Uncertainty Estimation Cheat Sheet for Probabilistic Risk Assessment

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Introduction

When uncertainty estimates are expected to inform decision-makers, it is especially important to carefully consider, understand, and communicate the significance of the statistical parameters used in the characterization of failure probability distributions.

Uncertainty analysis aims to make a technical contribution to decision-making through the quantification of uncertainties in the relevant variables as well as through the propagation of these uncertainties up to the result.

We will illustrate key principles as we step through the quantification of uncertainty.

Finally, the risk implications of uncertainty estimation are summarized in a convenient reference card: Uncertainty Estimation Cheat Sheet.

Probability Distributions

- Informally, a **probability distribution** is a mathematical function that assigns probabilities to each element of the **sample space** (the set of all possible outcomes in an experiment).

- A **random variable** is a function that maps outcomes of an experiment to numerical quantities.

- For a continuous distribution, the **probability density function (pdf)** is the function that is used to generate the probability that a random variable ***X*** lies within an interval [*a*, *b*]:

$$\Pr[a \le \boldsymbol{X} \le b] = \int_a^b f(x)\,dx$$

Probability Distributions

- The probability density of the **exponential distribution** is:

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

- The pdf of the **normal (or Gaussian) distribution** is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- The pdf of the **lognormal distribution** parameterized with the mean (μ) and standard deviation (σ) of the underlying normal distribution is given as:

$$f(\lambda) = \frac{1}{\lambda\sigma\sqrt{2\pi}} \exp\left(-\frac{[\ln(\lambda) - \mu]^2}{2\sigma^2}\right), (0 < \lambda < \infty)$$

Central Tendency

- For a continuous distributions, the arithmetic **mean** is:

$$E[\boldsymbol{X}] = \int xf(x)\,dx,$$

where the weighting function $f(x)$ is the pdf of $\boldsymbol{X}$.

- The **median** or 50th percentile is the midpoint where half of the probability (area under the pdf) lies to either side.

$$\int_{-\infty}^{median} f(x)\,dx = \int_{median}^{\infty} f(x)\,dx = \frac{1}{2}$$

- The **mode** is a local maximum or peak of the pdf.

Dispersion

- The **variance** is the expected value of the squared deviations about the mean:

$$Var[\boldsymbol{X}] = E[(\boldsymbol{X} - E[\boldsymbol{X}])^2]$$

- The square root of the variance is the **standard deviation**:

$$\sigma = \sqrt{variance}$$

- The **error factor (EF)** is defined as the square root of the 95th percentile divided by the 5th percentile. Equivalently, the EF is equal to the 50th divided by the 5th and the 95th divided by the 50th as summarized in the following equivalence:

$$EF = \sqrt{\frac{95^{th}\ percentile}{5^{th}\ percentile}} = \frac{95^{th}\ Percentile}{50^{th}\ Percentile} = \frac{50^{th}\ Percentile}{5^{th}\ Percentile}$$

Failure Rate Uncertainty

- Component failure rates (λ) are not physical quantities; that is, they cannot be measured directly but must be inferred.
- Previous research evaluated different distributions to represent the uncertainty of the parameter λ [1]. They found the lognormal distribution was appropriate for simple components with a single failure mode.
- Uncertainty has many sources in addition to variation among individuals within a population and lack knowledge due to sparse data. However, this paper examines the implications of applying uncertainty around central tendency estimates in order to quantify degree of belief – in particular when expressing degree of belief via the shape of the lognormal pdf.

The Bayesian Approach

- Application of classical life data analysis requires component data in the form of failures and exposure time or number of demands.
- Highly reliable components produced in small quantities, such as in space applications, do not have enough operating time and failure history to yield useful confidence bounds using classical statistical data analysis
- Bayesian approach is able to address the challenges described above because it admits prior experience into the estimation procedure in the form of a prior degree of belief about the likely values of the component in the form of a prior distribution.
- In our experience, engineers with specific discipline expertise are generally familiar with the normal probability distribution, but have little direct experience with skewed distributions, such as the lognormal.
- Subject matter experts who often assist PRA analysts in the quantification of the prior failure rate distribution must be educated to develop an intuitive understanding of how the lognormal distribution morphs as its centrality and dispersion measures are varied.

The Bayesian Approach

One of the main purposes of this paper is to illustrate with specific examples the effects of varying one of the parameters, such as the dispersion while holding another fixed to show the effect on the remaining parameters.

Specifying any two parameter values completely specifies the lognormal distribution. Thus we can solve for μ and σ and then fill in the remaining parameter values in the table using the formulas.

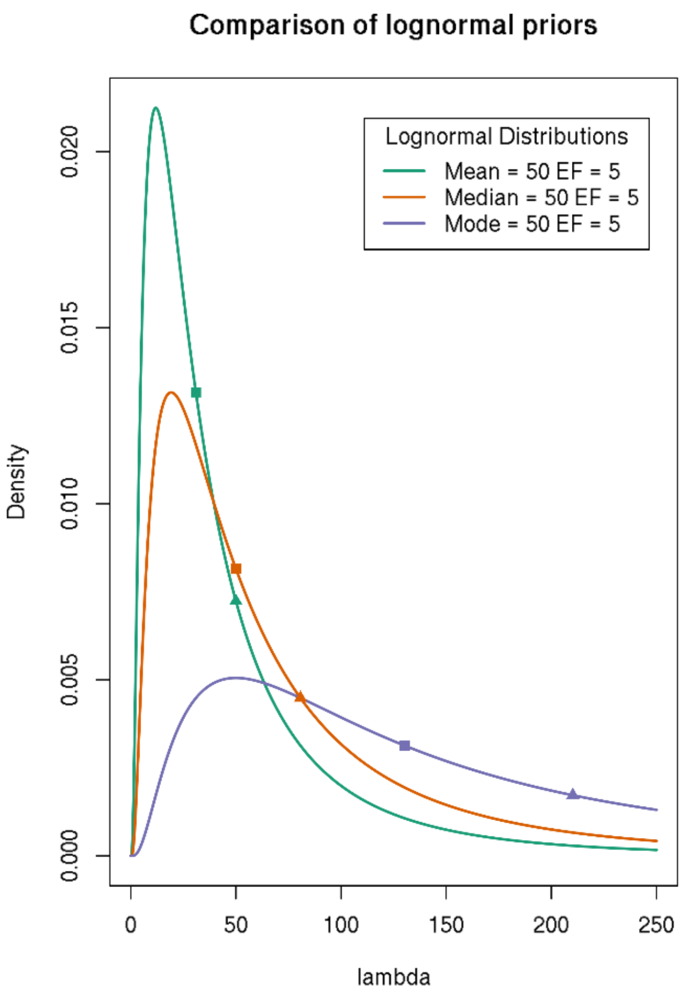
Parameter	**As a function of μ and σ**
Mean	$\exp\left(\mu + \frac{\sigma^2}{2}\right)$
Median	$\exp(\mu)$
Mode	$\exp(\mu - \sigma^2)$
Standard Deviation	$\sqrt{[\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2)}$
Error Factor	$\exp(1.64485\sigma)$

Uncertainty Estimation Examples

- By fixing each of the mode, mean and median, while varying the error factor, we demonstrate the effect on the other measures of centrality as well as the risk implications to results via uncertainty propagation.

Uncertainty Estimation Example 1

- We begin with an engineering judgment prior example. What is given is a single central value for lambda and an estimate for the error factor.



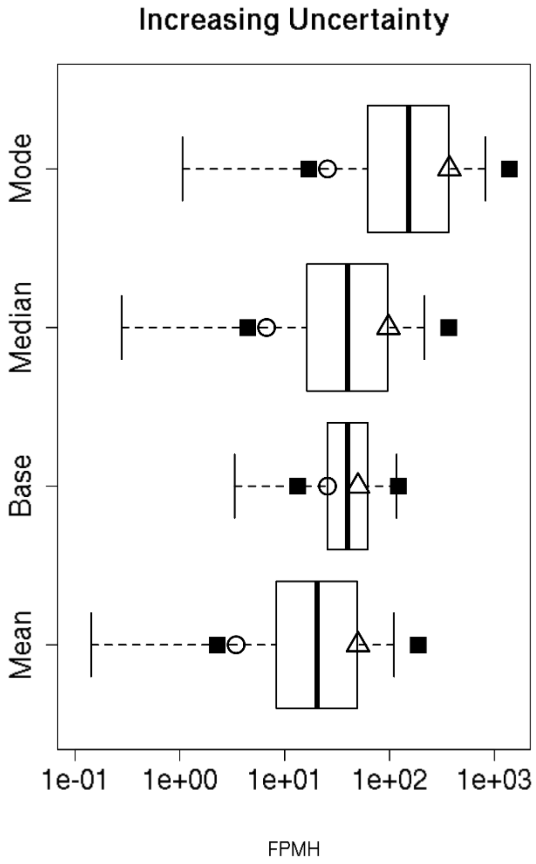
Uncertainty Estimation Example 2

- Next we will examine the development of a prior by similarity. This begins with a given distribution: lognormal(mean = 50, EF = 3) as our base case. We calculate the median and mode from the base case, then fix each of the mean, mode and median while increasing the error factor from 3 to 9.

Listed below are the features of the boxplots in the following slide

- The box represents the IQR.
- Thick bar through the middle of the box is the Median.
- Respectively, the low and upper whiskers are located at Q1 – 1.5 * IRQ and Q3 + 1.5 * IRQ.
- Solid square symbols are the 5th and 95th percentiles.
- Circle is the mode.
- Triangle is the mean.

Uncertainty Estimation Example 2

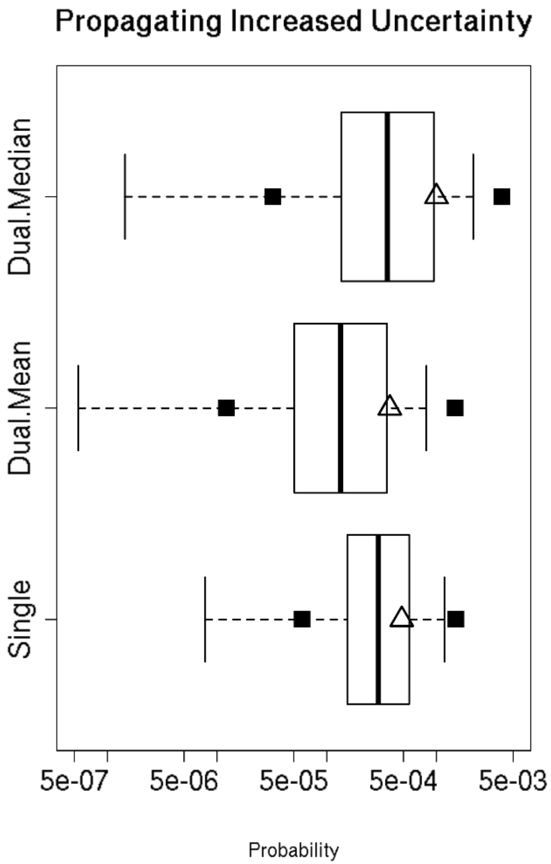


Several observations are apparent:

1. The mean value moves from the 63rd percentile to the 75th percentile as the error factor increases from 3 to 9.
2. The mode moves from the 25th percentile to the 9th percentile as the error factor is increased from 3 to 9.
3. The 95th percentile moves above the upper whisker as the error factor moves from 3 to 9.
4. Fixing the mean lowers the 75th percentile and the median while stretching the 4th quartile to the left and right.
5. Fixing the median lowers the 25th percentile, the mode and the 5th percentile. The mean, the 75th percentile and the 95th percentile are increased, as well.

Fixing the mode increases the percentiles from the 5th on up as well as increasing the mean.

Uncertainty Estimation Example 3



In this example we compare a highly reliable heritage zero failure tolerant design with a retrofitted redundant option is not only susceptible to common cause failaure modes where each leg of redundancy is considered less reliable than the heritage design.

These results demonstrate that the effects on risk-informed decisions by the mere choice of the central parameter about which uncertainty is estimated can, in fact, be pivotal.

Uncertainty Estimation Cheat Sheet

Lognormal Uncertainty Estimation Cheat Sheet

Uncertainty Increased	
Fixed	**Risk**
Mean	Lower
Median	Neutral
Mode	Higher

Uncertainty Decreased	
Fixed	**Risk**
Mean	Higher
Median	Neutral
Mode	Lower

The purpose of the cheat sheet is to reinforce an understanding of the cause and effect relationships between the adjusting of parameters (that measure centrality and dispersion) and their risk implications throughout the churning of the Bayesian approach.

Conclusion

- Although many cases are presented, the typical case for aerospace PRA is to increase uncertainty while fixing a central value.
- It is often the case and without strong rationale, the parameter chosen to be fixed is the mean.
- Theoretical distributions do not always behave intuitively. Care must be taken when adjusting the parameters of a distribution as part of a heuristic or other method.
- One ought to understand the relationships and effects of all relevant parameters as well as the risk implications.
- It is our hope that the Uncertainty Estimation Cheat Sheet (for Lognormal Uncertainty) will help those involved in the PRA process (such as managers, subject matter experts and PRA analysts) make effective technical contributions to decision-making.

Questions?

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