A fast Monte Carlo method for model-based prognostics based on stochastic calculus

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Introduction

Ingredients of model-based prognostic:

- state-space formulations $\dot{x} = f_{\theta}(x, u, \omega)$
- Monte Carlo (MC) methods $x^{(i)} \sim p(X)$

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Some considerations:

- Monte Carlo methods are computationally expensive
- state-space formulations are differential equations with stochastic terms (SDE)

Introduction

Contribution of this work

Try to take advantage of stochastic calculus and SDE solutions to accelerate model-based prognostic using Monte Carlo simulations (?).

Potential

reducing computational time preserving (enhancing) the precision of estimations

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Summary of model-based prediction

$$\dot{x} = f_\theta(x, u, \omega)$$

$$\downarrow$$

$$x_k = x_{k-1} + f_\theta(x_{k-1}, u_{k-1}, \omega_{k-1})\,\Delta t_k$$

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Input: $x_k^{(i)} \sim p(X_k)$, l
Output: $x_{k+l}^{(i)} \sim p(X_{k+l})$
for *each* $x_k^{(i)} \sim p(X_k)$ **do**
  for *each* $\tau \in \{1, \ldots, l\}$ **do**
    $\omega_{k+\tau-1}^{(i)} \sim p(\Omega_{k+\tau-1})$
    $x_{k+\tau}^{(i)} = x_{k+\tau-1}^{(i)} + f_\theta(x_{k+\tau-1}^{(i)}, u_{k+\tau-1}, \omega_{k+\tau-1}^{(i)})\Delta t_{k+\tau-1}$
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Using stochastic calculus properties in model-based prediction

State-space model utilized in prognostic (additive noise case)

$$\dot{x}_t = f_\theta(x_t, u_t) + \omega_t$$
$$x_k = x_{k-1} + f_\theta\left(x_{k-1}, u_{k-1}\right)\Delta t_k + \omega_{k-1}\Delta t_k$$

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Typical SDE formulation

$$\dot{X}_t = f_{\theta}(X_t, U_t) + \sigma(t, X_t)\,\xi_t$$
$$X_t = X_0 + \int_0^t f_{\theta}(X_s, U_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s$$
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Considerations

under certain assumptions (e.g., $\sigma \neq \sigma(X_t)$), we can compute the SDE terms separately and we can find similarities between noise term and the diffusion term

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Using stochastic calculus properties in model-based prediction

Let us consider $\sigma(t, X_t) = \sigma$ in the SDE:

$$\int_0^t \sigma \mathrm{d}B \approx \sum_{s=0}^{k-1} \sigma \Delta B_s$$

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$$\int_0^t \sigma \mathrm{d}B \approx \sum_{s=0}^{k-1} \sigma \Delta B_s$$

If we assume $\omega \sim \mathcal{N}(0, \sigma)$ in the state-space model:

$$\omega_{k-1} \Delta t_k = \sigma \, z_{k-1} \Delta t_k = \sigma \, \Delta B_k$$

Using stochastic calculus properties in model-based prediction

Some useful properties of Brownian motion B:

- $\mathrm{d}B \sim \mathcal{N}(0, \mathrm{d}t) \rightarrow \mathrm{d}B^{(i)} = \sqrt{\mathrm{d}t}\, z^{(i)}$
- $B_{t_2} - B_{t_1} \sim \mathcal{N}(0, t_2 - t_1)$

Applications

Case study 1: prognostic of electrolytic capacitors[1]:

$$\dot{C}_l = \alpha C_l - \alpha\beta + \omega$$

$$C_l(t) = e^{\alpha t}\left(-\beta + \beta e^{-\alpha t} + \int_0^t \sigma e^{-\alpha s}\mathrm{d}B_s\right)$$

[1] Celaya J, Kulkarni C, Biswas G, Saha S, Goebel K. A model-based prognostic methodology for electrolytic capacitors based on electrical overstress accelerated aging. Annual Conference of the PHM Society 2011; 25-29 Sept. 2011

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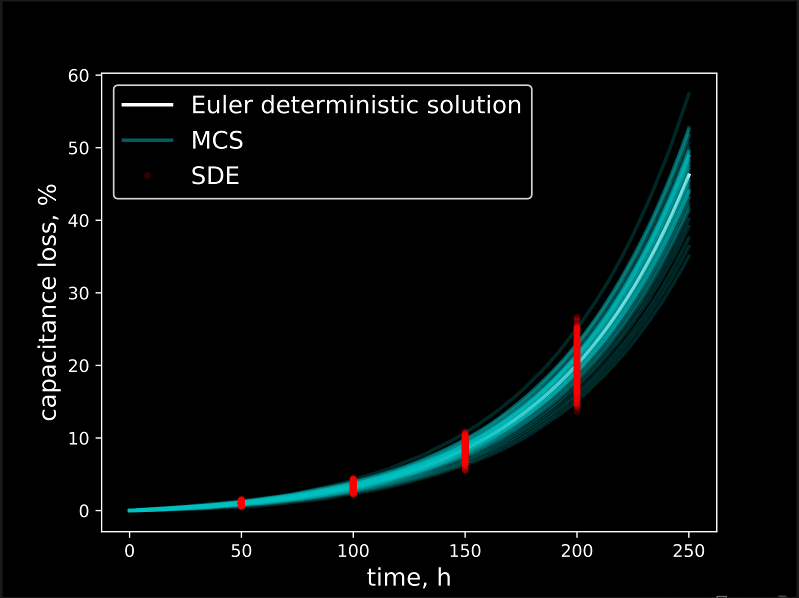
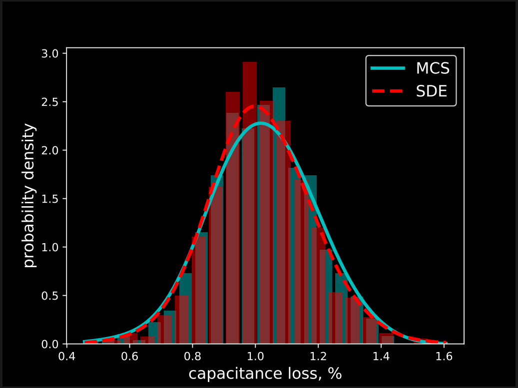
Case study 1: prognostic of electrolytic capacitors

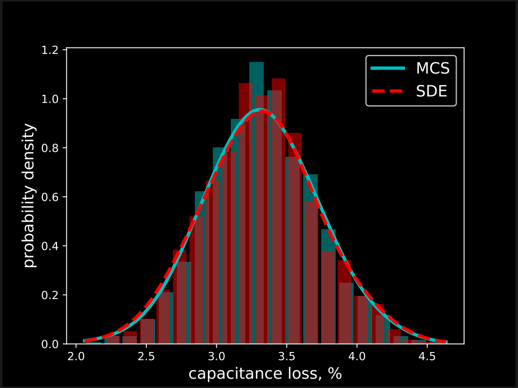


Case study 1: prognostic of electrolytic capacitors

Capacitance loss at t = 50 h

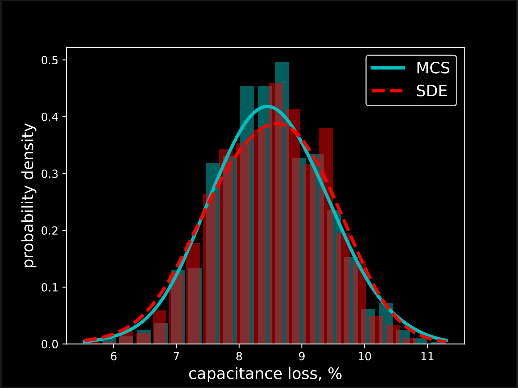


Capacitance loss at t = 100 h

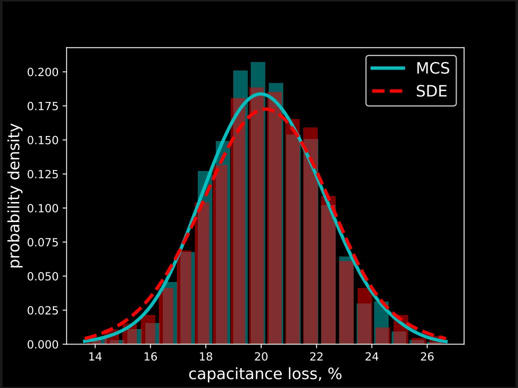


Case study 1: prognostic of electrolytic capacitors

Capacitance loss at t = 150 h



Capacitance loss at t = 200 h



Case study 1: prognostic of electrolytic capacitors

time [h]	$\mathrm{KL}(p_{\mathrm{MCS}}\|\|p_{\mathrm{SDE}})$	Hyp. test @ $\nu = 0.05$ $H_0 : \mu_{C_l,\mathrm{MCS}} = \mu_{C_l,\mathrm{SDE}}$ $H_1 : \mu_{C_l,\mathrm{MCS}} \neq \mu_{C_l,\mathrm{SDE}}$ T	$t_{\nu/2,2N-2}$	computing time [s] MCS	SDE
50	0.00405	0.305	1.961	0.230	0.243
100	0.000703	0.441		0.427	0.468
150	0.00429	0.095		0.627	0.700
200	0.00514	1.187		0.819	0.990

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Case study 1: prognostic of electrolytic capacitors

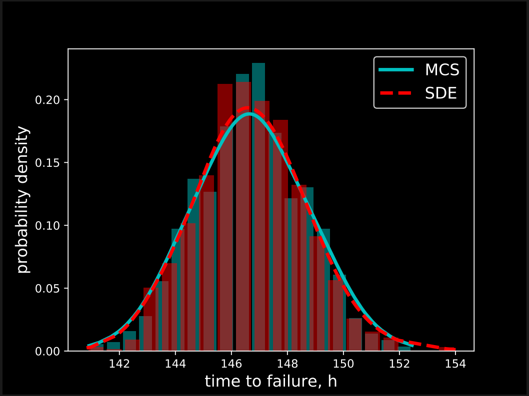
Time-to-failure (TTF) prediction:

$$\mathbb{E}[\mathrm{T_F}] = \frac{1}{\alpha} \ln \left(1 - \frac{C_{l,th}}{\beta} \right)$$

$$\mathrm{T_F}^{(i)} = \frac{1}{\alpha} \ln \frac{C_{l,th} - \beta}{\int_0^{\mathbb{E}[T_F]} \sigma e^{-\alpha s} \mathrm{d} B_s - \beta} \qquad \forall\, i = 1, \ldots, N$$

Case study 1: prognostic of electrolytic capacitors

time-to-failure pdf, $C_{l,th} = 8\%$



$\mathrm{KL}(p_{\mathrm{MCS}} \| p_{\mathrm{SDE}})$	Hyp. test @ $\nu = 0.05$ $H_0: \mu_{C_l,\mathrm{MCS}} = \mu_{C_l,\mathrm{SDE}}$ $H_1: \mu_{C_l,\mathrm{MCS}} \neq \mu_{C_l,\mathrm{SDE}}$ T	$t_{\nu/2,2N-2}$
0.001513	0.049	1.961

computing time [s]	
MCS	SDE
706.792	0.592

Case study 2: predicting the remaining time to discharge of Lithium-ion batteries using a simple state-of-charge (SOC) model[2]:
R = internal resistance, E = total energy delivered, S = SOC, $\omega_{\cdot} \sim \mathcal{N}(0, \sigma_{\cdot}^2)$

$$\dot{R} = \omega_R$$
$$\dot{S} = -\frac{P}{E} + \omega_S$$
$$\dot{E} = \omega_E$$

[2] Sierra G, Orchard M, Goebel K, Kulkarni C. Battery Health Management for Small-size Rotary-wing Electric Unmanned Aerial Vehicles: An Efficient Approach for Constrained Computing Platforms. Reliability Engineering & System Safety 2018

Case study 2: remaining time to discharge of Lithium-ion batteries

We can directly sample from the pdfs of R and E at time T:

$$R_T^{(i)} \sim \mathcal{N}(R_0, \sigma_R^2\, T)$$
$$E_T^{(i)} \sim \mathcal{N}(E_0, \sigma_E^2\, T)$$

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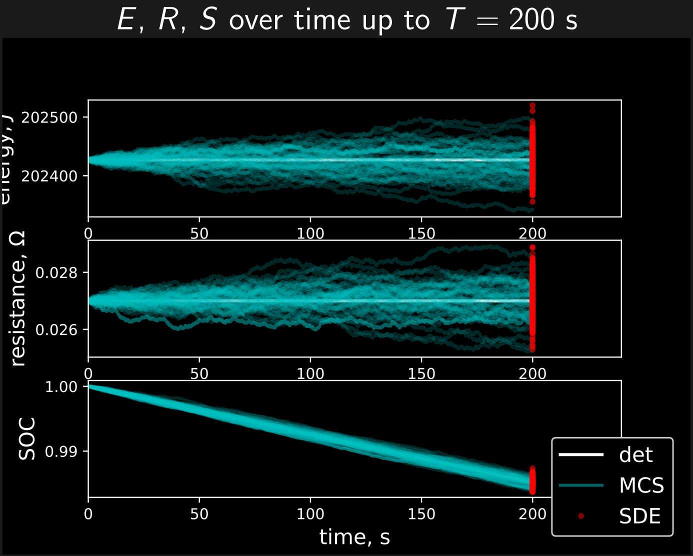
the i-th SOC sample becomes:

$$S_T^{(i)} = S_0 - \frac{P}{E_T^{(i)}}\, t + \sigma_S \sqrt{T} z^{(i)}$$

Current i_T and voltage V_T are then estimated from R and S

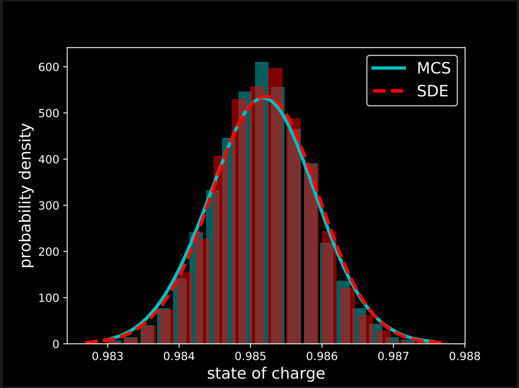
$$V_T = v_{oc,T}(S_T) - i_T(R_T, P)\, R_T + \omega_V$$

Case study 2: remaining time to discharge of Lithium-ion batteries

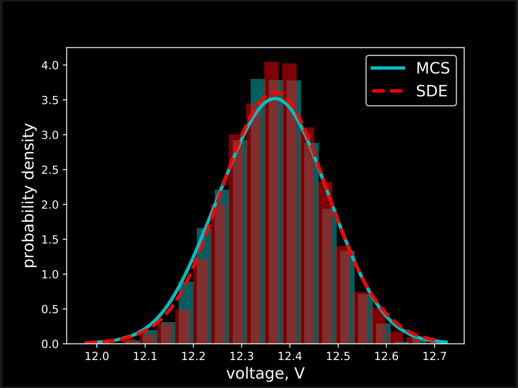


Case study 2: remaining time to discharge of Lithium-ion batteries

SOC pdf



Voltage pdf



Case study 2: remaining time to discharge of Lithium-ion batteries

Comparing SOC distributions at t = 200 s

$\mathrm{KL}(p_{\mathrm{MCS}} \| p_{\mathrm{SDE}})$	$\tilde{t}$	$t_{\nu/2,2N-2}$	computing time [s] MCS	SDE
0.00138	1.454	1.961	2.978	0.009

Applications

Case study 3: fatigue damage prognosis of cracked structure under constant amplitude fatigue loading, using Paris' law:

$$\frac{\mathrm{d}a}{\mathrm{d}n} = C' a^{\gamma} e^{\omega}$$

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$$\frac{1}{C'} \int_{a_0}^{a_F} \frac{1}{a^{\gamma}} \mathrm{d}a = \int_0^{n_F} e^{\omega_s} \mathrm{d}s$$

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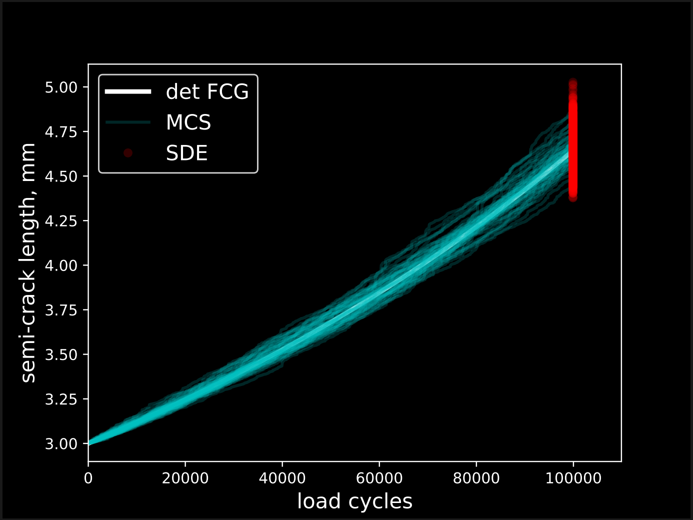
$$\int_0^{\mathbb{E}[n_F]} e^{\omega_s} \mathrm{d}s \approx \sum_{s=0}^{k-1} e^{\omega^{(i)}} \Delta n_s \quad \forall i = 1, \ldots, N$$

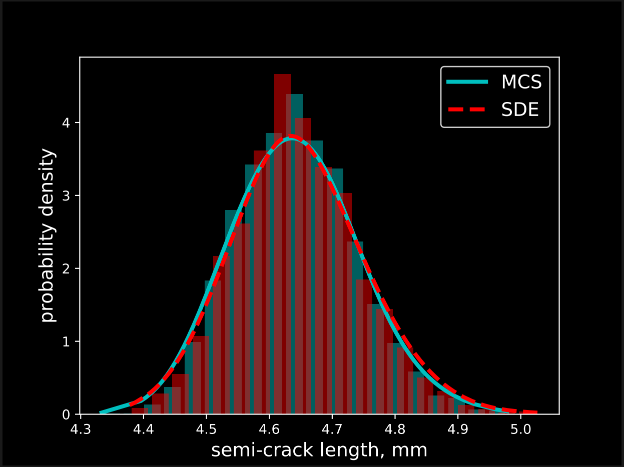
Case study 3: fatigue damage prognosis of cracked structure

FCG over time (up to $n = 100000$ load cycles)



Case study 3: fatigue damage prognosis of cracked structure

crack length pdf, $n = 100000$ load cycles



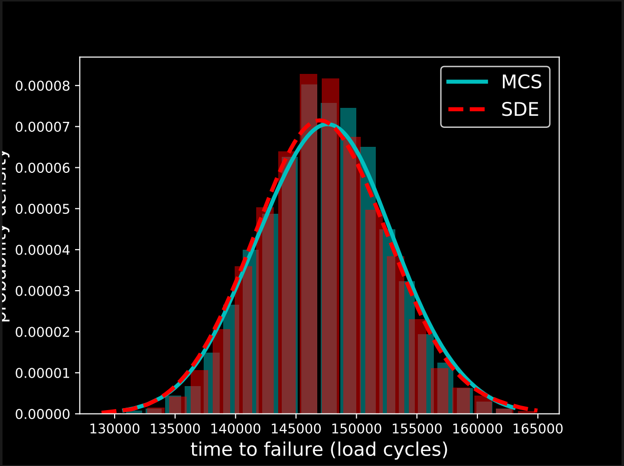
SDE method does not help in this case:

$\mathrm{KL}(p_{\mathrm{MCS}} \| p_{\mathrm{SDE}})$	$\tilde{t}$	$t_{\nu/2,2N-2}$
0.00198	1.748	1.961

computing time [s]	
MCS	SDE
0.180	0.205

Case study 3: fatigue damage prognosis of cracked structure

time-to-failure pdf, $a_{th} = 6\ \mathrm{mm}$



$\mathrm{KL}(p_{\mathrm{MCS}} \| p_{\mathrm{SDE}})$	$\tilde{t}$	$t_{\nu/2,2N-2}$
0.00180	1.579	1.961

computing time [s]	
MCS	SDE
29.617	0.156

Conclusions

To summarize

- Fast MC approximation of prediction distributions using stochastic calculus
- Pro: pdfs of interest can be computed much faster
- Cons: limited to relatively simple models
- Cons: does not generalize easily, performance are model-dependent

Future works

- generalize to $x_0 \sim p(X_0)$ and $\boldsymbol{\theta} \sim p(\boldsymbol{\theta})$ before deployment.
- extension to vector SDEs and other model classes, whenever possible
- sensitivity analysis: number of samples, number of prediction steps, etc.

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