Pc Uncertainty Update

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Probability of Collision Calculation

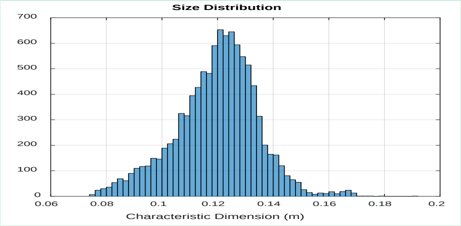
- **Pc is only a nominal solution for the conjunction**
  - Derived from estimates of the mean
    - If error distributions non-Gaussian, then this is not an expression of central tendency
  - Does not include uncertainties on the inputs
    - "Uncertainty of uncertainty volumes" or HBR
- **Thus, while representing the risk, nominal Pc is just a point estimate**
- **Want to know how much variation or uncertainty in the Pc calculated for any given conjunction**
  - Determine uncertainty PDFs for the Pc calculation inputs
  - Through Monte Carlo trials, vary above inputs to the Pc calculation
  - Generate a probability density of resultant Pc values
  - Characterize this distribution empirically

Potential Sources of Pc Uncertainty

- **Sources of uncertainty and potential modeling approaches**
  - Attempted in earlier construction of Pc uncertainty tool
  - Present examination a reprise of this approach

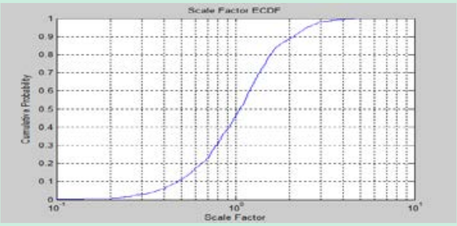
Underlying Uncertainties

HBR Uncertainty



Draw from projected area distribution (primary) and RCS PDF (secondary)

Covariance Uncertainty

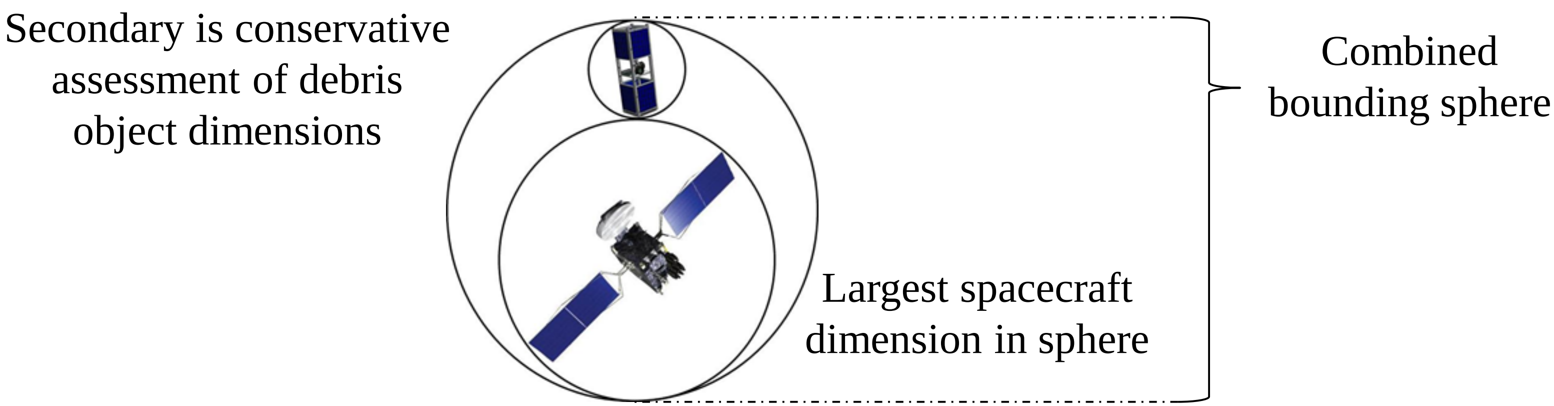


Draw from scale factor distributions for both objects

Generate P_c distribution

Hard-Body Radius

- **HBR typically established by circumscribing both objects in spheres and combining the objects into one bounding sphere**
  - Size of the secondary is typically not known, so added as a large estimate of debris object dimensions

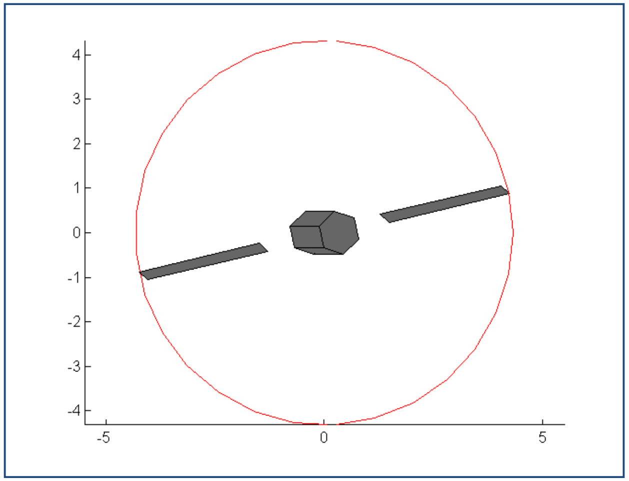


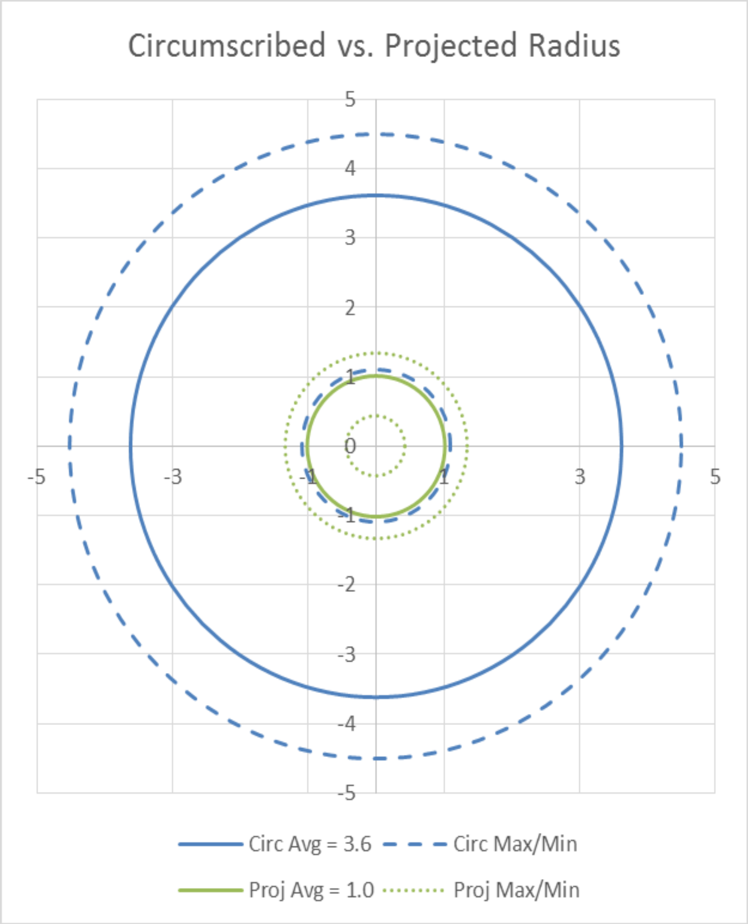
- **In most conjunctions, primary HBR dominates**
- **Two approaches for more realistic HBR values for primary**
  - Use CAD model to generate PDF of HBR from random orientation of primary
  - Use CAD model and conjunction geometry to determine spacecraft projected area into conjunction plane

Primary Object HBR Uncertainty: PDF of More Realistic Values

- **Uncertainty estimated by the projected area of the spacecraft in a random orientation on the conjunction plane**
  - Simplified geometric model of the spacecraft
  - Save the projection areas to a PDF
    - Projected area expressed as a circular radius
  - Large variation in both approaches

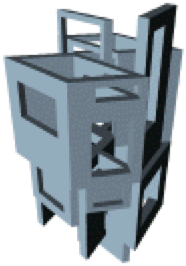
Geometric model of OCO-2 in arbitrary orientation on conjunction plane



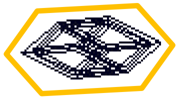


Primary Object HBR Improvement: Projection into Conjunction Plane

- **If steering law of primary known, then precise projection of satellite area into conjunction plane can be determined**
  - Satellite aspect function and vector to conjunction plane both known
  - Complex shape reduced to particular area region (here black set of hexagons), with rather little uncertainty



- **"Combined" area can be determined by adding estimated radius of secondary to all the exterior dimensions**
  - Amber line about projected area

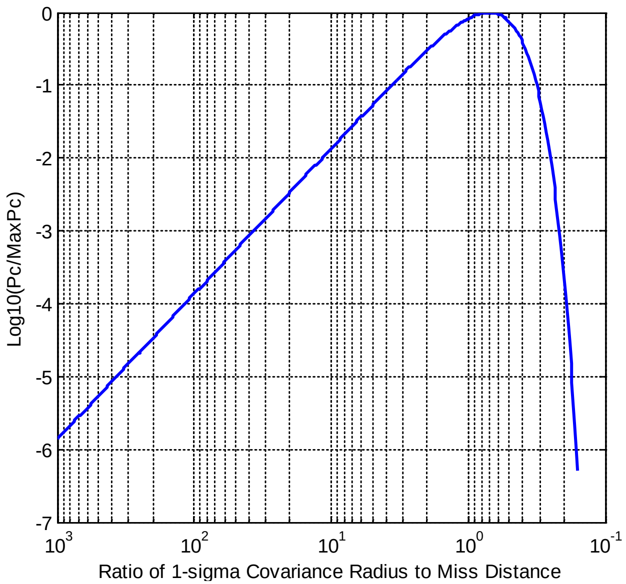


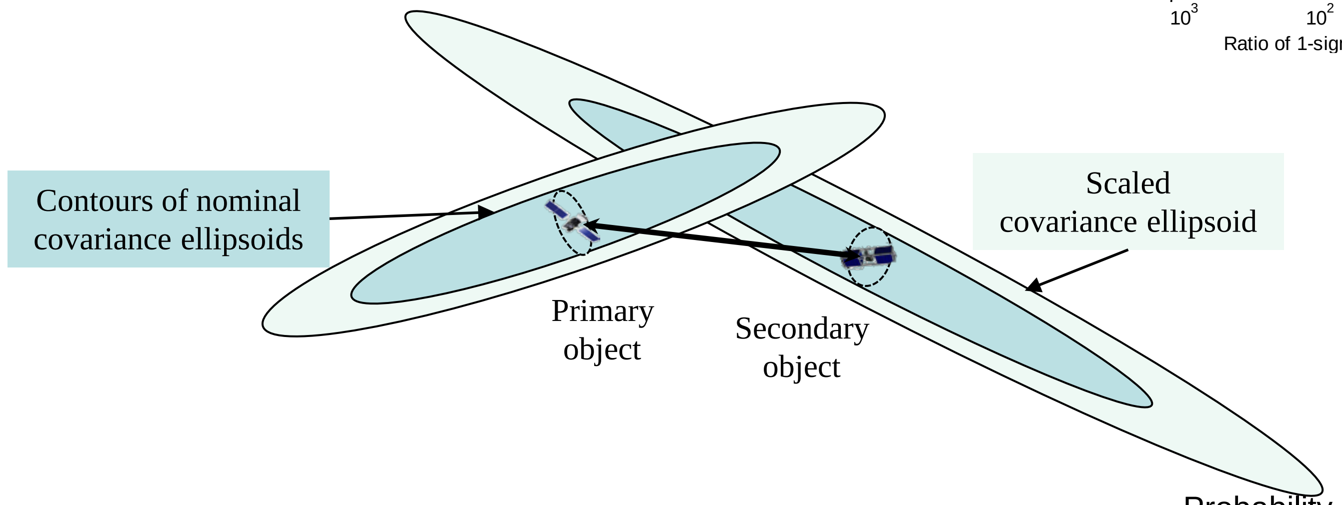
- **This area then used as HBR area**
  - Can transform to equivalent circular area (easier approach)
  - Can perform contour integral along area boundary (more difficult but more accurate)
  - Preferred approach can be determined by sensitivity analysis
- **Projection approach is recommended method**
  - Dr. Alinda Mashiku presently performing enabling analysis

Covariance Uncertainty

- **Changes in covariance sizes can change calculated Pc, sometimes substantially**
  - Especially if on right side of canonical curve
- **Need to know range of values for appropriate scale factors for covariances**
  - Typical applied range is from 0.2 to 5, but this is unrealistically large for nearly all cases
  - Should be object-specific
  - Should include probabilistic element

Relative Pc vs Covariance "Size"



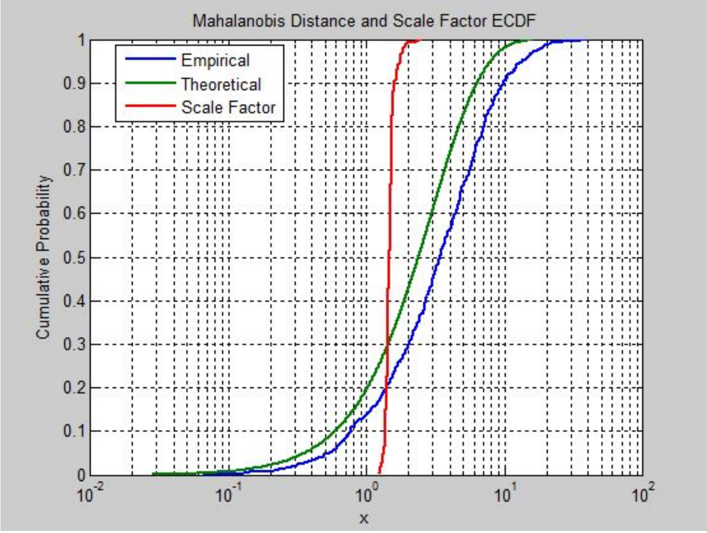


Covariance Uncertainty: Evaluation Products

- **JSpOC-resident utility generates reference orbits for every satellite**
  - Similar methodology as that used for SLR precision ephemerides
  - Covariance data from generating ODs preserved
- **Second utility compares each generated SP vector to reference orbit at propagation points of interest**
  - 1, 2, 3, 5, and 7 days from epoch
  - Calculates position residuals and combined covariance, which is combination of propagated vector covariance and reference orbit covariance
- **With position residuals and combined covariance, can compute covariance "realism" factor for each vector at each prop point**
  - For each vector, can calculate $\varepsilon C^{-1}\varepsilon^{T}$ (M^2, square of Mahalanobis distance)
    - ε is the vector of position residuals; C is the combined covariance
  - If covariance realistic, M^2 set should produce a 3-DoF chi-squared distribution

Mahalanobis Distances to Scale Factors: "Percentile Matching" Approach

- **Presume set of 100 M2 factors generated for a satellite**
  - Rank-order the 100 factors
  - Align each with the 3-DoF chi-squared value for that given percentile
    - E.g., factor #20 aligned with 20th percentile chi-squared value
  - Empirical / ideal value is scale factor for each instance
    - Value by which covariance would need to be multiplied to produce ideal chi-squared value for that percentile point
  - Set of 100 scale factors now available for Monte Carlo draws
- **Sets of these calculated for every satellite for propagation points of interest**



Percentile Matching: Issues

- **Percentile matching approach criticized by CNES researchers**
  - Tends to be variance-low, probably due to the forced alignment of order statistics
  - Probably a legitimate criticism
- **CNES alternative approach: presume M^2 distribution a mixture of *k* single-factor-scaled chi-square distributions**
  - E.g., M^2 overall distribution mixture of
    - 60% scaled by 1.3
    - 30% scaled by 2.0
    - 10% scaled by 0.9
- **Claim is that this approach more likely to align with causes of non-theoretical behavior**

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Covariance matrix uncertainty analysis and correction

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Abstract

This paper studies the possibility to increase the realism of the state estimate covariances used for the computation of the probability of collision via the application of scaling factors. The determination of these factors is done through a statistical analysis of past collision risk alerts. The model proposed here allows to characterize a set of alerts using only a few scaling factors and their associated weights. Beyond the presentation of the method, a first validation on known entries is presented, followed by an application of the method on real collision risk alerts.

Keywords: Collision risks, Covariance realism, Mahalanobis distance, Probability of collision.

1. Introduction

Since the first orbital launch in 1957, the number of artificial objects in Earth orbit has been steadily increasing. This has led to a corresponding increase in the number of threats to active satellites from hyper-velocity collision, putting in jeopardy crucial services that benefit human society. To mitigate such risks, one of the approaches commonly adopted consists in the planning and the execution of collision avoidance maneuvers for operational satellites. This approach relies on the analysis of the probability of collision (P_c) between the satellite of interest and any space object entering a safety volume around it. A realistic estimation of this probability of collision is thus primordial for the management of the spacecrafts population. An under-estimated P_c could lead to serious risks being ignored, while an over-estimated P_c could trigger unnecessary maneuvers, leading to mission perturbations and reducing of the operational lifespan of the satellite.

Although the computation of the P_c itself have been extensively studied [1, 2, 3, 4, 5, 6], it relies on inputs generated by an orbit determination (OD) process and propagated to the estimated date of the encounter. They are thus subject to approximation errors and may possibly misrepresent the actual states of the objects. The resulting uncertainty on the input parameters should thus be modeled and taken into account for the computation of the P_c. Since multiple factors can contribute to this uncertainty (invalid observations, modeling and propagation of the uncertainty, ...), this can be a complex task. Its scope can however be reduced when considering the problem from an operational point of view.

When an alert for a collision risk is received (as a Conjunction Data Message or CDM [7]), it contains the mean state vectors of the primary and the secondary and their associated uncertainty, all defined at the time of closest approach (TCA). Each state vector uncertainty is supposed to follow a normal distribution and is defined by a covariance matrix in the local orbital frame (LOF) of the corresponding object. Assuming the mean and shape of these distributions are correct, the uncertainty on the covariance matrices resides mainly on their scale. Applying the correct scaling factors to them should thus improve their realism. The difficulty resides of course in determining these factors. Currently, the CNES (Centre National d'Études Spatiales) computes its operational probability of collision by scanning a range of scaling factors (k_p for the primary, k_s for the secondary), giving access to the variation of the P_c over the domain considered and highlighting the maximal probability found [8]. The k_p/k_s domain is however chosen arbitrarily (albeit using empirical knowledge from operations) and the method presumes an equal likelihood between all the factors considered. This paper presents the efforts done at the CNES to refine the definition of these correction factors by analyzing past conjunctions between satellites and space objects.

The idea behind this study is to define a population of objects with common characteristics and to compare the orbits received from past collision alerts to their orbit of reference. This comparison is done through the Mahalanobis distance [9], which can be described as the distance between a point and the mean of a distribution in terms of standard deviation. This metric is dimensionless, invariant by change of scale and is known to follow a χ^2 distribution when the underlying distribution is Gaussian. Because of these properties, it is often used to assess covariance realism (i.e. uncertainty realism for normal distribution). This is usually done by sampling a number of random particle states from the initial distribution and propagating them to the epoch of interest. These propa-

CNES Alternative: CARA Evaluation

- **Arbitrariness in number of mixtures and percent constitution**
  - Uniqueness of particular solutions an open question
    - Solutions with, say, 2, 4, and 25 mixtures give similar GOF results; what principle other than Ockham's Razor to choose among them?
    - More than just an abstract question if mixtures believed to be linked to external causes
- **Can linkages of mixtures to external causes be demonstrated?**
- **Difficult to determine upper and lower bounds for covariance sizes**
  - If, say, only two mixtures with two scale factors, difficult to model upper and lower tails decisively

Covariance Error Modeling: New CARA Approach

- **Developed by C. Elrod of Baylor University**
- **Uses Bayesian methods to estimate parameters that will allow statistically-representative sets of covariances to be produced, informed by actual covariance realism data**
  - Will thus produce a set of “realistic” covariances

Bayesian Covariance Error Modeling: Basics

- **Nomenclature**
  - $\mathbf{\Sigma}$ = propagated covariance matrix
  - $\mathbf{\Lambda}$ = information matrix; inverse of covariance matrix
  - $\mathbf{r}$ = residual (difference between predicted and actual position); obtained from SuperCODAC runs
- **Function sought:**
  - $\mathcal{E}(\mathbf{\Sigma})$ such that $\mathbf{r} \sim \mathcal{N}(0, \mathcal{E}(\mathbf{\Sigma}))$
- **Proposed form of solution**
  - $\mathcal{E}(\mathbf{\Sigma},\theta) = \mathbf{U}\theta\mathbf{U}'$, in which $\mathbf{\Sigma} = \mathbf{U}\mathbf{U}'$
    - Inner-product scaling matrix; opposite of Todd Cerven's proposal
    - $\mathbf{U}$ is upper triangular, a kind of "reverse Cholesky" factorization
    - More desirable because simplifies switching between Wishart and inverse-Wishart distributions (covariances modeled by inverse-Wishart, but Wishart easier to work with)
  - Bayesian technique: θ will be a distribution

Bayesian Covariance Error Modeling: Priors and Posteriors

- **For our sample of N observations**
  - $\mathbf{r}_i \sim \mathcal{N}(0, \mathbf{U}_i\theta\mathbf{U}_i')$, which is equivalent to
  - $\mathbf{U}_i^{-1}\mathbf{r}_i = \mathcal{N}(0,\theta)$
- **Bayesian priors**
  - $\theta \sim$ Inverse-Wishart$_{v_o}$ $(\mathbf{\Lambda}_0^{-1})$, which for ease of calculation can be formulated as
  - $\theta^{-1} \sim$ Wishart$_{v_o}$ $(\mathbf{\Lambda}_0)$
  - Reasonably uninformative priors set v_0 to 7 and $\mathbf{\Lambda}_0$ to diag([3 3 3])
    - With hundreds of data points per satellite, priors quite unimportant
- **Bayesian posteriors**
  - Closed-form solution (virtue of inner-product scaling matrix); no MCMC needed
  - $v_1 = v_0 + \mathrm{N}$
  - $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_0 + \sum_{i=1}^{N}(\mathbf{U}_i^{-1}\mathbf{r}_i)(\mathbf{U}_i^{-1}\mathbf{r}_i)'$

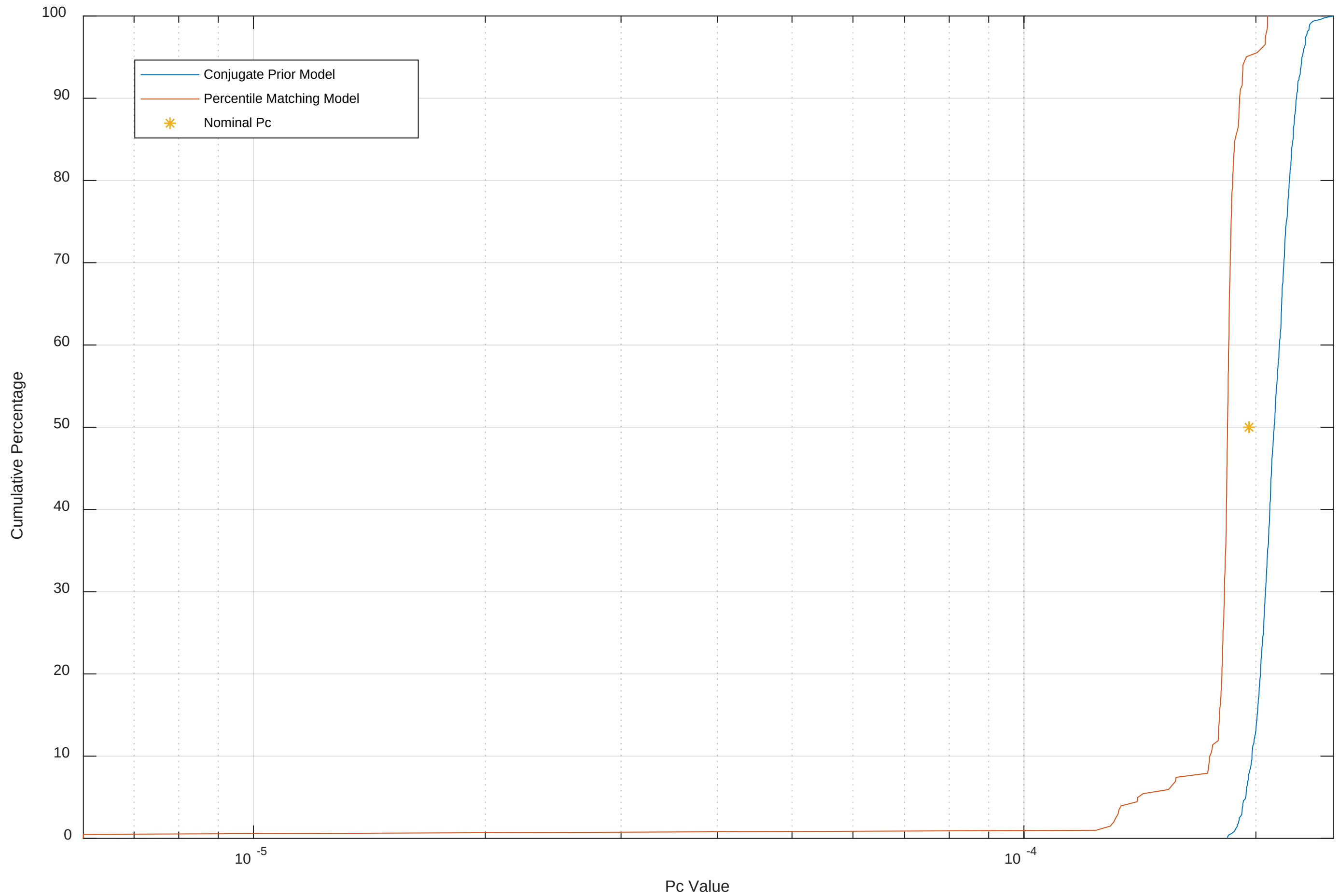
Rubric for Generating Pc Probability Density

- **Create $\boldsymbol{\theta}$ estimators for every object**
  - v_1 and $\boldsymbol{\Lambda}_1$; can be calculated from SuperCODAC data
  - Need these for different propagation states; can choose by nearest neighbor
- **Generate $\boldsymbol{\theta}$ set for each object (perhaps 1000 scaling matrices)**
  - $\boldsymbol{\theta}$ = Inverse-Wishart-Rnd (v_1, $\boldsymbol{\Lambda}_1^{-1}$)
- **Generate covariance set for each object (perhaps 1000 covariances)**
  - $\boldsymbol{\Sigma}$ set = $\mathbf{U}\boldsymbol{\theta}\mathbf{U}'$, in which **U** here is the reverse-Cholesky decomposition of the current covariance and $\boldsymbol{\theta}$ is as above
- **Calculate (perhaps 1000) Pcs as usual, using matrices from above**
  - C. Elrod also created an ERF-based Pc calculator using Gaussian quadrature / Chebyshev polynomial integration (with nodal symmetry), which is vectorizable and outperforms MATLAB quad2d by factor of 250

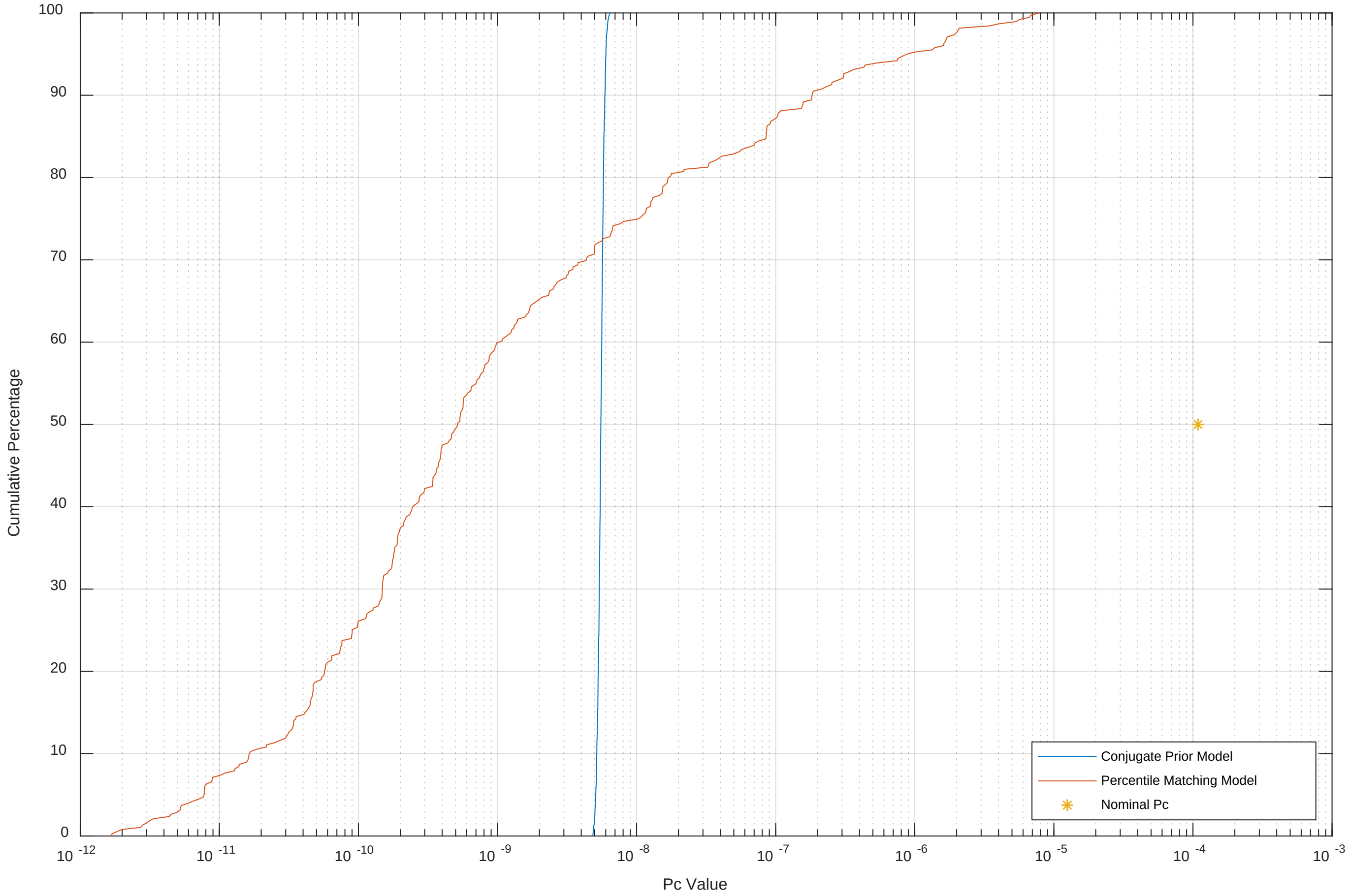
Pc Uncertainty Sample Output

- **Cumulative Distribution Function (CDF) plots of Pc using both new technique (blue) and old technique (orange)**
- **Only secondary object's covariance perturbed for this analysis**
  - Larger run will take place this week-end and perturb both covariances
- **In the main, both techniques produce similar results**
- **Difference with nominal Pc value sometimes striking**
  - Often both techniques will depart from it substantially
  - Can happen when covariance for a particular object not very realistic
- **Additionally, can use technique to determine upper and lower scale factors to bound probabilistic dataset**
  - Somewhat disappointing activity, but may be easiest to convey to users

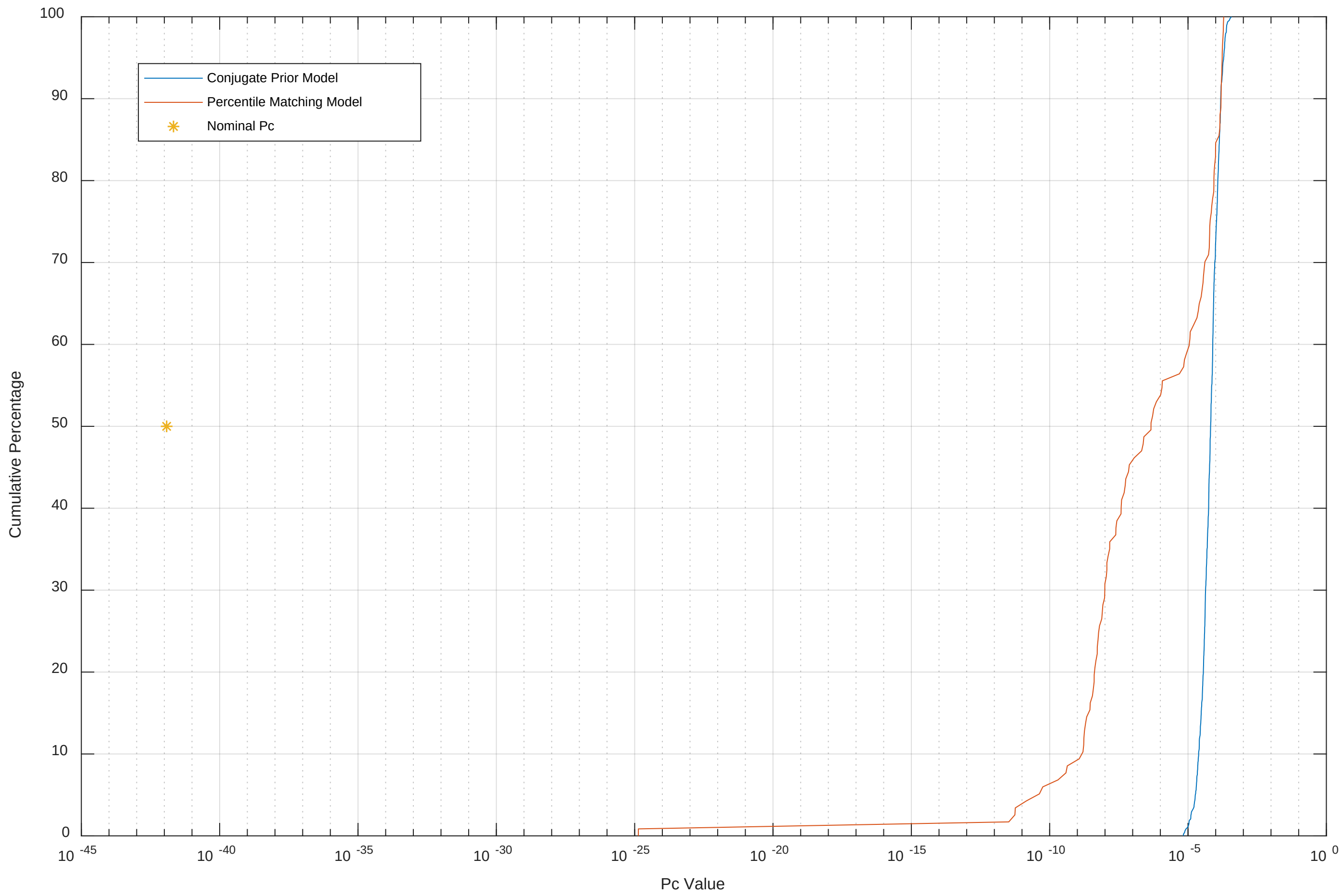
Sample Result: Good Agreement Between Methods and Nominal



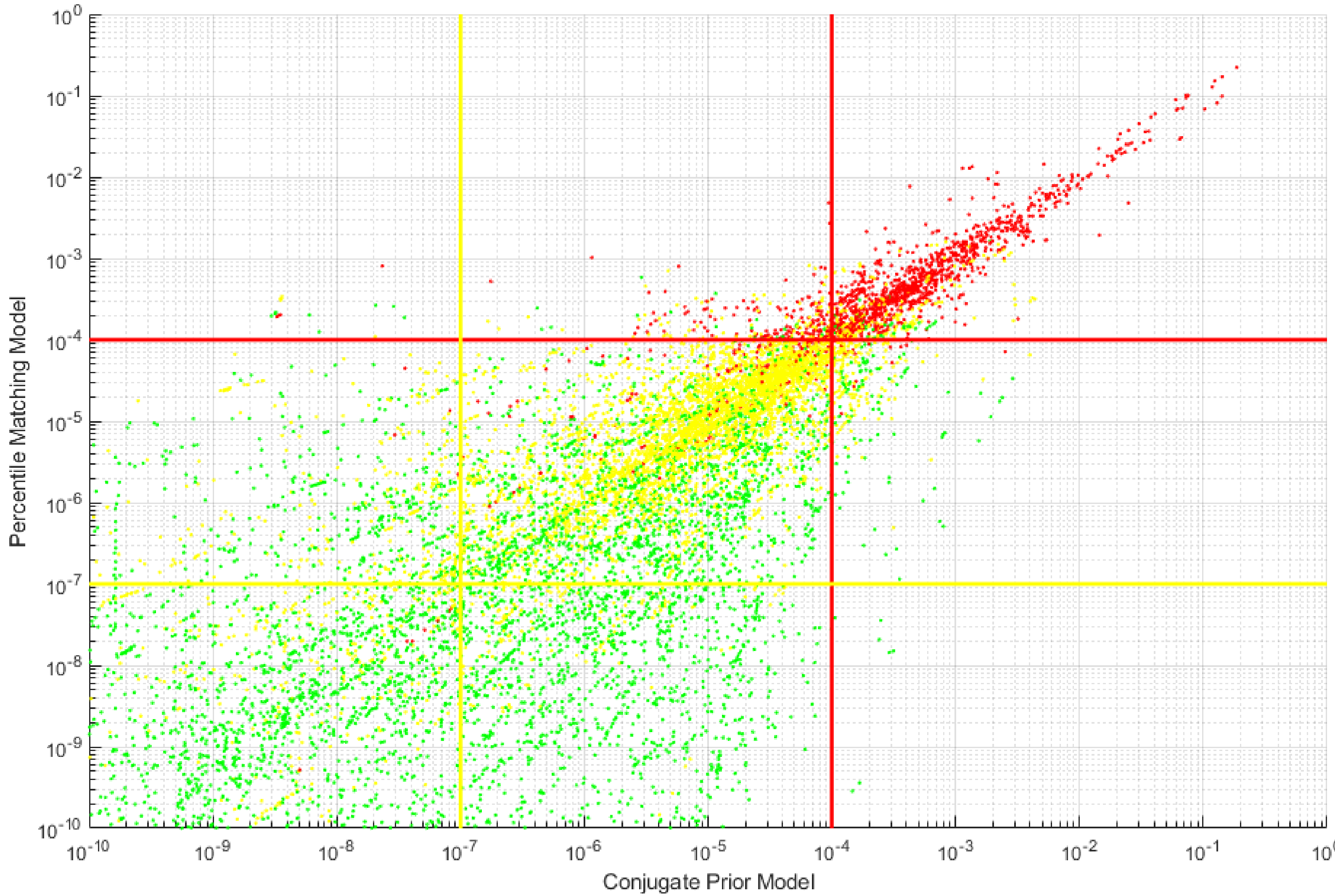
Sample Result: Model Agreement but not with Nominal



Sample Result:
Model Agreement but not with Nominal



Model Performance:
All CARA Conjunctions JUL-DEC 2017



Model Performance and Comparison: Tabular Summary

	Nominal Pc Color	Green	Green	Green	Yellow	Yellow	Yellow	Red	Red	Red
	Revised Color (50ile)	Green	Yellow	Red	Green	Yellow	Red	Green	Yellow	Red
Percentage	Conjugate Model	88.8	3.7	0.2	0.8	4.6	0.4	0.0	0.4	1.2
	Percentile Matching	90.3	2.2	0.1	0.6	4.9	0.4	0.0	0.2	1.3
Counts	Conjugate Model	96303	3980	192	891	5044	430	20	398	1252
	Percentile Matching	98001	2407	67	657	5315	393	10	242	1418

	Nominal Pc Color	Green	Green	Green	Yellow	Yellow	Yellow	Red	Red	Red
	Revised Color (95ile)	Green	Yellow	Red	Green	Yellow	Red	Green	Yellow	Red
Percentage	Conjugate Model	87.8	4.5	0.2	0.5	4.8	0.6	0.0	0.3	1.2
	Percentile Matching	87.2	5.2	0.2	0.2	5.1	0.6	0.0	0.1	1.4
Counts	Conjugate Model	95305	4932	238	518	5242	605	16	322	1332
	Percentile Matching	94619	5621	235	217	5493	655	0	147	1523

Model Performance: Summary

- **In the main, both models not wildly different**
- **However, differences are large enough to require additional study**
  - Conjugate Prior model tends to recategorize more frequently
    - Not entirely unexpected, since percentile-matching approach probably variance low
- **Additional study rubric**
  - Thorough scrub of both methodologies by Bayesian theory expert
  - Simulation approach
    - Generate synthetic Mahalanobis distance histories
    - Determine which approach best recovers Mahalanobis distance generating functions
- **Bounded operational use also possible**
  - When both models recategorize the same way, recommendation can be followed
- **Overall, considering covariance error histories does suggest non-trivial conjunction event severity recategorizations**

Next Steps

- **Fully accredit covariance uncertainty evaluation technique**
- **Integrate both above technique and Mashiku projected-area HBR into Pc Uncertainty tool**
  - Develop tool so that either or both features can be enabled
  - Refine displays
  - Perhaps include Carpenter miss distance confidence interval code and display
- **Decrement other functionality**
- **Plan is for tool and key algorithms to be available to missions by September 2018**