Conjunction Assessment Risk Analysis

Remediating Non-Positive Definite State Covariances for Collision Probability Estimation

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Agenda

• Introduction
  – Motivation and objectives
  – Geometrical visualizations of covariances

• The observed frequency of NPD covariances
  – Analysis of 830,000 actual conjunctions spanning 2 years
  – Low-precision vs. high-precision covariances
  – Low-eccentricity vs. high-eccentricity orbits

• Covariance remediation for Pc estimation
  – Covariance requirements for Pc-related calculations
  – *Spectrum shifting, Higham, and eigenvalue clipping* methods
  – Pc estimation with *eigenvalue clipping* remediation
Motivation and Objectives

• Motivation:
  – The probability of collision (Pc) between two Earth-orbiting satellites requires estimates of their orbital trajectories and associated uncertainties
  – Pc estimation requires processing conjunction state vectors and covariance matrices
  – CARA sometimes encounters non-positive definite (NPD) covariances that can potentially prevent or adversely affect Pc estimation

• Objectives:
  – Investigate the frequency of NPD covariances
  – Implement method(s) to remediate NPD covariances when necessary
The 1-sigma ellipsoid of the PDF defined by the 3x3 covariance matrix

Major Eigenvector = $V_{\text{max}}$
Major Eigenvalue = $\lambda_{\text{max}}$
Semi-major axis = $\sigma_{\text{max}} = (\lambda_{\text{max}})^{1/2}$

Minor Eigenvector = $V_{\text{min}}$
Semi-minor axis = $\sigma_{\text{min}}$

Medium Eigenvector = $V_{\text{med}}$
Semi-principal axis = $\sigma_{\text{med}}$
Transition from Positive Definite to Semi-Positive Definite 3x3 Covariances

Positive definite (PD) covariances have positive $\sigma_{\min}$ values.
Positive semi-definite (PSD) covariances have zero $\sigma_{\min}$ values.
Non-positive definite (NPD) covariances have imaginary $\sigma_{\min}$ values.

| $\lambda_{\min} = \lambda_{\text{med}}$ | $\lambda_{\min} = \lambda_{\text{med}} / 4$ | $\lambda_{\min} = \lambda_{\text{med}} / 100$ | $\lambda_{\min} = 0$ |
| $\sigma_{\min} = \sigma_{\text{med}}$ | $\sigma_{\min} = \sigma_{\text{med}} / 2$ | $\sigma_{\min} = \sigma_{\text{med}} / 10$ | $\sigma_{\min} = 0$ |
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The Frequency of NPD State Covariance Matrices Over Time

- Analysis of 830,509 events
  - 2015-04-01 to 2017-04-06

- NPD covariances for primary objects decreased markedly just before 2016-05-01
  - Coincides with an increase in the number of significant figures used for covariances
  - Better precision leads to fewer NPD state covariances

- This same pattern is seen for 3x3 position covariances
  - But at frequencies reduced by a factor of 100 or more
The Frequency of NPD State Covariances: High Eccentricity vs. Low Eccentricity

- Analysis of 428,589 conjunctions
  - 2016-05-01 to 2017-04-06
  - High precision covariances

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6x6 ECI state covariance matrices at TCA
6x6 ECI state correlation matrices at TCA

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9 primary objects

72 primary objects
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Covariance Requirements for Numerical Stability in Pc-Related Estimations

- **Mahalanobis distance estimation**
  - The marginalized 3x3 relative position covariance $A(t) = A_p(t) + A_s(t)$ needs to be positive definite

- **2D Pc estimation**
  - A marginalized 2x2 projection of $A(t_{ca}) = A_p(t_{ca}) + A_s(t_{ca})$ needs to be positive definite

- **3D Pc estimation**
  - The marginalized 3x3 relative position covariances $A(t_i)$ need to be positive definite at all ephemeris times $t_i$ used in the calculation

- **Monte Carlo Pc estimation**
  - The full NxN state covariances at the sampling epoch time $C(t_{ep})$ need to be at least positive semi-definite

Pc-related calculations don’t always require fully PD state covariances for both objects, because they often use combined and marginalized covariances with reduced dimensions
<table>
<thead>
<tr>
<th>Description</th>
<th>Spectrum Shifting</th>
<th>Higham Remediation*</th>
<th>Eigenvalue Clipping</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Description</strong></td>
<td>Add an offset to all eigenvalues, and then reconstruct covariances using original eigenvectors</td>
<td>Find closest PSD covariance and/or correlation matrix in terms of Frobenius norm</td>
<td>Clip the eigenvalues at a minimum limit, and then reconstruct covariances using original eigenvectors</td>
</tr>
<tr>
<td><strong>Advantages</strong></td>
<td>- Relatively simple to implement</td>
<td>- Mathematically well defined, as used by the financial industry - Algorithms and codes posted on-line</td>
<td>- Simplest to implement - Constrained by physics-based considerations - Produces fully PD position covariances</td>
</tr>
<tr>
<td><strong>Disadvantages</strong></td>
<td>- Assumes original eigenvectors can be used for matrix reconstruction - Offset applied to all eigenvalues, even if they don’t need remediation - Not constrained using any physics-based considerations</td>
<td>- Most complicated to implement in software - Only designed to produce PSD matrices - The “closest Frobenius norm” criterion, while well mathematically defined, is not a physics-based criterion</td>
<td>- Assumes original eigenvectors can be used for matrix reconstruction</td>
</tr>
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Pc Estimates for Conjunctions with Remediated NPD Covariances

TCA 6x6 ECI covariance status: \( C_p = PD \quad C_s = NPD \quad C_p+C_s = PD \)

<table>
<thead>
<tr>
<th>Remediation Method</th>
<th>2D Pc</th>
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<tr>
<td>No covariance remediation</td>
<td>( 1.1137 \times 10^{-4} )</td>
</tr>
<tr>
<td>Spectrum shifting remediation</td>
<td>( 1.1137 \times 10^{-4} )</td>
</tr>
<tr>
<td>Higham remediation</td>
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Most often, remediating 6x6 ECI covariances changes Pc negligibly.

*Primary: NOAA 15    Secondary: BREEZE-M DEB*
**Pc Estimates for Conjunctions with Remediated NPD Covariances**

29499_conj_38481_20160727_045045_20160815_125105*

TCA 6x6 ECI covariance status: \( C_p = \text{NPD} \quad C_s = \text{NPD} \quad C_p+C_s = \text{NPD} \)

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</tr>
<tr>
<td>Higham remediation</td>
<td>( 1.3464 \times 10^{-3} )</td>
</tr>
<tr>
<td>Eigenvalue clipping</td>
<td>( 1.3217 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

This \(~1.8\% Pc\) difference was the largest seen among 430,000 events

*Primary: METOP-A    Secondary: COSMOS 2251 DEB*
Remediating NPD Position Covariances using the Eigenvalue Clipping Method

Eigen - decomposition: $C = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ V_1 & V_2 & V_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \leftarrow V_1 \rightarrow \\ \leftarrow V_2 \rightarrow \\ \leftarrow V_3 \rightarrow \end{bmatrix}$

Clipped eigenvalues: $\lambda_{i,rem} = \max[\lambda_i, \lambda_{clip}]$  
$\lambda_{clip} \geq 0 = \begin{cases} = 0 & \text{Remediation to PSD} \\ > 0 & \text{Remediation to PD} \end{cases}$

Remediated matrix: $C_{rem} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ V_1 & V_2 & V_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_{1,rem} & 0 & 0 \\ 0 & \lambda_{2,rem} & 0 \\ 0 & 0 & \lambda_{3,rem} \end{bmatrix} \begin{bmatrix} \leftarrow V_1 \rightarrow \\ \leftarrow V_2 \rightarrow \\ \leftarrow V_3 \rightarrow \end{bmatrix}$

What is a sensible value for $\lambda_{clip}$ - the eigenvalue clipping limit? Can physics be used to constrain this value?
Collision Probability as a Function of $\sigma_{\text{min}}$

The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere.

These three conjunctions produce similar $P_c$ values (because they’ve carved out similar fractions of the PDF):

- $\sigma_{\text{min}} = 2 \text{HBR}$
- $\sigma_{\text{min}} = \text{HBR}/10$
- $\sigma_{\text{min}} << \text{HBR}$
Collision Probability Visualization: \( \text{Pc as a Function of } \sigma_{\text{min}} \)

The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere.

\( \sigma_{\text{min}} = 2 \text{ HBR} \)  \( \sigma_{\text{min}} = \frac{\text{HBR}}{10} \)  \( \sigma_{\text{min}} \ll \text{HBR} \)

The two conjunctions on the right will produce similar \( \text{Pc} \) values.

The one on the left will produce a smaller \( \text{Pc} \) value.

\( \sigma_{\text{min}} = 2 \text{ HBR} \)

\( \sigma_{\text{min}} = \frac{\text{HBR}}{10} \)

\( \sigma_{\text{min}} \ll \text{HBR} \)
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere. 

\[ \sigma_{\text{min}} = 2 \text{ HBR} \]

\[ \sigma_{\text{min}} = \text{HBR/10} \]

\[ \sigma_{\text{min}} << \text{HBR} \]

\[ P_c \] values are insensitive to the \( \sigma_{\text{min}} \) value whenever \( 0 < \sigma_{\text{min}} << \text{HBR} \). This can be used to set a sensible eigenvalue clipping level.
Conclusions

• The frequency of NPD state covariance matrices decreased markedly in mid-2016
  – After increasing the covariance precision
• NPD state covariances occur much more frequently for objects in high-eccentricity orbits
  – Likely due to covariance interpolation inaccuracies
• Estimating Pc values doesn’t always require fully positive definite state covariances
  – Because the calculations use marginalized covariances
• Remediating ECI state covariances doesn’t change Pc values
  – Three remediation methods produce equivalent results
• Eigenvalue clipping can be used for remediation
  – It is the simplest to implement, and can be constrained when required using physics-based considerations