Conjunction Assessment Risk Analysis

Quantifying Shortcomings in the 2-D Pc Calculation

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Agenda

- **Motivation and Objectives**
- **Methodology**
  - Critically examine the three assumptions required for 2D-Pc estimates
- **Analysis**
  - Temporal variations of cumulative Pc and Pc rate
  - Mahalanobis distance variations as an efficient proxy for Pc variations
  - Proposed testing and test statistics for evaluating the viability of 2D-Pc assumptions using Mahalanobis distances
- **Conclusions and Ongoing/Future Work**

Motivation and Objective

- **Motivation** The probability of collision, Pc, between two Earth-orbiting satellites can often, **but not always**, be approximated adequately using the “2D-Pc” formulation
- **Objective** Find a set of “boundary conditions” that ensure the 2D-Pc approximation be sufficiently accurate, so that it may be determined when computationally-intensive *Brute Force Monte Carlo*[1] (BFMC) Pc estimates are required

[1]D.Hall *et al* “High-Fidelity Collision Probabilities Estimated Using Brute Force Monte Carlo Simulations” AAS 18-244, Aug. 2018

Methodology

- **Critically examine the assumptions used in the formulation of the 2D-Pc approximation**
  - Assumption 1: During the conjunction, the satellite trajectories can be approximated as linear
  - Assumption 2: During the conjunction, the TCA relative position covariance can be approximated as constant
  - *Ancillary Assumption: The input TCA states and covariances for the primary and secondary satellites are valid*
- **Formulate “2D-Pc boundary condition” tests to check if these assumptions are satisfied adequately**
  - Examine assumptions one at a time, yielding different tests
  - Base the tests on Mahalanobis distances, used here as a computationally-efficient proxy measure

The 2D-Pc Formulation

- **Foster and Estes[1] presented the original 2D-Pc formulation in 1992**
  - Through marginalized probability analysis reduces dimension of conjunction
  - Performs numerical integration of joint covariance probability density over circular region that represents hard-body radius

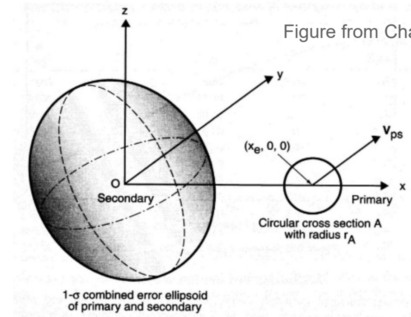


Figure from Chan (2008)

- **Akella and Alfriend[2] used the same assumptions to reformulate the 2D-Pc derivation in 2000, showing that**

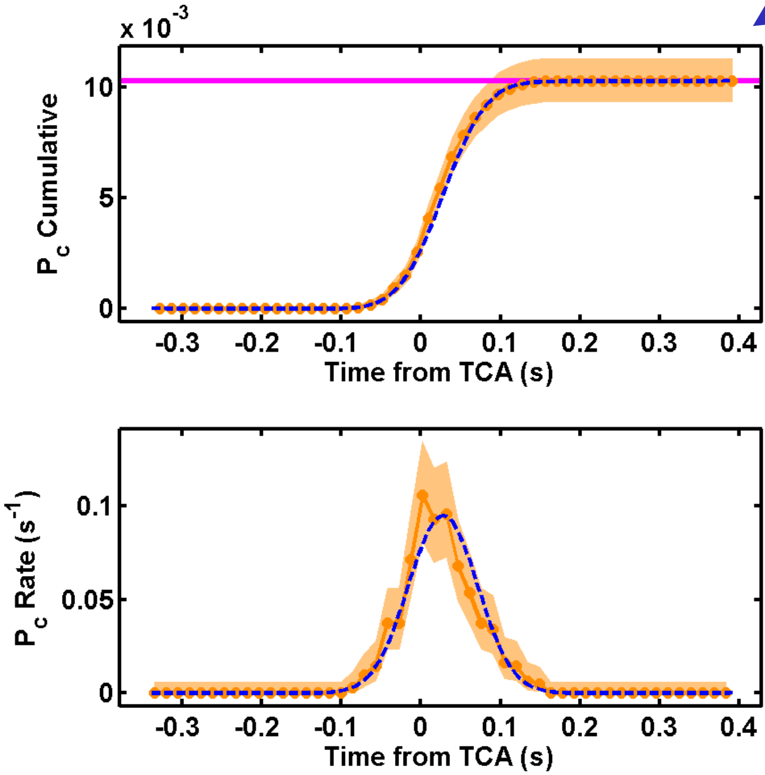
$$P_c = \int_{-\infty}^{+\infty} \dot{P}_c(t)\, dt$$

- **The probability rate, $\dot{P}_c(t)$, peaks during the conjunction near the time of closest approach (TCA)**
  - Exactly at TCA for spherical relative-position uncertainty PDFs
  - Offset from TCA for ellipsoidal uncertainty PDFs

[1]J.L. Foster and H.S. Estes, "A Parametric Analysis of Orbital Debris Collision Probability and Maneuver Rate for Space Vehicles," NASA/JSC-25898, Aug. 1992.
[2]M.R. Akella and K.T. Alfriend, "The Probability of Collision Between Space Objects," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 5, pp. 769-772, 2000.

Temporal Risk Analysis Plots for a CARA Conjunction with a Valid 2D-Pc Estimate

Conjunction Duration[1]



The cumulative collision probability grows during the event up to the overall Pc value for the conjunction

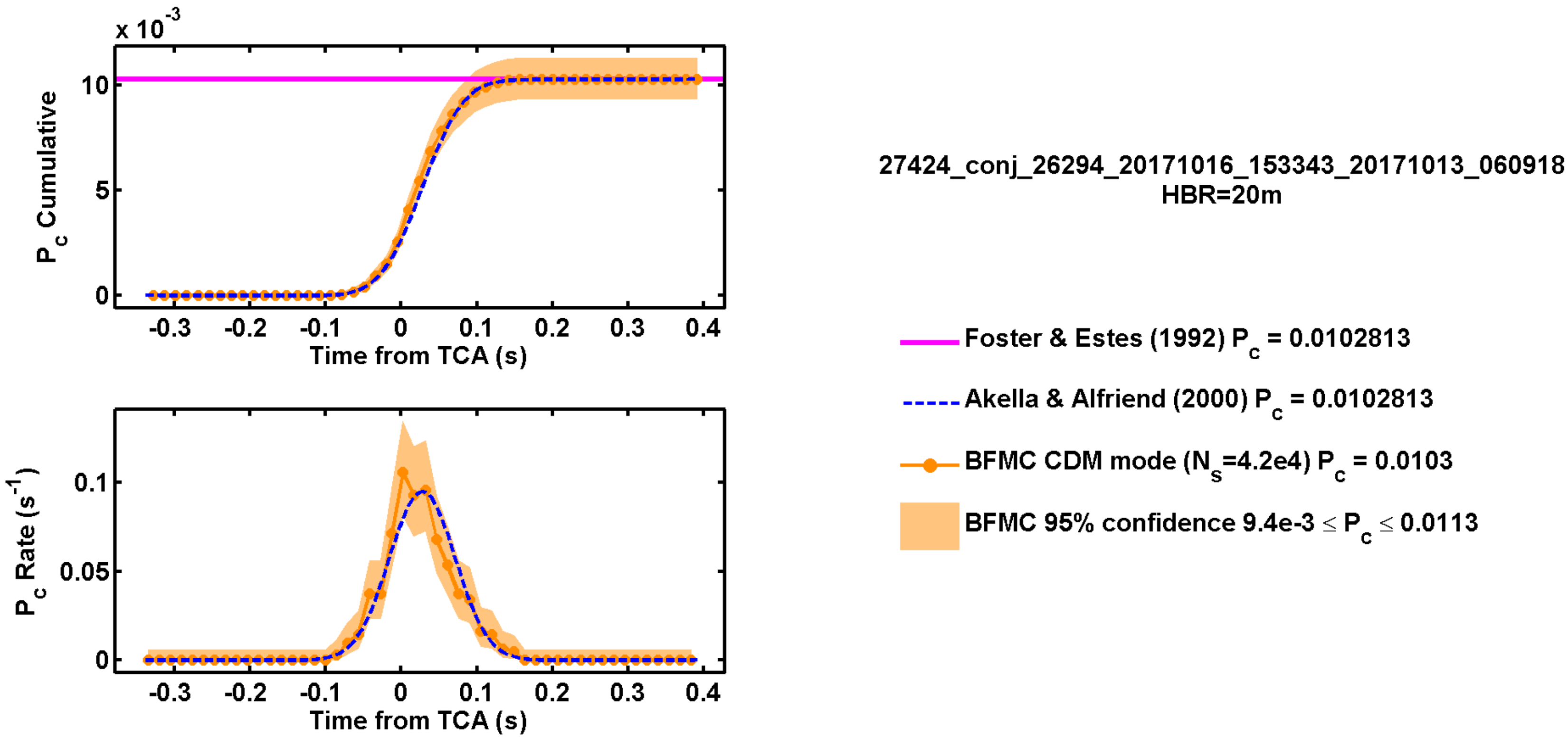
27424_conj_26294_20171016_153343_20171013_060918
HBR=20m

- Foster & Estes (1992) P_c = 0.0102813
- Akella & Alfriend (2000) P_c = 0.0102813
- BFMC CDM mode (N_s=4.2e4) P_c = 0.0103
- BFMC 95% confidence 9.4e-3 $\leq P_c \leq$ 0.0113

The probability rate, $\dot{P}_c(t)$, peaks during the conjunction, at a time that can be offset from the TCA

[1]V.Coppola "Evaluating the Short Encounter Assumption of the Probability of Collision Formula" AAS 12-247, Feb. 2012

Well-Behaved 2-D Pc Conjunction: Temporal Analysis Curves

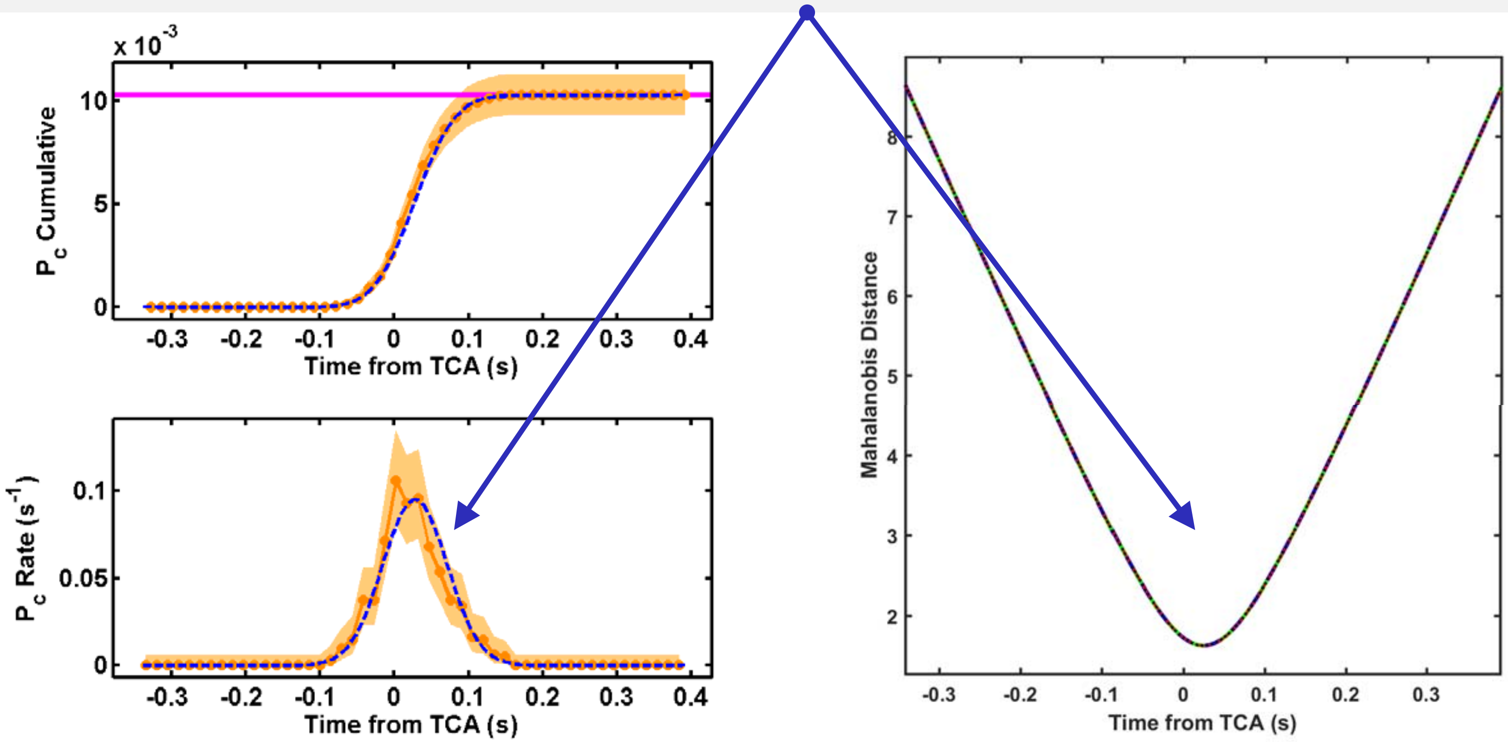


Example conjunction with valid 2D-Pc estimate

Mahalanobis Distance Time-History Plots

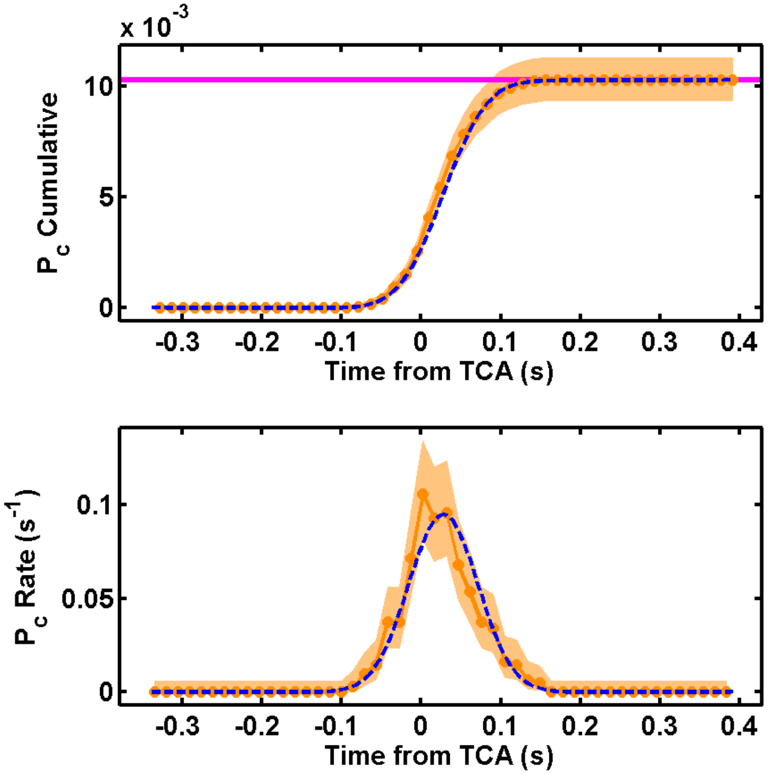
$\dot{P}_c(t)$ peaks very close to the time that the relative-position Mahalanobis distance reaches its minimum value[1]

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathrm{MD}(t) = (\mathbf{r} - \bar{\mathbf{r}})^T \mathbf{C}^{-1} (\mathbf{r} - \bar{\mathbf{r}})$$

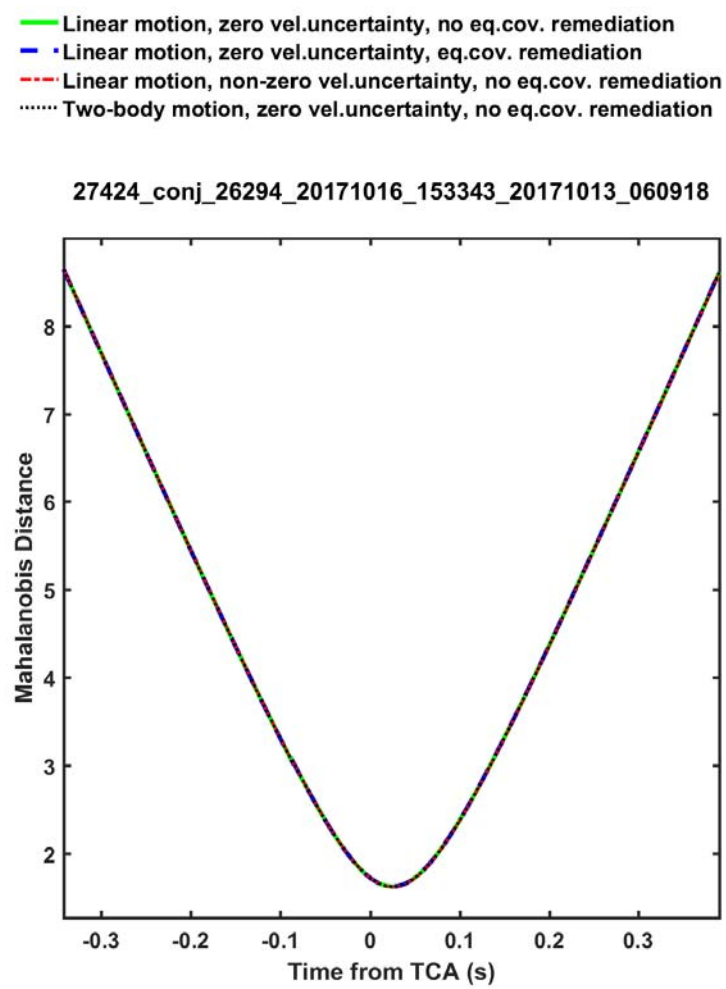


[1]D.Hall *et al* "Time Dependence of Collision Probabilities During Satellite Conjunctions" AAS 17-271, Feb. 2017

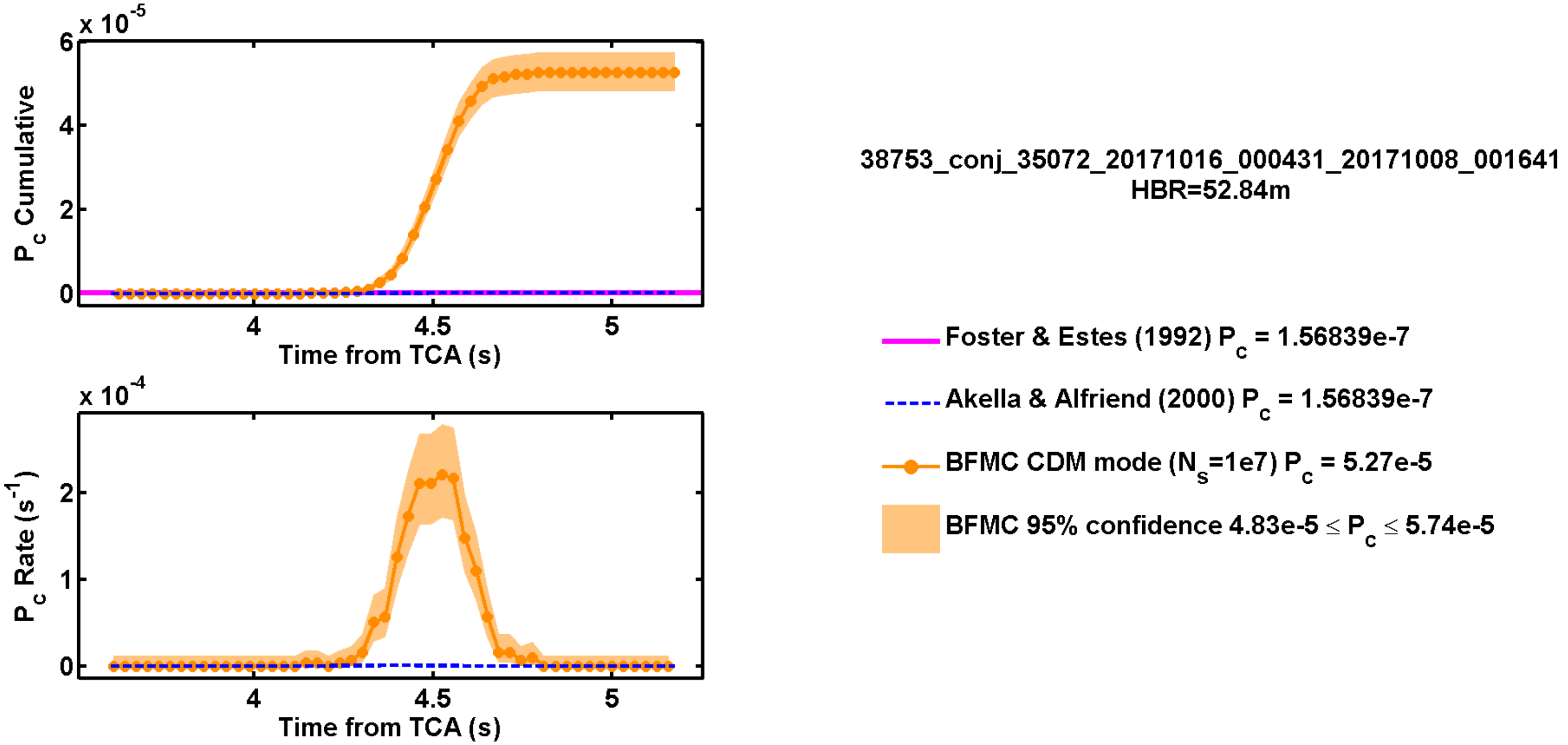
Well-Behaved 2-D Pc Conjunction: Alignment of Mahalanobis Distance Curves



No significant Mahalanobis distance differences

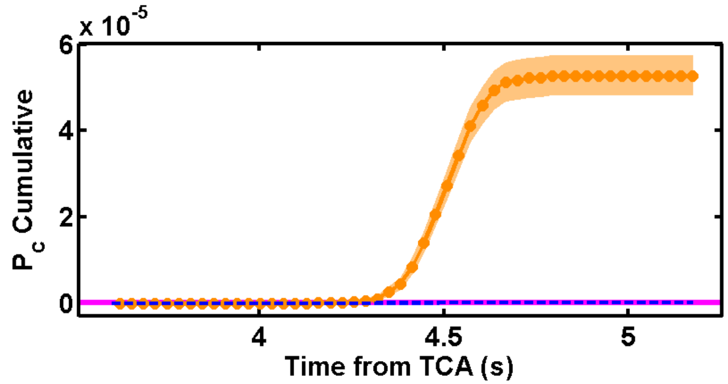


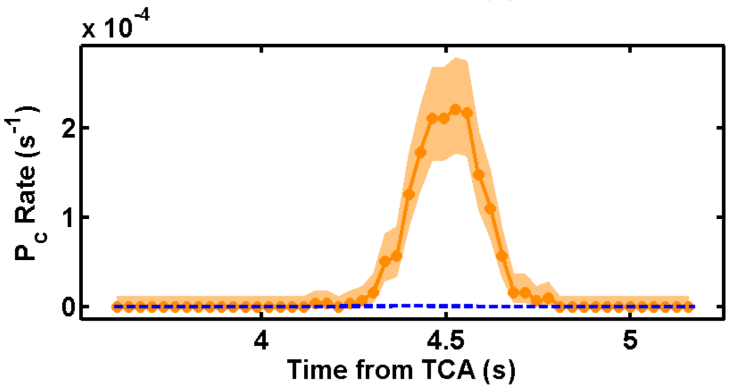
Large Velocity Covariance Situation: BFMC-Pc ≈ 300×2D-Pc >> 2D-Pc



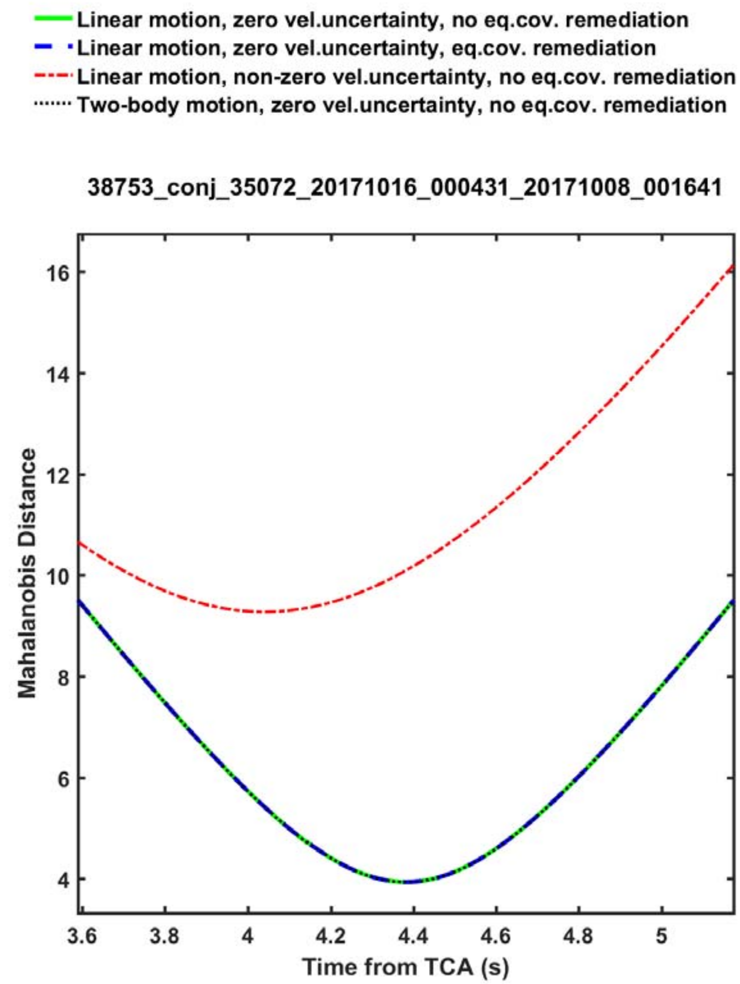
Example conjunction with invalid 2D-Pc estimate

Large Velocity Covariance Situation: Mahalanobis Distance Curve Divergence





Mahalanobis distance minimum is shifted



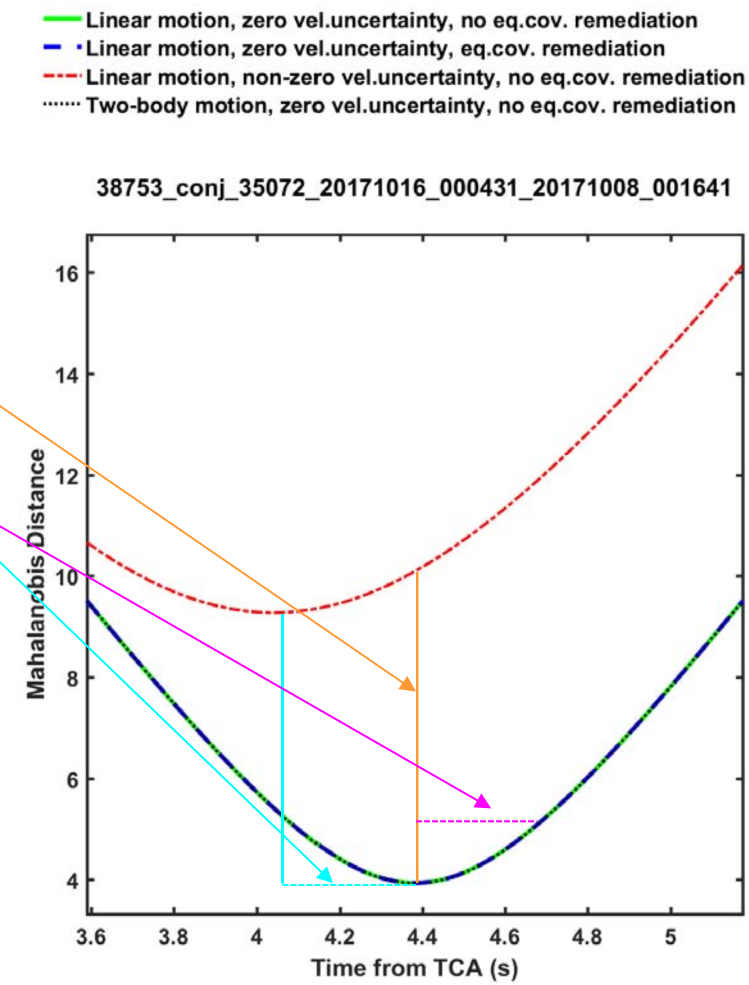
Mahalanobis Distance Minimum Shift: Proposed Test Statistics

- **1st M-Distance shift indicator:**

$$VIa = |\Delta MD1| \ll a \text{ (perhaps 1)}$$

- **2nd M-Distance shift indicator:**

$$VIb = \left|\frac{\Delta t(offset)}{\Delta t(1\sigma\, width)}\right| \ll a$$



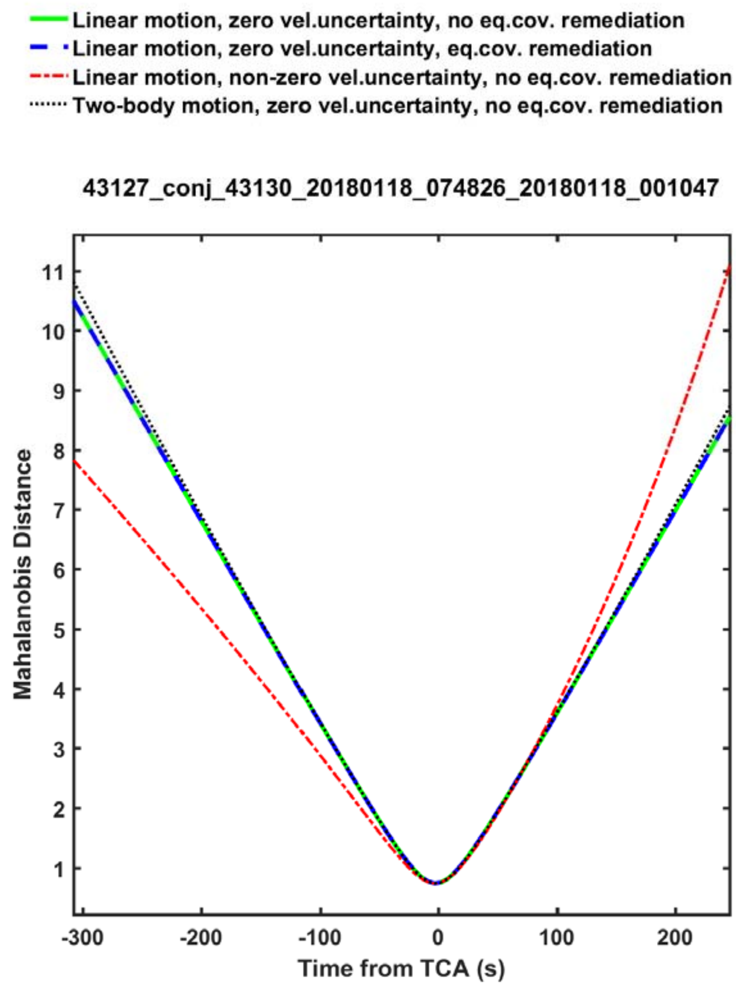
Combined M-Distance Shift Test:

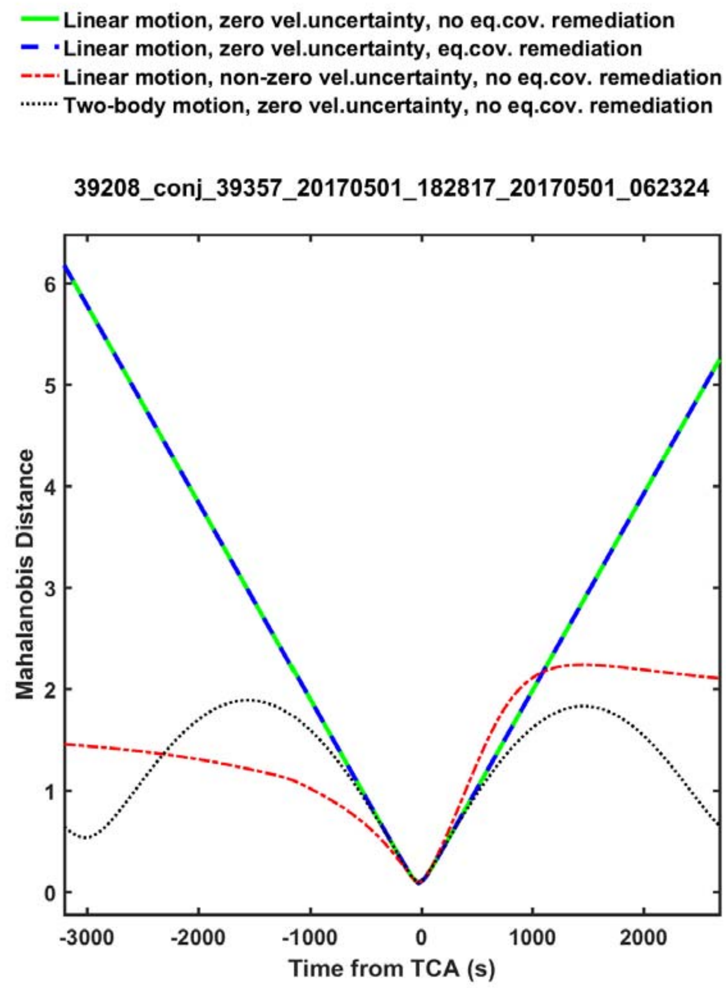
$$VI = max(VIa, VIb) \ll a$$

$$\Rightarrow$$

2D-Pc approximation is not adversely affected by velocity covariances

Mahalanobis Distance Differences: Long-Duration and Repeating Conjunctions





Mahalanobis Distance Curve Alignment: Proposed Test Statistics

- 1st M-curve divergence indicator:

$$V2a = |\Delta MD| \ll q$$

- 2nd M-curve divergence indicator

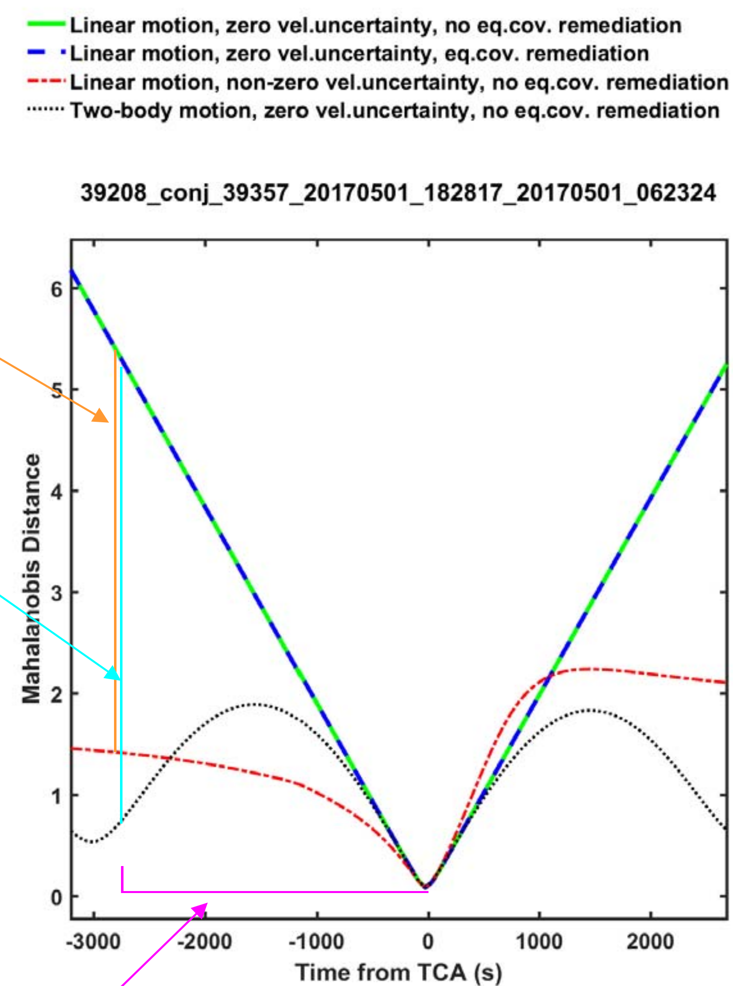
$$V2b = |\Delta MD| \ll q$$

Combined divergence test:

$$V2 = max(V2a, V2b) \ll q$$

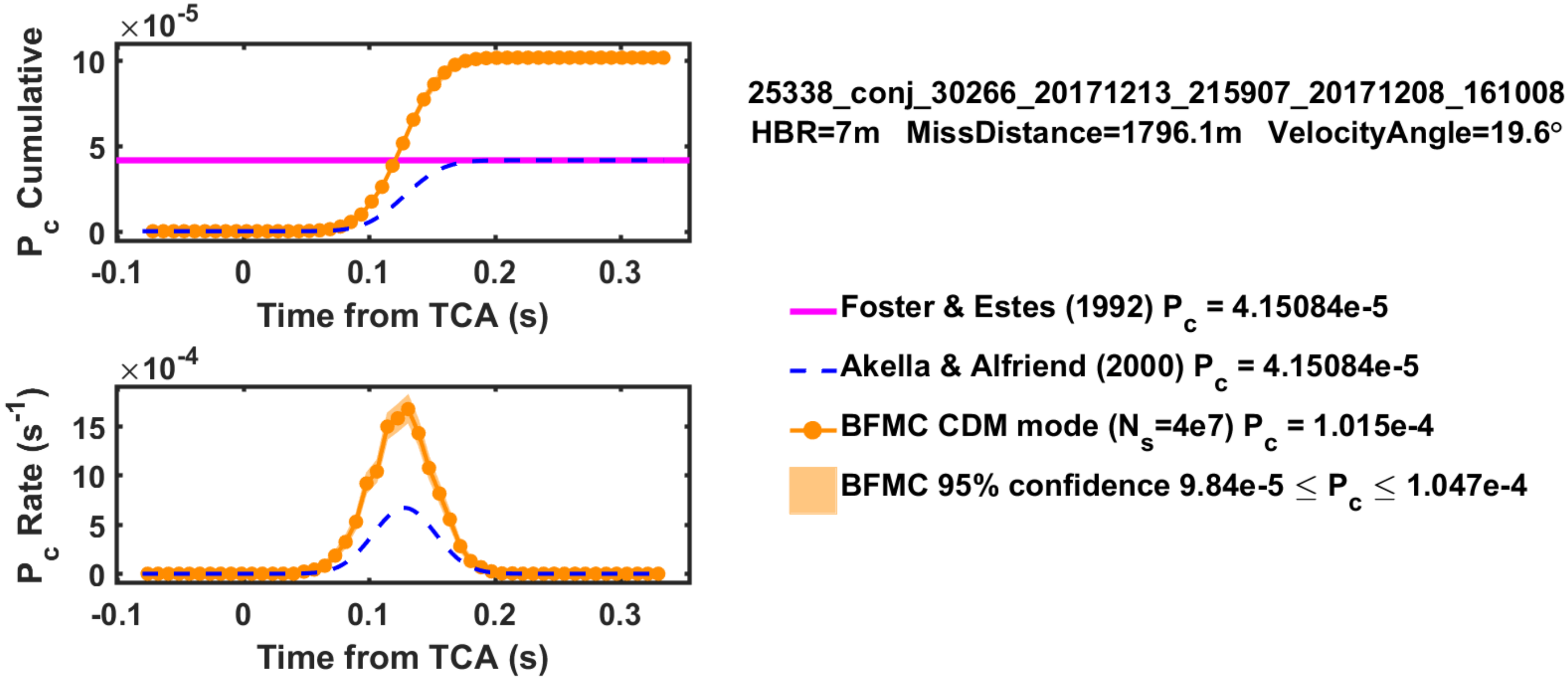
$$\Rightarrow$$

2D-Pc approximation is not adversely affected by velocity covariances or rectilinear motion assumption



Coppola Conjunction Boundary

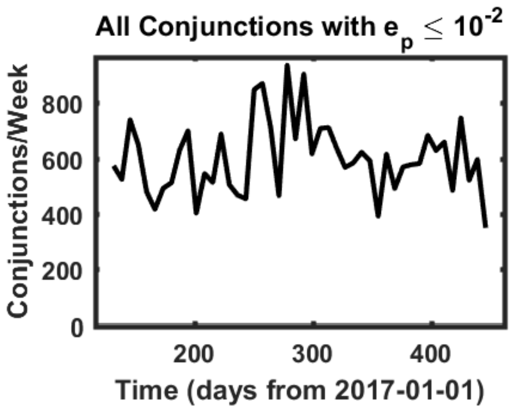
Non-Positive-Definite Covariance Event: BFMC-Pc ≈ 2.4×2DPc

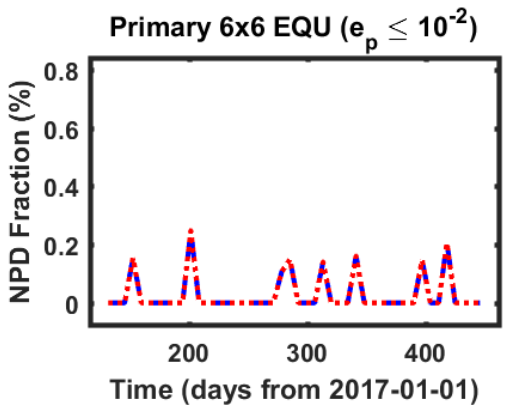


Example conjunction with invalid 2D-Pc estimate; difference introduced by remediating NPD condition

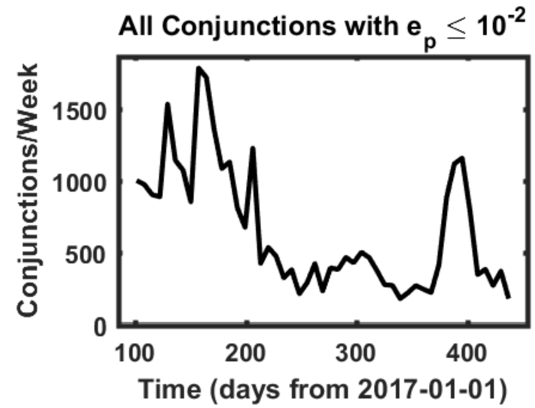
Frequency of NPD TCA Equinoctial States

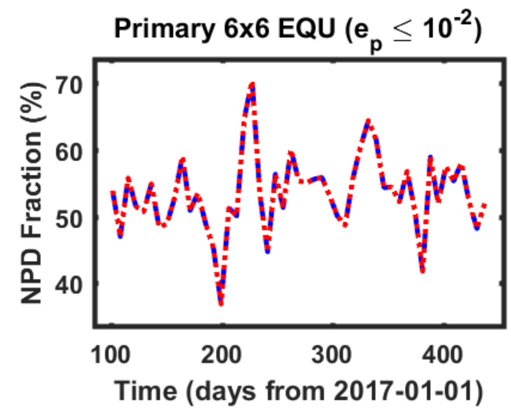
CARA Conjunction Data
~28,000 events with 2DPc > 1e-7
(from high-precision CDMs)



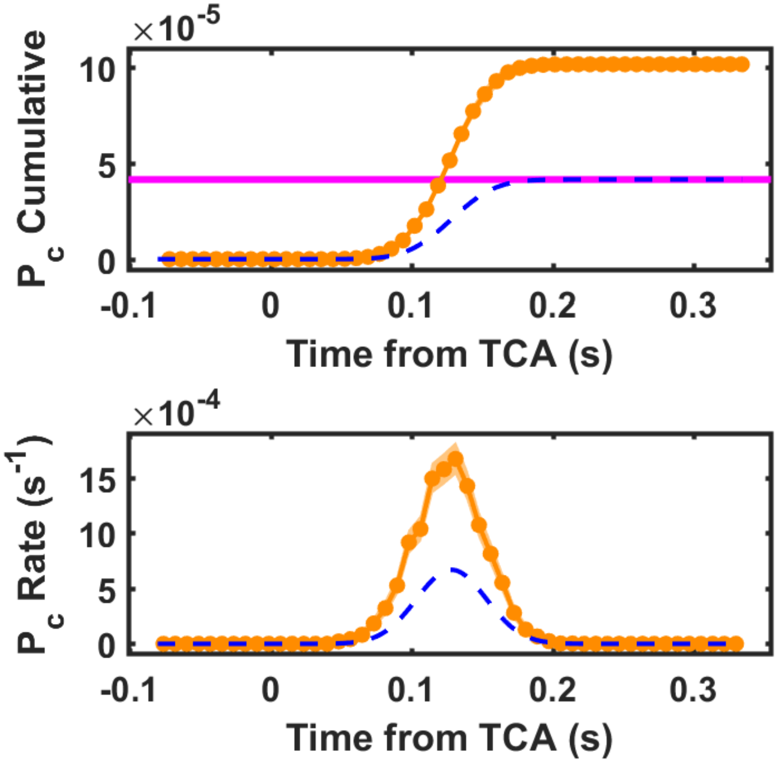


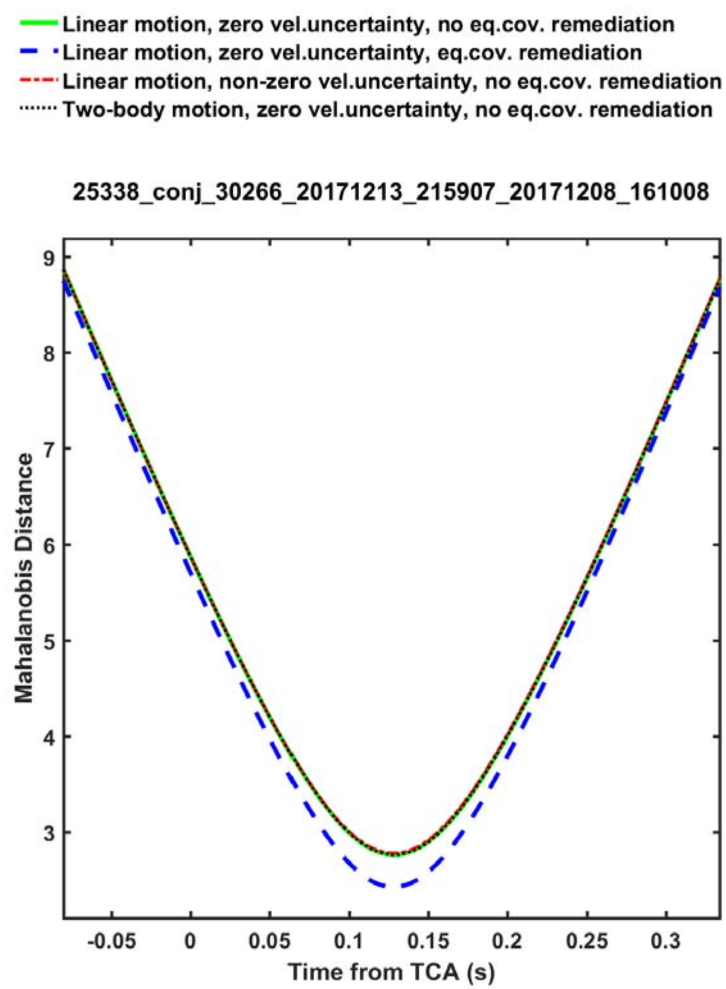
SSPAT Low Rel. Velocity Data
~32,000 events with 2DPc > 1e-7
(from low-precision CDMs)





Non-Positive-Definite Covariance Event: Mahalanobis Distance Curve Alignment





Mahalanobis distance differences caused by NPD TCA equinoctial state covariances

Conclusions

- **2D-Pc boundary condition metrics can be developed to test the three assumptions used in the 2D-Pc formulation**
  - **1) Linear trajectories 2) Constant covariances *0) Valid input data***
- **Proposed boundary condition metrics based on Mahalanobis distances (MD) variations**
  - **Examine MD differences that occur when invoking/relaxing the three assumptions one at a time: $\{\Delta MD_1, \Delta MD_2, \Delta MD_3\}$**
  - **Conjunctions with small MD differences fall within the 2D-Pc boundaries:**
    - **$Max(\Delta MD_n) << 1$ $\rightarrow$ 2D-Pc is reliably accurate**
    - **Otherwise $\rightarrow$ 2D-Pc is potentially inaccurate**
  - **Tests thus identify 2D-Pc failure candidates but also capture cases in which 2-D Pc is acceptable calculation**
- **Ongoing and Future Work:**
  - **Set test thresholds for producing acceptable Pmd and Pfa levels**
    - **Final results to be published next year**
  - **Reinforce fact that some Monte Carlo necessary for Pc calculation**