ALGORITHMS FOR THE COMPUTATION OF DEBRIS RISK

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ABSTRACT

Determining the risks from space debris involve a number of statistical calculations. These calculations inevitably involve assumptions about geometry - including the physical geometry of orbits and the geometry of satellites. A number of tools have been developed in NASA's Orbital Debris Program Office to handle these calculations; many of which have never been published before. These include algorithms that are used in NASA's Orbital Debris Engineering Model ORDEM 3.0, as well as other tools useful for computing orbital collision rates and ground casualty risks. This paper presents an introduction to these algorithms and the assumptions upon which they are based.

1 INTRODUCTION

Computation of collisions and other long-term behaviours of orbiting objects are often computed using statistical tools. That is because it is often difficult to predict where a particular satellite will be at some time far into the future, especially the detailed position within its orbit. Instead, we rely on the location probability of a satellite at any particular time in the future. This information is computed in terms of probability distributions in the relevant parameters, which can be converted to spatial density.

Spatial density is a useful quantity, because it can be converted directly into collision probability under the assumptions of kinetic gas theory. This assumes that orbiting objects have dimensions much smaller than the scale of the changes in the spatial density. Under such conditions, the flux of a distribution of objects can be computed as

$$flux = spatial\ density * velocity \tag{1}$$

And number of collisions is

$$Number\ of\ collisions = flux * area * time \tag{2}$$

where "area" is the area of the orbiting object. In order for this relation to hold, the area/size of the satellite must be much smaller than the scale over which the spatial density varies. In this paper, we will discuss cases where this assumption breaks down, and how to compute the collision rate in those cases.

For much of the work in this paper, we will assume that objects orbit in idealized Kepler orbits. In reality, the effects of atmospheric drag, the non-spherical gravity of the Earth, and the effects of the gravity of the Sun and Moon mean that this assumption is only an approximation.

2 KESSLER EQUATION

Kessler [1] used a spatial density equation to compute the collision rates of Jupiter's moons. These equations have been used uncounted times since then to compute the collision probabilities of debris striking Earth-orbiting satellites. In order to develop the tools for this paper, we must first re-derive the Kessler equations, but in a way that opens up other possibilities for computation. Note that many of the equations used in this section parallel those in Dennis [2].

The probability distribution function (PDF) of a periodic parameter can be computed using the localized time rate of change of that parameter. For instance, for any periodic parameter *x*,

$$p(x)dx \propto \frac{dx}{|\dot{x}|} \tag{3}$$

For example, the radial velocity of a Kepler orbit is

$$\dot{r}(r) = \pm\sqrt{\frac{\mu}{a}}\,\frac{\sqrt{(r_A - r)(r - r_P)}}{r} \tag{4}$$

where a is the semi-major axis, μ the gravitational constant, and r_A and r_P are the apoapsis and periapsis radii, respectively. The radius of a satellite oscillates between r_A and r_P during each orbit period, so we can apply equation 3 to determine the normalized PDF of the orbit radius over a long period of time.

$$p(r)dr = \frac{r\,dr}{\pi\,a\,\sqrt{(r_A - r)(r - r_P)}} \tag{5}$$

Unlike the approximation used by Kessler, there is an exact analytic solution to the integral of this equation.

$$\int p(r')\,dr' = \frac{\sqrt{(r_A - r)(r - r_P)}}{\pi\,a} - \frac{1}{\pi} arcsin\left(\frac{2(a - r)}{r_A - r_P}\right) + C \tag{6}$$

Of course, this only applies for r between r_P and r_A; the PDF is zero otherwise. This radial equation is useful in computing how much time a satellite spends between two altitudes for computing equivalent spatial density.

Kessler used the assumption that the ascending node and argument of periapsis of an orbit are uniformly distributed from 0° to 360°. This is a good assumption for many types of orbits, especially if averaging over long periods of time.

A similar formula for PDF of the latitude λ can be computed for an orbit with inclination i, assuming that the argument of periapsis is uniformly distributed. This equation is applicable for $sin^2(\lambda) \leq sin^2(i)$ (the PDF is zero otherwise)

$$p(\lambda)d\lambda = \frac{cos(\lambda)\,d\lambda}{\pi\,\sqrt{sin^2(i) - sin^2(\lambda)}} \tag{7}$$

This equation can also be integrated analytically

$$\int p(\lambda')\,d\lambda' = -\frac{1}{\pi} arcsin\left(\frac{sin(\lambda)}{sin(i)}\right) + C \tag{8}$$

These latitude equations are used to compute the re-entry risk to populations on the ground for satellites decaying in an uncontrolled manner. Equation 8 is used to compute the fraction of time a satellite spends at each latitude, which will correspond to the probability the satellite will re-enter at that location. Combined with a model of how humans are distributed on the Earth, an estimate of the average density of people beneath a particular orbit can be computed [3].

The third parameter is the longitude ϕ. Because the ascending node is assumed to be uniformly distributed, the longitude distribution is simply

$$p(\phi)\,d\phi = \frac{d\phi}{2\pi} \tag{9}$$

In order to compute the spatial density, we need the equation for a small element of volume in spherical coordinates

$$dV = r^2 \cos(\lambda)\;dr\,d\lambda\,d\phi \tag{10}$$

The spatial density is, then

$$\rho(r, \lambda, \phi) = \frac{p(r)\,dr\,p(\lambda)\,d\lambda\,p(\phi)\,d\phi}{dV} = \frac{1}{2\pi^3 ra\sqrt{sin^2(i) - sin^2(\lambda)}} * \frac{1}{\sqrt{(r_A - r)(r - r_P)}} \tag{11}$$

and is valid for radius $r_P \leq r \leq r_A$ and latitude $sin^2(\lambda) \leq sin^2(i)$, otherwise the density is zero. This is equivalent to Kessler's spatial density equation [1].

For completeness, the velocity components are given by the radial velocity equation 4, and by

$$v_\phi(r, \lambda) = \frac{1}{r}\sqrt{\frac{\mu\,r_A\,r_P}{a}}\frac{\cos(i)}{\cos(\lambda)} \tag{12}$$

$$v_\lambda(r, \lambda) = \pm\frac{1}{r}\sqrt{\frac{\mu\,r_A\,r_P}{a}}\sqrt{1 - \left(\frac{\cos(i)}{\cos(\lambda)}\right)^2} \tag{13}$$

3 THE TRUE ANOMALY PDF

It is a good point in this discussion to introduce a very useful equation that describes the PDF for position within a Kepler orbit, using the true anomaly ν. Using equation 3 for the time rate of change of the true anomaly,

$$p(\nu)\,d\nu = \frac{(1-e^2)^{3/2}}{2\pi\,(1+e\cos(\nu))^2}\,d\nu \tag{14}$$

What is so useful about this equation is that there is no need to integrate over mean anomaly and convert to true anomaly at each step using Kepler's equation in order to solve for the time spent at each location. We can now simply integrate over the more "natural" geometric true anomaly and apply this weighting function.

So, consider a satellite A moving along its orbit through a spatial density of another satellite B. The position of satellite A is a function of the true anomaly, as well as other parameters of satellite A's orbit, which we will define as $\vec{r}(\nu)$. It will also have a three-dimensional velocity vector dependent on its position in the orbit $\vec{V}_A(\vec{r}(\nu))$. Using the spatial density of object B, ρ_B, and its velocity $\vec{V}_B$ which are both a function of position $\vec{r}(\nu)$, the two can be combined to find the integrated flux on A from B due to its travels through B's spatial density

$$flux = \int_0^{2\pi} \left\|\vec{V}_B(\vec{r}(\nu)) - \vec{V}_A(\vec{r}(\nu))\right\| * \rho_B(\vec{r}(\nu))\,p_A(\nu)\,d\nu \tag{15}$$

Note that the norm of the difference between the two vector velocity terms is simply the relative speed of the two objects at that point in space.

Equation 15 only needs to have a way to the handle the spatial density of object B and its velocity to solve the flux equation. Much of the rest of this paper consists of methods of determining ρ_B for a variety of different types of orbits.

Note that in general there are several velocity values $\vec{V}_B$ associated with each location $\vec{r}(\nu)$. As described in Kessler [1], the total flux will be computed for each velocity case and averaged to get the total flux.

4 AVERAGING THE KESSLER EQUATION

It has long been recognized that at the periapsis and apoapsis, and at the northernmost and southernmost latitudes of an orbit, the PDFs and the corresponding spatial density gives infinite values. PDFs often have this property, and as long as the integrals of the PDF are finite (which these are), this does not introduce any unrealistic behaviour. However, if undertaking a numerical integration of equation 15 and you encounter an infinite density, this could result in an incorrect flux.

One technique is to deliberately choose a small finite "box" in space, and compute the average spatial density. This has the effect of "blurring" the distribution so that the "infinities" go away. Using equations 6 and 8, a precise analytical solution for this "averaged" spatial density can be computed. When the "box" is between radii r_1 and r_2, and between latitudes λ_1 and λ_2, the "averaged" spatial density is

$$\langle\rho(r,\lambda,\phi)\rangle = \left\{\frac{\sqrt{(r_A-r)(r-r_P)}}{\pi\,a} - \frac{1}{\pi}arcsin\left(\frac{2(a-r)}{r_A-r_P}\right)\right\}_{r_1}^{r_2} * \frac{\left\{-\frac{1}{\pi}arcsin\left(\frac{sin(\lambda)}{sin(i)}\right)\right\}_{\lambda_1}^{\lambda_2}}{\frac{2\pi}{3}(r_2{}^3-r_1{}^3)(\sin(\lambda_2)-\sin(\lambda_1))} \tag{16}$$

One problem with computing this way is that the averaging "box" is not clearly defined.

A more self-consistent way to handle these infinite values is to make use of the fact that we are often dealing with families of orbits with a distribution of orbital parameters.

In Matney and Kessler [4], Divine's [5] meteoroid model was reformulated to implement these distributions in orbital parameters.

$$\rho(r,\lambda,\phi,r_P,r_A,i)\,dr_P\,dr_A\,di = \frac{D(r_P,r_A,i)\,dr_P\,dr_A\,di}{2\pi^3 ra\sqrt{sin^2(i)-sin^2(\lambda)}} * \frac{1}{\sqrt{(r_A-r)(r-r_P)}} \tag{17}$$

D(r_P,r_A,i) represents the PDF that describes how the satellite or satellites are distributed in orbital elements, either because they represent a family of objects, or the time average of a single satellite as its orbit evolves over time. Note that the orbit distribution can be in terms of any similar combination of orbital elements appropriate to describe the system of interest. For instance, we could use r_P, e (the orbit eccentricity), and i, which is the distribution used by the NASA ORDEM3.0 Engineering Model.

Equation 11 actually represents the special case of equation 17 where the orbit distribution PDF consists of Dirac delta functions in the orbital elements.

The reason why equation 17 is superior to equation 16 is that the averaging is now correctly computed over a known distribution of orbit elements. By integrating over orbit elements, the probability of having an orbit where the density at a point in space has exactly an infinite value (such as $r = r_P$) becomes vanishingly small, and the infinite values of the spatial density are replaced by finite values.

For an example of how distributions of orbits would be implemented, consider a simplified version using the radial density equation 5. Assume a semi-major axis of 1.0, and an eccentricity of 0.5, such that the periapsis is 0.5 and the apoapsis is 1.5. Now consider a similar case, with fixed apoapsis at 1.5, but with a uniform distribution of periapses from $r_{P1} = 0.4$ to $r_{P2} = 0.6$. The radial density equation would be

$$p(r)dr = \int_{Minimum(r_{P_1},r)}^{Minimum(r_{P_2},r)} D(r_p) * \left(\frac{r\ dr}{\pi\ a \sqrt{(r_A - r)(r - r_P)}} \right) dr_P \tag{18}$$

Note that the limits of integration must be adjusted to make sure $r_P \leq r$. Fig. 1 shows the radial PDFs of these two cases. Note that when the radial PDF is computed by integrating over a distribution of periapses, the infinite value of the PDF near perigee goes away and the new PDF remains finite. For the full 3D density or PDF, all of the infinite values disappear when distributions over the orbital elements are considered.

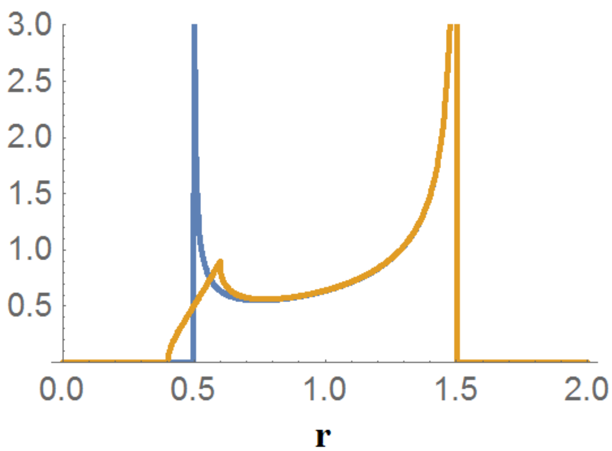

Figure 1. These two curves show the radial PDFs for the two cases described in the text. The blue curve is for the case with fixed periapsis 0.5 and apoapsis 1.5. The orange curve has the same apoapsis, but with a uniform distribution of periapses between 0.4 and 0.6. The blue curve has infinite values at periapsis and apoapsis, but the orange curve has finite PDF values throughout the periapsis region.

For practical numerical computation, integration of equation 17 over a distribution in orbital elements can still result in computing points with an infinite value of density. It is suggested that the integral be evaluated using the Second Euler-Maclaurin summation formula [6]. One can always break up any integral into the sum of multiple integrals such that the infinite values are at the limits of each sub-integration. This numerical technique samples points arbitrarily close to the limits of integration, yet avoids evaluating the troublesome points right at the limits.

5 VARIATIONS ON THE KESSLER EQUATION

One problem that arises with Kessler's equation is when the orbit in question is circular. Then the radial equation 5 becomes a Dirac delta function. While accurate, this is not a very useful equation. A more useful form is to assume that one or more satellites are distributed in circular orbits with some semi-major axis distribution, given as a density per unit altitude.

$$p(r)dr = D(a)\ da = D(r)\ dr \tag{19}$$

This results in the following useful spatial density equation for a distribution of purely circular orbits

$$\rho(r, \lambda, \phi) = \frac{D(r)}{2\pi^2\ r^2 \sqrt{sin^2(i) - sin^2(\lambda)}} \tag{20}$$

One could use this density in equation 15 to integrate the flux point-by-point. However, it is also possible to integrate this flux in a different way.

Consider two circular orbits oriented in such a way that the ascending nodes are uniformly distributed relative to one another. This only requires that at least one of the two orbits has randomized nodes; for instance one orbit can be polar with fixed ascending node, while the other is not and has randomized nodes over time.

Given the two inclinations i_1 and i_2, and the two ascending nodes Ω_1 and Ω_2, the angle α between the two planes is computed by taking the dot product of the two angular momentum vectors

$$\cos(\alpha) = \hat{h}_1 \cdot \hat{h}_2 \tag{21}$$
$$= (\sin(i_1)\sin(\Omega_1)\,\hat{x} - \sin(i_1)\cos(\Omega_1)\,\hat{y} + \cos(i_1)\,\hat{z}) \cdot (\sin(i_2)\sin(\Omega_2)\,\hat{x} - \sin(i_2)\cos(\Omega_2)\,\hat{y} + \cos(i_2)\,\hat{z})$$
$$= \cos(i_1)\cos(i_2) + \sin(i_1)\sin(i_2)\cos(\Omega_1 - \Omega_2)$$
$$= \cos(i_1)\cos(i_2) + \sin(i_1)\sin(i_2)\cos(\Delta\Omega)$$

So the angle between the orbit planes is a function of the two inclinations, and the difference between the ascending nodes. For our random node case, we simply integrate $\Delta\Omega$ from 0 to 2π to sample all possible orientations.

One further trick is to realize that we can place one orbit in the x-y plane with "inclination" zero, and the second orbit tilted with "inclination" α. If all the orbits involved are circular orbits, the "ascending node" of the "inclined" orbit is irrelevant, so we can treat this case as a circular orbit with "inclination" zero at "latitude" zero encountering another orbit with "inclination" α. Assuming our "inclined" orbit has altitude distribution *D(r)*, the other circular orbit "sees" density (cf. equation 20)

$$\rho = \frac{D(r)}{2\pi^2\, r^2 \sin(\alpha)} \tag{22}$$

To compute the flux, the relative velocity is

$$v_{REL} = \sqrt{\frac{\mu}{r}}\sqrt{(1-\cos(\alpha))^2 + (\sin(\alpha))^2} = \sqrt{\frac{\mu}{r}}\sqrt{2 - 2\cos(\alpha)} \tag{23}$$

So, the flux integral would be

$$flux = \int_0^{2\pi} \frac{D(r)}{2\pi^2\, r^2 \sin(\alpha)} * \sqrt{\frac{\mu}{r}}\sqrt{2-2\cos(\alpha)}\; d(\Delta\Omega) = \frac{D(r)}{\pi^2\, r^2}\sqrt{\frac{\mu}{r}}\; 2\sqrt{2} * \frac{K\left(\frac{-2\sin(i_1)\sin(i_2)}{1+\cos(i_1)\cos(i_2)-\sin(i_1)\sin(i_2)}\right)}{\sqrt{1+\cos(i_1)\cos(i_2)-\sin(i_1)\sin(i_2)}} \tag{24}$$

where K is the incomplete elliptical integral of the first kind. This expression is similar to that computed by Su & Kessler [7]. Fig. 2 shows a plot of this analytic flux expression for the full range of possible inclinations. Note that a high-flux "ridge" exists when the sum of that two inclinations equals 180° (the angles are supplementary). This is the case where the two orbits can be coplanar and counter-rotating (depending on the ascending nodes). This creates an infinite spatial density problem when performing numerical calculations.

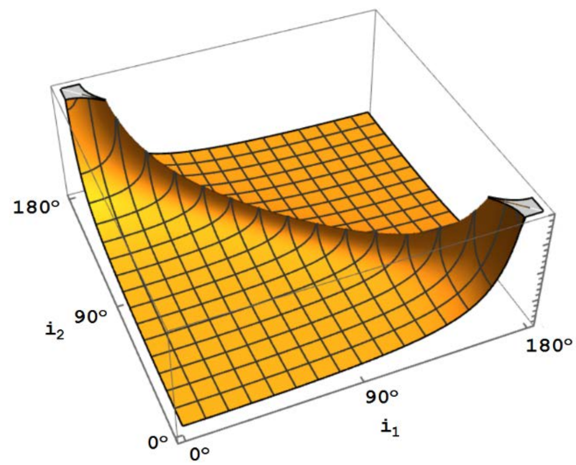

Figure 2. This chart shows the relative flux between two sets of circular orbits with inclinations i_1 and i_2. There is a high-flux "ridge" when the two inclination angles are supplementary, which corresponds to the case where the two orbits can be coplanar and counter-rotating.

Whenever the angle between the orbit planes approaches zero, the $sin(\alpha)$ term in the denominator drives up the spatial density. Therefore, there should be a similar high-density "ridge" for $i_1 = i_2$. In that case, the orbits can be co-rotating and coplanar. However, in the special analytic case with circular orbits, the relative velocity drops to zero, and the product of the relative velocity with the infinite density results in a finite flux. However, any eccentricity at all will introduce a similar "ridge" along the $i_1 = i_2$ line where the flux values approach infinity. As above, the solution to this problem is to compute spatial density with a distribution in inclination values.

6 THE 2D DENSITY

The discussion in the last section introduces another useful idea, that of describing the PDF within an orbit plane. In this case, we preserve the assumption that the argument of perigee is uniformly distributed, but now the ascending node is fixed. Because the orbit is a 2D distribution, we use an areal density function, assembled from equation 5.

The areal density for an elliptical orbit will be

$$\Sigma(r) = \frac{p(r)}{2\pi\, r} = \frac{1}{2\pi^2\, a\sqrt{(r_A - r)(r - r_P)}} \tag{25}$$

Similarly, for a collection of circular orbits with distribution D(r) with altitude

$$\Sigma(r) = \frac{D(r)}{2\pi\, r} \tag{26}$$

This distribution is "disc" that would look like a common CD or DVD, with a large "hole" in the center for $r < r_P$. We can choose the "disc" orbit B as being in the x-y plane. The other orbit A will cross the disc twice (unless they are coplanar), and the flux is computed at each crossing.

The encounter geometry is shown in Fig. 3. For calculation purposes, the "disc" of orbit B is temporarily assumed to have a small thickness δ, much smaller than the scale of the orbits, but much larger than the size of the satellites. The relative velocity is obtained by subtracting the velocity of B from A. This is the velocity in the frame where B has zero local velocity. This results in an angle ξ that the relative velocity makes to the B orbit plane.

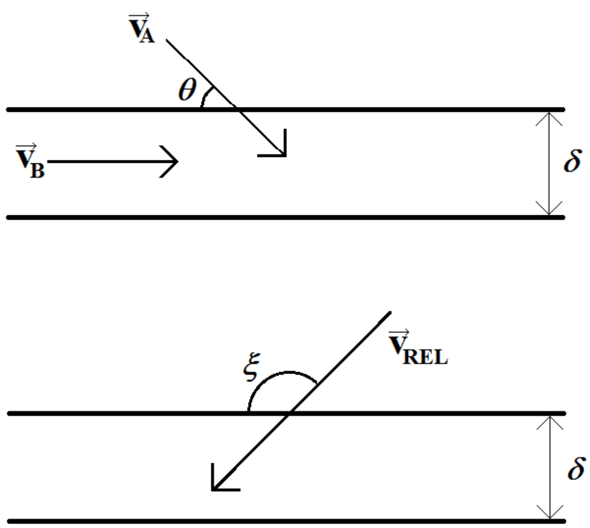

Figure 3. This is the geometry of the case described in section 6. The "disc" of object B's orbit has a small thickness δ. In the lower panel, the same encounter geometry is show from the locally stationary frame of object B. In this frame the relative velocity $\vec{v}_{REL}$ makes an angle ξ with respect to the disc.

The local 3D density is

$$\rho = \frac{\Sigma}{\delta} \tag{27}$$

and the path length of A through the disc is

$$L = \frac{\delta}{\sin(\xi)} \tag{28}$$

Using the period of satellite A, τ_A, the flux for each transit through the disc will be

$$Flux = \rho\, \frac{L}{\tau_A} = \frac{\Sigma}{\delta}\, \frac{1}{\tau_A}\, \frac{\delta}{\sin(\xi)} = \frac{\sigma}{\tau_A \sin(\xi)} \tag{29}$$

Where the temporary thickness δ disappears from the equation. The total flux is computed by summing over the flux at the two points where orbit A transits the "disc".

In actuality, there are two radial velocities in the disc at any point (equation 4) corresponding to the inbound and outbound cases for B, and will, in general, result in different ξ values. The relative velocity and the flux will need to be computed for each case and the results averaged for each plane crossing. The total flux is then the sum of the fluxes for each of the two plane crossings.

7 THE GENERALIZED ORBIT CASE

The discussions so far have dealt with distributions that are randomized in argument of perigee or ascending node, or where one satellite or another is in a circular orbit. How is the flux handled between two orbits with fixed nodes where the only parameter randomized is the position in the orbit?

To handle this case, we take a page from the collision avoidance community. They actually follow an uncertainty covariance associated with an object around its orbit. If we could "smear" out that probability ellipse over time, the orbit probability distribution and spatial density would look something like a long "fuzzy caterpillar" in space. The "thickness" of this "caterpillar" could be thought to represent the differences between a pure Kepler orbit and a real orbit, uncertainty in the orbit elements, or perhaps just a tool to create a spatial density much larger than the satellites themselves.

The first thing we will need is the linear density of a satellite along its orbit $\mathcal{L}$. This is computed using the following relation

$$\mathcal{L}(\nu)\, V(\nu) = \frac{1}{\tau} \qquad (30)$$

where $V(\nu)$ is the speed of the satellite at true anomaly ν and τ is the period of the orbit. This comes from the fact that if the object has one object in it, we ought to see that object once per orbit period.

Using the equation for the speed of an orbit as a function of r,

$$\mathcal{L}(\nu) = \frac{\sqrt{1-e^2}}{2\pi\, a \sqrt{1 + 2e\cos(\nu) + e^2}} \qquad (31)$$

This equation represents the number of satellites per unit length along a stretch of orbit at position ν averaged over time. The linear density is lower near periapsis where the orbit velocity is higher, and higher near apoapsis.

The next step is to add the "caterpillar fuzz" along this orbit. This is accomplished by adding a symmetric 2D normal distribution perpendicular to the orbit with standard deviation σ much smaller than the scale of the orbit but much larger than the satellites themselves.

$$\rho(\vec{r}) = \frac{\mathcal{L}(\nu)}{2\pi\,\sigma^2} Exp\left[\frac{-\|\vec{r} - \vec{q}(\vec{r})\|^2}{2\,\sigma^2}\right] \qquad (32)$$

Here the vector $\vec{q}(\vec{r})$ is the point along the orbit closest to the point $\vec{r}$, and the term $\|\vec{r} - \vec{q}(\vec{r})\|$ represents the perpendicular distance from the orbit to point $\vec{r}$. In general, it must be computed numerically.

As with the other densities discussed in this chapter, the density in equation 32 can be used in computing flux in equation 15. Experience has shown that numerically calculating the flux using equation 32 is extremely computer intensive if integrated over time, because only during specific geometries do two orbits usually overlap enough to give non-trivial "caterpillar" density.

One last thought – these distributions form a hierarchy. If we summed over many arguments of perigee, the "caterpillar" distribution would look very much like the 2D "disc" distribution. If we tilted the 2D "disc" and rotated it around all possible ascending nodes, the resulting density would be the Kessler spatial density.

8 CONCLUSIONS

The discussions in this paper have dealt with some tools to make accurate flux calculations for a variety of assumptions. Hopefully, many of these will provide useful tools for the space debris community, as they have for NASA's Orbital Debris Program Office.

The use of distributions of orbital elements (section 4) has allowed the ability to compute accurate fluxes for NASA's ORDEM3.0 model, especially near GEO where the ascending node of the satellites cannot be assumed to be uniformly distributed. While sometimes these computations are slow, we are always finding ways to increase their speed and accuracy.

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