A new formulation of time domain boundary integral equation for acoustic wave scattering in the presence of a uniform mean flow

Fang Q. Hu* and Michelle E. Pizzo†
Old Dominion University, Norfolk, Virginia 23529
and
Douglas M. Nark‡
NASA Langley Research Center, Hampton, Virginia 23681

It has been well-known that under the assumption of a constant uniform mean flow, the acoustic wave propagation equation can be formulated as a boundary integral equation, in both the time domain and the frequency domain. Compared with solving partial differential equations, numerical methods based on the boundary integral equation have the advantage of a reduced spatial dimension and, hence, requiring only a surface mesh. However, the constant uniform mean flow assumption, while convenient for formulating the integral equation, does not satisfy the solid wall boundary condition wherever the body surface is not aligned with the uniform mean flow. In this paper, we argue that the proper boundary condition for the acoustic wave should not have its normal velocity be zero everywhere on the solid surfaces, as has been applied in the literature. A careful study of the acoustic energy conservation equation is presented that shows such a boundary condition in fact leads to erroneous source or sink points on solid surfaces not aligned with the mean flow. A new solid wall boundary condition is proposed that conserves the acoustic energy and a new time domain boundary integral equation is derived. In addition to conserving the acoustic energy, another significant advantage of the new equation is that it is considerably simpler than previous formulations. In particular, tangential derivatives of the solution on the solid surfaces are no longer needed in the new formulation, which greatly simplifies numerical implementation. Furthermore, stabilization of the new integral equation by Burton-Miller type reformulation is presented. The stability of the new formulation is studied theoretically as well as numerically by an eigenvalue analysis. Numerical solutions are also presented that demonstrate the stability of the new formulation.

I. Introduction

The numerical solution of sound scattering by an acoustically large body remains a significant challenge due to the high demand on computational resources that are required to resolve the acoustic waves of short wavelengths. It is well-known that under the assumption of a constant mean flow, the acoustic wave propagation is governed by the convective wave equation that, in turn,

*Professor, Department of Mathematics and Statistics, Associate Fellow AIAA
†Graduate Student, Department of Mathematics and Statistics, Member AIAA
‡Senior Research Scientist, Research Directorate, Structural Acoustics Branch, Associate Fellow AIAA

can be converted into a boundary integral equation. The boundary integral equation approach has the advantage of reducing the spatial dimensions of the problem by one, making it an attractive computational method for calculating sound scattering and shielding at mid to high frequencies. In this paper, we consider the problem of acoustic scattering by rigid bodies in the presence of a uniform flow using the boundary integral equation approach. The present approach is based on the time domain boundary integral equation. The time domain approach has some distinct advantages over a frequency domain approach. Most notably, scattering solutions at all frequencies are obtained within one single computation. In addition, broadband noise sources and time dependent transient signals can be simulated and studied. The time domain approach also couples naturally with nonlinear computations where many frequencies are generated.

Previously, scattering of sound waves by rigid bodies with flow has been studied, in both the frequency domain and the time domain. In [30], acoustic radiation in moving flow was formulated as a boundary integral equation in the frequency domain. The nonuniqueness of the exterior problem was dealt with by applying the Burton-Miller reformulation procedure.[1] The time domain boundary integral equation approach for scattering by moving surfaces was first formulated and studied in [28]. More recent studies of the time domain approach in the presence of a mean flow can be found in [13, 16, 19].

A major difference between the current approach and those taken previously is in the treatment of the boundary condition at solid surfaces in the presence of flow. While the linear acoustic problem as a perturbation over the mean flow can be considered separately from the mean flow, an implicit condition is that the mean flow itself satisfies the solid wall boundary condition. The assumption of a constant mean flow is an approximation to the actual mean flow and this assumption is made to make the formulation of a boundary integral equation possible. While this facilitates the conversion of the partial differential equation to the boundary integral equation, the simplified mean flow itself obviously can not satisfy the physical boundary condition at solid boundaries wherever the surface is not aligned with the assumed constant mean flow. As pointed out in [28], the boundary integral equation derived based on such an assumption would be formally valid when $M_n \ll 1$ where M_n is the Mach number of mean flow normal to the body surface. In this paper, we take a closer look at the boundary condition to be used for scattering of acoustic waves at solid surfaces where M_n is not zero. In all the previous studies, a boundary condition of normal acoustic velocity being zero has been applied everywhere including the surfaces where $M_n \neq 0$. However, an analysis of the acoustic energy equation shows that the usual boundary condition would lead to nonzero energy flux at surfaces where $M_n \neq 0$. A new formulation is derived based on this acoustic energy consideration, and an alternative boundary condition is proposed by the requirement that energy flux be zero at solid surfaces. As we will see, due to the structure of the integral equation, the new formulation also becomes much simpler than those found in the literature for scattering with flow, which is of great benefit for computation.

In addition to the modification of boundary condition at solid surfaces, a Burton-Miller type reformulation of the integral equation consistent with the new boundary condition is also presented. It is well-known that the direct solution of the boundary integral equation for exterior scattering problems is prone to numerical instabilities.[1,2,6,13,17,18,30] In the time domain, the instability is more easily excited because all frequencies within the numerical resolution are present. There are generally two approaches for dealing with this instability. One is the Burton-Miller reformulation which has been widely used for frequency domain exterior scattering problems. Recently, it has been shown that Burton-Miller reformulation is effective for time domain as well.[2,6,18] Another method for the removal of the instability is the CHIEF method.[17,29] In the present study, we apply the Burton-Miller technique to our new formulation for the elimination of instabilities.

The rest of the paper is organized as follows. In Section 2, an integral relation for acoustic wave propagation is derived for a constant mean flow in a general direction. Then, the time domain boundary integral equation for scattering by rigid bodies is derived in Section 3. In Section 4, a Burton-Miller type reformulation of time domain boundary integral equation is presented and a discussion on the stability of the new formulation is given in Section 5. Numerical methods for the time domain boundary integral equation are discussed in Section 6. Stability of the current formulation is demonstrated in Section 7 by analyzing the eigenvalues of the discretized system. Section 8 contains the conclusions.

II. Integral representation of acoustic waves in the presence of a uniform mean flow

The current problem is considered in the context of solving the wave equation in a moving medium exterior of certain specified surface S, such as the scattering of sound field by an object as shown in Figure 1. Acoustic waves are assumed to be disturbances of small amplitudes. Linear acoustic problems are frequently formulated using a velocity potential function $\phi(\boldsymbol{r}, t)$ where the acoustic velocity $\boldsymbol{u}$ and pressure p are related to ϕ as follows:

$$\boldsymbol{u} = \nabla\phi, \quad p = -\rho_0 \left(\frac{\partial \phi}{\partial t} + \boldsymbol{U} \cdot \nabla\phi \right) \tag{1}$$

where ρ_0 is the mean density. With a constant mean flow $\boldsymbol{U}$, the acoustic disturbances are governed by the convective wave equation.[25] In the present study, we consider the solution of the following equation for the velocity potential:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla \right)^2 \phi - c^2 \nabla^2 \phi = q(\boldsymbol{r}, t) \tag{2}$$

with homogeneous initial conditions

$$\phi(\boldsymbol{r}, 0) = \frac{\partial \phi}{\partial t}(\boldsymbol{r}, 0) = 0, \quad t = 0 \tag{3}$$

In the above, c is the speed of sound, $\boldsymbol{U}$ is the constant mean velocity, and $q(\boldsymbol{r}, t)$ represents the known acoustic sources. Furthermore, in addition to the radiation condition at the far field, (2) and (3) are to be supplemented with boundary conditions on the scattering surface S. The suitable boundary conditions to be applied on solid surfaces will be discussed in Section 3.

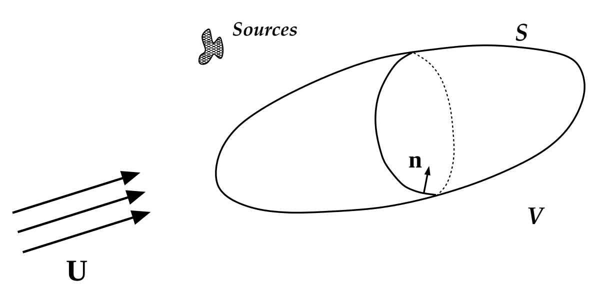


Figure 1. A schematic showing the scattering body and mean flow. Scattering surface is denoted by S and the solution domain exterior of S is denoted by V. The surface normal vector $\boldsymbol{n}$ is taken to be inward toward the interior of the body.

It is well-known that the convective wave equation (2) and the initial condition (3) as well as the boundary conditions can be reformulated into an integral equation. In the literature, the integral representation of sound waves in a moving flow is often derived by making use of generalized functions in a setting of moving bodies in an otherwise undisturbed medium.[4,7,8,20,23,26] Here, we present a derivation using a free-space Green's function $\tilde{G}(\boldsymbol{r}, t; \boldsymbol{r}', t')$ that, for convenience of discussion, is defined as follows:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla\right)^2 \tilde{G} - c^2 \nabla^2 \tilde{G} = \delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t') \tag{4}$$

with initial conditions

$$\tilde{G}(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{\partial \tilde{G}}{\partial t}(\boldsymbol{r}, t; \boldsymbol{r}', t') = 0, \quad t > t'. \tag{5}$$

Note that the time domain Green's function $\tilde{G}(\boldsymbol{r}, t; \boldsymbol{r}', t')$ defined above is nonzero for $t \in (-\infty, t']$. The solution to (4) and (5) is well-known (see, e.g., [4, 10, 25]) and, for a mean flow of a general direction, can be written as

$$\tilde{G}(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{G_0}{4\pi c^2}\delta\left(t' - t + \boldsymbol{\beta} \cdot (\boldsymbol{r}' - \boldsymbol{r}) - \frac{\bar{R}}{c\alpha^2}\right) \tag{6}$$

where

$$G_0 = \frac{1}{\bar{R}(\boldsymbol{r}, \boldsymbol{r}')}, \quad \text{and} \quad \bar{R}(\boldsymbol{r}, \boldsymbol{r}') = \sqrt{[\boldsymbol{M} \cdot (\boldsymbol{r} - \boldsymbol{r}')]^2 + \alpha^2 |\boldsymbol{r} - \boldsymbol{r}'|^2} \tag{7}$$

in which

$$\boldsymbol{M} = \frac{\boldsymbol{U}}{c}, \quad \alpha = \sqrt{1 - M^2}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{U}}{c^2 - U^2} = \frac{\boldsymbol{U}}{c^2\alpha^2} = \frac{\boldsymbol{M}}{c\alpha^2}, \quad U = |\boldsymbol{U}|, \quad M = |\boldsymbol{M}| \tag{8}$$

By an operation of $\tilde{G}\times(2) - \phi\times(4)$ and by integrating over the volume V exterior of the scattering surface S for space and an interval $[0^-, t'^+]$ for time t, it is straight-forward to show that we will get

$$\int_{0^-}^{t'^+} \int_V \left\{ \frac{\partial}{\partial t}\left[\tilde{G}\left(\frac{\partial \phi}{\partial t} + \boldsymbol{U} \cdot \nabla\phi\right) - \phi\left(\frac{\partial \tilde{G}}{\partial t} + \boldsymbol{U} \cdot \nabla\tilde{G}\right)\right] + \nabla \cdot \left[\left(\tilde{G}\left(\frac{\partial \phi}{\partial t} + \boldsymbol{U} \cdot \nabla\phi\right) - \phi\left(\frac{\partial \tilde{G}}{\partial t} + \boldsymbol{U} \cdot \nabla\tilde{G}\right)\right)\boldsymbol{U}\right]\right.$$

$$\left. - c^2 \nabla \cdot \left[\tilde{G}\nabla\phi - \phi\nabla\tilde{G}\right]\right\} d\boldsymbol{r}dt = \int_{0^-}^{t'^+} \int_V \left[\tilde{G}q(\boldsymbol{r}, t) - \phi(\boldsymbol{r}, t)\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t')\right] d\boldsymbol{r}dt$$

Integration of the first term in the above will be zero by initial conditions thus defined for ϕ and $\tilde{G}$. Then, upon using the divergence theorem and the condition at infinity, we get an expression for pressure ϕ at an arbitrary point $\boldsymbol{r}'$ in V and time t' as follows:

$$\phi(\boldsymbol{r}',t') = \int_{0^-}^{t'+} \int_V \tilde{G} q(\boldsymbol{r},t) d\boldsymbol{r} dt + c^2 \int_{0^-}^{t'+} \int_S (\tilde{G}\frac{\partial \phi}{\partial n} - \phi \frac{\partial \tilde{G}}{\partial n}) d\boldsymbol{r}_s dt$$

$$-c \int_{0^-}^{t'+} \int_S \left[\tilde{G} \left(\frac{\partial \phi}{\partial t} + \boldsymbol{U} \cdot \nabla \phi \right) - \phi \left(\frac{\partial \tilde{G}}{\partial t} + \boldsymbol{U} \cdot \nabla \tilde{G} \right) \right] M_n d\boldsymbol{r}_s dt \quad (9)$$

where $\boldsymbol{r}_s$ denotes points on surface S, and

$$M_n = \boldsymbol{n} \cdot \boldsymbol{M} = \boldsymbol{n} \cdot \boldsymbol{U}/c$$

is the normal component of the mean velocity Mach number on surface point $\boldsymbol{r}_s$. Here, the unit normal vector $\boldsymbol{n}$ is assumed to be outward from the solution domain. For the exterior scattering problem considered in the present study, the normal vector is then the one that is inward to the body as noted in Figure 1.

For convenience of discussion, we define a *modified normal derivative* (denoted by an overbar) as

$$\frac{\partial}{\partial \bar{n}} = \frac{\partial}{\partial n} - M_n (\boldsymbol{M} \cdot \nabla) \quad (10)$$

Then, equation (9) can be written as

$$\phi(\boldsymbol{r}',t') = \int_{0}^{t'+} \int_V \tilde{G} q(\boldsymbol{r},t) d\boldsymbol{r} dt + c^2 \int_{0}^{t'+} \int_S (\tilde{G}\frac{\partial \phi}{\partial \bar{n}} - \phi \frac{\partial \tilde{G}}{\partial \bar{n}}) d\boldsymbol{r}_s dt - c \int_{0}^{t'+} \int_S \left[\tilde{G}\frac{\partial \phi}{\partial t} - \phi \frac{\partial \tilde{G}}{\partial t} \right] M_n d\boldsymbol{r}_s dt \quad (11)$$

Furthermore, if we introduce a *combined normal derivative* (denoted by a tilde) as

$$\frac{\partial}{\partial \tilde{n}} = \frac{\partial}{\partial n} - \frac{M_n}{c} \left(\frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla \right) = \frac{\partial}{\partial \bar{n}} - \frac{M_n}{c} \frac{\partial}{\partial t} \quad (12)$$

we get another expression:

$$\phi(\boldsymbol{r}',t') = \int_{0^-}^{t'+} \int_V \tilde{G} q(\boldsymbol{r},t) d\boldsymbol{r} dt + c^2 \int_{0^-}^{t'+} \int_S (\tilde{G}\frac{\partial \phi}{\partial \tilde{n}} - \phi \frac{\partial \tilde{G}}{\partial \tilde{n}}) d\boldsymbol{r}_s dt \quad (13)$$

Equation (9), (11) or (13) is the Kirchhoff integral representation of the acoustic field in the presence of a uniform mean flow. The integral relation can be further expressed as the integration of retarded values by utilizing $\tilde{G}$ as given in (6). In particular, note that we have

$$\frac{\partial \tilde{G}}{\partial \tilde{n}} = \frac{1}{4\pi c^2} \frac{\partial G_0}{\partial \bar{n}} \left[\delta \left(t' - t + \boldsymbol{\beta} \cdot (\boldsymbol{r}' - \boldsymbol{r}) - \frac{\bar{R}}{c\alpha^2} \right) + \frac{\bar{R}}{c\alpha^2} \delta' \left(t' - t + \boldsymbol{\beta} \cdot (\boldsymbol{r}' - \boldsymbol{r}) - \frac{\bar{R}}{c\alpha^2} \right) \right] \quad (14)$$

where G_0 and $\bar{R}$ are those defined in (7). Then equation (13) can be written as

$$\phi(\boldsymbol{r}',t') = \frac{1}{4\pi c^2} \int_{V_s} \frac{1}{\bar{R}} q(\boldsymbol{r},t'_R) d\boldsymbol{r} + \frac{1}{4\pi} \int_S \left[G_0 \frac{\partial \phi}{\partial \tilde{n}}(\boldsymbol{r}_s,t'_R) - \frac{\partial G_0}{\partial \bar{n}} \left(\phi(\boldsymbol{r}_s,t'_R) + \frac{\bar{R}}{c\alpha^2} \frac{\partial \phi}{\partial t}(\boldsymbol{r}_s,t'_R) \right) \right] d\boldsymbol{r}_s \quad (15)$$

where V_s denotes the region of acoustic sources and the retarded time for t' is defined as

$$t'_R = t' + \boldsymbol{\beta} \cdot (\boldsymbol{r}' - \boldsymbol{r}) - \frac{\bar{R}}{c\alpha^2} \tag{16}$$

The modified normal derivative for G_0 is found to be the following:

$$\frac{\partial G_0}{\partial \bar{n}} = -\frac{1}{\bar{R}^2}\frac{\partial \bar{R}}{\partial \bar{n}} = -\alpha^2 \frac{\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}')}{\bar{R}^3} \tag{17}$$

Equation (15) relates the solution at point $\boldsymbol{r}'$ and time t' to the direct contribution from source function q and a surface contribution involving the retarded values of ϕ and their normal derivatives. As shown in [13], this form is equivalent to previous such formulations appearing in the literature, e.g., in [23,26], where the relationship had been derived under the assumption of a mean flow that is aligned with the x-axis.

When both $\phi(\boldsymbol{r}_s, t)$ and $\frac{\partial \phi}{\partial \tilde{n}}(\boldsymbol{r}_s, t)$ on surface S are known, $\phi(\boldsymbol{r}', t')$ at any field point $\boldsymbol{r}'$ can be computed by using (15). However, $\phi(\boldsymbol{r}_s, t)$ and $\frac{\partial \phi}{\partial \tilde{n}}(\boldsymbol{r}_s, t)$ are not independent. They have to satisfy the boundary integral equation formed when $\boldsymbol{r}'$ is taken to be a boundary point $\boldsymbol{r}'_s$ as we will discuss next.

III. Time domain boundary integral equation for scattering with solid surfaces

A Boundary Integral Equation (BIE) is formed by taking the limit $\boldsymbol{r}' \to \boldsymbol{r}'_s$ in the integral relation (15), where $\boldsymbol{r}'_s$ is a point on the boundary. The integral in (15) involving $\frac{\partial G_0}{\partial \bar{n}}$ is weakly-singular and, by using equation (55) given in the Appendix (assuming $\boldsymbol{r}'_s$ is a smooth boundary collocation point), it can be shown that

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s} \int_S \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}') \phi(\boldsymbol{r}_s, t'_R) d\boldsymbol{r}_s = \int_S \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}'_s) \phi(\boldsymbol{r}_s, t'_R) d\boldsymbol{r}_s - 2\pi\phi(\boldsymbol{r}'_s, t') \tag{18}$$

Applying this limit to (15), we get the following Time Domain Boundary Integral Equation (TD-BIE):

$$2\pi\phi(\boldsymbol{r}'_s, t') - \int_S \left(G_0 \frac{\partial \phi}{\partial \tilde{n}}(\boldsymbol{r}_s, t'_R) - \frac{\partial G_0}{\partial \bar{n}} \left[\phi(\boldsymbol{r}_s, t'_R) + \frac{\bar{R}}{c\alpha^2} \frac{\partial \phi}{\partial t}(\boldsymbol{r}_s, t'_R) \right] \right) d\boldsymbol{r}_s = Q(\boldsymbol{r}'_s, t') \tag{19}$$

where $Q(\boldsymbol{r}'_s, t')$ denotes the contribution from the external sources to the surface point $\boldsymbol{r}'_s$:

$$Q(\boldsymbol{r}'_s, t') = \frac{1}{c^2} \int_{V_s} \frac{1}{\bar{R}} q(\boldsymbol{r}, t'_R) d\boldsymbol{r} \tag{20}$$

For sound scattering problems, $\phi(\boldsymbol{r}'_s, t')$ on the scattering surface S is to be determined by (19) when the boundary condition for ϕ on S is given. A customary boundary condition on rigid surfaces is that the normal component of the acoustic velocity is zero, i.e., $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, which, considering (1), leads to

$$\boldsymbol{n} \cdot \nabla \phi = \frac{\partial \phi}{\partial n}(\boldsymbol{r}_s, t) = 0, \quad \boldsymbol{r}_s \in S \tag{21}$$

Indeed, in all the previous literature on wave scattering with a uniform mean flow (e.g., [5, 10, 13, 15, 19, 28, 30]), in both the frequency domain and the time domain, boundary conditions of type (21) have been assumed at solid wall boundaries. To implement such a boundary condition, the combined normal derivative appearing in (19) would then be separated into the normal and tangential components as

$$\frac{\partial \phi}{\partial \tilde{n}} = \left(1 - M_n^2\right) \frac{\partial \phi}{\partial n} - M_n \left(\frac{1}{c} \frac{\partial \phi}{\partial t} + \boldsymbol{M}_T \cdot \nabla \phi \right) \tag{22}$$

where $\boldsymbol{M}_T$ is the tangential component of the mean flow Mach number $\boldsymbol{M}$.

In the present paper, however, we propose an alternative boundary condition to be used at solid surfaces when solving TDBIE (19) in the presence of a uniform flow. The new boundary condition is based on a consideration of the acoustic energy.

It can be shown that the convective wave equation (2) without the source term has an associated energy equation:

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{23}$$

where

$$E = \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2c^2} \left| \frac{D\phi}{Dt} \right|^2 - \frac{\boldsymbol{U} \cdot \nabla \phi}{c^2} \frac{D\phi}{Dt}, \quad \boldsymbol{J} = -\frac{\partial \phi}{\partial t} \left(\nabla \phi - \frac{1}{c^2} \frac{D\phi}{Dt} \boldsymbol{U} \right), \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla \tag{24}$$

Equation (23) can be validated directly by using the expressions defined in (24). When substituted by the acoustic velocity and pressure defined in (1), $\rho_0 E$ is the usual acoustic energy density in a uniform flow.[22,24,27]

By (24), it is immediately clear that the energy flux at a surface of normal $\boldsymbol{n}$ is the following:

$$J_n = \boldsymbol{J} \cdot \boldsymbol{n} = -\frac{\partial \phi}{\partial t} \left(\frac{\partial \phi}{\partial n} - \frac{M_n}{c} \frac{D\phi}{Dt} \right) = -\frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial \tilde{n}} \tag{25}$$

Clearly, on a surface where the normal component of the mean velocity M_n is not zero, i.e., where the surface is not aligned with the mean flow, application of boundary condition (21) will result in nonzero energy flux, i.e., $J_n \neq 0$ and, consequently, cause the surface to be acting like an acoustic energy source or sink according to (25). This will apparently lead to nonconservation of the total acoustic energy.

Alternatively, the boundary condition on the solid surface may be defined by the requirement that no energy flows into or out of the surface. By (25) and to ensure energy flux $J_n = 0$ on solid surfaces, we propose that the boundary condition be modified such that the *combined normal derivative* of ϕ, defined in (12), is zero:

$$\frac{\partial \phi}{\partial \tilde{n}}(\boldsymbol{r}_s, t) = \frac{\partial \phi}{\partial n} - \frac{M_n}{c} \frac{D\phi}{Dt} = 0, \quad \boldsymbol{r}_s \in S \tag{26}$$

The total acoustic energy will be conserved under this new condition.

Now by applying boundary condition (26) to (19), a new formulation of the TDBIE for $\phi(\boldsymbol{r}_s', t')$ with solid surfaces is found as follows:

$$2\pi \phi(\boldsymbol{r}_s', t') + \int_S \frac{\partial G_0}{\partial \bar{n}} \left(\phi(\boldsymbol{r}_s, t_R') + \frac{\bar{R}}{c\alpha^2} \frac{\partial \phi}{\partial t}(\boldsymbol{r}_s, t_R') \right) d\boldsymbol{r}_s = Q(\boldsymbol{r}_s', t') \tag{27}$$

Equation (27) is one of the main results of the present paper. It is a new formulation for the time domain boundary integral equation for acoustic scattering by rigid surfaces in a constant mean flow. It is different from those in the literature in several aspects. First, the boundary condition used for (27) is one that is based on the acoustic energy flux consideration instead of the acoustic normal velocity. The two approaches differ on the part of the boundary where the mean flow itself does not satisfy the slip boundary condition. Second, the new equation is much simpler than those of the previous formulations in which tangential derivatives of the solution on the scattering surface are required to be kept as part of the integral equation. Of course, boundary condition (26) reduces to the usual one (21) wherever the mean flow does satisfy the solid wall boundary condition, i.e., $M_n = 0$.

IV. Burton-Miller type reformulation in time domain with a mean flow

Direct solution of boundary integral equations for exterior scattering problems, however, is known to suffer numerical instabilities. The instability is generally attributed to the existence of resonance frequencies for the interior domain.[1,2,6,30] In time domain solutions, the instability is more easily triggered as a continuous spectrum of frequencies within the numerical resolution are present in the computation. This instability is one of the major difficulties that have hindered the use of time domain integral equations. Recently, Burton-Miller type reformulation that has been widely used for exterior scattering problems in the frequency domain has shown to be effective in eliminating the instability in the time domain as well.[1,6,29] In [2], a theoretical justification has been provided for the extension of the Burton-Miller formulation to the time domain for the wave equation without flow. In this section, we propose the Burton-Miller reformulation for the TDBIE (27). An analysis on its stability similar to that in [2] is given in the next section.

For convenience of discussion, we define the following time domain double layer potential:

$$\mathcal{D}[\phi](\boldsymbol{r}', t') = \int_0^{t'+} \int_S \frac{\partial \tilde{G}}{\partial \tilde{n}}(\boldsymbol{r}_s, t; \boldsymbol{r}', t')\phi(\boldsymbol{r}_s, t) d\boldsymbol{r}_s dt = \int_S \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}') \left(\phi(\boldsymbol{r}_s, t'_R) + \frac{\bar{R}}{c\alpha^2} \frac{\partial \phi}{\partial t}(\boldsymbol{r}_s, t'_R) \right) d\boldsymbol{r}_s \tag{28}$$

The Burton-Miller type reformulation is carried out by applying a linear combination of the time and certain normal derivatives to the time domain integral equation. In earlier studies of the Burton-Miller formulation for scattering with a flow, the modified normal derivative (10) had been used.[13,30] Here, we propose that the normal derivative to be used for the Burton-Miller formulation be the *combined normal derivative* defined in (17). Specifically, the Burton-Miller reformulation is obtained by applying the following derivative operator to the boundary integral equation at surface points $\boldsymbol{r}'_s$:

$$\tilde{a} \frac{\partial}{\partial t'} + \tilde{b} c \frac{\partial}{\partial \tilde{n}'} \tag{29}$$

where $\tilde{a}$ and $\tilde{b}$ are constants and c is the speed of sound, namely,

$$\tilde{a} \frac{\partial}{\partial t'} \Big(2\pi\phi(\boldsymbol{r}'_s, t') + \mathcal{D}[\phi](\boldsymbol{r}'_s, t') \Big) + \tilde{b} c \, \frac{\partial}{\partial \tilde{n}'} \Big(4\pi\phi(\boldsymbol{r}', t') + \mathcal{D}[\phi](\boldsymbol{r}', t') \Big) \Big|_{\boldsymbol{r}'=\boldsymbol{r}'_s} = \tilde{a} \frac{\partial Q}{\partial t'}(\boldsymbol{r}'_s, t') + \tilde{b} c \frac{\partial Q}{\partial \tilde{n}'}(\boldsymbol{r}'_s, t') \tag{30}$$

Applying again the solid surface boundary condition (26), equation (30) is expanded to be the following:

$$\tilde{a}\left[2\pi\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s', t') + \int_S \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}')\left(\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s, t_R') + \frac{\bar{R}}{c\alpha^2}\frac{\partial\phi}{\partial t^2}(\boldsymbol{r}_s, t_R')\right)d\boldsymbol{r}_s\right]$$

$$+\tilde{b}c\left[\frac{\partial}{\partial\tilde{n}'}\int_S \frac{\partial G_0}{\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}')\left(\phi(\boldsymbol{r}_s, t_R') + \frac{\bar{R}}{c\alpha^2}\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s, t_R')\right)d\boldsymbol{r}_s\right]_{\boldsymbol{r}'=\boldsymbol{r}_s'} = \tilde{a}\frac{\partial Q}{\partial t'}(\boldsymbol{r}_s', t') + \tilde{b}c\frac{\partial Q}{\partial\tilde{n}'}(\boldsymbol{r}_s', t') \tag{31}$$

Note that an integral with a kernel $\frac{\partial^2 G_0}{\partial\tilde{n}'\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_s')$ is hyper-singular when $\boldsymbol{r}_s$ coincides with $\boldsymbol{r_s}'$. In particular, we have

$$\frac{\partial^2 G_0}{\partial\tilde{n}'\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_s') = \frac{\partial}{\partial\tilde{n}'}\left[-\alpha^2\frac{\boldsymbol{n}\cdot(\boldsymbol{r}_s - \boldsymbol{r}_s')}{\bar{R}^3}\right] = \frac{\alpha^2}{\bar{R}^3}\left[\boldsymbol{n}\cdot\boldsymbol{n}' - M_{n'}M_n\right] + 3\alpha^4\frac{\left[\boldsymbol{n}\cdot(\boldsymbol{r}_s - \boldsymbol{r}_s')\right]\left[\boldsymbol{n}'\cdot(\boldsymbol{r}_s' - \boldsymbol{r}_s)\right]}{\bar{R}^5} \tag{32}$$

Thus, $\frac{\partial^2 G_0}{\partial\tilde{n}'\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_s')$ is of order $O(1/|\boldsymbol{r}_s - \boldsymbol{r}_s'|^3)$ as $\boldsymbol{r}_s \to \boldsymbol{r}_s'$.

We consider the following regularization process for the hyper-singular integral in (31) that adds and subtracts a term involving the value at the collocation point $\phi(\boldsymbol{r}_s', t')$:

$$\frac{\partial}{\partial\tilde{n}'}\left[\int_S \frac{\partial G_0}{\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_s')\left(\phi(\boldsymbol{r}_s, t_R') + \frac{\bar{R}}{c\alpha^2}\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s, t_R')\right)d\boldsymbol{r}_s\right]$$

$$= \frac{\partial}{\partial\tilde{n}'}\left[\int_S \frac{\partial G_0}{\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_s')\left(\phi(\boldsymbol{r}_s, t_R') - \phi(\boldsymbol{r}_s', t') + \frac{\bar{R}}{c\alpha^2}\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s, t_R')\right)d\boldsymbol{r}_s\right] + \phi(\boldsymbol{r}_s', t')\frac{\partial}{\partial\tilde{n}'}\left[\int_S \frac{\partial G_0}{\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_s')d\boldsymbol{r}_s\right] \tag{33}$$

The first integral is now integrable by Cauchy Principal Value (Appendix B) and the second integral is zero according to (55) given in the Appendix A. Upon carrying out the derivatives inside the first integral shown above, we get the following Burton-Miller reformulation of the time domain boundary integral equation (BM-TDBIE):

$$2\pi\tilde{a}\frac{\partial\phi(\boldsymbol{r}_s', t')}{\partial t} + \tilde{a}\int_S \frac{\partial G_0}{\partial\bar{n}}\left(\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s, t_R') + \frac{\bar{R}}{c\alpha^2}\frac{\partial^2\phi}{\partial t^2}(\boldsymbol{r}_s, t_R')\right)d\boldsymbol{r}_s - \frac{\tilde{b}}{c\alpha^4}\int_S \bar{R}^3\frac{\partial G_0}{\partial\tilde{n}'}\frac{\partial G_0}{\partial\bar{n}}\frac{\partial^2\phi}{\partial t^2}(\boldsymbol{r}_s, t_R')d\boldsymbol{r}_s$$

$$+\tilde{b}c\int_S \frac{\partial^2 G_0}{\partial\tilde{n}'\partial\bar{n}}\left(\phi(\boldsymbol{r}_s, t_R') - \phi(\boldsymbol{r}_s', t') + \frac{\bar{R}}{c\alpha^2}\frac{\partial\phi}{\partial t}(\boldsymbol{r}_s, t_R')\right)d\boldsymbol{r}_s = \tilde{a}\frac{\partial Q}{\partial t'}(\boldsymbol{r}_s', t') + \tilde{b}c\frac{\partial Q}{\partial\tilde{n}'}(\boldsymbol{r}_s', t') \tag{34}$$

The proper values for the coefficients $\tilde{a}$ and $\tilde{b}$ will be given in the next section where stability of (34) will be discussed.

V. Stability of the time domain Burton-Miller formulation in the presence of a mean flow

Following closely the work in [2] for the case without flow, we demonstrate in this section that the Burton-Miller type reformulation presented in the previous section eliminates the nontrivial solutions of the homogeneous integral equation in the case with a flow as well.

Suppose that there is a nontrivial solution $\phi_0(\boldsymbol{r}_s, t)$ to the homogeneous formulation (34). We will show in what follows that such a solution is not possible. Consider the double layer potential (28) extended to domains both exterior and interior of surface S:

$$\mathcal{D}[\phi_0](\boldsymbol{r}', t') = \int_S \frac{\partial G_0}{\partial \tilde{n}}(\boldsymbol{r}_s, \boldsymbol{r}') \left(\phi_0(\boldsymbol{r}_s, t'_R) + \frac{\bar{R}}{c\alpha^2} \frac{\partial \phi_0}{\partial t}(\boldsymbol{r}_s, t'_R) \right) d\boldsymbol{r}_s \equiv \begin{cases} w^+, & \boldsymbol{r}' \in V, \ \textit{exterior of } S \\ w_0, & \boldsymbol{r}' = \boldsymbol{r}'_s \ \text{on } S \\ w^-, & \boldsymbol{r}' \in V^-, \ \textit{interior of } S \end{cases}$$

We note that w^+ and w^- satisfy the homogeneous convective wave equation in the exterior and interior domains of S, respectively. It can also be shown that

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s} w^+ = w_0 - 2\pi \phi_0(\boldsymbol{r}'_s, t') \tag{35}$$

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s} w^- = w_0 + 2\pi \phi_0(\boldsymbol{r}'_s, t') \tag{36}$$

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s} \frac{\partial w^+}{\partial \tilde{n}'} = \lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s} \frac{\partial w^-}{\partial \tilde{n}'} \tag{37}$$

Equations (35) and (36) can be found by using the limits given in (55) in the Appendix, and equation (37) follows after an application of the regularization process (33) to both sides of the equation.

Now since $\phi_0(\boldsymbol{r}_s, t)$ satisfies the homogeneous Burton-Miller formulation (30), we have, at $\boldsymbol{r}' = \boldsymbol{r}'_s$,

$$\tilde{a} \frac{\partial}{\partial t'} \left(2\pi \phi_0 + w_0 \right) + \tilde{b} c \, \frac{\partial}{\partial \tilde{n}'} \left(4\pi \phi_0 + w^+ \right) \bigg|_{\boldsymbol{r}'_s} = 0$$

By the jump conditions (35)-(37) as well as the boundary condition (26), the above yields

$$\tilde{a} \frac{\partial w^-}{\partial t'} + \tilde{b} c \frac{\partial w^-}{\partial \tilde{n}'} = 0 \tag{38}$$

On the other hand, since w^- satisfies the convective wave equation and by the energy equation (23) of the convective wave equation, we have

$$\frac{\partial}{\partial t} \int_{V^-} \left[\frac{1}{2} |\nabla w^-|^2 + \frac{1}{2c^2} \left| \frac{D w^-}{Dt} \right|^2 - \frac{\boldsymbol{U} \cdot \nabla w^-}{c^2} \frac{D w^-}{Dt} \right] d\boldsymbol{r} = \int_{V^-} \nabla \cdot \left[\frac{\partial w^-}{\partial t} \left(\nabla w^- - \frac{1}{c^2} \frac{D w^-}{Dt} \boldsymbol{U} \right) \right] d\boldsymbol{r}$$

which, with an application of the divergence theorem, becomes

$$\int_{V^-} \left[\frac{1}{2} |\nabla w^-|^2 + \frac{1}{2c^2} \left| \frac{D w^-}{Dt} \right|^2 - \frac{\boldsymbol{U} \cdot \nabla w^-}{c^2} \frac{D w^-}{Dt} \right] d\boldsymbol{r} = - \int_0^{t^+} \int_S \frac{\partial w^-}{\partial t} \frac{\partial w^-}{\partial \tilde{n}} d\boldsymbol{r}_s dt \tag{39}$$

where V^- represents the volume interior of S. The minus sign on the right hand side has been added due to the fact that the normal derivative used in (39) is still the one that is inward of the body surface. Note that, for subsonic flows where $|\boldsymbol{U}| < c$, the left hand side of (39) is nonnegative:

$$\frac{1}{2}|\nabla w^-|^2 + \frac{1}{2c^2}\left|\frac{Dw^-}{Dt}\right|^2 - \frac{\boldsymbol{U}\cdot\nabla w^-}{c^2}\frac{Dw^-}{Dt} = \frac{1}{2}\left(|\nabla w^-| - \frac{1}{c}\left|\frac{Dw^-}{Dt}\right|\right)^2 + \frac{1}{c}|\nabla w^-|\left|\frac{Dw^-}{Dt}\right| - \frac{\boldsymbol{U}\cdot\nabla w^-}{c}\frac{Dw^-}{Dt} \geq 0$$

On the other hand, using (38), the right hand side of (39) will be nonpositive:

$$-\int_0^{t^+}\int_S \frac{\partial w^-}{\partial t}\frac{\partial w^-}{\partial \tilde{n}} d\boldsymbol{r}_s = \frac{1}{c^2}\int_0^{t^+}\int_S \frac{\tilde{a}}{\tilde{b}c}\left|\frac{\partial w^-}{\partial t}\right|^2 d\boldsymbol{r}_s \leq 0$$

provided

$$\frac{\tilde{a}}{\tilde{b}} < 0 \tag{40}$$

The above implies that w^- has to be a trivial solution, i.e., $w^- \equiv 0$ under the condition (40). A simple choice for $\tilde{a}$ and $\tilde{b}$ is $\tilde{a} = -\tilde{b} = 1$.

VI. Time Domain Boundary Element Method

In this section, we describe a numerical solution of (34) by the Time Domain Boundary Element Method (TDBEM) and demonstrate numerical stability of the new formulation.

Let surface S be discretized by surface elements E_j, $j = 1, 2, ..., N_e$, where N_e is the total number of elements, and the time be discretized by $t_n = n\Delta t$, where Δt is the time step. The time domain numerical solution on the surface can be expanded as

$$\phi(\boldsymbol{r}_s, t) = \sum_{n=0}^{N_t}\sum_{j=1}^{N_e} u_j^n \varphi_j(\boldsymbol{r}_s)\psi_n(t) \tag{41}$$

where $\varphi_j(\boldsymbol{r}_s)$ is the surface basis function for element E_j and $\psi_n(t)$ is the temporal basis function for time node t_n. Here N_t is the total number of time steps. For simplicity, we consider only constant elements where collocation node $\boldsymbol{r}_j$ is located at the center of the element and the nodal basis function is

$$\varphi_j(\boldsymbol{r}_s) = \begin{cases} 1, & \boldsymbol{r}_s \text{ on element } E_j \text{ that contains node } \boldsymbol{r}_j \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

The temporal basis function is taken to be the third-order shifted Lagrange basis polynomial that is commonly used for time domain boundary element methods:[14,18]

$$\psi_n(t) = \Psi\left(\frac{t - t_n}{\Delta t}\right) \tag{43}$$

where

$$\Psi(\tau) = \begin{cases} 1 + \frac{11}{6}\tau + \tau^2 + \frac{1}{6}\tau^3 & -1 < \tau \leq 0 \\ 1 + \frac{1}{2}\tau - \tau^2 - \frac{1}{2}\tau^3 & 0 < \tau \leq 1 \\ 1 - \frac{1}{2}\tau - \tau^2 + \frac{1}{2}\tau^3 & 1 < \tau \leq 2 \\ 1 - \frac{11}{6}\tau + \tau^2 - \frac{1}{6}\tau^3 & 2 < \tau \leq 3 \\ 0 & \text{other} \end{cases} \tag{44}$$

For example, at a point $\boldsymbol{r}_s$ on element E_j and at an off-nodal time $t = t_n - \eta\Delta t$, $0 \le \eta < 1$, the value for $\phi(\boldsymbol{r}_s, t)$ is found by

$$\phi(\boldsymbol{r}_s, t) = \varphi_j(\boldsymbol{r}_s)\left[u_j^n\Psi(-\eta) + u_j^{n-1}\Psi(1-\eta) + u_j^{n-2}\Psi(2-\eta) + u_j^{n-3}\Psi(3-\eta)\right] \tag{45}$$

With nodal spatial and temporal basis functions defined above, the expansion coefficient u_j^n in (41) represents the value of ϕ at the collocation node $\boldsymbol{r}_j$ on element E_j at time level t_n. By substituting expansion (41) into boundary integral equation (34) and evaluating the equation at collocation points $\boldsymbol{r}_i$ of all elements and at time level t_n, a March-On-in-Time scheme (MOT) is obtained that can be expressed in a matrix form as

$$\boldsymbol{B}_0\boldsymbol{u}^n = \boldsymbol{q}^n - \boldsymbol{B}_1\boldsymbol{u}^{n-1} - \boldsymbol{B}_2\boldsymbol{u}^{n-2} - \cdots \boldsymbol{B}_J\boldsymbol{u}^{n-J} \tag{46}$$

where $\boldsymbol{u}^k$ denotes a vector that contains all unknowns $\left\{u_j^k, j = 1, 2, ..., N_e\right\}$ at time level t_k. The nonzero entries for matrices $\boldsymbol{B}_k$, $k = 0, 1, 2, ..., J$, in (46) are:

$$\{\boldsymbol{B}_k\}_{ij} = 2\pi\tilde{a}\delta_{ij}\psi'_{n-k}(t_n) + \tilde{a}\int_{E_j}\frac{\partial G_0}{\partial\bar{n}}\left(\psi'_{n-k}(t_R^n) + \frac{\bar{R}}{c\alpha^2}\psi''_{n-k}(t_R^n)\right)d\boldsymbol{r}_s + \tilde{b}c\delta_{ij}\delta_{k0}D_i$$

$$+\tilde{b}c\int_{E_j}\frac{\partial^2 G_0}{\partial\bar{n}'\partial\bar{n}}\left(\psi_{n-k}(t_R^n) - \delta_{ij}\psi_{n-k}(t_n) + \frac{\bar{R}}{c\alpha^2}\psi'_{n-k}(t_R^n)\right)d\boldsymbol{r}_s + \frac{\tilde{b}}{c\alpha^4}\int_{E_j}\bar{R}^3\frac{\partial G_0}{\partial\bar{n}'}\frac{\partial G_0}{\partial\bar{n}}\psi''_{n-k}(t_R^n)d\boldsymbol{r}_s \tag{47}$$

for $i, j = 1, 2, ..., N_e$, where δ_{ij} and δ_{k0} are Kronecker delta functions and a prime in the above denotes the derivative with respect to time and

$$t_R^n = t_n + \boldsymbol{\beta}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_s) - \frac{\bar{R}(\boldsymbol{r}_s, \boldsymbol{r}_i)}{c\alpha^2}, \quad D_i = -\int_{S-E_i}\frac{\partial^2 G_0}{\partial\bar{n}'\partial\bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}_i)d\boldsymbol{r}_s \tag{48}$$

It is easy to see that the entry $\{\boldsymbol{B}_k\}_{ij}$ represents contributions to the value at node $\boldsymbol{r}_i$ from the nodal value of element E_j of time level t_{n-k}. The integrals in (47) are to be evaluated using high-order quadrature on each element. For the computational results reported in this paper, each element is mapped into a standard element of $[-1, 1] \times [-1, 1]$ and Legendre-Gauss quadrature rule of degree 6 is used for integration in each dimension. Integration on the singular elements when $i = j$ is detailed in Appendix B.

The index J in (46) denotes the maximum time history of the solution required for (46) and is dependent on the length of the scattering surface and the mean flow as

$$J = \frac{\bar{L}}{c\alpha^2\Delta t} + 3, \quad \bar{L} = \max_{\boldsymbol{r}_s, \boldsymbol{r}'_s \in S}\left[-\boldsymbol{M}\cdot(\boldsymbol{r}'_s - \boldsymbol{r}_s) + \bar{R}(\boldsymbol{r}_s, \boldsymbol{r}'_s)\right] \tag{49}$$

Due to the limited temporal stencil width shown in (44) and (45), the $\boldsymbol{B}$ matrices are sparse. In particular, we note that matrix $\boldsymbol{B}_0$ in (46) is a very sparse matrix and represents interactions within the same element or between nearby nodes at the same time level t_n. $\boldsymbol{B}_0$ is also found to be diagonally dominant. Solutions for $\boldsymbol{u}^n$ in (46) can be found efficiently by an iterative method, such as the Jacobi iterative method, with rapid convergence.[9,18]

VII. Stability and eigenvalue analysis of the new integral equation

As mentioned in previous sections, direct numerical solution of the time domain boundary integral equation (27) is prone to numerical instabilities. In Figure 2, we first show an example of scattering by a parabolic wing in a mean flow of Mach number 0.5, $\boldsymbol{M} = (0.5, 0, 0)$, to demonstrate the elimination of numerical instability by the Burton-Miller reformulation of TDBIE (27). The geometry of the scattering surface is a convex parabolic wing and is defined as follows:

$$z = 0.1L_x(1 - x^2/L_x^2), \quad -L_x \leq x \leq L_x, \quad -L_y \leq y \leq L_y \tag{50}$$

where $L_x = L_y = 0.5$. The scattering surface is discretized by 2316 quadrilateral elements as shown in Figure 2. The source function is a broadband point source defined as the following:

$$q(\boldsymbol{r}, t) = e^{-\sigma t^2}\delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{51}$$

where $\boldsymbol{r}_0 = (0, 0, 1)$ and $\sigma = 1.42/(6\Delta t)^2$. The time history of the solution on a surface collocation point is plotted in Figure 2 for the cases without and with Burton-Miller reformulation. The top figure shows the result obtained by directly solving the TDBIE (27). It is seen that the solution behaves well initially but eventually becomes unstable. On the other hand, the solution obtained by the BM-TDBIE (34), shown in the bottom figure, remains stable.

To further study the stability of the MOT scheme (46), we conduct a numerical eigenvalue study of the discretized system of equations.[3] We look for solutions of the form

$$\boldsymbol{u}^n = \lambda^n \boldsymbol{e}_0 \tag{52}$$

to the corresponding homogeneous system of (46). By substituting (52) into (46), we obtain a polynomial eigenvalue problem

$$\left[\boldsymbol{B}_0\lambda^J + \boldsymbol{B}_1\lambda^{J-1} + \boldsymbol{B}_2\boldsymbol{\lambda}^{J-2} + \cdots + \boldsymbol{B}_{J-1}\lambda + \boldsymbol{B}_J\right]\boldsymbol{e}_0 = 0 \tag{53}$$

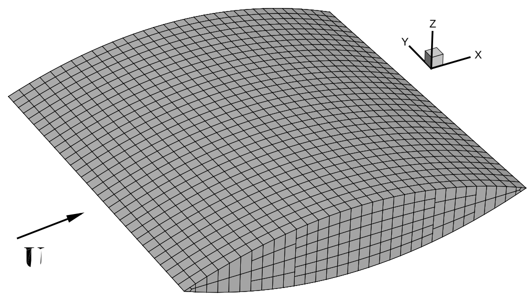
which can be cast into a generalized eigenvalue problem as follows:

$$\begin{bmatrix} -\boldsymbol{B}_1 & -\boldsymbol{B}_2 & \cdots & \cdots & -\boldsymbol{B}_{J-1} & -\boldsymbol{B}_J \\ \boldsymbol{I} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \boldsymbol{I} & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \boldsymbol{I} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{J-1} \\ \boldsymbol{e}_{J-2} \\ \cdot \\ \cdot \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_0 \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{B}_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \boldsymbol{I} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_{J-1} \\ \boldsymbol{e}_{J-2} \\ \cdot \\ \cdot \\ \boldsymbol{e}_1 \\ \boldsymbol{e}_0 \end{bmatrix} \tag{54}$$

where $\boldsymbol{e}_j = \lambda^j \boldsymbol{e}_0$. Numerical scheme (46) is stable if $|\lambda| \leq 1$ for all eigenvalues of (54).

Eigenvalue analyses of scattering by two geometric shapes are presented in Table 1. One of the geometries is the parabolic wing as described previously in (50). The other is a sphere of radius 0.5. The surface of the sphere is first discretized by 512 unstructured triangular elements each of which is then subdivided into three quadrilateral surface elements resulting in a total of 1536 surface elements. The mean flow Mach number varies from 0 to 0.9. A total of eight cases are considered in Table 1.

Eigenvalues of the generalized eigenvalue problem (54) can be found via a sparse eigenvalue solver available in MATLAB and Python, or by a matrix power iteration method detailed in Appendix C.



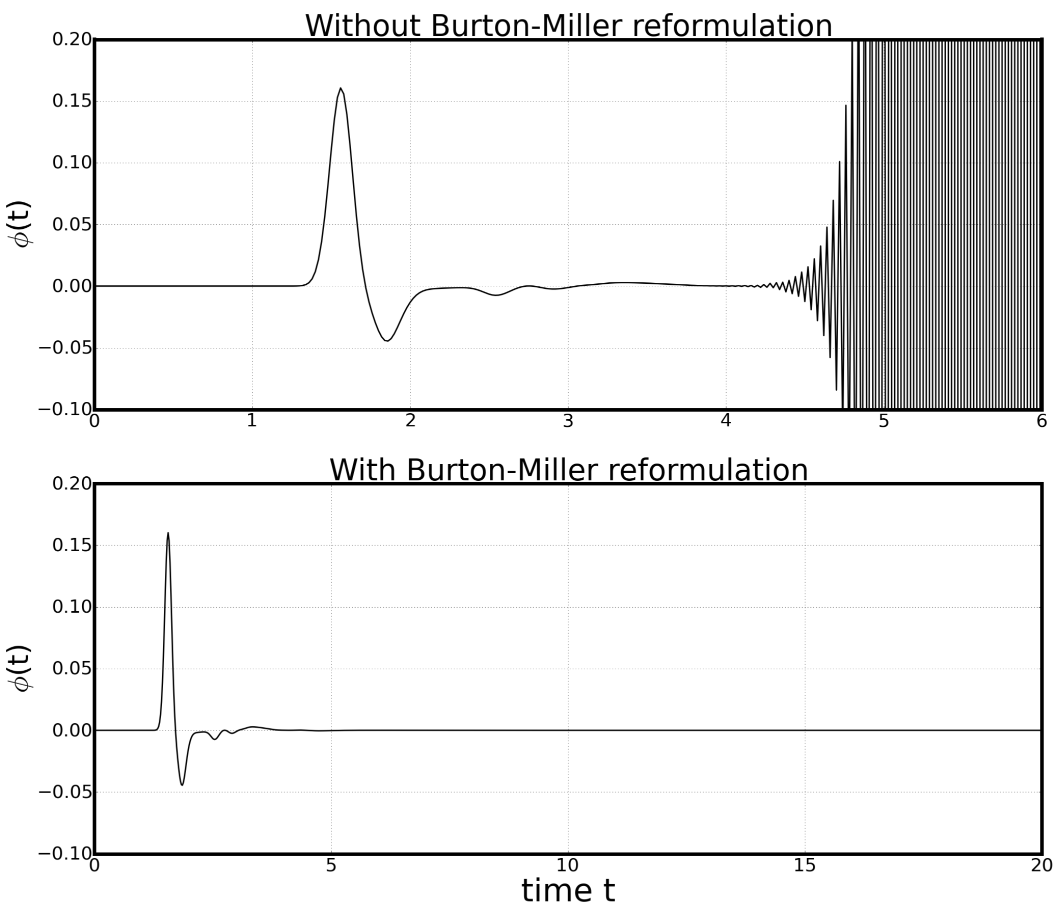


Figure 2. Time history of numerical solution on a surface collocation point, showing the elimination of instability by Burton-Miller reformulation of TDBIE. $M = (0.5, 0, 0)$, $\Delta t = 0.02$. Top: solution of (27) without Burton-Miller reformulation; bottom: solution by BM-TDBIE (34).

The values of the largest eigenvalue for the eight cases are listed in Table 1. For the Burton-Miller formulation BM-TDBIE (34), all eigenvalues are no greater than unity and stability is observed. In contrast, direct solution of the equation (27) results in eigenvalues greater than unity in all but two of the eight cases studied, indicating that equation (27) without Burton-Miller reformulation can lead to unstable solutions.

Table 1. Maximum eigenvalue, $|\lambda|_{max}$, computed by (54) for scattering by a parabolic wing and a sphere, for cases with and without Burton-Miller (B-M) reformulation. N_e is the total number of elements and M is the mean flow Mach number.

Parabolic Wing				Sphere			
		$\|\lambda\|_{max}$				$\|\lambda\|_{max}$	
N_e	M	with B-M Eq. (34)	without B-M Eq. (27)	N_e	M	with B-M Eq. (34)	without B-M Eq. (27)
2316	0.0	1.000000	1.095949	1536	0.0	1.000000	1.007840
2316	0.3	1.000000	1.160628	1536	0.3	1.000000	1.000000
2316	0.6	1.000000	1.129116	1536	0.6	1.000000	0.999968
2316	0.9	1.000000	1.582909	1536	0.9	1.000000	1.003901

VIII. Conclusions

In this paper, we have considered the boundary condition to be used in time domain boundary integral equation analysis of acoustic scattering by solid bodies under a constant mean flow assumption. After an examination of the energy equation associated with the convective wave equation, it is proposed that the boundary condition be defined by the requirement that the energy flux be zero at solid boundaries. A new TDBIE is derived based on this new solid wall boundary condition. The new formulation differs from those found in the literature on the part of the boundary where the constant mean flow itself does not satisfy the solid surface boundary condition. In addition to conserving the acoustic energy, another significant advantage of the new equation is that it is considerably simpler than previous formulations. In particular, tangential derivatives of the solution on the solid surfaces are no longer needed in the new formulation, which greatly simplifies numerical implementation. To stabilize the TDBIE, Burton-Miller reformulation is also derived. Numerical solutions and eigenvalue analysis are presented that demonstrate stability of the new formulation.

Naturally, from a physical point of view, the null acoustic energy flux condition at rigid surfaces should be equivalent to, or a direct consequence of, the condition that the normal acoustic velocity becomes zero on rigid surfaces. The fact that the two now differ in the formulation of the boundary integral equation for scattering with flow is due to the inconsistency on the part of the underlying mean flow itself when the constant flow simplification is made. Thus, boundary integral equation approaches with a constant mean flow would be applicable only to problems where such a simplification is acceptable or justified, such as in scattering with flow over slender bodies. From a computational point of view, however, the current formulation based on the energy flux condition is significantly simpler than those based on the normal velocity condition. As such, as a result of the present analysis, the enforcement of normal acoustic velocity being zero on boundary points where the mean flow itself does not satisfy such a condition, and the computational complications it brings in with the separation of normal and tangential derivatives of the solution, become unnecessary.

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Appendix

A. Limit of weakly-singular integral

By equations (17) and (32), it is easy show that the modified normal derivative $\frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}'_s)$ and $\frac{\partial^2 G_0}{\partial \bar{n}' \partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}'_s)$ have a singularity of order $O(1/|\boldsymbol{r}_s - \boldsymbol{r}'_s|)$ and $O(1/|\boldsymbol{r}_s - \boldsymbol{r}'_s|^3)$, respectively, which makes their surface integrals weakly-singular and hyper-singular, respectively. In this appendix, we state some useful results.

For surface integrals involving $\frac{\partial G_0}{\partial \bar{n}}$, we have

$$\frac{1}{4\pi}\int_S \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}')d\boldsymbol{r}_s = \begin{cases} 0 & \boldsymbol{r}' \in V, \text{ exterior of } S \\ \frac{1}{2} & \boldsymbol{r}' = \boldsymbol{r}'_s \in S \\ 1 & \boldsymbol{r}' \in V^-, \text{ interior of } S \end{cases} \tag{55}$$

The first and third equations in (55) can be obtained by the fact that any constant can be a solution to the homogeneous convective wave equation with homogeneous normal derivative on the boundary for the interior domain V^- enclosed by S. By substituting $p = 1$ to equation (15) and noting the choice of the normal direction and the placement of $\boldsymbol{r}'$, the first and third equation in (55) follow immediately.

The second integral in (55) becomes weakly singular when $\boldsymbol{r}'$ approaches a point on surface S. This particular limit has been studied previous in the literature for a mean flow that is aligned with the x-coordinate.[21,26] Here, we show the calculation for a general mean flow. Assuming $\boldsymbol{r}'_s$ is a smooth point on S, consider modifying surface S by a spherical surface of radius ϵ and centered at $\boldsymbol{r}'_s$ as shown in Figure 3. The surface is assumed to be smooth at $\boldsymbol{r}'_s$. If we denote the small hemispherical surface as S_ϵ, we have

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s}\int_S \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}')d\boldsymbol{r}_s = \lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s}\int_{S-S_\epsilon} \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}')d\boldsymbol{r}_s + \lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s}\int_{S_\epsilon} \frac{\partial G_0}{\partial \bar{n}}(\boldsymbol{r}_s, \boldsymbol{r}')d\boldsymbol{r}_s \tag{56}$$

Note that, for the surface integral on S_ϵ, using (10), we have

$$\frac{\partial G_0}{\partial \bar{n}} = -\alpha^2 \frac{n_1(x_s - x'_s) + n_2(y_s - y'_s) + n_3(z_s - z'_s)}{\bar{R}^3} = -\alpha^2 \frac{\epsilon}{\bar{R}^3}$$

By the symmetry of $\bar{R}$ with respect to hemispheres S_ϵ and S'_ϵ, the complementary hemisphere of S_ϵ, and by using a local spherical coordinate system centered at $\boldsymbol{r}'_s$ and its local z direction coincides with mean flow $\boldsymbol{M}$, namely $x_s - x'_s = \epsilon \sin\nu \cos\theta$, $y_s - y'_s = \epsilon \sin\nu \sin\theta$, $z_s - z'_s = \epsilon\cos\nu$, we have

$$\lim_{\boldsymbol{r}' \to \boldsymbol{r}'_s}\int_{S_\epsilon} \frac{\partial G_0}{\partial \bar{n}} d\boldsymbol{r}_s = -\alpha^2 \int_{S_\epsilon} \frac{\epsilon}{\bar{R}^3} d\boldsymbol{r}_s = -\frac{\alpha^2}{2}\int_{S_\epsilon + S'_\epsilon} \frac{\epsilon}{\bar{R}^3} d\boldsymbol{r}_s = -\frac{\alpha^2}{2}\int_0^{2\pi}\int_0^{\pi} \frac{\epsilon^3 \sin\nu}{\left(\epsilon^2\cos^2\nu + \epsilon^2\alpha^2\sin^2\nu\right)^{3/2}} d\nu d\theta$$

$$= -\pi\alpha^2 \int_{-1}^{1} \frac{1}{(\alpha^2 + (1-\alpha^2)\chi^2)^{3/2}} d\chi = -2\pi$$

The last integral above can be found by direct integration. The second equation in (55) follows as $\epsilon \to 0$ and by noting that, for $\boldsymbol{r}' \in V$, the limit on the left hand side of (56) is zero.

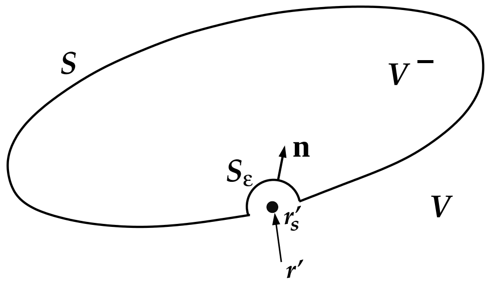

Figure 3. A schematic diagram for a hemisphere that caps a surface point $\boldsymbol{r}_s'$. Note that the normal vector is in the direction outward from the region of solution and into the body.

B. Evaluation of hyper-singular integral

We consider the numerical evaluation of the regularized integral involving the double normal derivative of G_0 in (34). Note that as $\boldsymbol{r}_s \to \boldsymbol{r}_s'$,

$$\phi(\boldsymbol{r}_s, t_R') - \phi(\boldsymbol{r}_s', t') + \frac{\bar{R}}{c\alpha^2}\frac{\partial \phi}{\partial t}(\boldsymbol{r}_s, t_R') = \nabla\phi(\boldsymbol{r}_s', t')\cdot(\boldsymbol{r}_s - \boldsymbol{r}_s') + \boldsymbol{\beta}\cdot(\boldsymbol{r}_s' - \boldsymbol{r}_s)\frac{\partial \phi}{\partial t}(\boldsymbol{r}_s', t') + O(|\boldsymbol{r}_s - \boldsymbol{r}_s'|^2) \quad (57)$$

Let a surface element E_j be mapped to a local coordinate $(\xi, \eta) \in [-1, 1] \times [-1, 1]$, which is then in turn converted into a local polar coordinate (r, θ) centered at the collocation point $\boldsymbol{r}_s'$. Denote the integrand for the integral in (r, θ) as

$$F(r,\theta) = \left(\frac{\partial^2 G_0}{\partial \bar{n}' \partial \bar{n}}\right)\left(\phi(\boldsymbol{r}_s, t_R') - \phi(\boldsymbol{r}_s', t') + \frac{\bar{R}}{c\alpha^2}\frac{\partial \phi}{\partial t}(\boldsymbol{r}_s, t_R')\right)|\mathbf{r}_\xi \times \mathbf{r}_\eta| \quad (58)$$

By (57), $F(r, \theta)$ is of order $O(1/r^2)$ as $r \to 0$. Let

$$\lim_{r \to 0} r^2 F(r,\theta) = G(\theta) \quad (59)$$

It is easy to show that $\int_0^{2\pi} G(\theta) d\theta = 0$. Then we have the following for the integral on surface element E_j:

$$\lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^{r(\theta)} F(r,\theta) r dr d\theta = \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^{r(\theta)} [\frac{r^2F(r,\theta) - G(\theta)}{r} + \frac{G(\theta)}{r}] dr d\theta$$

$$= \int_0^{2\pi}\int_0^{r(\theta)} \frac{r^2F(r,\theta) - G(\theta)}{r} dr d\theta + \lim_{\epsilon\to 0}\int_0^{2\pi} G(\theta)[\ln r(\theta) - \ln\epsilon] d\theta$$

$$= \int_0^{2\pi}\int_0^{r(\theta)} \frac{r^2F(r,\theta) - G(\theta)}{r} dr d\theta + \int_0^{2\pi} G(\theta)\ln r(\theta) d\theta$$

The final integrals can be evaluated using high-order numerical quadrature.

C. Eigenvalue by matrix power iteration method

We describe a matrix power iteration method for finding the largest eigenvalue of (54). Let

$$\boldsymbol{A} = \begin{bmatrix} -\boldsymbol{B_0^{-1}}\boldsymbol{B}_1 & -\boldsymbol{B_0^{-1}}\boldsymbol{B_2} & \cdots & \cdots & -\boldsymbol{B_0^{-1}}\boldsymbol{B_{J-1}} & -\boldsymbol{B_0^{-1}}\boldsymbol{B}_J \\ I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & I & 0 \end{bmatrix}$$

Then, the power iteration method proceeds as follows:[9]

given an arbitrary unit vector $\boldsymbol{e}^{(0)}$, and for $k = 1, 2, ...,$ compute

$$\boldsymbol{v}^{(k)} = \boldsymbol{A}\boldsymbol{e}^{(k-1)} \tag{60}$$

$$\boldsymbol{e}^{(k)} = \frac{\boldsymbol{v}^{(k)}}{||\boldsymbol{v}^{(k)}||_2} \tag{61}$$

and eigenvalue

$$\lambda^{(k)} = \left[\boldsymbol{e}^{(k)}\right]^T \boldsymbol{A}\boldsymbol{e}^{(k)} = \left[\boldsymbol{e}^{(k)}\right]^T \boldsymbol{v}^{(k+1)} \tag{62}$$

The iteration is stopped when $\left|\lambda^{(k)} - \lambda^{(k-1)}\right| / \left|\lambda^{(k)}\right| < \epsilon$, where ϵ is the tolerance and set to be 10^{-12}. When the iteration is convergent, it converges to the largest eigenvalue of A.

In particular, if we denote

$$\boldsymbol{e}^{(k)} = \begin{bmatrix} \boldsymbol{e}_{J-1}^{(k)} \\ \boldsymbol{e}_{J-2}^{(k)} \\ \cdot \\ \cdot \\ \boldsymbol{e}_1^{(k)} \\ \boldsymbol{e}_0^{(k)} \end{bmatrix}, \quad \boldsymbol{v}^{(k)} = \begin{bmatrix} \boldsymbol{v}_{J-1}^{(k)} \\ \boldsymbol{v}_{J-2}^{(k)} \\ \cdot \\ \cdot \\ \boldsymbol{v}_1^{(k)} \\ \boldsymbol{v}_0^{(k)} \end{bmatrix} \tag{63}$$

then, equation (60) can also be computed through the following relations that save memory and storage:

$$\boldsymbol{v}_{J-1}^{(k)} = -\boldsymbol{B}_0^{-1}\left[\boldsymbol{B}_1\boldsymbol{e}_{J-1}^{(k-1)} + \boldsymbol{B}_2\boldsymbol{e}_{J-2}^{(k-1)} + \cdots + \boldsymbol{B}_{J-1}\boldsymbol{e}_1^{(k-1)} + \boldsymbol{B}_J\boldsymbol{e}_0^{(k-1)}\right], \boldsymbol{v}_{J-2}^{(k)} = \boldsymbol{e}_{J-1}^{(k-1)}, \cdots, \boldsymbol{v}_0^{(k)} = \boldsymbol{e}_1^{(k-1)} \tag{64}$$

We note that the iterative step shown in (64) is the same as the MOT iteration (46) without the source term. Therefore, it can also carried out using the same computational scheme for (46).

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