Skew-Symmetric Splitting and Stability of High Order Central Schemes

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(Joint work: B. Sjogreen, D. Kotov)

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Challenges in Numerical Method Development

(Multiscale DNS & LES, and Aeroacoustic Turbulence Applications)

- **Schemes developed for short time integration might suffer from nonlinear instability for longer time integration**
- **Stable & Accurate Temporal & Spatial Low Dissipative & Dispersive methods applicable to long time integration are required**
- **Numerical stability & accuracy requirements are an intricate balancing act**
  - *More stable schemes usually contain more numerical dissipation than their higher accuracy schemes counterparts*
  - *Turbulence cannot tolerate numerical dissipation*
  - *Proper amount of numerical dissipation is required for stability in the vicinity of discontinuities*
  - *Flows containing stiff source terms:*
    Numerical dissipation & under-resolved grid may lead to incorrect shock speed

Recent developments:

Yee & Sjogreen, 2007-2009, Sjogreen & Yee, 2016-2017,
Wang et al., 2009-2015, Kotov et al., 2011-2014

Outline

- **Methods to Improve Nonlinear Stability & Accuracy for Long Time Wave Propagation & Long Time Integration of Complex Compressible Flows**
- **Skew-Symmetric Splitting of the Inviscid Flux Derivatives**
  (This talk concentrates on one of 5 improvements)
- **Selected Numerical Results**
- **Concluding Remarks**

Remark

Schemes with improve nonlinear stability can benefit short time & long time integrations of nonlinear fluid flows

Five Methods to Improve Nonlinear Stability & Accuracy

(Long Time Wave Propagation & Long Time Integration
of Complex Compressible Fluids & Plasma)

Under the Yee et al. nonlinear filter approach framework:

- Standard High Order Linear Filters are to be Replaced by High Order Nonlinear Filters
- Smart Flow Sensors to Provide Locations & Amount of Needed Numerical Dissipation
- Skew-Symmetric Splitting of the Inviscid Flux Derivative Before the Application of Non-Dissipative Centered Schemes
- DRP (Dispersion Preservation-Relation) Schemes as Alternatives to Classical High Order Central Schemes
- Stable High-Order Entropy Conservative Numerical Fluxes with Entropy Satisfying Properties - *Numerical solution satisfies an additional discretized conservation law*

Remaining 4 methods can be found in Yee & collaborators published work (2009-2017)

Remark

Present numerical method development for gas dynamics
***with* Modification**
can carry over to MHD for short time & long time integration

Yee et al., 2000-2013, Yee & Sjogreen, 2007-2009, Sjogreen & Yee, 2016-2017,
Wang et al., 2009-2015, Kotov et al., 2011-2014

Skew-Symmetric Splitting of Inviscid Flux Derivatives

(Improve nonlinear stability for high order central schemes)

Olsson & Oliger 1994, Yee et al. 1999, Ducros et al. 2000, Pirozzoli 2009

- **Entropy splitting:** Semi-conservative splitting for shock-free turbulence
  (Olsson & Oliger 1994, Yee et al. 1999-2007, Sandham et al. 2002-present)
- **Natural Splitting:** Linearized Euler & Non-conservative Systems
- **Splitting to Preserve Discrete Momentum and/or Energy Conservation:**
  (Arakawa 1966, Blaisdell et al. 1996, Mansour 1980, etc.)
- **Ducros et al. Type Conservative Splitting:** Euler & MHD
- **Generalized Skew-Symmetric Splitting:** 3-parameter family
  (Pirozzoli 2009)

→ **This talk concentrates only on Ducros et al. type conservative splitting**

Ducros et al. Splitting

(Improve nonlinear stability for high order central schemes)

Split the derivative of a product into conservative & non-conservative parts:

$$(ab)_x = \frac{1}{2}(ab)_x + \frac{1}{2}ab_x + \frac{1}{2}a_x b$$

Approximation of the split form can be written in conservative form: e.g.,

$$\frac{1}{2}D_0(ab)_j + \frac{1}{2}a_j D_0 b_j + \frac{1}{2}b_j D_0 a_j = \frac{1}{4}D_+(a_j + a_{j-1})(b_j + b_{j-1})$$

D_0: 2^{nd}-order central, $D_+ u_j = (u_{j+1} - u_j)/\Delta x$

The above can be generalized to $2p^{th}$-order accurate: *Ducros et al. 2000*

$$D_{0p} u_j = \sum_{k=1}^{p} \alpha_k^{(p)} D_0(k) u_j \qquad D_0(k) u_j = (u_{j+k} - u_{j-k})/(2k\Delta x)$$

$$\sum_{k=1}^{p} \alpha_k^{(p)} = 1 \qquad \sum_{k=1}^{p} \alpha_k^{(p)} k^{2n} = 0, \;\; n = 1, \ldots, p-1$$

Ducros et al. Splitting (Cont.)

(Improve nonlinear stability for high order central schemes)

Approximation of the 2pth-order split form in conservation form:

$$\frac{1}{2}D_p(ab) + \frac{1}{2}D_p(a)b + \frac{1}{2}aD_p(b) =$$

$$\frac{1}{\Delta x}\sum_{k=1}^{p}\frac{1}{2}\alpha_k\left((a_{j+k}b_{j+k} - a_{j-k}b_{j-k}) + a_j(b_{j+k} - b_{j-k}) + (a_{j+k} - a_{j-k})b_j\right)$$

$$= \frac{1}{\Delta x}\sum_{k=1}^{p}\frac{\alpha_k}{2}\left(\sum_{m=0}^{k-1}(a_{j-m} + a_{j+k-m})(b_{j-m} + b_{j+k-m})\right.$$

$$\left.-\sum_{m=0}^{k-1}(a_{j-1-m} + a_{j-1+k-m})(b_{j-1-m} + b_{j-1+k-m})\right) = \frac{1}{\Delta x}(h_{j+1/2} - h_{j-1/2})$$

$2p^{th}$-order Central Ducros et al. Splitting Numerical Flux for 3D Gas Dynamics

3D Inviscid Flux Derivative in x-Direction:

$$\mathbf{f} = ([\rho u, \rho u^2 + p, \rho uv, \rho uw, (e+p)u]^T$$

$2p^{th}$-order Numerical Flux in x-Direction $\mathbf{h}_{j+1/2}$:

$$\mathbf{h}_{j+1/2} =$$

$$\frac{1}{2}\sum_{k=1}^{p}\alpha_k \sum_{m=1}^{k-1} \begin{pmatrix} (\rho_{j-m} + \rho_{j+k-m})(u_{j-m} + u_{j+k-m}) \\ (\rho_{j-m}u_{j-m} + \rho_{j+k-m}u_{j+k-m})(u_{j-m} + u_{j+k-m}) + p_{j-m} + p_{j+k-m} \\ (\rho_{j-m}v_{j-m} + \rho_{j+k-m}v_{j+k-m})(u_{j-m} + u_{j+k-m}) \\ (\rho_{j-m}w_{j-m} + \rho_{j+k-m}w_{j+k-m})(u_{j-m} + u_{j+k-m}) \\ (e_{j-m} + p_{j-m} + e_{j+k-m} + p_{j+k-m})(u_{j-m} + u_{j+k-m}) \end{pmatrix}$$

Well-Balanced High Order Nonliner Filter Schemes Non-Reacting & Reacting Flows

Yee & Sjögreen, *1999-2010,* **Wang et al.**, *2009-2010*

Preprocessing step

Condition (equivalent form) the governing equations by, e.g., *Yee et al. Entropy Splitting, Ducros et al. Splitting, Tadmor Splitting* to improve numerical stability

High order low dissipative base scheme step (Full time step)

- High order **central, DRP, or entropy conser. num. flux** scheme
- SBP numerical boundary closure, matching spatial & temporal order conservative metric evaluation *Vinokur & Yee, Sjögreen & Yee, Yee & Vinokur 2014*

Nonlinear filter step

- Filter the base scheme step solution by a dissipative portion of **any** positive high-order shock capturing scheme, e.g., **7^{th}-order positive WENO**
- Use local flow sensor to control the amount & location of the nonlinear numerical dissipation to be employed

Well-balanced scheme: preserve certain non-trivial physical steady state solutions of reactive eqns exactly

Note: "Nonlinear Filter Schemes" not to be confused with "LES filter operation"

Nonlinear Filter Step $(U_t + F_x(U) = 0)$

- **Denote the solution by the base scheme (e.g. 6^{th} order central, $4th$ order RK)**

$$U^* = L^*(U^n)$$

- **Solution by a nonlinear filter step**

$$U_j^{n+1} = U_j^* - \frac{\Delta t}{\Delta x}\left[H_{j+1/2} - H_{j-1/2}\right]$$
$$H_{j+1/2} = R_{j+1/2}\overline{H}_{j+1/2}$$

$\overline{H}_{j+1/2}$ **- numerical flux,** $R_{j+1/2}$ **- right eigenvector, evaluated at the Roe-type averaged state of** U_j^*

- **Elements of $\overline{H}_{j+1/2}$:**

$$\overline{h}_{j+1/2} = \frac{\kappa_{j+1/2}^m}{2}\left(s_{j+1/2}^m\right)\left(\phi_{j+1/2}^m\right)$$

$\phi_{j+1/2}^m$ - ***Dissipative portion of a shock-capturing scheme***
$s_{j+1/2}^m$ - ***Local flow sensor (indicates location where dissipation needed)***
$\kappa_{j+1/2}^m$ - ***Controls the amount of*** $\phi_{j+1/2}^m$

Improved High Order Filter Method

Form of nonlinear filter

$$\bar{h}_{j+1/2} = \frac{\kappa^m_{j+1/2}}{2} \left(s^m_{j+1/2} \right) \left(g^m_{j+1/2} - b^m_{j+1/2} \right)$$

Control amount of dissipation based on local flow condition

Local flow sensor *(Shock Sensor, ACM (Harten), Ducros et al, Multiresolution wavelet, etc.)*

Any High Order Shock capturing numerical flux *(e.g. WENO5)*

High order central numerical flux *(e.g. 6th order central)*

2007 – $\kappa =$ global constant
2009 – $\kappa_{j+1/2} =$ local, evaluated at each grid point
Simple modification of κ *(Yee & Sjögreen, 2009)*

$$\kappa = f(M) \cdot \kappa_0$$

$$f(M) = min \left(\frac{M^2}{2} \frac{\sqrt{4 + (1 - M^2)^2}}{1 + M^2}, 1 \right)$$

For other forms of $\kappa_{j+1/2}, s_{j+1/2}$, see *(Yee & Sjögreen, 2009)*

Performance of High Order Nonlinear Filter Scheme

(Skew-Symmetric Splitting of Inviscid Flux Derivative)

Rapidly Developing Flows: *(subsonic, transonic, supersonic & hypersonic)*

- > ***Smooth flows,*** *Yee et al., 1999*
- > ***Flows with discontinuities,*** *Yee et al., Sjogreen & Yee, Sandham et al., 2000-2004*
- > ***Supersonic Mixing & Richtmyer-Meshkov Instability,*** *Yee & Sjogreen, 2004, 2012*
- > ***Extreme Flows -*** *positivity-preserving nonlinear filter scheme, Kotov et al., 2014*
- > ***Flows with stiff source terms – Wrong shock speed***
  High order well-balanced subcell resolution schemes
  Wang et al., Yee et al., Kotov et al., 2009-2015

Long Time Integrations, DNS & LES:

- > ***Shock Free Compressible Turbulence*** *(Kotov et al., 2016)*
- > ***Low Speed Turbulence with Shocklets*** *(Kotov et al., 2016)*
- > ***LES of Temporally Evolving Mixing Layers*** *(Yee et al., 2012)*
- > ***DNS & LES of Turbulence Interacting with a Stationary Supersonic Shock --***
  One-sided SGS model & subcell resolution to locate the shock within one grid cell (Kotov et al., 2016)
- > ***3D Forced Turbulence (Time Varying Forcing)*** *(Sjorgreen et al., 2016)*
- > ***Dual & Direct Cascade Study of 2D Turbulence with Random Forcing***
  (Astrophysical Applications, Kritsuk et al., 2016)

3D Taylor-Green vortex
(Inviscid & Viscous Shock-Free Turbulence)

Computational Domain: 2π square cube, 64^3 grid.
(Reference solution on 256^3 grid)

Initial condition

$\rho = 1,$
$p = 100 + ([\cos(2z) + 2][\cos(2x) + \cos(2y)] - 2)/16,$
$u_x = \sin x \cos y \cos z$
$u_y = -\cos x \sin y \cos z$
$u_z = 0.$
Initial turbulent Mach number: $M_{t,0} = 0.042$
Final time: $t = 10$

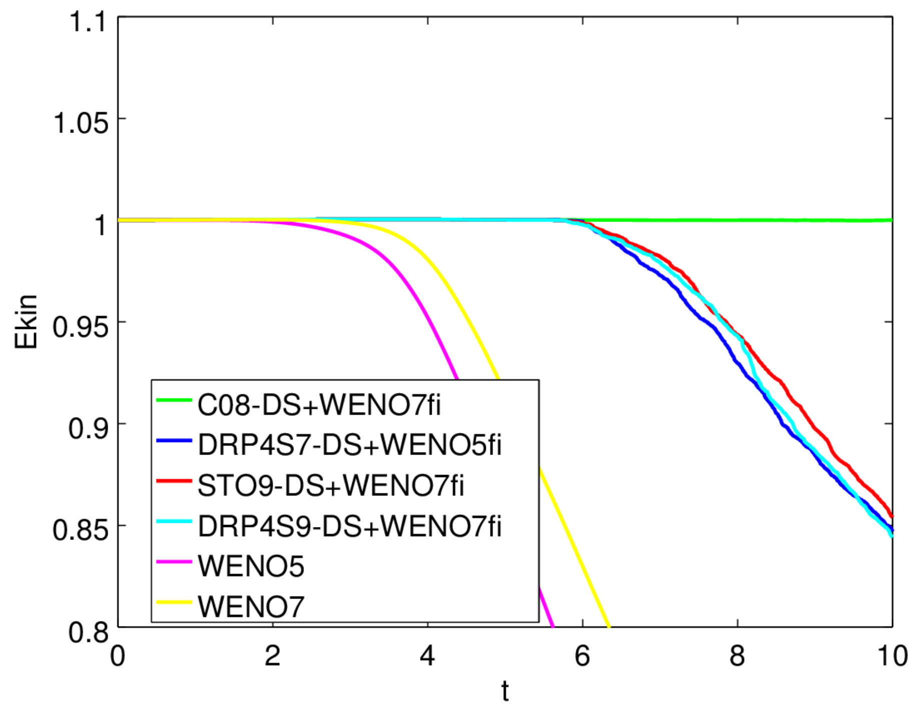
Viscous case

$\mu/\mu_{ref} = (T/T_{ref})^{3/4}$
$\mu_{ref} = 0.005, T_{ref} = 1, Re_0 = 2040$

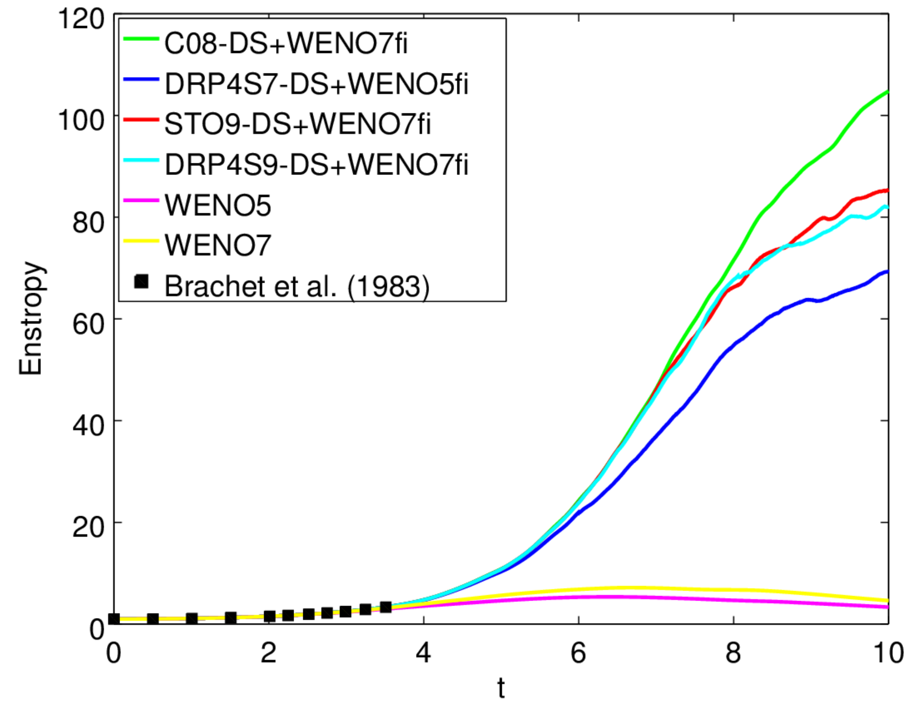
3D Taylor-Green Vortex (Compressible & Inviscid)

(Comparison of 6 Methods, 64^3 grids)

Kinetic Energy



Enstrophy



C08-DS+WENO7fi: 8th-order central + Ducros split +WENO7fi

DRP4S7-DS+WENO5fi: Tam & Webb 4th-order DRP, 7pt grid stencil + Ducros split +WENO5fi

ST09-DS+WENO7fi: Bogey & Bailly 4th-order DRP, 9pt grid stencil + Ducros split +WENO7fi

DRP4S9-DS+WENO7fi: Tam & Webb 4th-order DRP, 9pt grid stencil + Ducros split +WENO7fi

Compressible Isotropic Turbulence

(Low Speed Turbulence with Shocklets)

Computational Domain: 2π square cube, 64^3 grid.
(Reference solution on 256^3 grid)

Problem Parameters

Root-mean-square velocity: $u_{rms} = \sqrt{\frac{\langle u_i u_i \rangle}{3}}$

Turbulent Mach number: $M_t = \frac{\sqrt{\langle u_i u_i \rangle}}{\langle c \rangle}$

Taylor-microscale: $\lambda = \sqrt{\frac{\langle u_x^2 \rangle}{\langle (\partial_x u_x)^2 \rangle}}$

Taylor-microscale Reynolds number: $Re_\lambda = \frac{\langle \rho \rangle u_{rms} \lambda}{\langle \mu \rangle}$

Eddy turnover time: $\tau = \lambda_0 / u_{rms,0}$

Initial Condition: **Random solenoidal velocity field with the given spectra**

$E(k) \sim k^4 \exp(-2(k/k_0)^2)$

$\frac{3}{2} u_{rms,0}^2 = \frac{\langle u_{i,0} u_{i,0} \rangle}{2} = \int_0^\infty E(k) dk$

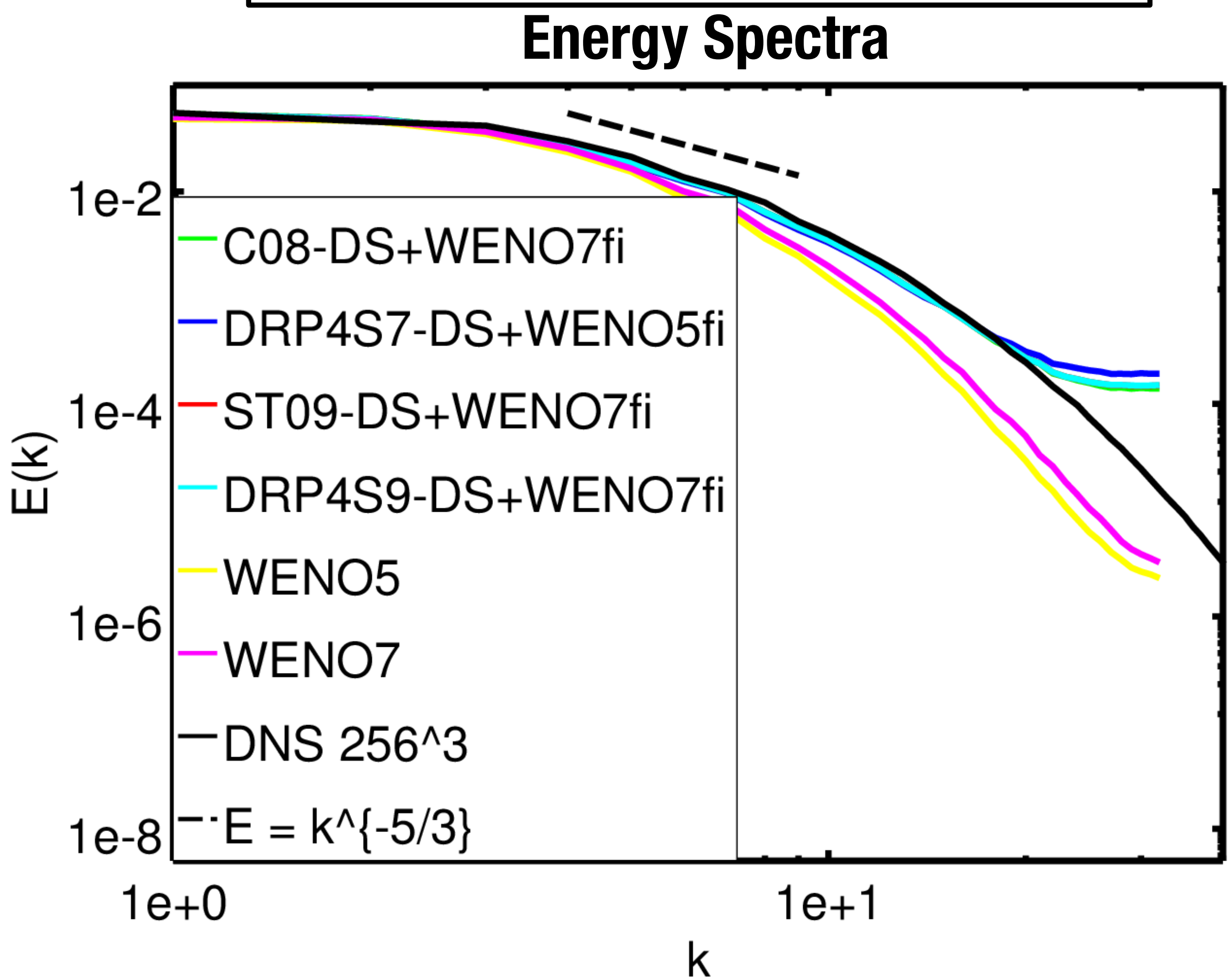
$u_{rms,0} = 1$, $k_0 = 4$, $\tau = 0.5$, $M_{t,0} = 0.6$, $Re_{\lambda,0} = 100$

Final time: $t = 2$ **or** $t/\tau = 4$

3D Isotropic Turbulence with Shocklets

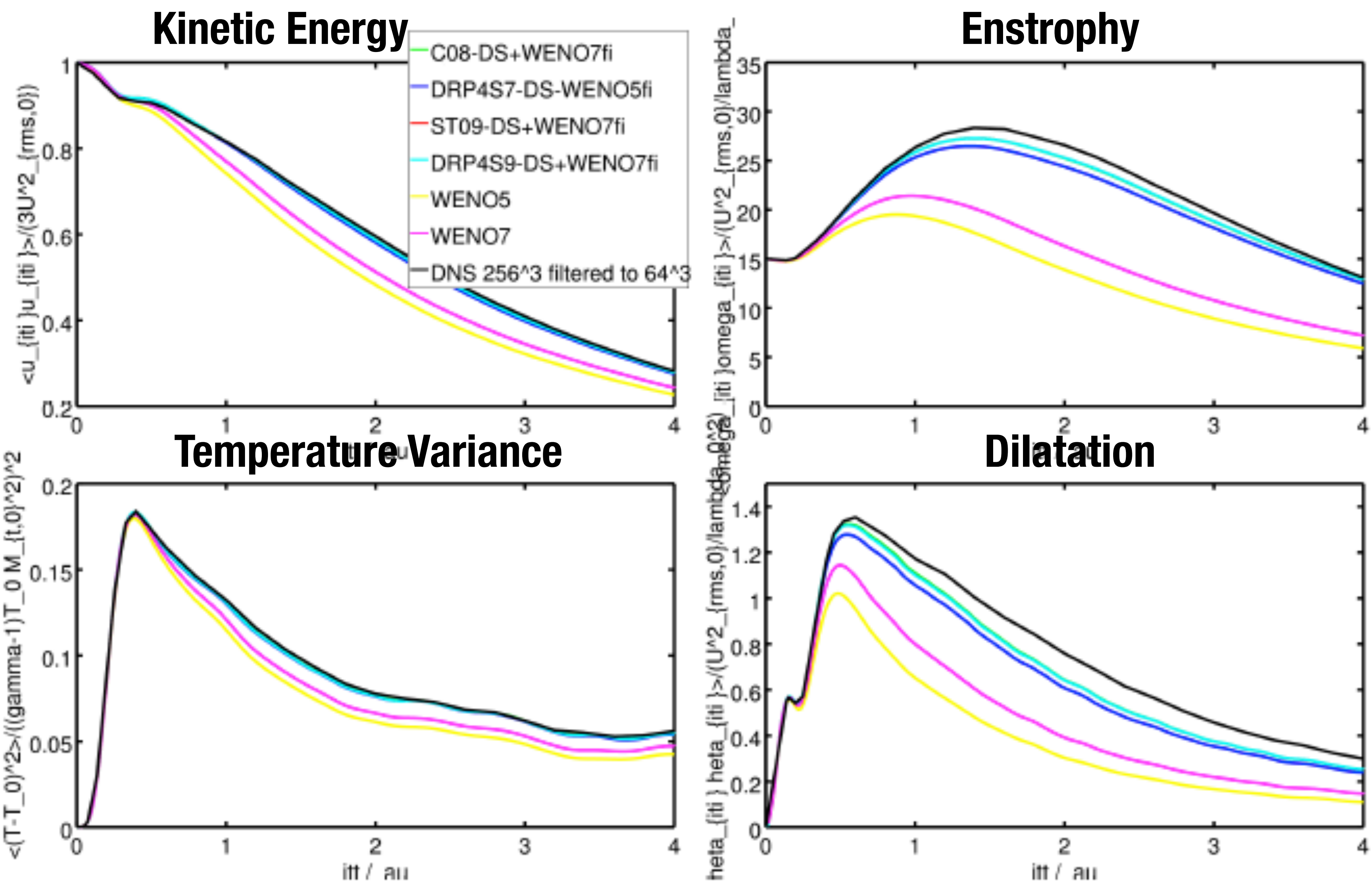
Compressible & Inviscid

Comparison of 6 Methods, 64^3 grids



3D Isotropic Turbulence with Shocklets

Comparison of 6 Methods, 64^3 grids



High Order Numerical Method Development in MHD

(Added Issues Beyond Compressible Gas Dynamics Developments)

MHD Equations:

> *Conservative Form* - *non-strictly hyperbolic system w/ degenerate identical eigenvalues*
> *Godunov/Powell Form (1972, 1994)* - *symmetrizable hyperbolic non-conservative system*
> *Janhunen Form (2000)*
> *Brackbill & Barnes (1980)*

Skew-symmetric Splitting of Inviscid Flux Derivatives: *Improve Stability & Minimize Num. Dissipation*

> *Yee et al. Entropy Splitting (2000) – Only for the gas dynamics portion*
> *Ducros et al. Splitting (2000) & Pirozzoli Generalization (2010) – Not unique*
> *High Order Extension of Tadmor Entropy Conservative Numerical Fluxes (Sjogreen & Yee, 2009) – can be viewed as a splitting*

Discrete Conservation Methods: *FV vs. FD & DG, etc; Low Order vs. High Order*

> *Entropy stable conservative numerical fluxes*
– *Low Order: Janhunen (2000), Winters & Gassner (2016), Chandrasekar-Klingenberg (2015)*
– *High Order: Sjogreen & Yee (2009) - Central, Fjordholm, Mishra & Tadmor (2012) - ENO, etc.*
> *Momentum conservation, Kinetic energy preservation, etc.*

Approximate Riemann Solver: *Extension of Roe's Average States*

> *Gallice average states (1997)*
> *Ismail & Roe (2009) – Logarithmic mean for entropy (not square root mean)*
. . .

Eigenvector Scaling: *(Roe & Balsara, 1996)*

Non-uniqueness of Ducros et al. Splitting for MHD

(Minimize the use of numerical dissipation for high order central schemes)

- MHD inviscid (ideal) flux derivatives consist of triple products of conservative variables & their derivatives
- No unique guidelines in splitting triple products of derivatives *(more choices than their gas dynamics counterparts)*
  (See Sjogreen & Yee, ICOSAHOM-2016 & Journal version for the chosen forms)
- 3-Forms: Split all 8 flux derivatives, partial or just the gas dynamic portion *(all recover to split form of gas dynamics when MHD not present)*
  (Results compare with no splitting)
- Four forms of the MHD Equations to be solved:
  - Conservative form
  - Godunov/Powell symmetrizable form (non-conservative)
  - Janhunen form: *(Div B) terms not included in the gas dynamics part of the equations)*
  - Brackbill & Barnes form

The above consists of 16 combinations for the current study

Concluding Remarks

(Compressible Gas Dynamics of a Wide Spectrum of Flow Types)

Stability Improvement by Skew-Symmetric Splitting

Smooth Flows: *Stable without added high order linear numerical dissipation*

> **Semi-Conservative Entropy Splitting with summation-by-part (SBP) boundary closure energy norm bound** *(Yee et al. 1999-2007, Sandham et al. 2002-present)*
> - *Most accurate & stable among the considered three splittings*

> **Ducros et al. splitting**
> - *Improved stability*
> - *Smaller improvement than Entropy Splitting*

Flows with shocks: *Under the Yee et al. nonlinear filter framework*

Ducros et al. Splitting Employs Two Types of Central Scheme:

> *Classical high order central (6^{th}-order & 8^{th}-order)*

> *Three DRP (4^{th}-order, 7-point & 9-point grid stencils)*

Among studied test cases
Classical central schemes provide slightly more improvement than DRP

Concluding Remarks

(Compressible Gas Dynamics of a Wide Spectrum of Flow Types)

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- **Ducros et al. splitting**
  - *Improved stability*
  - *Smaller improvement*

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Integrated Approach for Stability, Accuracy & *Reliable Simulations*

(Construction of High Order Low Dissipative Numerical Methods)

Yee et al. (1999-2003), Yee & Sjogreen (2004-2009), Kotov et al. (2013-2016)

Stability & Accuracy:

- *> Interior Scheme & Num. Boundary Scheme (non-periodic BC)*
  - *Summation-by-parts (SBP) boundary closures (Strand 1994, Olsson 1996)*
- *> Short Time Integration vs. Long Time Integration (DNS & LES)*
- *> High Order GCL (Geometric Conservation Law) metric evaluation (Sjogreen et al. 2014)*

Skew-symmetric Splitting of Inviscid Flux Derivatives:

Further improvement of Stability & Minimize Num. Dissipation

- *> Yee et al. Entropy Splitting (2000)*
- *> Ducros et al. Splitting (2000) & generalization*
- *> High Order Extension of Tadmor Entropy Conservative Numerical Fluxes (Sjogreen & Yee 2009)*

Discrete Conservation Methods: *FV vs. FD & DG, etc.*

- *> Entropy stable conservative high order numerical fluxes (Sjogreen & Yee 2009)*
- *> Momentum conservation, Kinetic energy preservation, etc.*

Numerical Dissipation Control: *Yee et al., Yee & Sjogreen, and Kotov et al. (1999-2016)*

- *> Turbulence cannot tolerate num. dissipation*
  - *Proper amount is needed in the vicinity of high shear, shocks & contacts*
- *> Different requirements in the minimization of num. dissipation for different flow types*
- *> Adaptive flow sensor to control the amount of num. dissipation*

Reacting Flow/Combustion: *Yee et al., Wang et al., Yee & Sweby, LeVeque & Yee (1990 – 2015)*

- *> Stiff source terms with shock - May lead to incorrect shock speed*
- *> Preserve certain physical steady states exactly – Well-balanced scheme*

Gas Dynamics vs. MHD scheme constructions

Some Gas Dynamics development can carry over to MHD (Items with an arrow)

Astrophysical Applications: 2D Turbulence

(Joint work with Alexei G. Kritsuk, U.C. San Diego)

Application: *Energetics of the ISM in Galactic Disks*

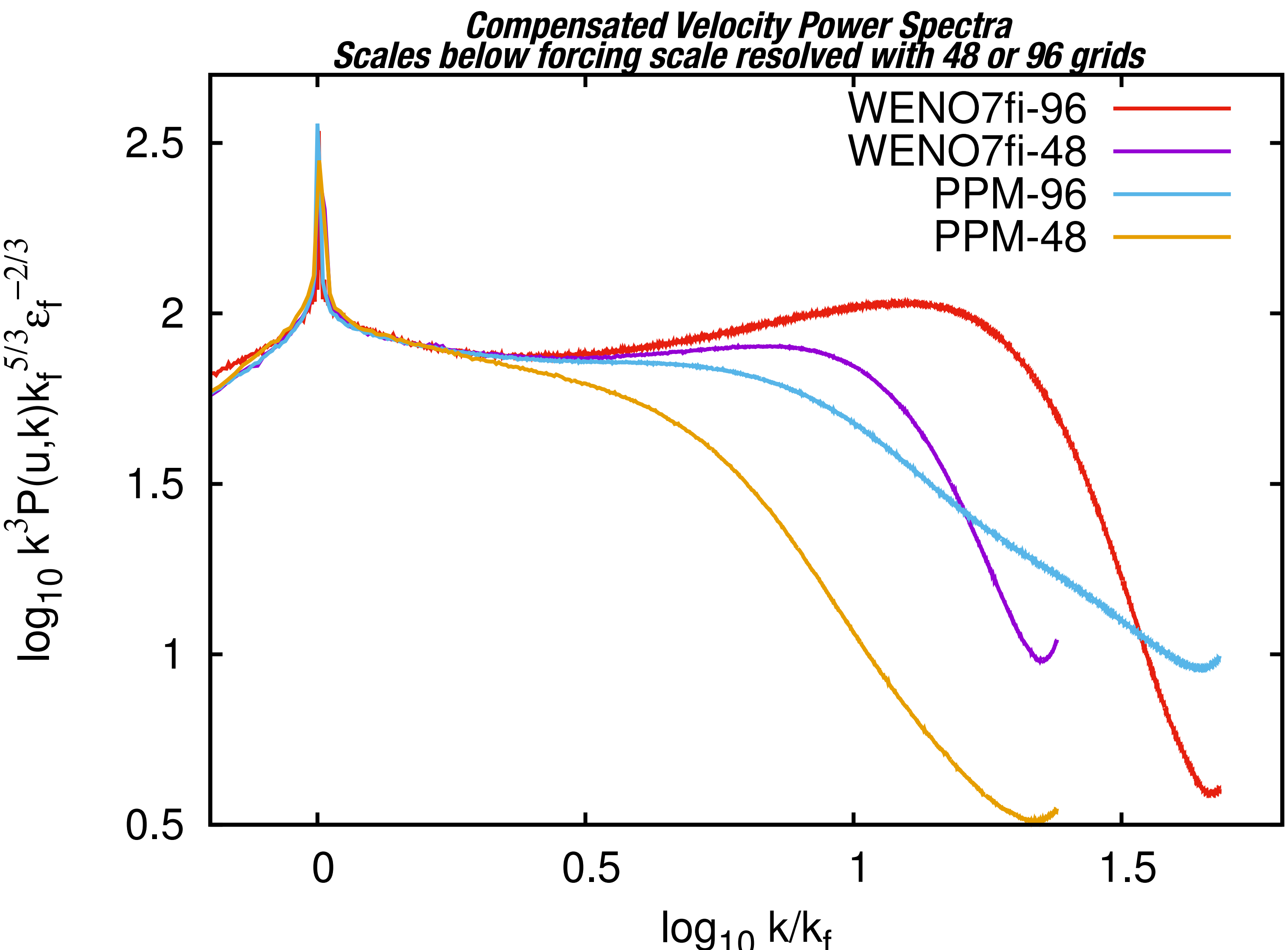
- *Dual energy cascade study*
- *Does the inverse energy cascade work in the compressible case?*
- *What are the corresponding scaling relations?*

Grid size:

- *Physics Study:* 512^2, $2{,}048^2$, $8{,}192^2$, $16{,}384^2$
- *Computation Grid Resolutions:* $2{,}048^2$, $8{,}192^2$, $16{,}384^2$

Scheme Comparison: PPM vs WENO7fi+split

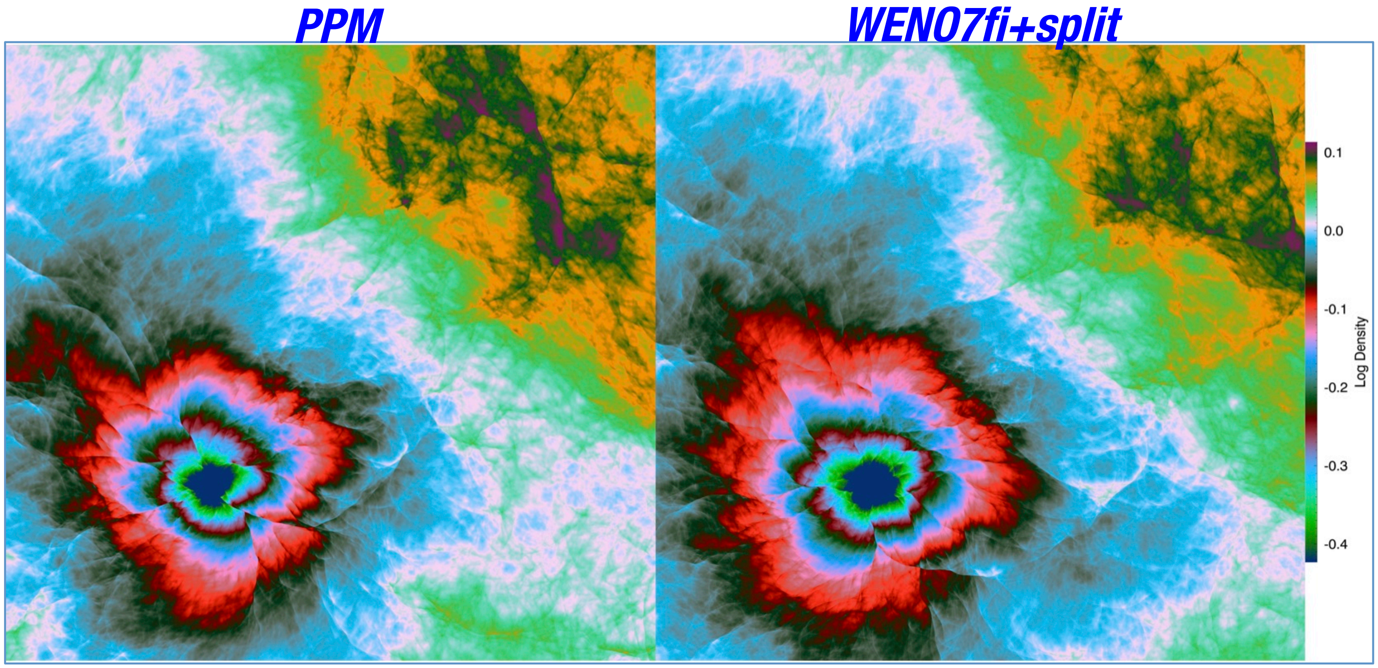
2D Compressible Turbulence: Isothermal γ=1.001, periodic BCs
Flow determined by grid N, energy injection rate & energy injection scale



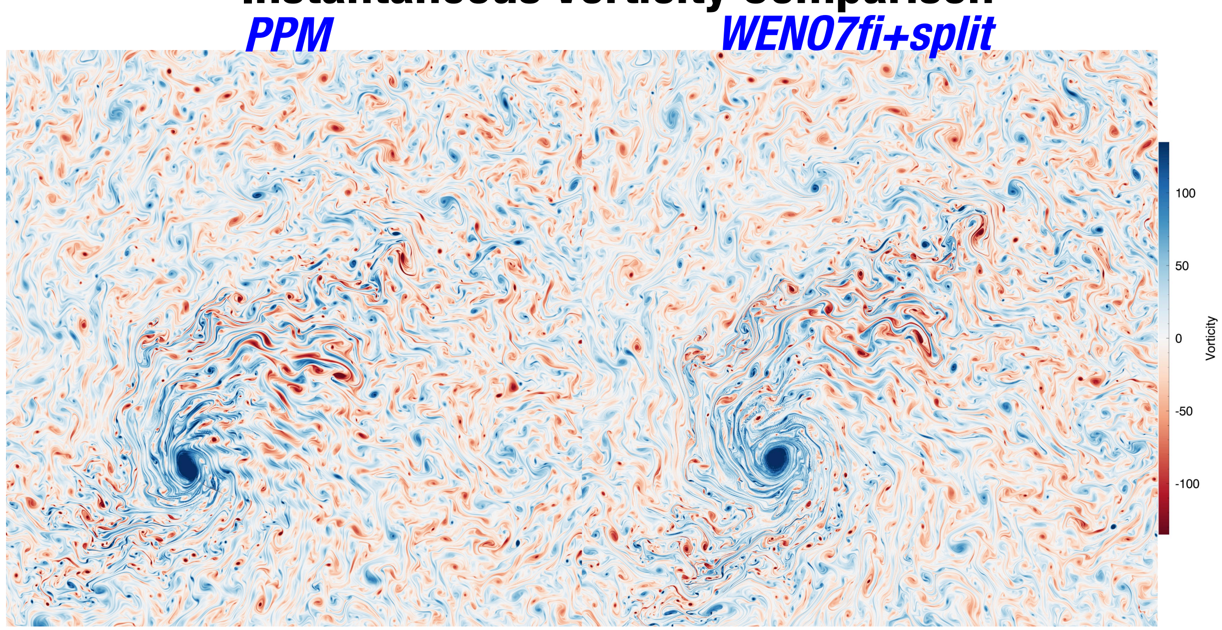
Spectral Bandwidth: WENO7fi+split 2.2 X > PPM; ~4 times less CPU in 2D for same resolution (assume 25%)

Note: If P(k) is a spectrum and P(k)~k^n, then the compensated spectrum is k^{-n}P(k)

Instantaneous Density Comparison



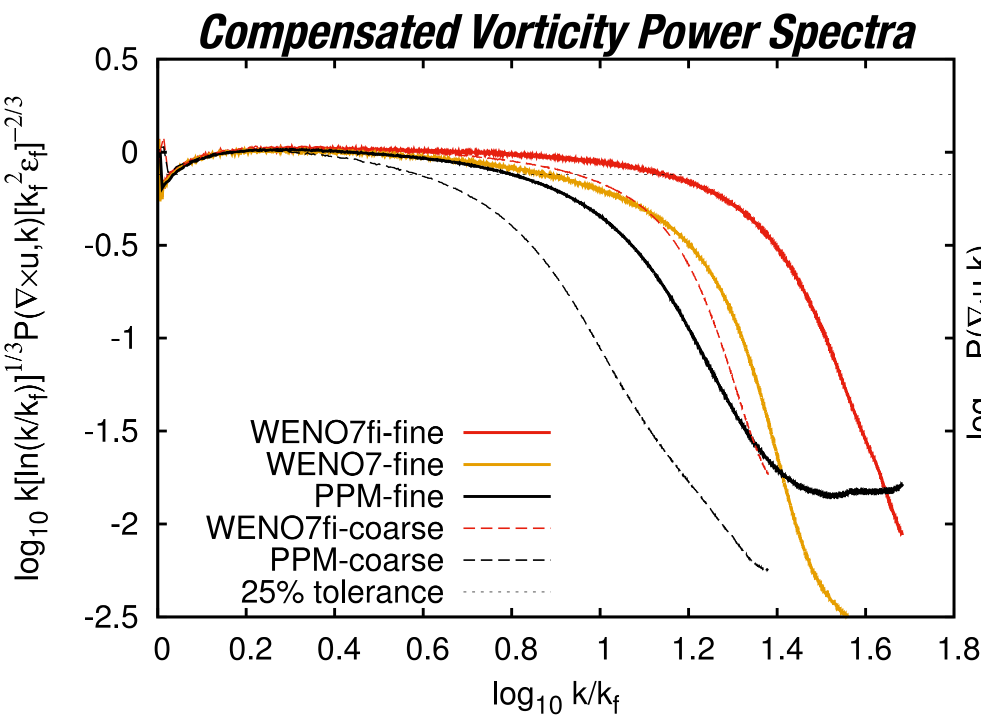
Instantaneous Vorticity Comparison

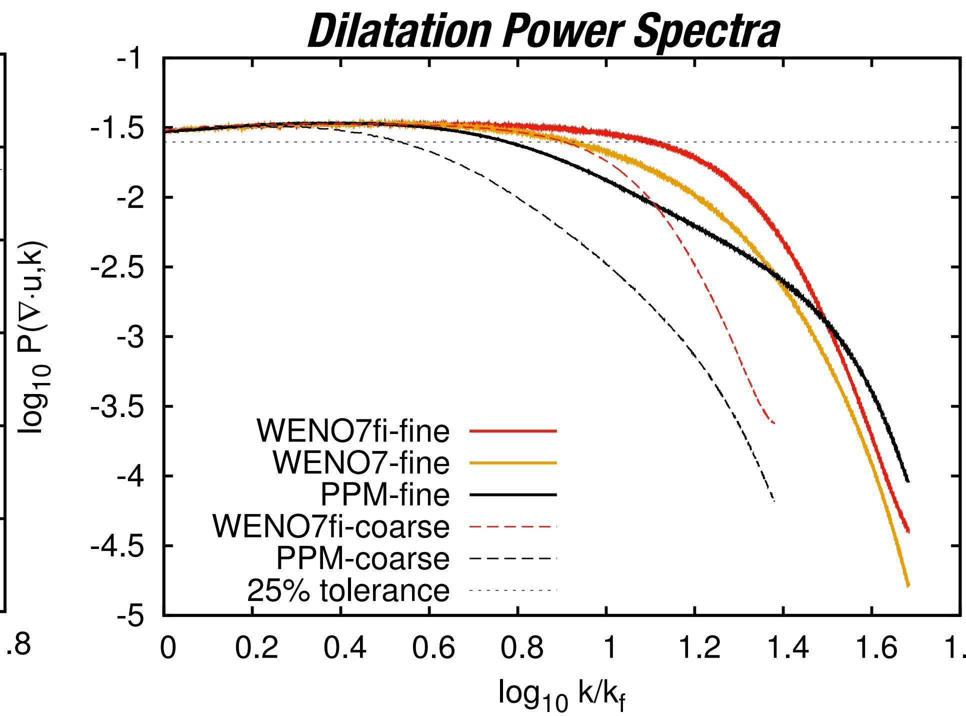


Scheme Comparison: PPM, WENO7, WENO7fi+split

2D Compressible Turbulence: Isothermal γ=1.001, periodic BCs
Flow determined by grid N, energy injection rate & energy injection scale

Direct Cascade study: Coarse vs. fine grids





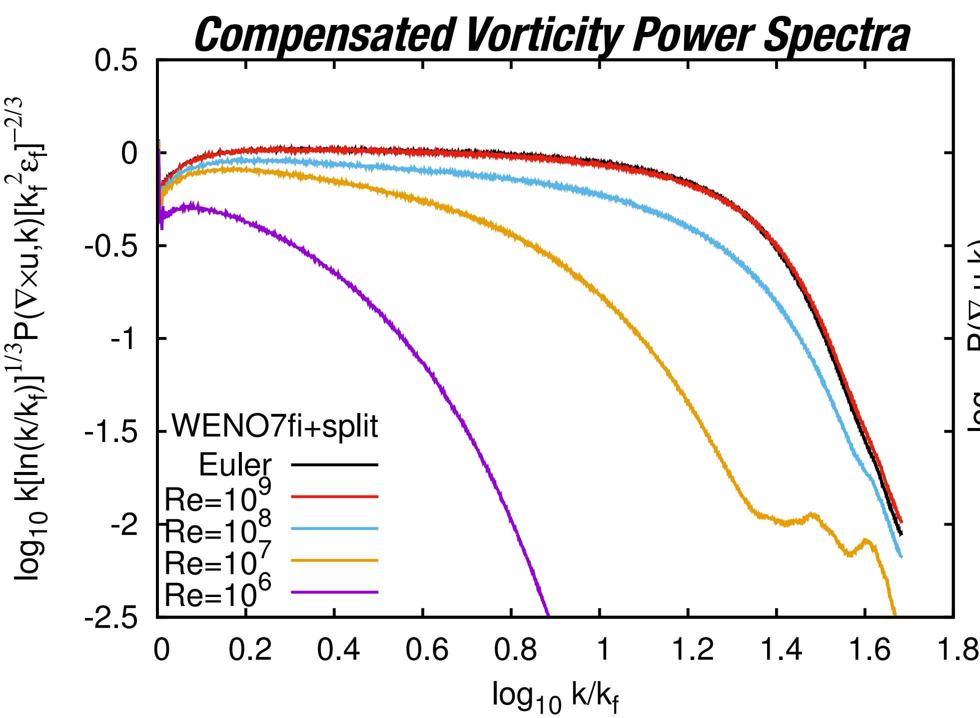
Conclusion:

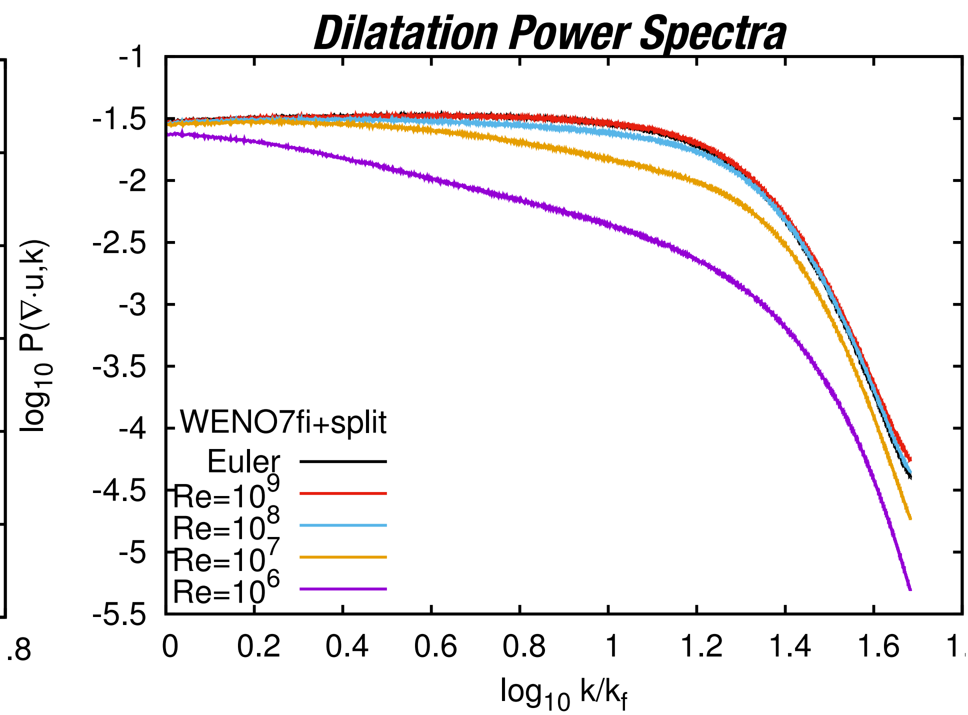
- **Vorticity bandwidth:** WENO7/PPM=1.2; WENO7fi/WENO7=1.8; WENO7fi/PPM=2.2
- **Dilatation bandwidth:** WENO7/PPM=1.5; WENO7fi/WENO7=1.5; WENO7fi/PPM=2.2
- **Absolute WENO7fi bandwidth:** for vorticity 68%; for dilatation 66%

Euler vs. NS Comparison: WENO7Fi+split

2D Compressible Turbulence: Isothermal γ=1.001, periodic BCs
Flow determined by grid N, energy injection rate & energy injection scale

Isothermal Fluids: $T=T_0$ Constant Dynamic Viscosity, $Re=10^6, 10^7, 10^8, 10^9$





Summary:

WENO7fi+split correctly captures theoretically predicted spectra for both incompressible & compressible diagnostics in the limit of vanishing controlled numerical dissipation

3-D Compressible MHD (Ideal)

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \\ B_x \\ B_y \\ B_z \end{pmatrix}_t + \mathrm{div} \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u}\mathbf{u}^T + (p + \frac{1}{2}B^2)I - \mathbf{B}\mathbf{B}^T \\ \mathbf{u}(e + p + \frac{1}{2}B^2) - \mathbf{B}(\mathbf{u}^T\mathbf{B}) \\ \mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \end{pmatrix} = 0$$

Conservative

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ e \\ B_x \\ B_y \\ B_z \end{pmatrix}_t + \mathrm{div} \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u}\mathbf{u}^T + (p + \frac{1}{2}B^2)I - \mathbf{B}\mathbf{B}^T \\ \mathbf{u}(e + p + \frac{1}{2}B^2) - \mathbf{B}(\mathbf{u}^T\mathbf{B}) \\ \mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \end{pmatrix} = -(\nabla \cdot \mathbf{B}) \begin{pmatrix} 0 \\ B_x \\ B_y \\ B_z \\ \mathbf{u}^T\mathbf{B} \\ u \\ v \\ w \end{pmatrix}$$

Non-conservative (Symmetrizable - *Godunov, Powell*)

$$\mathbf{u} = (u, v, w)^T$$

$$\mathbf{B} = (B_x, B_y, B_z)^T$$

$$B^2 = B_x^2 + B_y^2 + B_z^2$$

$$p = (\gamma - 1)(e - \frac{1}{2}\rho(u^2 + v^2 + w^2) - \frac{1}{2}(B_x^2 + B_y^2 + B_z^2))$$

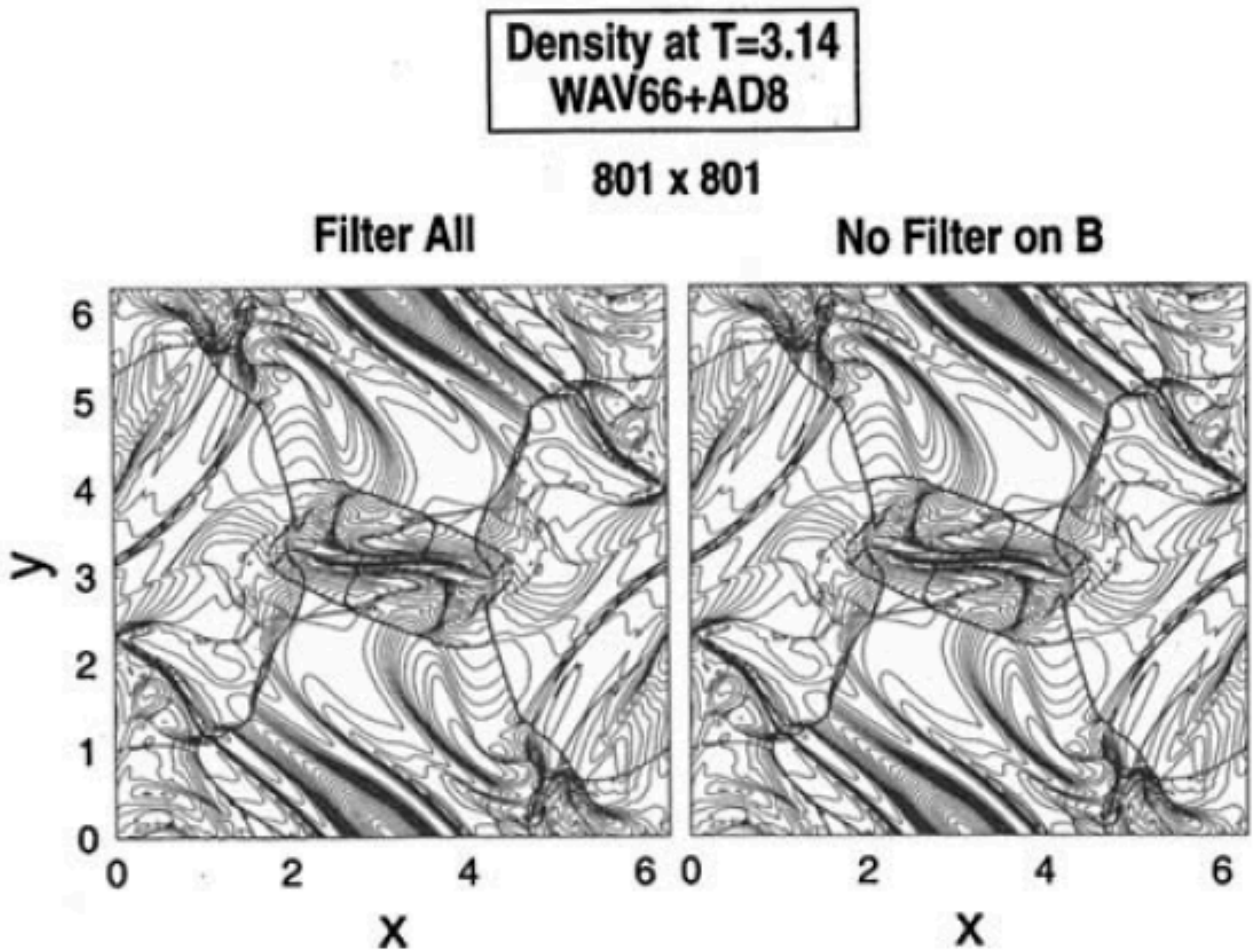
Compressible Orszag-Tang Vortex $(\gamma = 5/3)$

I.C.

$$\begin{pmatrix} \rho \\ u \\ v \\ w \\ p \\ B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 25/9 \\ -\sin y \\ \sin x \\ 0 \\ 5/3 \\ -\sin y \\ \sin 2x \\ 0 \end{pmatrix}$$

BC: Periodic

Domain: $0 < x < 2\pi$
$0 < y < 2\pi$

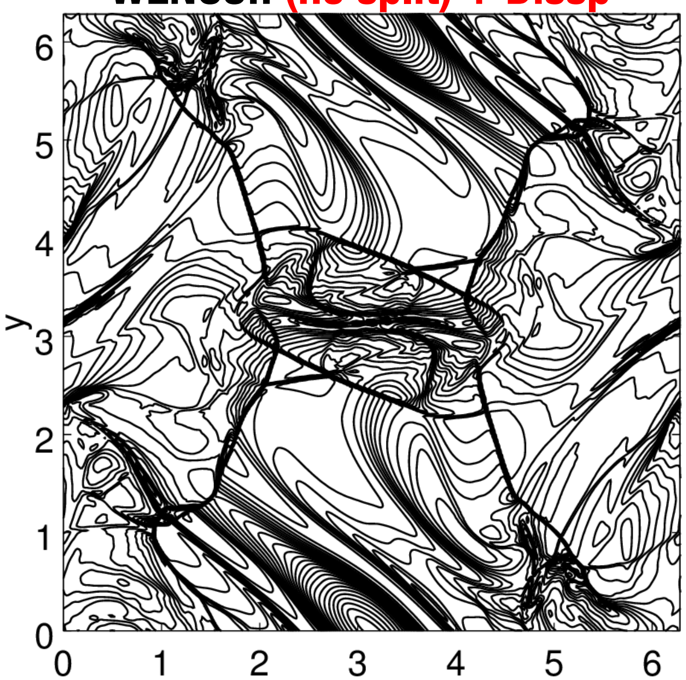


Ducros et al. Splitting *- Orszag-Tang Vortex Test case*

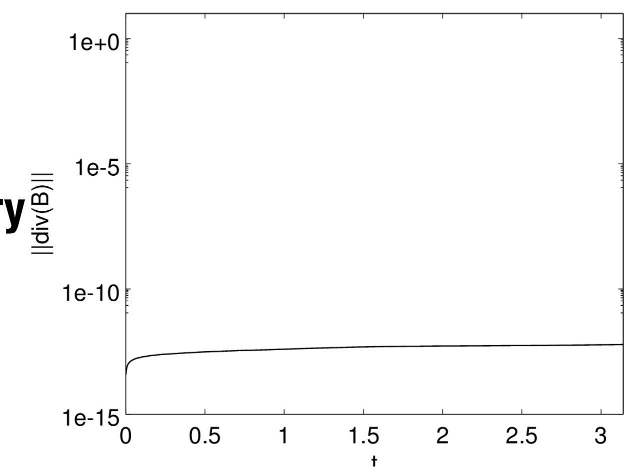
(Only on the Gas Dynamic Variables)

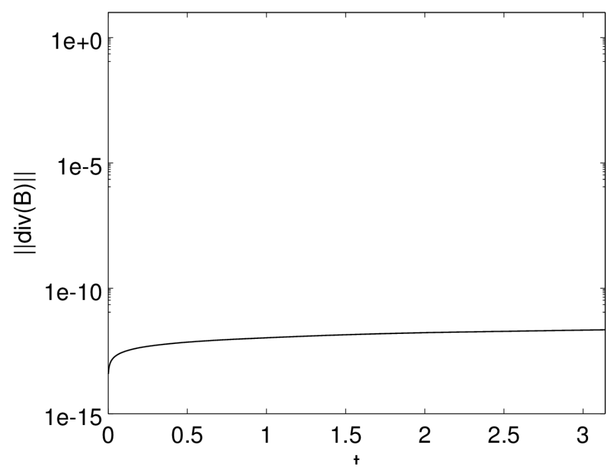
WENO5fi (no split) + Dissp | **WENO5fi+split**

Density



divB History





High Order Discrete Conservation Methods

(High Order Numerical Fluxes for compressible MHD)

- **Entropy stable conservative numerical fluxes**

  Low Order: ***Janhunen (2000), Winters-Gassner (2016), Chandrasekar-Klingenberg (2015)***

  High Order: Sjogreen-Yee (2009) - Central, Fjordholm, Mishra, &Tadmor (2012) - ENO, etc.

- **Momentum conservation, Kinetic energy preservation, etc.**

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