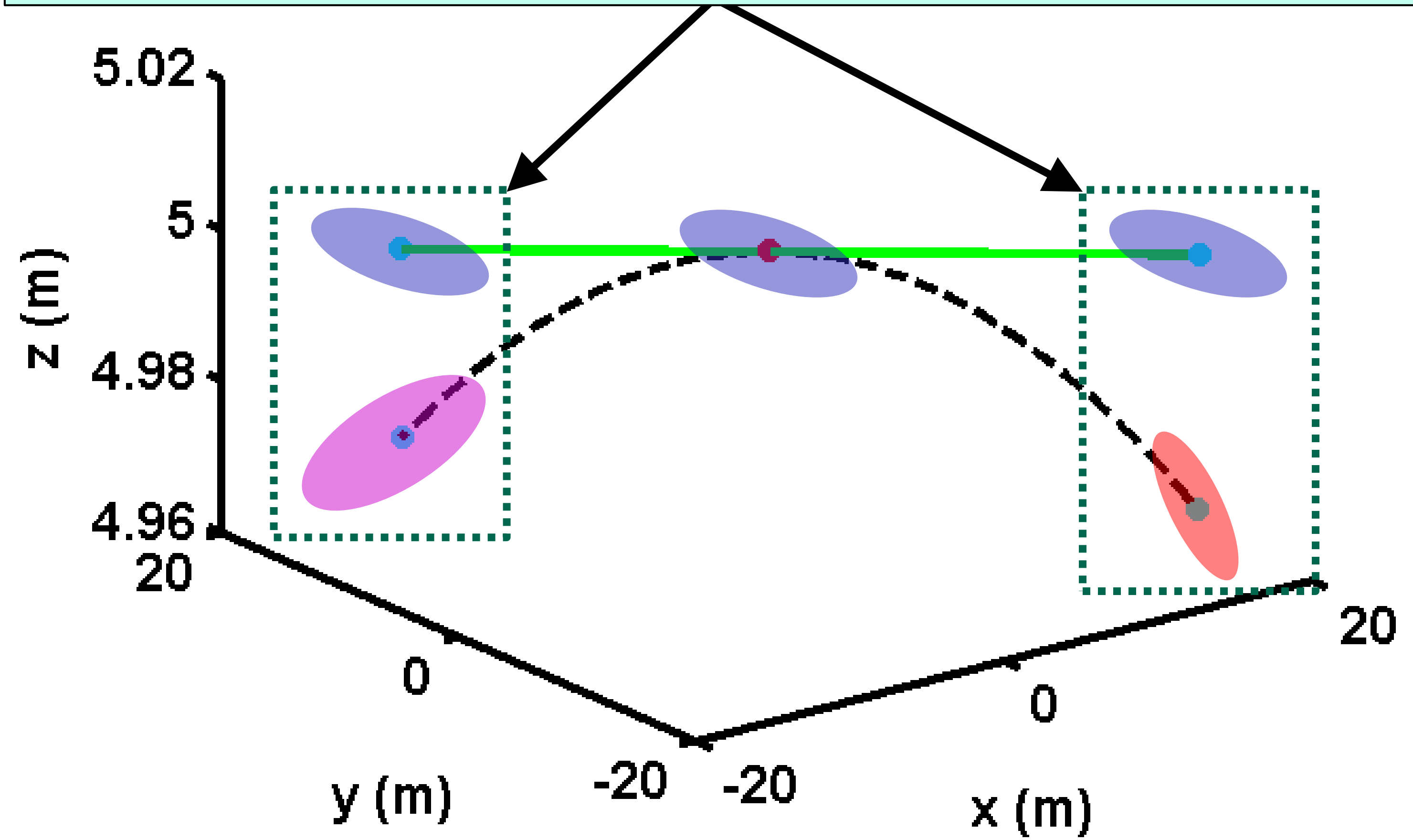


Illustration based on Alfano's test case #2

Mahalanobis Distance

The *Mahalanobis Distance* measures the difference between the positions of the primary and secondary objects, relative to the scale of their combined covariance:

$$M_D(t) = \left(\mathbf{r}^T \mathbf{A}^{-1} \mathbf{r}\right)^{1/2}$$

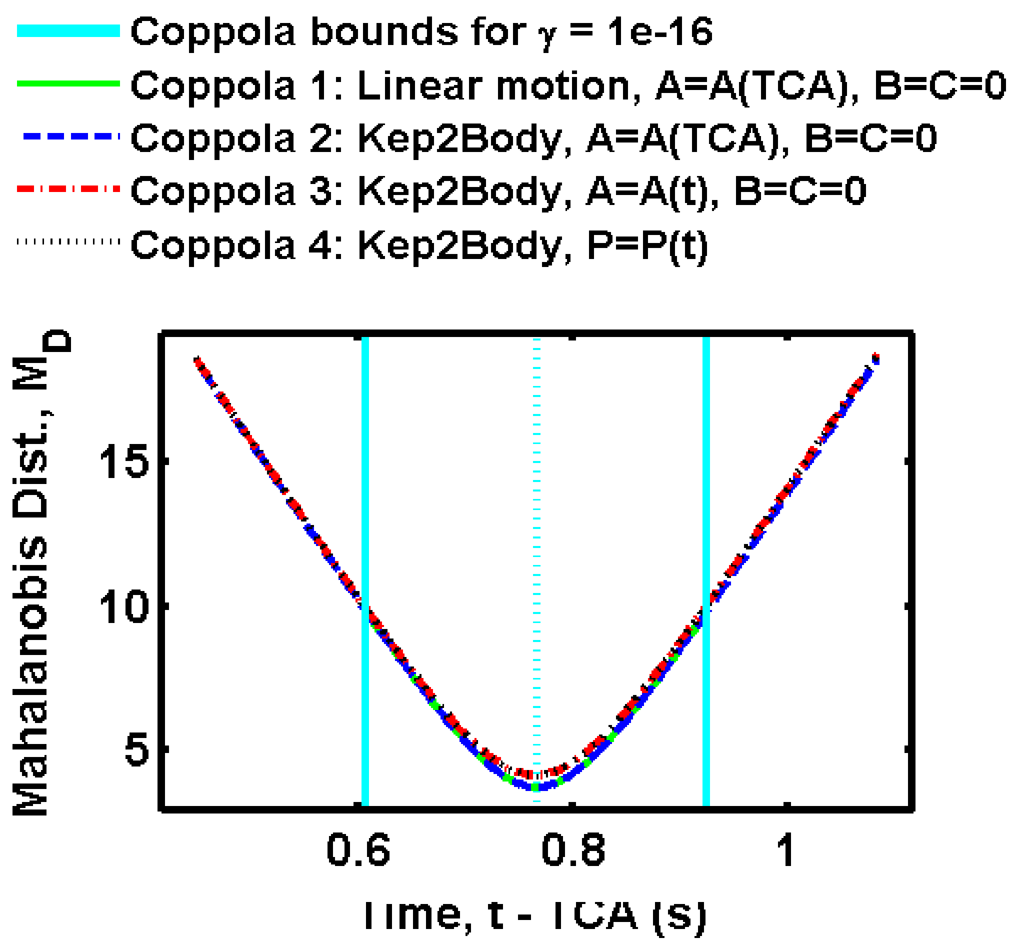
where

$$\mathbf{r} = \mathbf{r}(t) = \mathbf{r}_s - \mathbf{r}_p \qquad \mathbf{A} = \mathbf{A}(t) = \mathbf{A}_s + \mathbf{A}_p$$

$$\text{(relative position)} \qquad \text{(combined covariance)}$$

Minimum Mahalanobis Distance

- The Mahalanobis distance varies as a function of time during a conjunction
- The minimum value $(M_D)_{min}$ often occurs near the conjunction midpoint, but not always
- $(M_D)_{min}$ values vary significantly for different conjunction events



Monte Carlo P_c Estimation Procedure

1. **Sample the state PDFs for both the primary and secondary satellites**
2. **Propagate the sampled states over the desired time span, checking if the separation becomes less than the combined hard-body radii**
3. **If so, register a collision at the time the spheres defined by the hard-body radii make first first contact**
4. **Repeat steps 1-3 to improve statistical estimation accuracy**