On High-Order Upwind Methods for Advection

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Motivation

- Second-order methods are currently popular; for turbulent and unsteady flows, they are generally unreliable or impractical.
- Worldwide effort to improve accuracy and efficiency for such flow problems by employing high-order methods:
  1) The numerous papers at this conference.
  2) International Workshop on High-Order CFD Methods
  3) The TILDA project (Towards Industrial LES and DNS for Aeronautics) supported by the European Union.
- High-order methods need further development and improvement. The current work is along this direction.

Setup

- Goal: solve the Navier-Stokes equations numerically. Numerical methods toward this goal are typically first derived and analyzed for the advection equation.
- Van Leer (1977) introduced five schemes for advection in "Towards the ultimate conservative difference scheme, IV".
- Scheme I: least accurate but became most popular; widely known as the MUSCL scheme.
- Schemes III (piecewise linear, discontinuous) and V (piecewise parabolic, continuous): most accurate but least popular. They are the main focus of current work.

Setup

- (Van Leer and Nomura 2005): “When trying to extend these schemes beyond advection, viz., to a nonlinear hyperbolic system like the Euler equations, the first author ran into insuperable difficulties because the exact shift operator no longer applies, and he abandoned the idea”.
- Scheme III was extended to systems of equations in (Huynh 2006). The approach was further analyzed and applied to hyperbolic-relaxation equations by Suzuki, Khieu, Van Leer (2007-2009). High-order extension was carried out by Lo (2011) and Huynh (2013).
- Scheme V is being extended to systems of equations by Roe, Eymann, and Fan (2013-present) and called the active flux scheme.

Main Findings of Current Work

- **Equivalence result:** Schemes III and V are shown to be equivalent in the sense that they yield identical solutions.
  - This equivalence is counter intuitive.
  - This finding also shows a key connection between the approaches of discontinuous (scheme III) and continuous (scheme V) polynomial approximations.
- **High-order extension:** introduce a projection-interpolation framework that simultaneously extends schemes III and V.

Outline

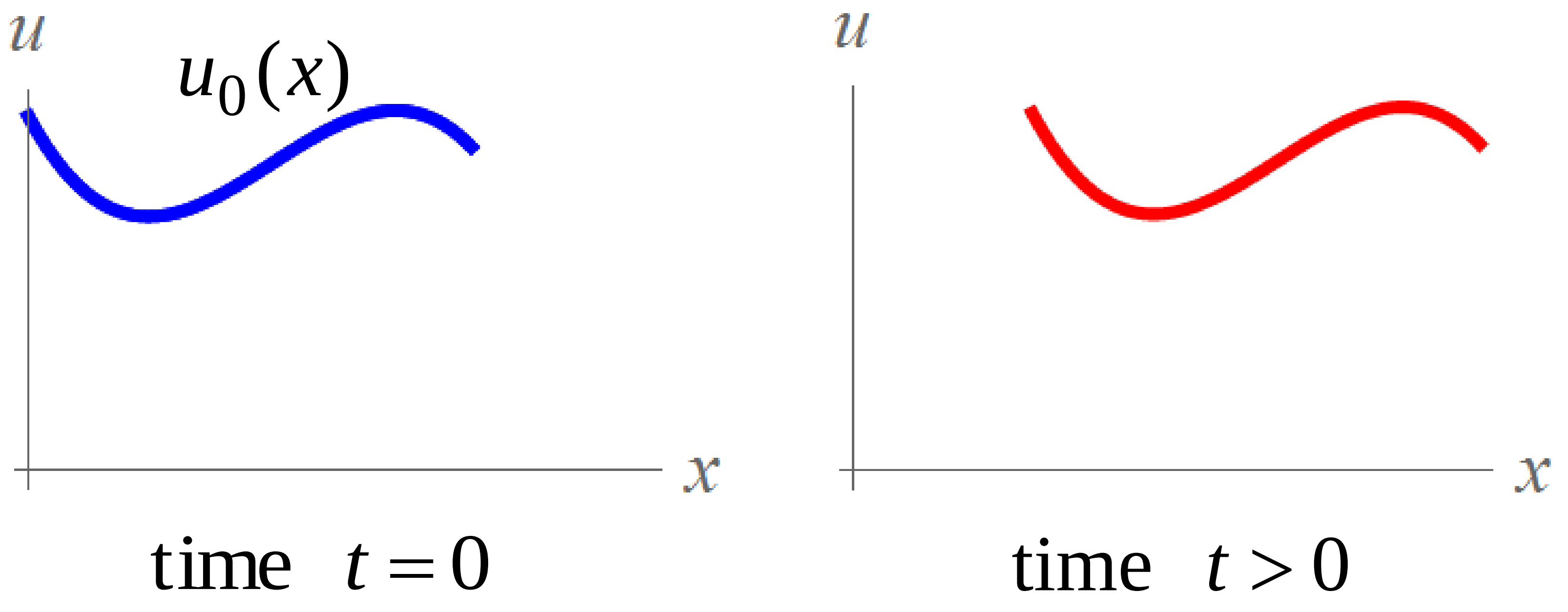
- Review MUSCL approach and schemes III and V.
- A new result: the equivalence of these two schemes.
- Introduce a projection-interpolation framework that simultaneously extends schemes III and V to arbitrary-order.
- Von Neumann (or Fourier) stability and accuracy analysis.
- Conclusions and discussion.

Advection Equation

$$u_t + a u_x = 0, \quad a \geq 0$$

Initial condition : at $t = 0$, $u(x) = u_0(x)$

Exact solution : $u(x,t) = u_0(x - at)$

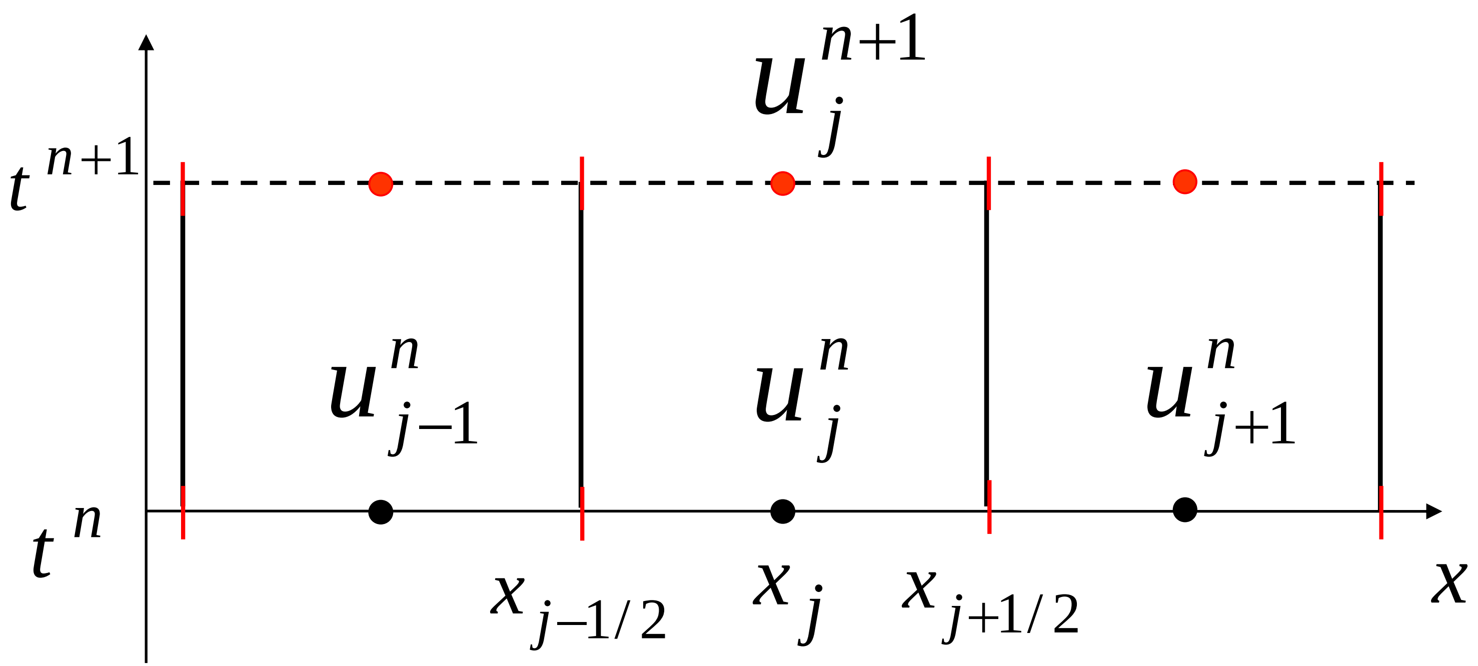


Advection Equation

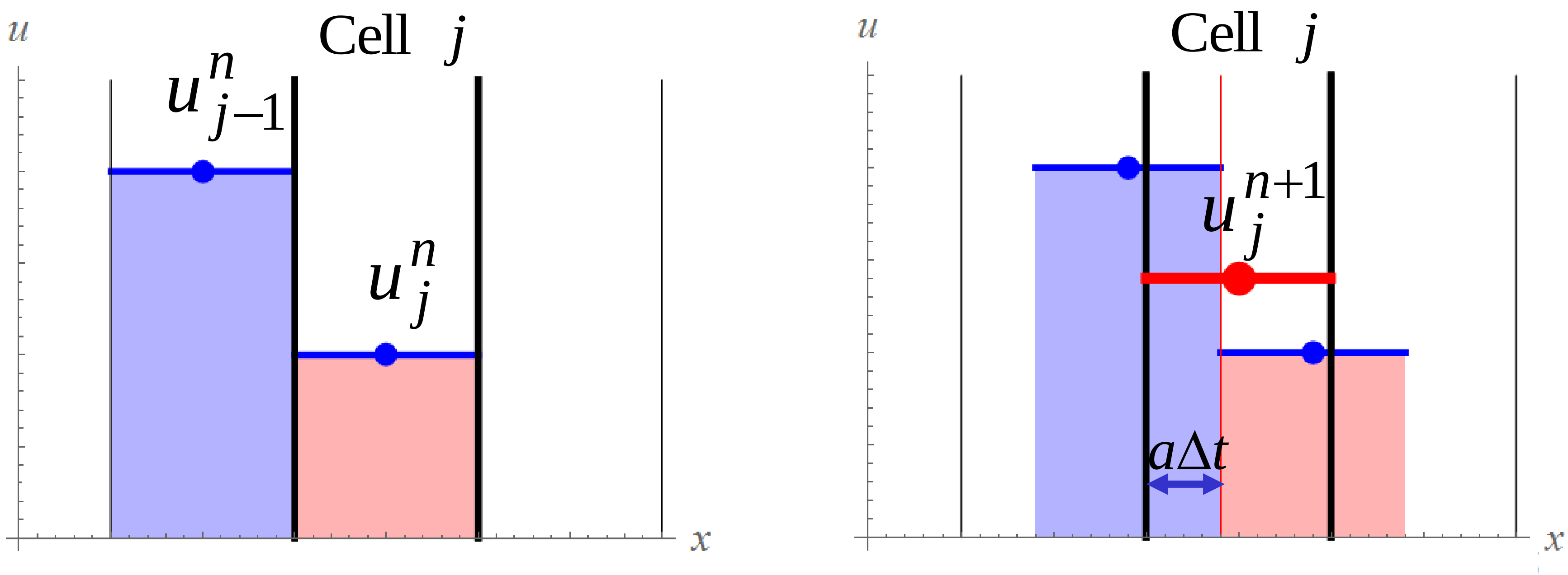
- Simplifies the derivation and description of CFD methods.
- Facilitates linear stability and accuracy analysis.
- Schemes derived for advection must then be extended to systems of equations, which is often not an easy task.

Discretization

The data u_j^n represents the average value of u in the cell $[x_{j-1/2}, x_{j+1/2}]$ at time t^n



First-Order Upwind Method for Advection: Shift Operator and Projection



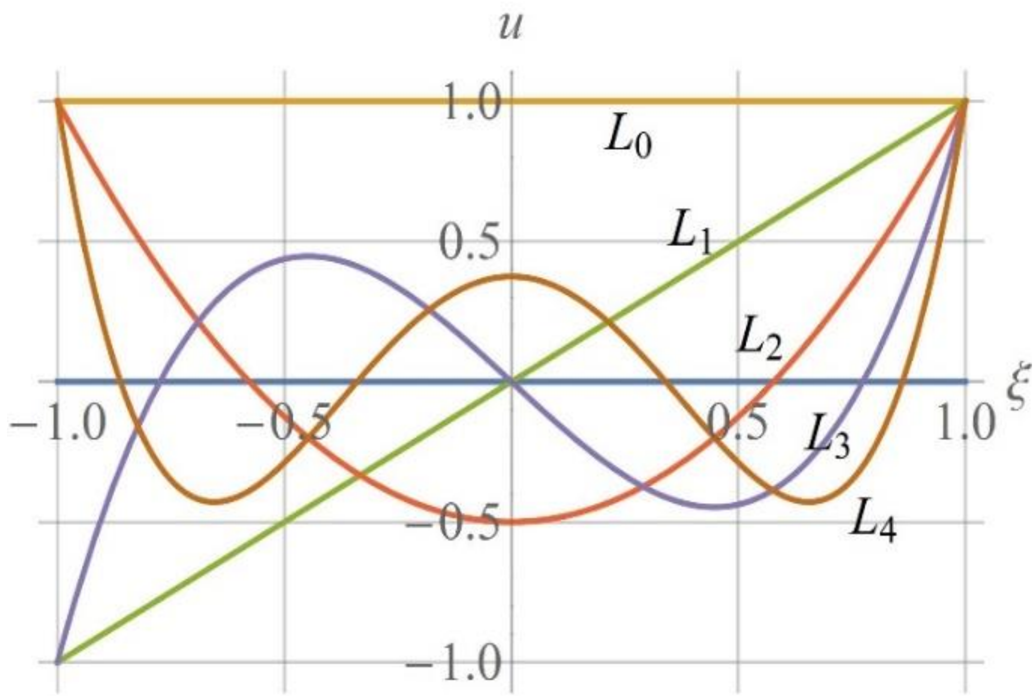
(a) Data

(b) First-order solution (red)

$$\sigma = a\frac{\Delta t}{\Delta x},$$

$$u_j^{n+1} = \sigma u_{j-1} + (1-\sigma) u_j$$

High-Order Extension: Legendre Polynomials

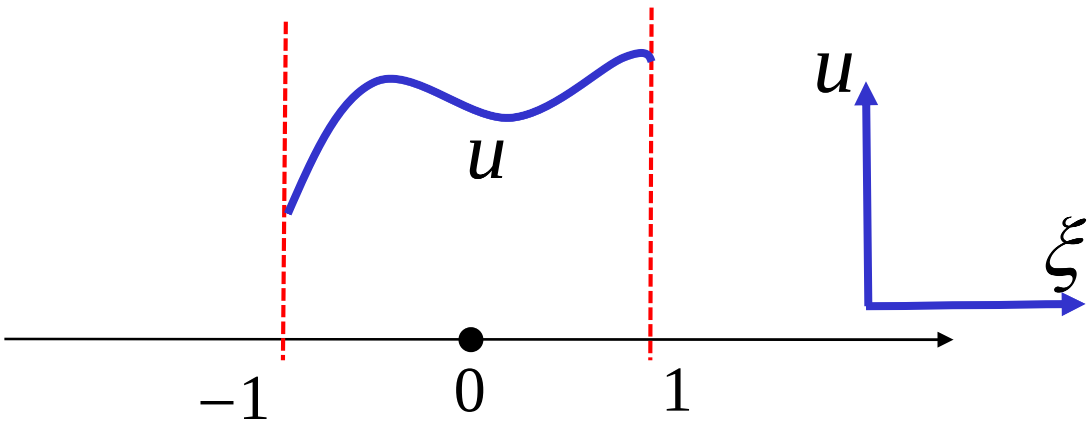


On $I = [-1,1]$, by orthgonalizing the monomials ξ^k, $k = 0,1,2, \ldots,$

via Gram-Schmidt process, we obtain the Legendre polynomials

$$L_0 = 1, \quad L_1 = \xi, \quad L_2 = \frac{1}{2}(3\xi^2 - 1), \quad L_3 = \frac{1}{2}(5\xi^3 - 3\xi), \ldots$$

Projection using Legendre Polynomials



On $I = [-1, 1]$, approximate u by $\sum_{k=0}^{p} u_{j,k} L_k$ where

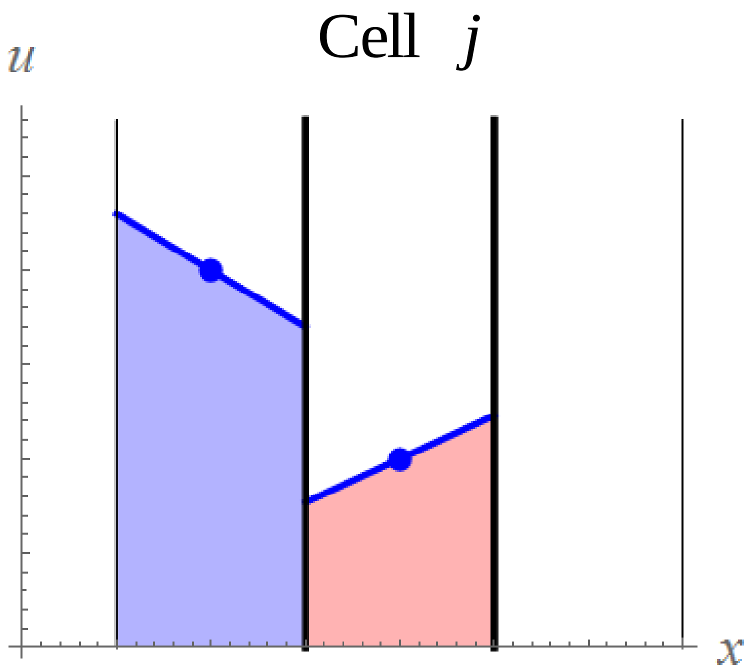
$$u_{j,0} = \int_{-1}^{1} L_0(\xi) u(\xi)\, d\xi \Big/ \|L_0\|^2 = \frac{1}{2} \int_{-1}^{1} 1\, u(\xi)\, d\xi\,,$$

$$u_{j,1} = \int_{-1}^{1} L_1(\xi) u(\xi)\, d\xi \Big/ \|L_1\|^2,$$

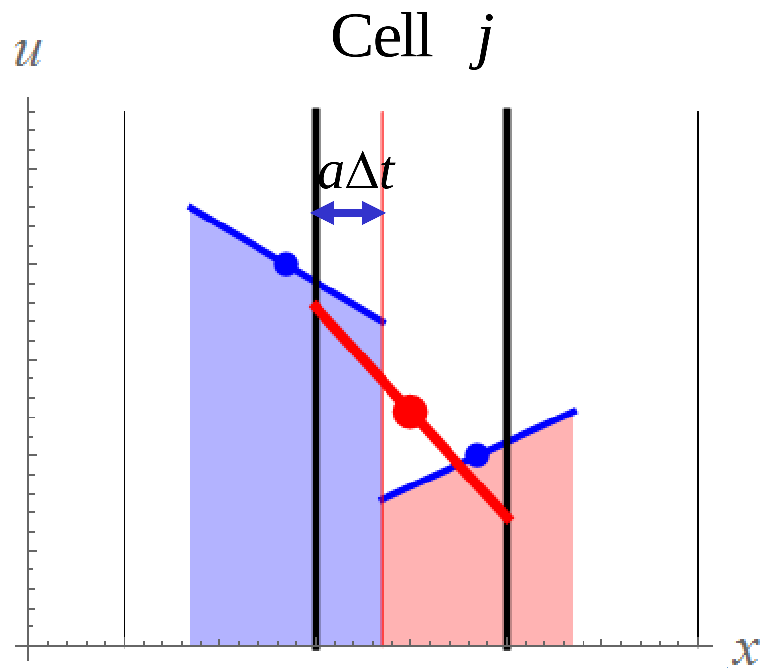
$$\dots$$

$$u_{j,k} = \int_{-1}^{1} L_k(\xi) u(\xi)\, d\xi \Big/ \|L_k\|^2$$

Van Leer's Scheme III Employing Shift and Projection

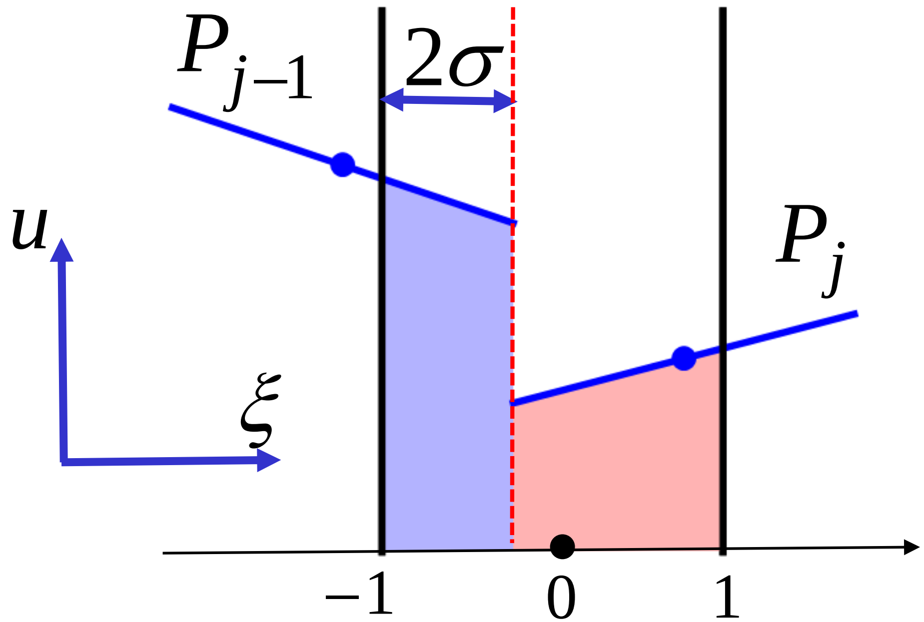


(a) Data



(b) Linear solution

Van Leer's Scheme III



$$u_{j,k}^{n+1} = \left(\begin{array}{l} \int_{-1}^{-1+2\sigma} P_{j-1}(\xi-\sigma+1)\, L_k(\xi)\; d\xi \\ + \int_{-1+2\sigma}^{1} P_j(\xi-\sigma)\, L_k(\xi)\; d\xi \end{array} \right) \Bigg/ \left\| L_k \right\|^2$$

$$P_j^{n+1}(\xi) = \sum_{k=0}^{p} u_{j,k}^{n+1}\, L_k(\xi)$$

Scheme III

$$\begin{pmatrix} u_{j,0}^{n+1} \\ u_{j,1}^{n+1} \end{pmatrix} = \begin{pmatrix} \sigma & \sigma(1-\sigma) \\ -3\sigma(1-\sigma) & -\sigma(3-6\sigma+2\sigma^2) \end{pmatrix} \begin{pmatrix} u_{j-1,0} \\ u_{j-1,1} \end{pmatrix} +$$

$$\begin{pmatrix} 1-\sigma & -\sigma(1-\sigma) \\ 3\sigma(1-\sigma) & (1-\sigma)(1-2\sigma-2\sigma^2) \end{pmatrix} \begin{pmatrix} u_{j,0} \\ u_{j,1} \end{pmatrix}$$

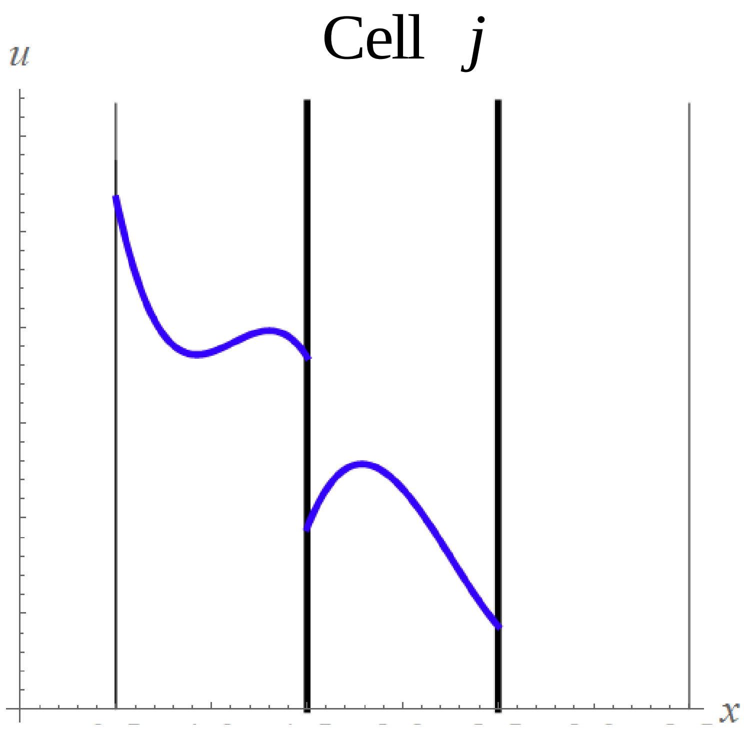
Set

$$U_j = \begin{pmatrix} u_{j,0} \\ u_{j,1} \end{pmatrix},$$

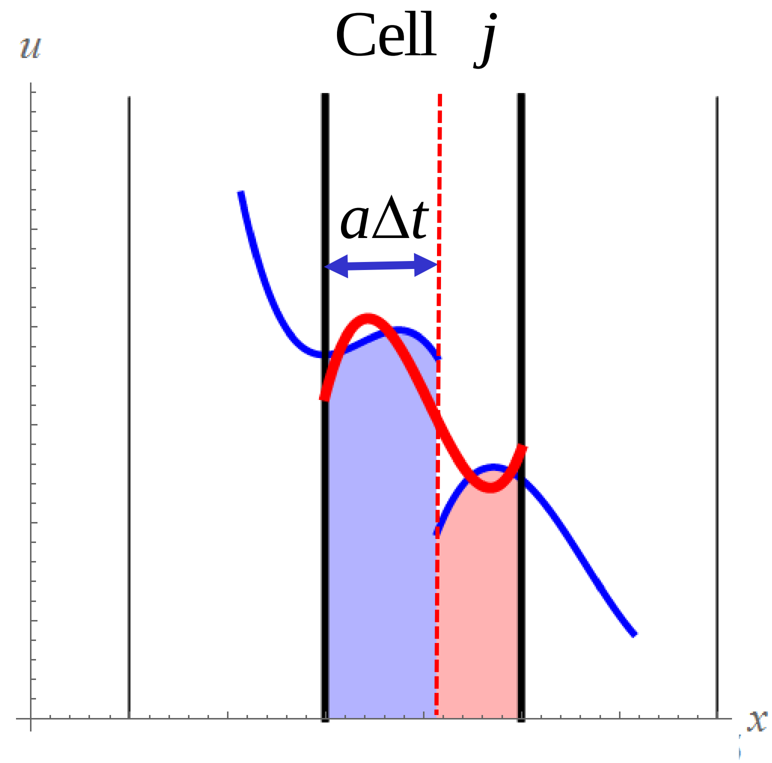
Then

$$U_j^{n+1} = C_{-1} U_{j-1} + C_0 U_j \,.$$

Projection to Arbitrary Order



(a) Data



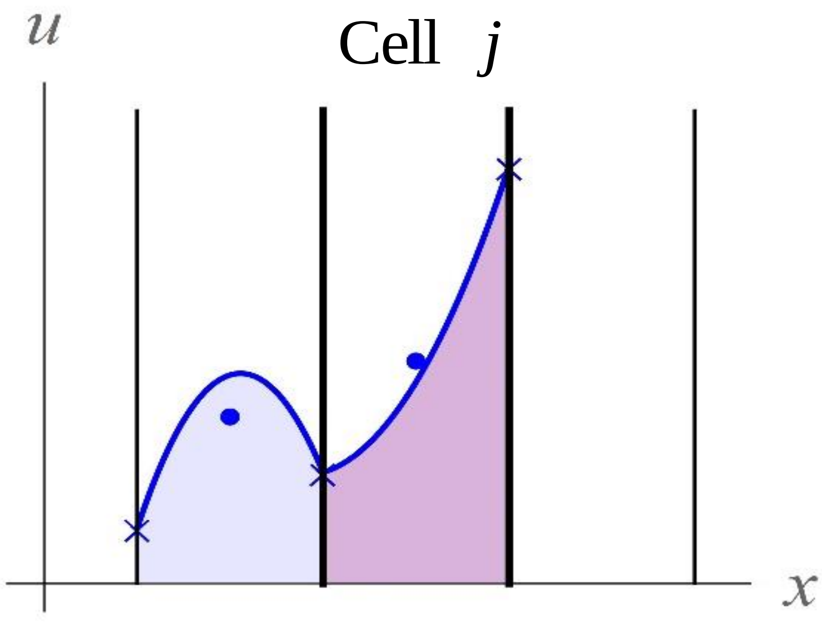
(b) Solution (red curve)

Scheme III can be considered as a precursor of the DG method.

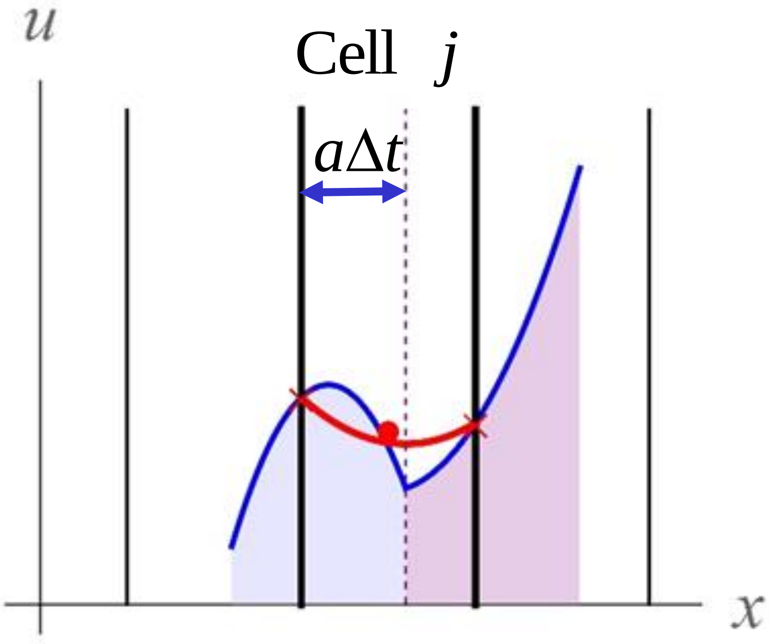
Van Leer's Scheme V

The cell average values $u_{j,0}$ and interface values $u_{j+1/2}$ are stored.

In each cell j, the values $u_{j-1/2}$, $u_{j,0}$, and $u_{j+1/2}$ define a parabola.



(a) Parabolic Data



(b) Parabolic solution (red)

Scheme V

$$\begin{pmatrix} u_{j,0}^{n+1} \\ u_{j+1/2}^{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^2(1-\sigma) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{j-2,0} \\ u_{j-3/2} \end{pmatrix} + \begin{pmatrix} \sigma^2(3-2\sigma) & \sigma(1-\sigma) \\ 0 & \sigma(-2+3\sigma) \end{pmatrix} \begin{pmatrix} u_{j-1,0} \\ u_{j-1/2} \end{pmatrix} + \begin{pmatrix} (1-\sigma)^2(1+2\sigma) & -\sigma(1-\sigma)^2 \\ 6\sigma(1-\sigma) & (1-\sigma)(1-3\sigma) \end{pmatrix} \begin{pmatrix} u_{j,0} \\ u_{j+1/2} \end{pmatrix}$$

$$U_j^{n+1} = C_{-2} U_{j-2} + C_{-1} U_{j-1} + C_0 U_j$$

Equivalence of Schemes III and V

- Assume that the cell average values $u_{j,0}$ and interface values $u_{j+1/2}$ for scheme V are known.
- Also assume that the CFL number σ is fixed.
- For scheme III, set

$$u_{j,1} = (1-\sigma)(u_{j+1/2} - u_{j,0}) + \sigma(u_{j,0} - u_{j-1/2})$$

- Then the cell average solutions at time t^{n+1} of schemes III and V are identical:

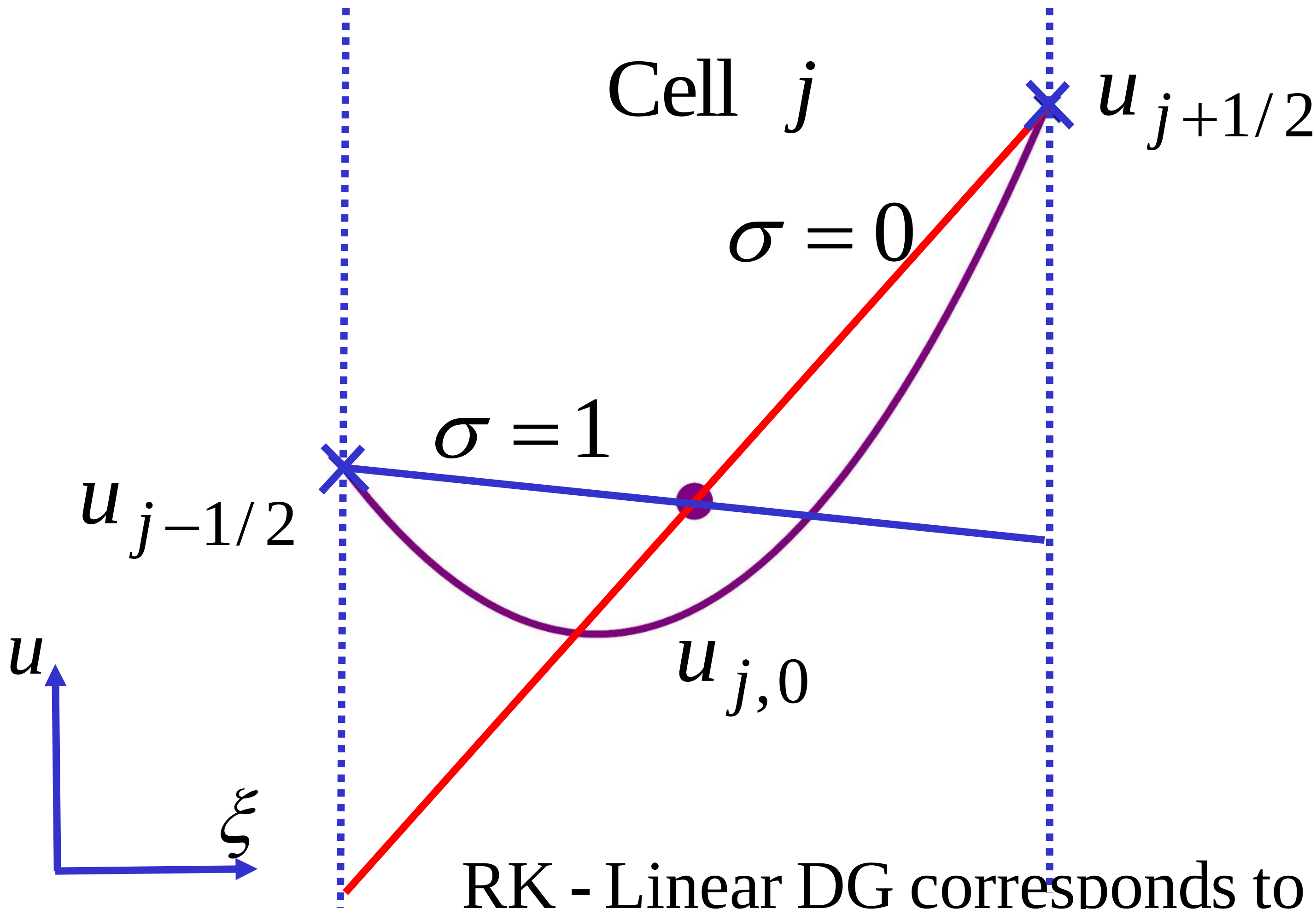
$$u_{j,0}^{\mathrm{III}} = u_{j,0}^{\mathrm{V}} = u_{j,0}$$

- In addition,

$$u_{j,1}^{n+1} = (1-\sigma)(u_{j+1/2}^{n+1} - u_{j,0}^{n+1}) + \sigma(u_{j,0}^{n+1} - u_{j-1/2}^{n+1})$$

Equivalence of Schemes III and V

$$u_{j,1} = (1-\sigma)(u_{j+1/2} - u_{j,0}) + \sigma(u_{j,0} - u_{j-1/2})$$



RK - Linear DG corresponds to red line

Equivalence of Schemes III and V

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* Then the cell average solutions at time t^{n+1} of schemes III and V are identical:

$$u^{\mathrm{III}}_{j,0} = u^{\mathrm{V}}_{j,0} = u_{j,0}.$$

* In addition,

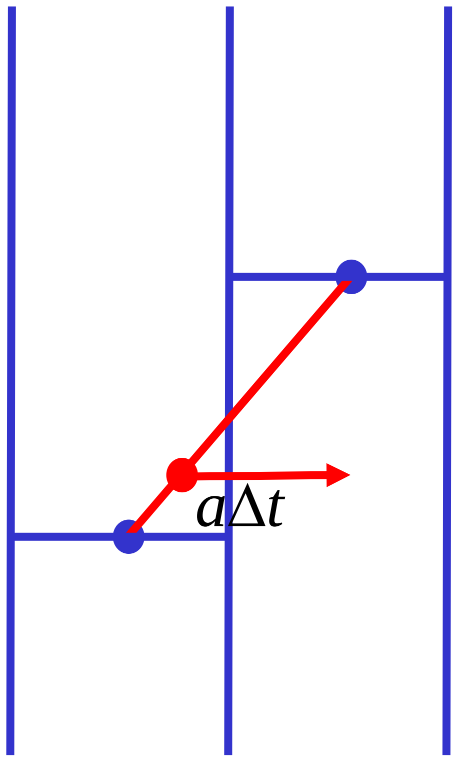
$$u^{n+1}_{j,1} = (1-\sigma)(u^{n+1}_{j+1/2} - u^{n+1}_{j,0}) + \sigma(u^{n+1}_{j,0} - u^{n+1}_{j-1/2}).$$

Projection-Interpolation Schemes

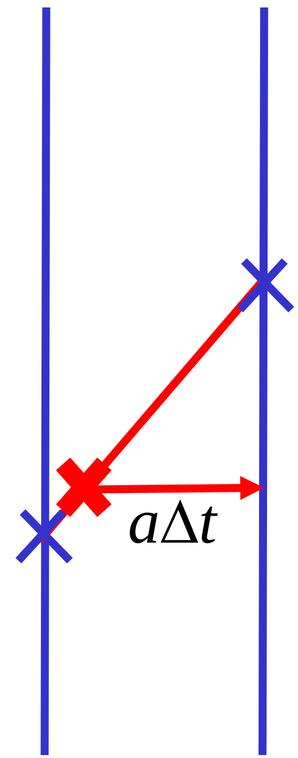
1 piece of data per cell, or $K = 1$

$$u_j^{n+1} = \sigma u_{j-1} + (1-\sigma) u_j$$

Projection P_0



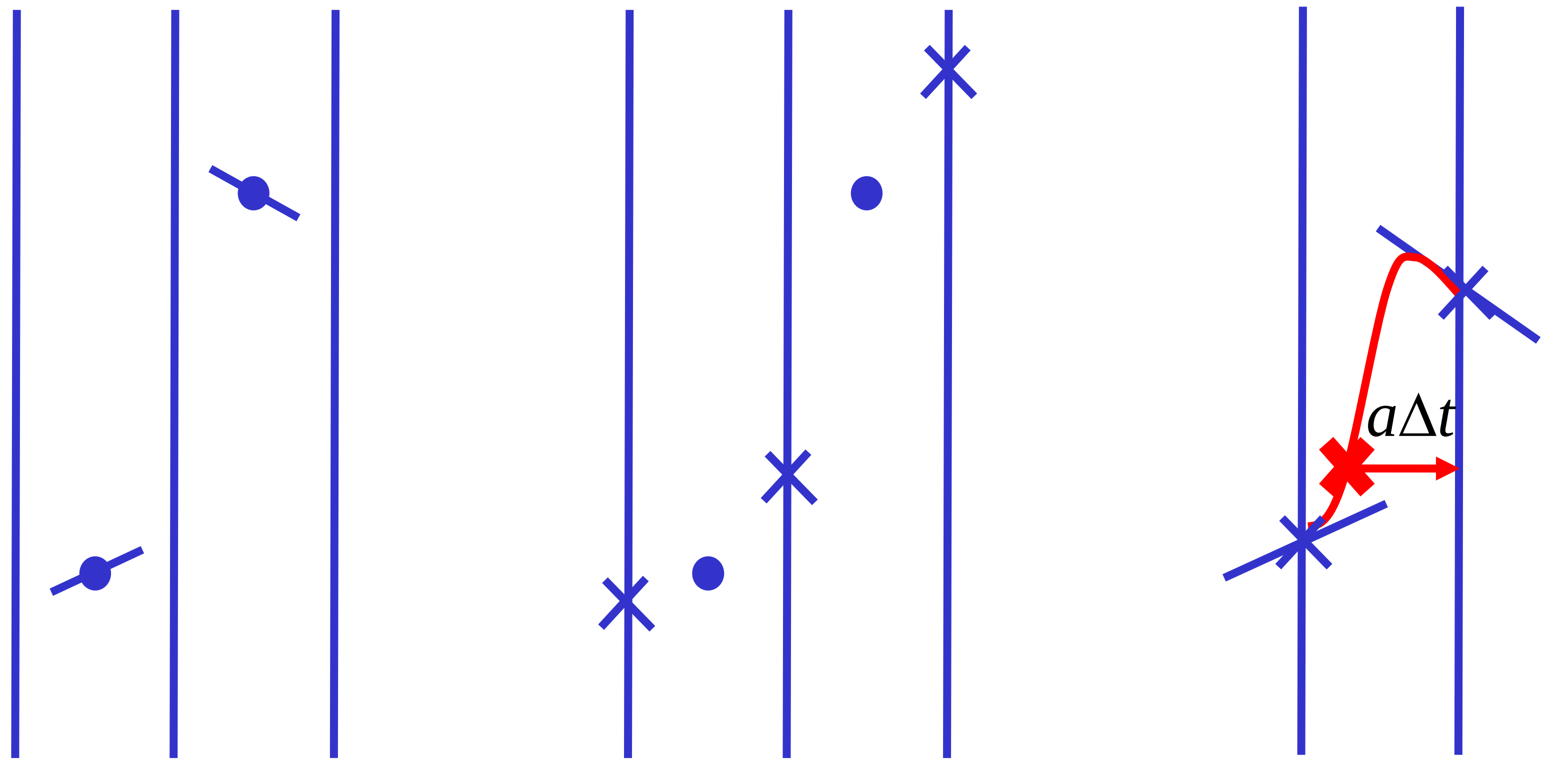
Interpolation I_0



Projection-Interpolation Schemes

2 pieces of data per cell, or $K = 2$

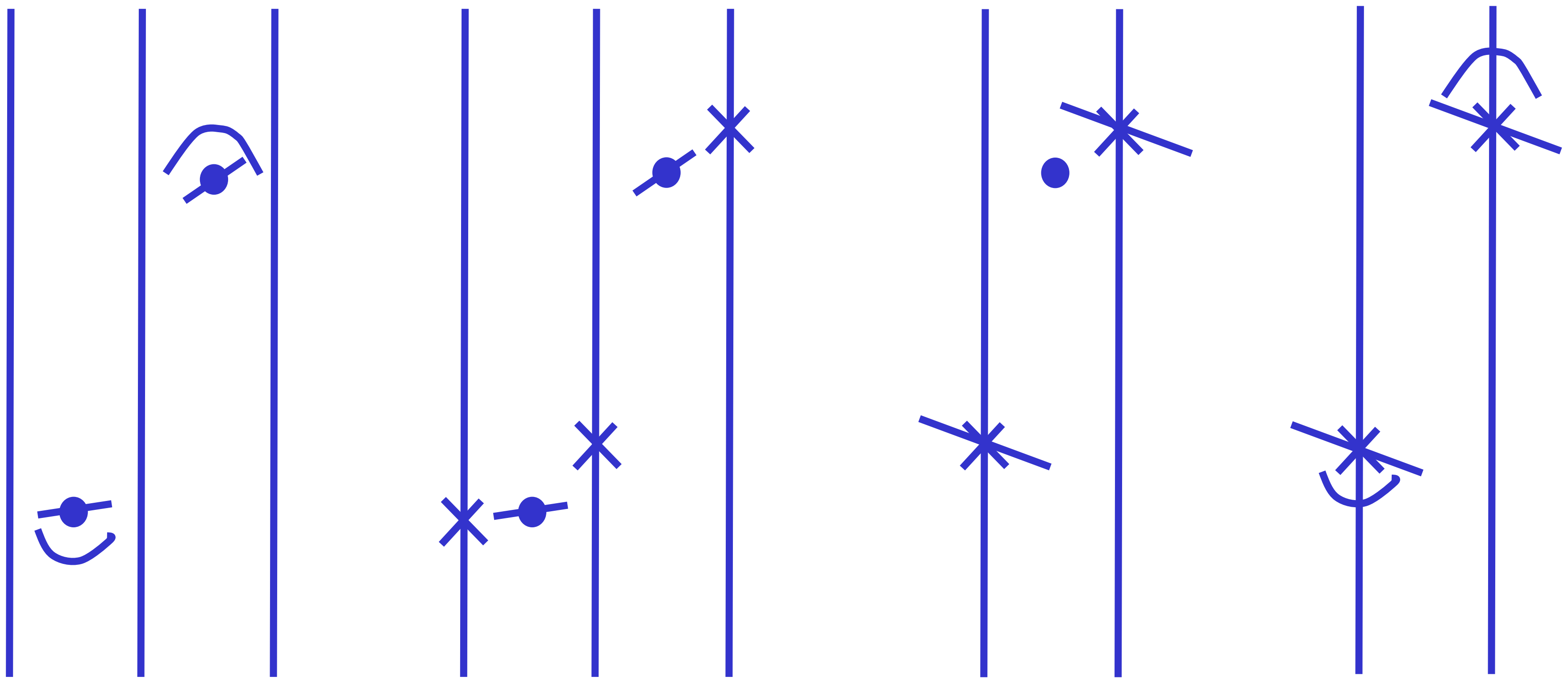
Projection P_1 Scheme III | P_0I_0 Scheme V | I_1 or Cubic Interpolation



Projection-Interpolation Schemes

3 pieces of data per cell, or $K = 3$

P_2 $P_1 I_0$ $P_0 I_1$ I_2



Family of $\mathrm{P}\mu\mathrm{I}\nu$ Schemes

* Projection part: Projecting the data onto P_μ results in $\mu+1$ quantities

$$u_{j,k}^n = u_{j,k}, \quad k = 0, \ldots, \mu$$

where

$$u_{j,k} = \frac{2k+1}{2}(u, L_k).$$

* Interpolation part: At each interface $x_{j+1/2}$, the interpolation part consists of approximations to $\frac{d^l u}{d\xi^l}$ to degree ν, i.e., the $\nu+1$ quantities

$$u_{j+1/2,l}^n = u_{j+1/2,l}, \quad l = 0, \ldots, \nu$$

Family of $\text{P}\mu\text{I}\nu$ Schemes

* For each cell, we have $\mu+1$ projection quantities in the cell and $2(\nu+1)$ interpolation quantities at the two interfaces.
* These $2\nu+\mu+3$ quantities for each cell define a polynomial of degree $2\nu+\mu+2$.
* Shift the polynomial data a distance $a\Delta t$.
* Update the projection part and the interpolation part.

Fourier Stability and Accuracy Analysis

Define U_j with $K = \mu + \nu + 2$ components by

$$U_j = \left(u_{j,0}, ..., u_{j,\mu}, u_{j+1/2,0} ..., u_{j+1/2,\nu}\right).$$

The solution can be written as

$$U_j^{n+1} = C_{-2} U_{j-2} + C_{-1} U_{j-1} + C_0 U_j .$$

Assume the data is a harmonic that satisfies

$$U_{j-1} = e^{-iw} U_j .$$

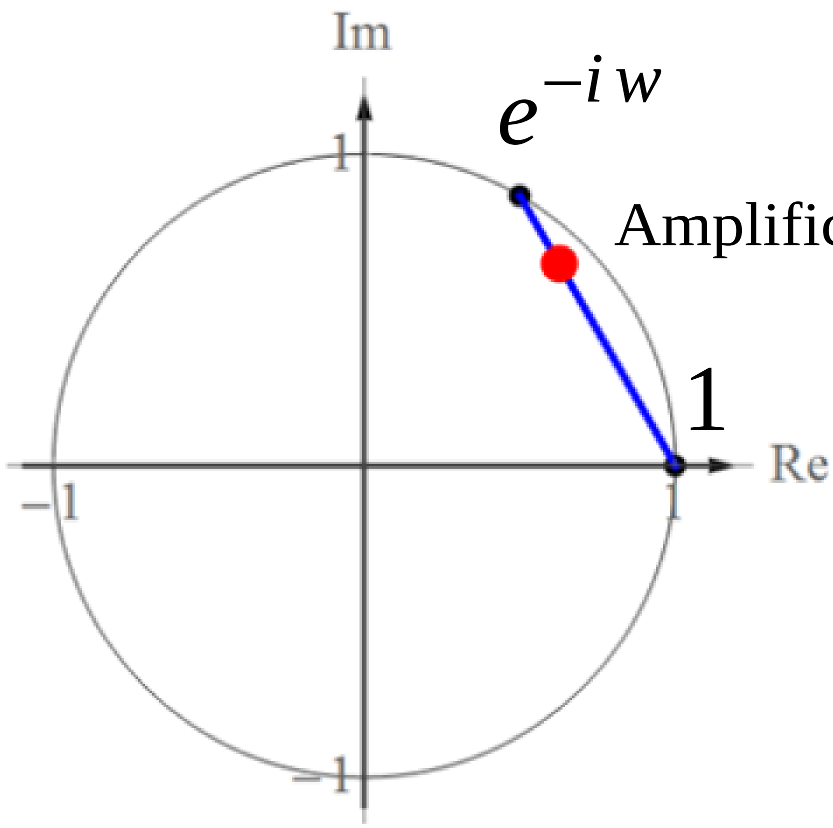
Then,

$$U_j^{n+1} = \left[e^{-2iw} C_{-2} + e^{-iw} C_{-1} + C_0\right] U_j .$$

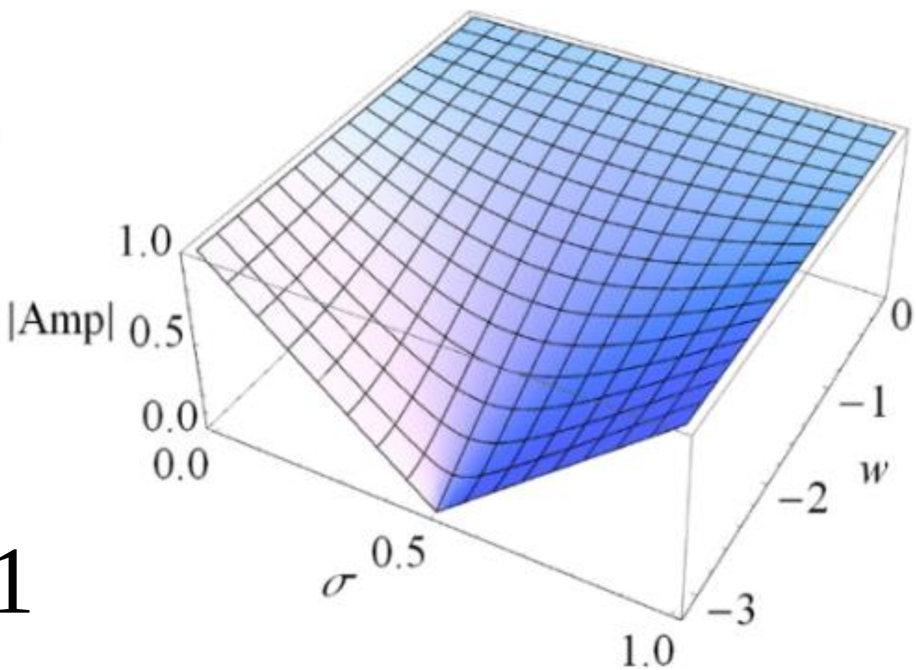
For the first-order upwind scheme,

$$U_j^{n+1} = \left[e^{-iw}\sigma + (1-\sigma)\right] U_j .$$

Fourier Analysis: Plots of Absolute Values of Eigenvalues

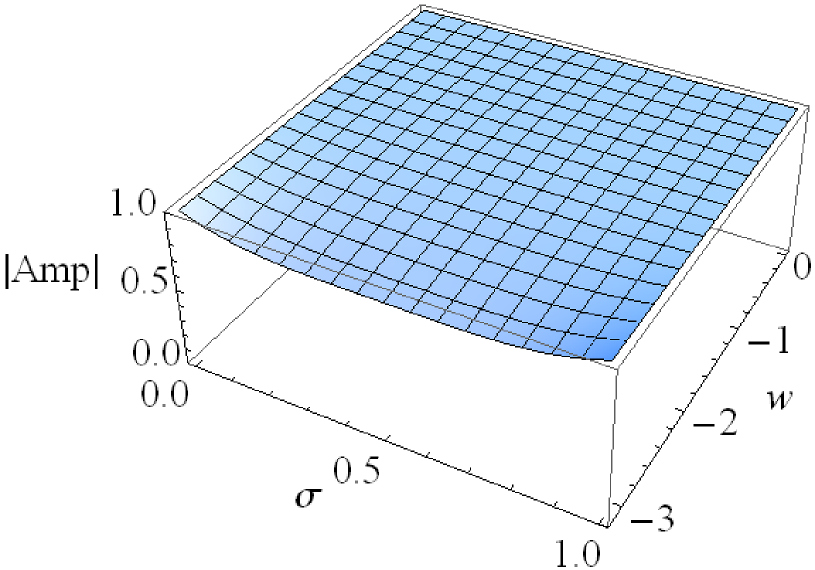


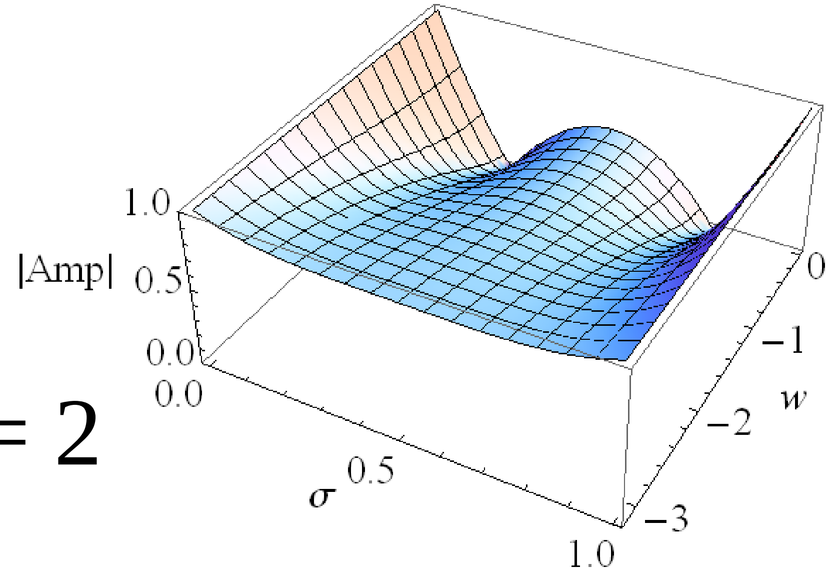
Amplification Factor $e^{-iw}\sigma + (1-\sigma)$



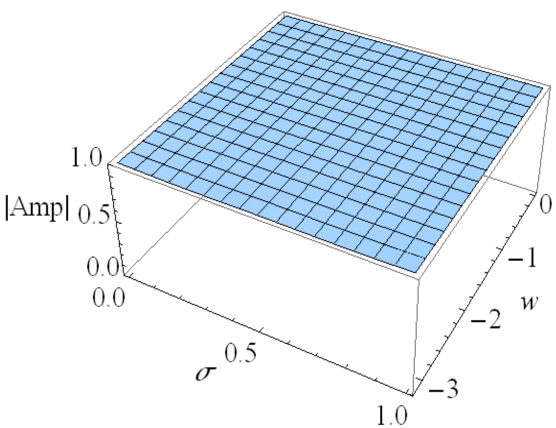
$$K = 1$$

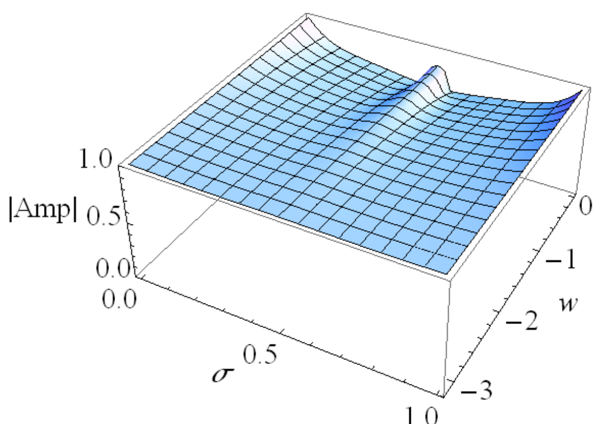
Fourier Analysis: Plots of Absolute Values of Eigenvalues

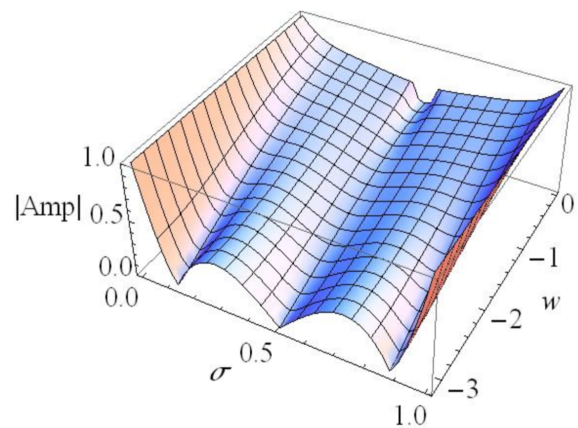




$K = 2$

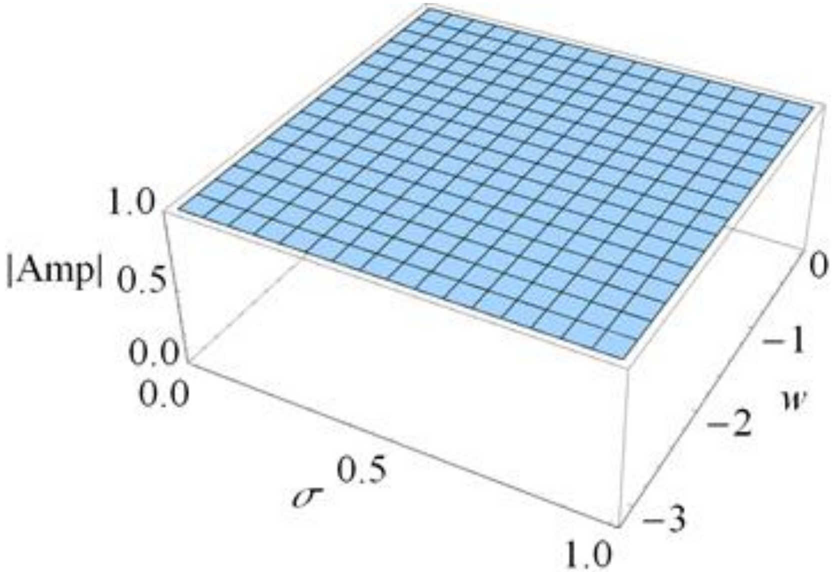


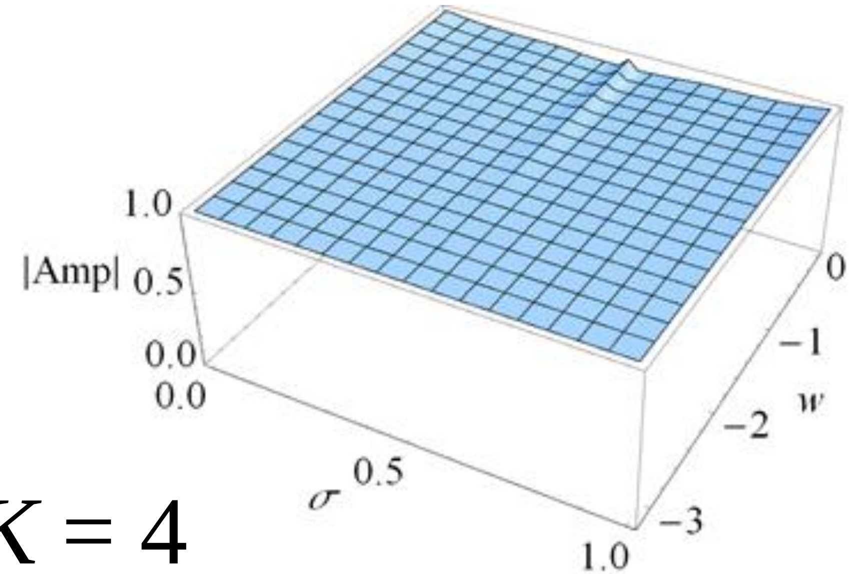




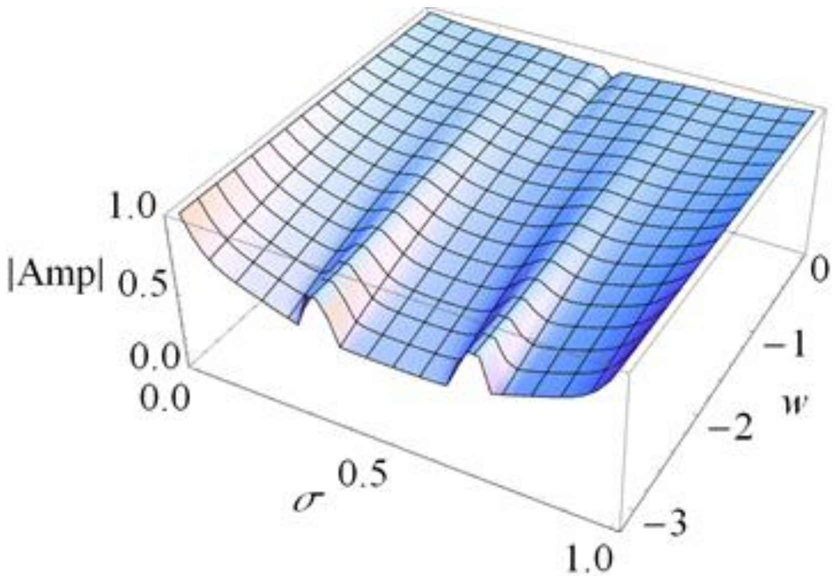
$K = 3$

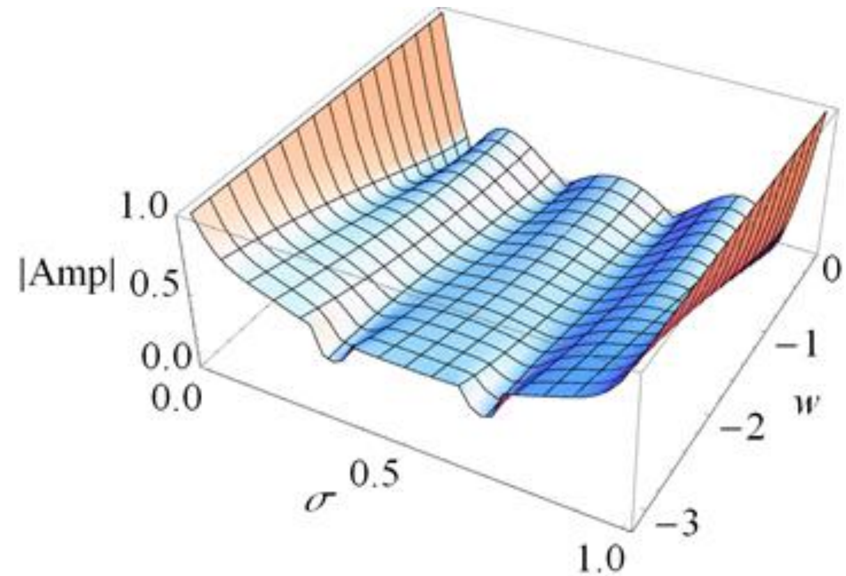
Fourier Analysis: Plots of Absolute Values of Eigenvalues





$K = 4$





Projection-Interpolation Schemes

- Derived for advection equation
- With K degrees of freedom per cell, the scheme is accurate to order $2K - 1$ (i.e., it is super accurate or super convergent)
- CFL condition is 1 as opposed to $\sim 1/K^2$ for explicit RK-DG.
- Extension to systems of equations in multiple dimensions remains an open problem.

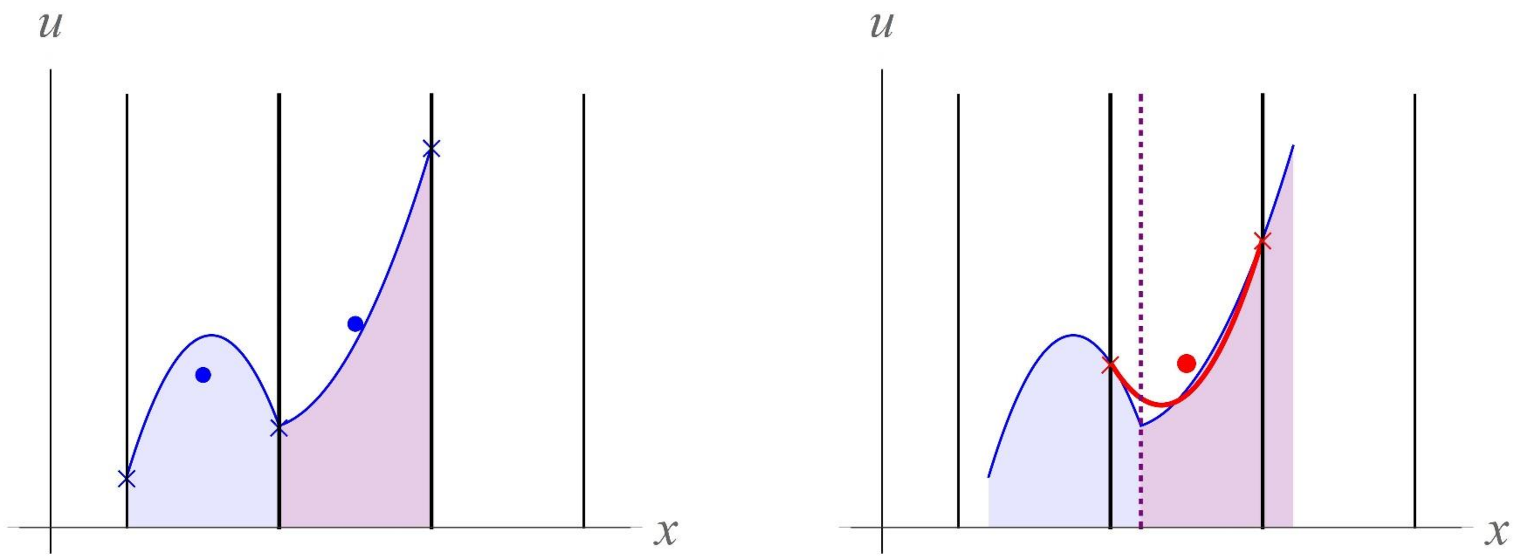
Conclusions

- Reviewed MUSCL approach for the advection equation.
- Presented a key new result: the equivalence of schemes III and V.
- The above result shows a key connection between continuous and discontinuous approaches.
- Introduced a projection-interpolation framework that simultaneously extends these schemes into a single family of high-order methods.
- Discussed Von Neumann (or Fourier) stability and accuracy analysis.

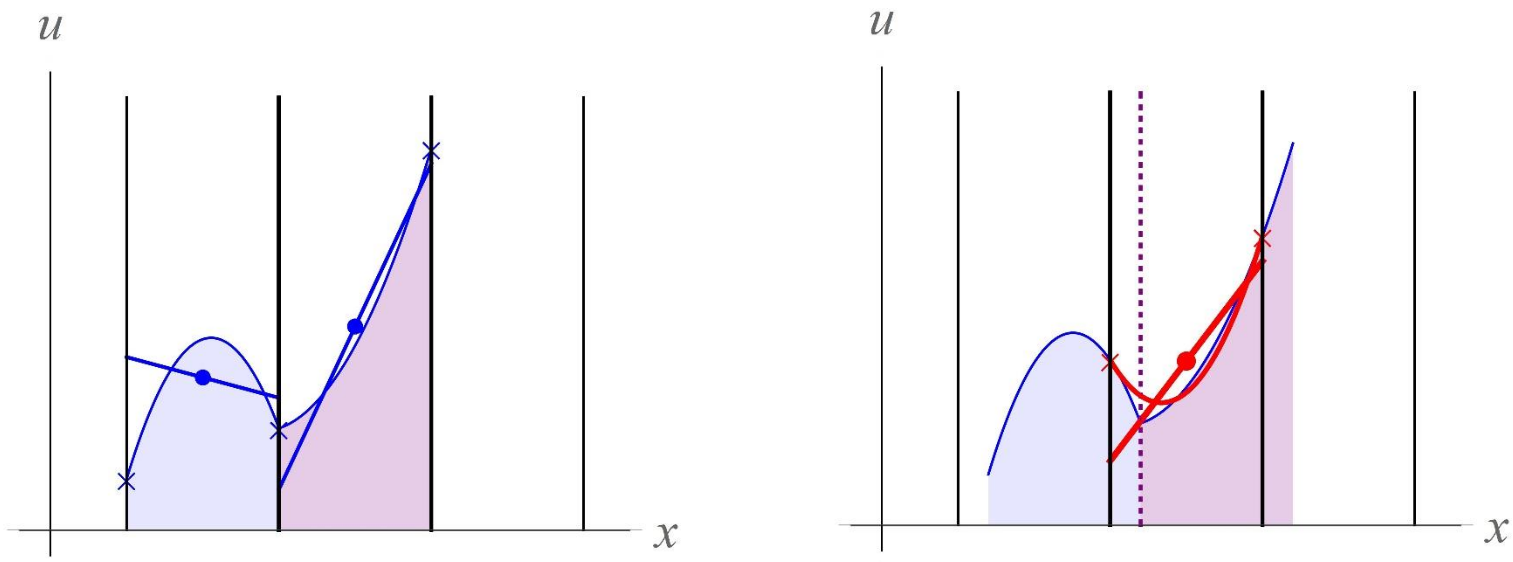
• Further research on these methods is needed.

Thank you
for your attention.

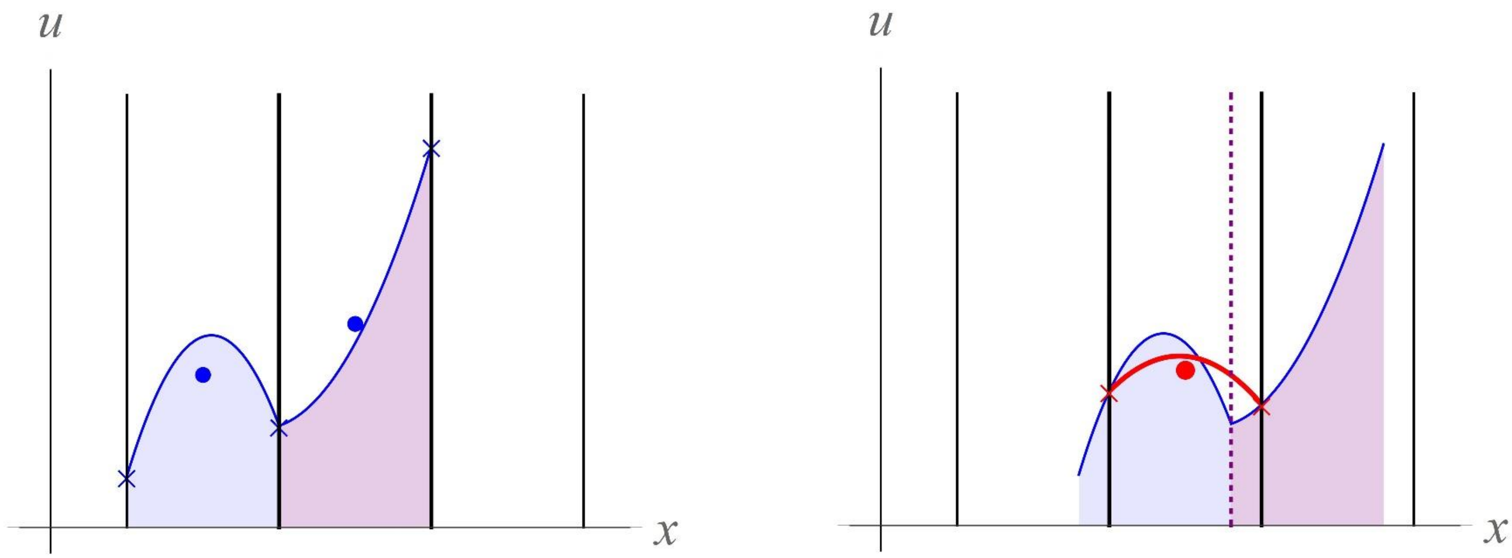
Examples for the Equivalence, $\sigma = 0.2$



$$u_{j,1} = (1-\sigma)(u_{j+1/2} - u_{j,0}) + \sigma(u_{j,0} - u_{j-1/2}) \text{ is close to } (u_{j+1/2} - u_{j,0})$$



Examples for the Equivalence, $\sigma = 0.8$



$$u_{j,1} = (1-\sigma)(u_{j+1/2} - u_{j,0}) + \sigma(u_{j,0} - u_{j-1/2}) \text{ is close to } (u_{j+1/2} - u_{j,0})$$