EFFICIENT ON-ORBIT SINGULARITY-FREE GEOPOTENTIAL ESTIMATION

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For vehicles in orbital cruise, either in orbit around a body or in a transfer orbit, gravity is the primary external force; therefore, the complexity of the geopotential model heavily influences the accuracy of the navigation state. As this type of spacecraft requires a highly accurate geopotential model and performs the computation numerous times a second, this calculation needs to be efficient. This calculation needs to include the first and second derivatives of the geopotential because navigation systems can require both the gravitational acceleration and gradient for inertial integration and state filtering. The most efficient method for calculating the geopotential, the forward column method, is discussed in detail, as well as a method to avoid the singularities that exist when using this method. This is shown to decrease the computation time of the geopotential compared to other popular methods. In addition, methods for first and second order propagation are discussed, which decrease the rate at which the full geopotential model needs to be calculated while maintaining the accuracy required of navigation systems. These estimation methods are shown to decrease the necessary computation of the gravity model by multiple orders of magnitude. This method decreases the computation time of the Exploration Upper Stage navigation system geopotential model by potentially an order of magnitude compared with the previous model without affecting the navigation error. A properly implemented geopotential model can have the accuracy of an 8x8 model while approaching the computational requirement as using only the J234 coefficients.

INTRODUCTION

The knowledge of the geopotential is an important aspect of a spacecraft's navigational system. It is used to determine the acceleration of the vehicle during launch or orbit, as well as the change in acceleration, which is then used in filters to estimate the velocity and position. The accuracy requirement of such a system is greater than that which is generated by assuming the body is a perfect sphere; the non-sphericalness of the body must be taken into account in order to meet the required accuracy. This is done by using geopotential models which model the body as a non-spherical object. For example, the EGM 2008 model uses up to a 2190 degree and 2159 order model of the Earth for this modeling.[1] Models such as this, with lower maximum degrees, have been made for other bodies in the Solar System, including the Moon and Mars.[2,3] These models can be truncated at any order or degree; the same model can be used for a lower degree model by using a lower number of terms.

Considerable work has gone into creating geopotential implementation models that are singularity free; for example, three implementations are compared in Reference 4. While this type of model has the advantage

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of avoiding singularities, the singularity that they are designed to avoid happens rarely; even in a polar orbit, a spacecraft generally only persists in a location that would generate a singularity for less than a millisecond during each orbit. During this fraction of a second that the geopotential model would generate a singularity, there is a very simple method to avoid this singularity, which is discussed in a later section.

Geopotential Overview

Geopotential models are based on the equation

$$V = \frac{\mu}{r}\sum_{n=0}^{N}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n}(\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda)\,\bar{P}_{nm}(\cos\theta) \tag{1}$$

Where: V is the gravitational potential, $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are the fully normalized coefficients provided by the model, μ is the gravitational parameter, a is the semi-major axis of the ellipsoid, r is the distance from the center of the body to where the potential is being calculated, N is the maximum degree, θ is the co-latitude, and $\bar{P}_{nm}$ is the fully normalized associated Legendre function.[1] As this is the equation for the gravitational potential, taking the first derivative of this results in the acceleration caused by gravity, and the second derivative of this results in the change of acceleration matrix with respect to position.

Coordinate System Transformation

There are two main coordinate systems used in the geopotential calculation. These are both in the Earth Centered Earth Fixed (ECEF) frame; this frame can be generalized as Planet Centered Planet Fixed (PCPF), since this method can be used on any body that has a geopotential model. The first is based on using spherical coordinates, and uses longitude λ, geodetic latitude φ, and distance from the center of the body r. In calculations, it can be more convenient to use the co-latitude θ, in which θ=90°-φ. This is the coordinate frame in which geopotential models are created, and it is the most efficient coordinate system in which to calculate the geopotential. However, the coordinate frame used in navigational systems is generally in Cartesian coordinates x, y, and z; therefore, equations to relate these two coordinate frames are required, which include the first and second sets of derivatives. For derivations of these equations, see References 5 and 6.

$$\ddot{x} = \frac{\partial V}{\partial x} = u\cos(\lambda)\frac{\partial V}{\partial r} - \frac{t}{r}\cos(\lambda)\frac{\partial V}{\partial\theta} - \frac{\sin(\lambda)}{u\,r}\frac{\partial V}{\partial\lambda} \tag{2}$$

$$\ddot{y} = \frac{\partial V}{\partial y} = u\sin(\lambda)\frac{\partial V}{\partial r} - \frac{t}{r}\sin(\lambda)\frac{\partial V}{\partial\theta} - \frac{\cos(\lambda)}{u\,r}\frac{\partial V}{\partial\lambda} \tag{3}$$

$$\ddot{z} = \frac{\partial V}{\partial z} = t\frac{\partial V}{\partial r} + \frac{r}{r}\frac{\partial V}{\partial\theta} \tag{4}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{u^2\cos^2\lambda}{r^2}\left(r^2\frac{\partial^2 V}{\partial r^2} + t\frac{\partial V}{\partial\theta^2} - \frac{2rt}{u}\frac{\partial^2 V}{\partial\theta\partial r}\right) + \frac{\sin^2\lambda}{r^2u^2}\frac{\partial^2 V}{\partial\lambda^2} + \frac{1}{r}(1 - u^2\cos^2\lambda)\frac{\partial V}{\partial r}$$
$$+ \frac{t}{ur^2}(3u^2\cos^2\lambda - 1)\frac{\partial V}{\partial\theta} + \frac{2\cos\lambda\sin\lambda}{r^2}\left(\frac{1}{u^2}\frac{\partial V}{\partial\lambda} + \frac{t}{u}\frac{\partial V}{\partial\theta\partial\lambda} - r\frac{\partial V}{\partial r\partial\lambda}\right) \tag{5}$$

$$\frac{\partial V}{\partial y^2} = \frac{u^2\sin^2\lambda}{r^2}\left(r^2\frac{\partial^2 V}{\partial r^2} + t\frac{\partial V}{\partial\theta^2} - \frac{2rt}{u}\frac{\partial^2 V}{\partial\theta\partial r}\right) + \frac{\cos^2\lambda}{r^2u^2}\frac{\partial^2 V}{\partial\lambda^2} + \frac{1}{r}(1 - u^2\sin^2\lambda)\frac{\partial V}{\partial r}$$
$$+ \frac{t}{ur^2}(3u^2\sin^2\lambda - 1)\frac{\partial V}{\partial\theta} - \frac{2\cos\lambda\sin\lambda}{r^2}\left(\frac{1}{u^2}\frac{\partial V}{\partial\lambda} + \frac{t}{u}\frac{\partial V}{\partial\theta\partial\lambda} - r\frac{\partial V}{\partial r\partial\lambda}\right) \tag{6}$$

$$\frac{\partial V}{\partial z^2} = t^2\frac{\partial^{2V}}{\partial r^2} + \frac{u^2}{r^2}\left(u^2\frac{1}{t}\frac{\partial^2 V}{\partial\theta} + 2rt\frac{\partial^2 V}{\partial t\partial r} + r\frac{\partial V}{\partial r} - \frac{3t}{u}\frac{\partial V}{\partial t}\right) \tag{7}$$

$$\frac{\partial^2 V}{\partial x\partial y} = \frac{\sin\lambda\cos\lambda}{r^2}\left\{u^2\left[t\frac{\partial^2 V}{\partial\theta^2} + r^2\frac{\partial V}{\partial r^2} - r\frac{\partial V}{\partial r} + \frac{t}{u}\left(3\frac{\partial V}{\partial\theta} - 2r\frac{\partial^2 V}{\partial\theta\partial r}\right)\right] - \frac{1}{u^2}\frac{\partial^2 V}{\partial\lambda^2}\right\}$$

$$+\frac{sin^2\lambda - cos^2\lambda}{r^2}\left(\frac{t}{u}\frac{\partial^2 V}{\partial\theta\partial\lambda}+\frac{1}{u^2}\frac{\partial V}{\partial\lambda}-r\frac{\partial^2 V}{\partial r\partial\lambda}\right) \quad (8)$$

$$\frac{\partial^2 V}{\partial x\partial z}=\frac{\cos\lambda}{r^2}\left\{ut\left[-\frac{u^2}{t}\frac{\partial^2 V}{\partial\theta^2}+r^2\frac{\partial V}{\partial r^2}-r\frac{\partial V}{\partial r}+\left([3t^2-1]\frac{\partial V}{\partial\theta}+r[u^2-t^2]\frac{\partial^2 V}{\partial\theta\partial r}\right)\right]\right\}$$
$$-\frac{sin\,\lambda}{r^2}\left(\frac{\partial^2 V}{\partial\theta\partial\lambda}+\frac{rt}{u}\frac{\partial^2 V}{\partial r\partial\lambda}\right) \quad (9)$$

$$\frac{\partial^2 V}{\partial y\partial z}=\frac{\sin\lambda}{r^2}\left\{ut\left[-\frac{u^2}{t}\frac{\partial^2 V}{\partial\theta^2}+r^2\frac{\partial V}{\partial r^2}-r\frac{\partial V}{\partial r}+\left([3t^2-1]\frac{\partial V}{\partial\theta}+r[u^2-t^2]\frac{\partial^2 V}{\partial\theta\partial r}\right)\right]\right\}$$
$$+\frac{cos\,\lambda}{r^2}\left(\frac{\partial^2 V}{\partial\theta\partial\lambda}+\frac{rt}{u}\frac{\partial^2 V}{\partial r\partial\lambda}\right) \quad (10)$$

In order to calculate $\frac{\partial^2 V}{\partial Z^2}$, Laplace's equation can be used. In Cartesian coordinates, this equation is

$$\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}+\frac{\partial^2 V}{\partial Z^2}=0 \quad (11)$$

STANDARD FORWARD COLUMN METHOD

There are numerous methods to implement the gravity potential equation; for a review of a few of these, see References 7, 8, 9, and 10. This paper goes into detail on calculating the second order derivatives using the forward column method, which has shown to be among the most efficient algorithms.[7]

The standard forward column method of implementing the spherical harmonic gravitational model is one of the simplest models, and it has been shown to be the most efficient. The method presented here is based on furthering the method presented by Kuga and Carrara, who presented this method for the first derivative of the potential.[11] The method presented by Kuga and Carrara is based on a paper by Holmes and Featherstone.[8] See Reference 11 for the derivation and equations of the first derivatives of potential.

The second derivatives of the potential, or the first derivatives of the acceleration, are derived by taking derivatives of the first derivatives in a similar manner to how those equations were derived. As the second derivative matrix is symmetric, and Laplace's equation holds in both spherical and Cartesian coordinates, only five terms need to be directly calculated, with the sixth term being calculated using Laplace's equation. For this calculation, the simplest method is to calculate $\frac{\partial^2 V}{\partial\theta^2}$ using Laplace's equation so that the second derivative of the Legendre equation doesn't have to be explicitly calculated.

Forward Column $\frac{\partial^2 V}{\partial\lambda^2}$. The second derivative of V with respect to λ is calculated by taking the derivative of $\frac{\partial V}{\partial\lambda}$. Similarly to how $\frac{\partial V}{\partial\lambda}$ was derived in Reference 11, this requires the derivative of a sine and cosine. This equation becomes

$$\frac{\partial^2 V}{\partial\lambda^2}=\frac{GM}{r}\sum_{m=0}^{M} m^2[-cos(m\lambda)X_{mC}-sin(m\lambda)\,X_{mS}] \quad (12)$$

Forward Column $\frac{\partial^2 V}{\partial r^2}$. The second derivative of *V* with respect to *r* is calculated by taking the derivative of $\frac{\partial V}{\partial r}$ with respect to *r*. Defining new terms as

$$X_{mC}^{rr}=\sum_{n=m}^{M}\left(\frac{a}{r}\right)^n(n+1)(n+2)\bar{C}_{nm}\bar{P}_{nm}(\theta) \quad (13)$$

$$X_{mS}^{rr} = \sum_{n=m}^{M} \left(\frac{a}{r}\right)^n (n+1)(n+2)\bar{S}_{nm}\bar{P}_{nm}(\theta) \tag{14}$$

yields the final expression for $\frac{\partial^2 V}{\partial r^2}$:

$$\frac{\partial^2 V}{\partial r^2} = \frac{GM}{r^3} \sum_{m=0}^{M} [cos(m\lambda)\, X_{mC}^{rr} + sin(m\lambda)\, X_{mS}^{rr}] \tag{15}$$

Forward Column $\frac{\partial^2 V}{\partial r \partial \lambda}$. The second derivative of V with respect to r and λ can be calculated by taking the derivative of $\frac{\partial V}{\partial r \partial \lambda}$ with respect to λ; this only requires a derivative of a sine and cosine. This equation becomes

$$\frac{\partial^2 V}{\partial r \partial \lambda} = -\frac{GM}{r^2} \sum_{m=0}^{M} m[-sin(m\lambda)\, X_{mC}^{r} + cos(m\lambda)\, X_{mS}^{r}] \tag{16}$$

Forward Column $\frac{\partial^2 V}{\partial \theta \partial \lambda}$. The second derivative of V with respect to λ and θ can be calculated by taking the derivative of $\frac{\partial V}{\partial \theta}$ with respect to λ so that it only requires a derivative of a sine and cosine. This equation becomes

$$\frac{\partial^2 V}{\partial \theta \partial \lambda} = \frac{GM}{r} \sum_{m=0}^{M} m\left[- sin(m\lambda)\, X_{mC}^{\theta} + cos(m\lambda)\, X_{mS}^{\theta}\right] \tag{17}$$

Forward Column $\frac{\partial^2 V}{\partial \theta \partial r}$. The second derivative of V with respect to r and θ can be calculated by taking the derivative of $\frac{\partial V}{\partial \mathrm{r}}$ with respect to θ so that it only requires a derivative of a X_{mC}^{r} and X_{mS}^{r}. These become

$$X_{mC}^{\theta r} = \frac{\partial}{\partial \theta} X_{mC}^{r} = \sum_{n=m}^{M} \left(\frac{a}{r}\right)^n (n+1)\bar{C}_{nm}\bar{P}_{nm}^{1}(\theta) \tag{18}$$

$$X_{mS}^{\theta r} = \frac{\partial}{\partial \theta} X_{mS}^{r} = \sum_{n=m}^{M} \left(\frac{a}{r}\right)^n (n+1)\bar{S}_{nm}\bar{P}_{nm}^{1}(\theta) \tag{19}$$

The second derivative of *V* with respect to *r* and *θ* then becomes:

$$\frac{\partial^2 V}{\partial \theta \partial \mathrm{r}} = -\frac{GM}{r^2} \sum_{m=0}^{M} \left[cos(m\lambda)\, X_{mC}^{\theta r} + sin(m\lambda)\, X_{mS}^{\theta r}\right] \tag{20}$$

Forward Column $\frac{\partial^2 V}{\partial \theta^2}$. The second derivative of V with respect to θ is calculated using Laplace's equation in spherical coordinates. This is rearranged to solve for $\frac{\partial^2 V}{\partial \theta^2}$ and becomes

$$\frac{\partial^2 V}{\partial \theta^2} = -\frac{tr^2}{u^2}\left(\frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} - \frac{2t}{ur^2}\frac{\partial V}{\partial \theta} + \frac{1}{u^2 r^2}\frac{\partial^2 V}{\partial \lambda^2}\right) \tag{21}$$

In order to avoid dividing by 0 if the spacecraft is at the equator, the quantity $\frac{1}{t}\frac{\partial^2 V}{\partial \theta^2}$ must be calculated separately and used to calculate $\frac{\partial V}{\partial z^2}$; this is given by

$$\frac{1}{t}\frac{\partial^2 V}{\partial \theta^2} = -\frac{r^2}{u^2}\left(\frac{2}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial r^2} - \frac{2t}{ur^2}\frac{\partial V}{\partial \theta} + \frac{1}{u^2 r^2}\frac{\partial^2 V}{\partial \lambda^2}\right) \tag{22}$$

Once these six second order derivatives and the three first order derivatives are known, these can be transformed into Cartesian coordinates using Eq. (2) through Eq. (11).[5,6]

Forward Column Optimization Methods

There are a few methods to increase the efficiency of using the forward column method. If these are used properly, this is one of the most efficient methods of gravity estimation, and the most efficient out of all the methods tested.

One of these methods is to calculate everything possible apriori, and then only reference these coefficients when needed. The coefficients that can be calculated ahead of time include a_{nm}, b_{nm}, d_m and f_{nm}. Each of these can stored in either *Nmax* by *Nmax* or *Nmax* by 1 arrays, then referenced when needed.

The quantity $\left(\frac{a}{r}\right)^n$ can be calculated recursively instead of each time it is called. This can be done as

$$q_0 = 1 \tag{23}$$

$$q = \left(\frac{a}{r}\right) \tag{24}$$

$$q_n = q_{n-1} q = \left(\frac{a}{r}\right)^n \tag{25}$$

The sine and cosine functions can also be calculated recursively for $\cos(m\lambda)$ and $\sin(m\lambda)$. These recursive properties are initially seeded with the first two (m=1 and m=2), and then can be calculated with

$$sin(m\lambda) = 2\, cos\{(m-1)\lambda\}\, cos(\lambda) - sin\{(m-2)\lambda\} \tag{26}$$

$$cos(m\lambda) = 2\, cos\{(m-1)\lambda\}\, cos(\lambda) - cos\{(m-2)\lambda\} \tag{27}$$

Laplace's equation can also be used in the calculation of the Cartesian first and second derivatives to simplify one of the equations for each set of derivatives.

PROPAGATION METHODS

The computation of a spherical harmonic geopotential can be very computationally intensive to run, even when using a truncated model. This can be due to the navigation algorithm needing to know the first two derivatives of the geopotential at a high rate, sometimes above 100 Hz. In order to reduce the computational requirement on the flight computer, a propagation method can be used so that the full model only has to be calculated at a lower rate, for example, once a second or even once every 10 or 20 seconds. Then at the higher rate, the current gravity can be estimated based on the last full model that was run. This has the potential to decrease the gravity function run rate from greater than 100 Hz to as low as 0.05 Hz, although this normally doesn't have to be less than 1 Hz.

First Order Taylor Series Expansion

The simplest gravity propagation method is to use a first order Taylor series expansion to propagate the geopotential derivatives between full model calculations. From Reference 12, this is calculated using the equation

$$\boldsymbol{g}(\boldsymbol{r}) \approx \boldsymbol{g}(\boldsymbol{r}^*) + \boldsymbol{G}(\boldsymbol{r}^*)[\boldsymbol{r} - \boldsymbol{r}^*] \tag{28}$$

Where: $\boldsymbol{r}$ is the location at which the algorithm is estimating gravity, $\boldsymbol{r}^*$ is the location at which the full gravity model was calculated, $\boldsymbol{g}$ is the gravity acceleration vector, and $\boldsymbol{G}$ is the partial derivative matrix of the gravity vector. The disadvantage of this method is that it assumes the gravity partial matrix to be constant, which will increase the error the longer this is used to propagate the gravity vector; however, it has been shown to be accurate when used to propagate gravity on the order of one or two seconds.[12]

Second Order Taylor Series Expansion

While the above first order expansion can be sufficient for short propagation times, it does not update the partial derivative matrix. This partial derivative matrix is used elsewhere in the navigation filter, so this matrix needs to be accurate. Two options on how to do this are presented below.

Point Mass Gravity Partial Update. The estimation accuracy of the first order Taylor series expansion can be increased by estimating the change in the gravity partial matrix. This can be done because the largest change of the gravity partial matrix is caused by first order terms. The change caused by the first order terms can be determined by calculating the partial gravity where the full model was calculated, as well as where it is being estimated, then applying this change to the partial derivative matrix from the full model. The point mass gravity partial matrix is calculated by taking a derivative of the gravity model where the body is a perfect sphere or a point mass; in this model the gravity vector is

$$\ddot{\boldsymbol{r}} = -\mu \frac{\boldsymbol{r}}{r^3} \tag{29}$$

Taking the derivative of this vector in Cartesian coordinates results in

$$\boldsymbol{G_{PM}}(\boldsymbol{r}) = \frac{\mu}{r^5}\begin{bmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3xy & 3y^2 - r^2 & 3yz \\ 3xz & 3yz & 3z^2 - r^2 \end{bmatrix} \tag{30}$$

This can then be used to update the gravity gradient matrix without having to run a full gravity model. Using this in Eq. (28), this becomes

$$\boldsymbol{g}(\boldsymbol{r}) \approx \boldsymbol{g}(\boldsymbol{r}^*) + \left[\boldsymbol{G}(\boldsymbol{r}^*) + \frac{\boldsymbol{G_{PM}}(\boldsymbol{r}) - \boldsymbol{G_{PM}}(\boldsymbol{r}^*)}{2}\right][\boldsymbol{r} - \boldsymbol{r}^*] \tag{31}$$

Or, to update the gravity gradient matrix at point $\boldsymbol{r}$, this becomes

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{G}(\boldsymbol{r}^*) + \frac{\boldsymbol{G_{PM}}(\boldsymbol{r}) - \boldsymbol{G_{PM}}(\boldsymbol{r}^*)}{2} \tag{32}$$

Hessian Estimation

Instead of needing to calculate the gravity partial matrix at the point where gravity is being estimated, the Hessian of the gravity acceleration components can be calculated at the same location as the full gravity model. As the Hessian matrix is only defined for a scalar value and not a matrix or vector, this requires separating the acceleration into the x, y, and z terms and results in three 3x3 matrixes that are the derivatives of Eq. (30), which are defined as

$$\boldsymbol{H_x} = \begin{bmatrix} \frac{\partial^2 \ddot{x}}{\partial x^2} & \frac{\partial^2 \ddot{x}}{\partial y \partial x} & \frac{\partial^2 \ddot{x}}{\partial z \partial x} \\ \frac{\partial^2 \ddot{x}}{\partial x \partial y} & \frac{\partial^2 \ddot{x}}{\partial y^2} & \frac{\partial^2 \ddot{x}}{\partial z \partial y} \\ \frac{\partial^2 \ddot{x}}{\partial x \partial z} & \frac{\partial^2 \ddot{x}}{\partial y \partial z} & \frac{\partial^2 \ddot{x}}{\partial z^2} \end{bmatrix}, \boldsymbol{H_y} = \begin{bmatrix} \frac{\partial^2 \ddot{y}}{\partial x^2} & \frac{\partial^2 \ddot{y}}{\partial y \partial x} & \frac{\partial^2 \ddot{y}}{\partial z \partial x} \\ \frac{\partial^2 \ddot{y}}{\partial x \partial y} & \frac{\partial^2 \ddot{y}}{\partial y^2} & \frac{\partial^2 \ddot{y}}{\partial z \partial y} \\ \frac{\partial^2 \ddot{y}}{\partial x \partial z} & \frac{\partial^2 \ddot{y}}{\partial y \partial z} & \frac{\partial^2 \ddot{y}}{\partial z^2} \end{bmatrix}, \boldsymbol{H_z} = \begin{bmatrix} \frac{\partial^2 \ddot{z}}{\partial x^2} & \frac{\partial^2 \ddot{z}}{\partial y \partial x} & \frac{\partial^2 \ddot{z}}{\partial z \partial x} \\ \frac{\partial^2 \ddot{z}}{\partial x \partial y} & \frac{\partial^2 \ddot{z}}{\partial y^2} & \frac{\partial^2 \ddot{z}}{\partial z \partial y} \\ \frac{\partial^2 \ddot{z}}{\partial x \partial z} & \frac{\partial^2 \ddot{z}}{\partial y \partial z} & \frac{\partial^2 \ddot{z}}{\partial z^2} \end{bmatrix} \tag{33}$$

Taking derivatives of Eq. (30), these become

$$\boldsymbol{H_x} = \frac{GM}{r^7}\begin{bmatrix} 5x(-2x^2 + y^2 + z^2) + 4xr^2 & 5y(-2x^2 + y^2 + z^2) - 2yr^2 & 5z(-2x^2 + y^2 + z^2) - 2zr^2 \\ 3yr^2 - 15x^2y & 3xr^2 - 15xy^2 & -15xyz \\ 3zr^2 - 15x^2z & -15xyz & 3xr^2 - 15xz^2 \end{bmatrix} \tag{34}$$

$$\boldsymbol{H_y} = \frac{GM}{r^7}\begin{bmatrix} 3yr^2 - 15x^2y & 3xr^2 - 15xy^2 & -15xyz \\ 5x(x^2 - 2y^2 + z^2) - 2xr^2 & 5y(x^2 - 2y^2 + z^2) + 4yr^2 & 5z(x^2 - 2y^2 + z^2) - 2zr^2 \\ -15xyz & 3zr^2 - 15y^2z & 3yr^2 - 15yz^2 \end{bmatrix} \tag{35}$$

$$\boldsymbol{H_z} = \frac{GM}{r^7}\begin{bmatrix} 3zr^2 - 15x^2z & -15xyz & 3xr^2 - 15xz^2 \\ -15xyz & 3zr^2 - 15y^2z & 3yr^2 - 15yz^2 \\ 5x(x^2 + y^2 - 2z^2) - 2xr^2 & 5y(x^2 + y^2 - 2z^2) - 2yr^2 & 5z(x^2 + y^2 - 2z^2) + 4zr^2 \end{bmatrix} \tag{36}$$

These are then used to estimate the gravitational acceleration through the equation

$$\boldsymbol{g}(\boldsymbol{r}) \approx \boldsymbol{g}(\boldsymbol{r}^*) + \boldsymbol{G}(\boldsymbol{r}^*)[\boldsymbol{r}-\boldsymbol{r}^*] + \frac{1}{2}\begin{bmatrix}[\boldsymbol{r}-\boldsymbol{r}^*]'\boldsymbol{H}_x(\boldsymbol{r}^*)[\boldsymbol{r}-\boldsymbol{r}^*] \\ [\boldsymbol{r}-\boldsymbol{r}^*]'\boldsymbol{H}_y(\boldsymbol{r}^*)[\boldsymbol{r}-\boldsymbol{r}^*] \\ [\boldsymbol{r}-\boldsymbol{r}^*]'\boldsymbol{H}_z(\boldsymbol{r}^*)[\boldsymbol{r}-\boldsymbol{r}^*]\end{bmatrix} \tag{37}$$

To update the gravity gradient matrix, this becomes

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{G}(\boldsymbol{r}^*) + \frac{1}{2}\begin{bmatrix}[\boldsymbol{r}-\boldsymbol{r}^*]'\boldsymbol{H}_x(\boldsymbol{r}^*) \\ [\boldsymbol{r}-\boldsymbol{r}^*]'\boldsymbol{H}_y(\boldsymbol{r}^*) \\ [\boldsymbol{r}-\boldsymbol{r}^*]'\boldsymbol{H}_z(\boldsymbol{r}^*)\end{bmatrix} \tag{38}$$

Three-Step Estimation Method

There are methods for further optimizing the propagation methods. Since the Jacobian of the acceleration doesn't change as significantly as acceleration, a three-step propagation method could be used. This requires running the full geopotential method at a low rate, such as 1 Hz. At a slightly higher rate, such as 10 Hz, a second order propagation method could be used to update the Jacobian matrix. Then at the rate needed by the navigation filter, such as 100 Hz or higher, the first order propagation method could be used to update the gravitational acceleration. This results in a geopotential model that takes approximately the same computation time as a first order model with approximately the accuracy of a second order model. Overall, this can result in the accuracy of an 8x8 model with close to the computation time of a J234 model.

SINGULARITY AT THE POLES

The forward column method discussed above has the flaw of needing to divide by the sine of the co-latitude; if the co-latitude is at ±90°, which occurs over the poles, this results in the equations being undefined due to dividing by zero. While there are relatively few satellites that will ever pass directly over either pole, accounting for this is relatively simple and requires few lines of additional code. This method is not as elegant as methods which convert everything into Cartesian coordinates prior to any calculations; however, this method maintains the increase in computation efficiency that comes from not having to use redundant variables while not including singularities.

One of the methods to avoid this singularity is to change the latitude in the full geopotential model to one that does not give a singularity. The distance away from the singularity can be decided ahead of time to provide the highest accuracy; if it is too small, there will be error from approaching the singularity; if it is too large, there will be increased error from propagation. Once this is known, the first or second order propagation methods can be used to propagate this to the actual spacecraft location. As the Hessian matrix is based off using the point mass method, this does not include a singularity at the poles; if desired, once the acceleration and partial matrix at the pole location are known, the Hessian matrixes can be calculated directly without propagation. This results in avoiding any singularity without sacrificing accuracy or computation time.

RESULTS

In order to test the above algorithms, five methods were created in MATLAB to test the run time of the full model at varying degrees/orders. These five were the forward column fully optimized, forward column non-optimized, Clenshaw fully optimized, Cunningham method as described in Reference 10, and the Pines method as described in Reference 9. These methods were run at varying locations at each maximum order/degree in order from 0 to 200 and every 100 orders/degrees from 200 to 2100 to calculate the average run time. To test the above propagation methods, two sample circular orbits were used: a Low Earth orbit (LEO) at approximately the same altitude as the International Space Station (408km), and an orbit close to geosynchronous at 34000km. These orbits were at an inclination of 52° to vary the latitude and longitude. To test the singularity avoidance method, a trajectory was generated near the poles at three different altitudes.

Full-Model Run Time Results

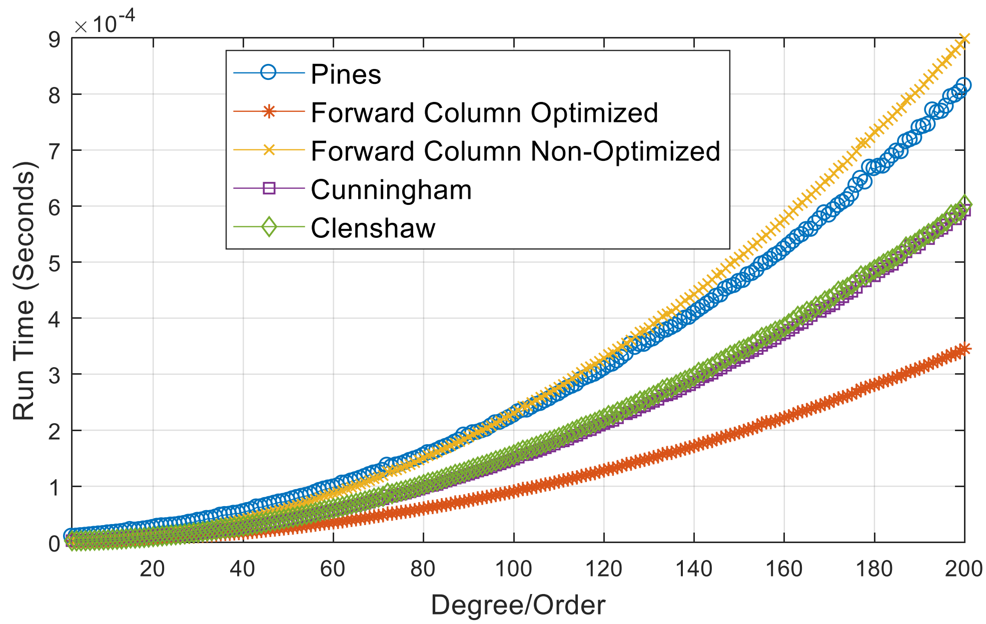

Fig.1 First derivative of the geopotential from degree/order 2 to 200.

Fig.1 shows the runtime of the five different methods up to degree 200 without the partial derivatives. This figure shows that the runtime of the forward column is very dependent on how it was optimized; without the optimization, it is not as efficient as other methods, while with the optimization, it is the most efficient.

Propagation Accuracy Results

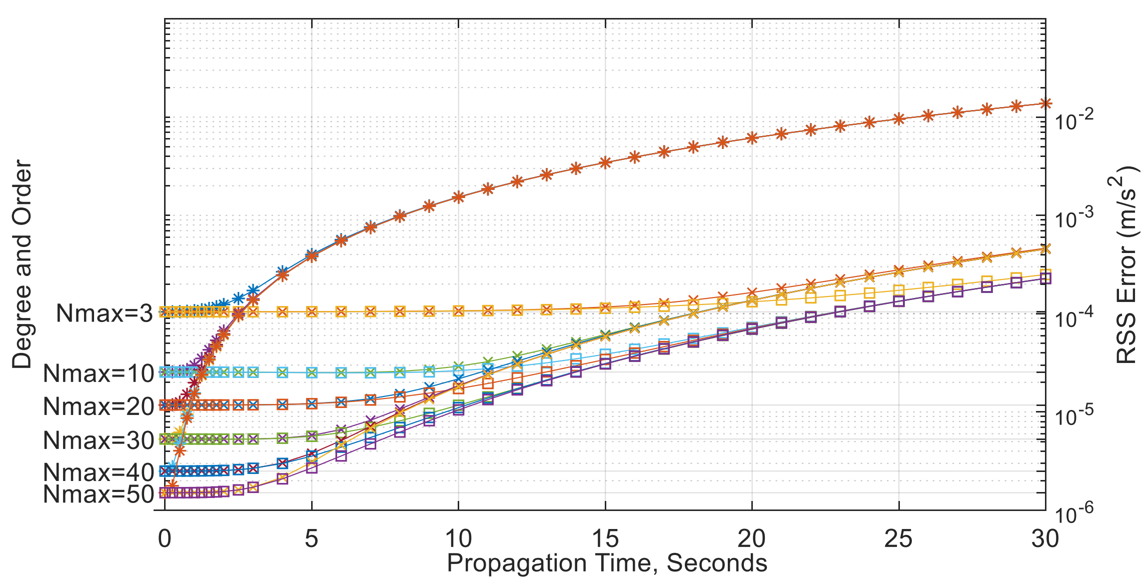

Fig. 2 Error caused by propagating the gravitational acceleration in a 400km orbit, with first order expansion (asterisk), second order expansion using the Hessian matrixes (x), and the second order expansion using the first order Jacobian update (square).

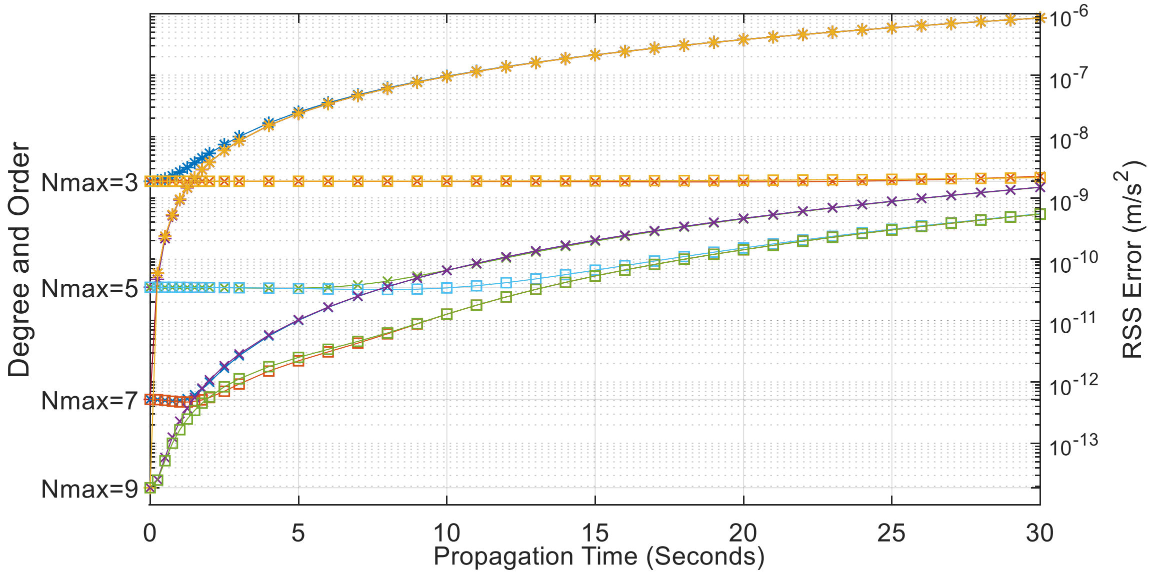

Fig. 3 Error caused by propagating the gravitational acceleration in a 34000 km orbit, with first order expansion (asterisk), second order expansion using the Hessian matrixes (x), and the second order expansion using the first order Jacobian update (square).

These results show that it is very feasible to use a first or second order expansion of the geopotential model in order to estimate the first and second derivatives of the geopotential. As shown in the above figures, the error caused by this propagation time widely varies between orbits, and is dependent on the altitude of the orbit. The lower the orbit, the higher the maximum order must be for the same accuracy levels.

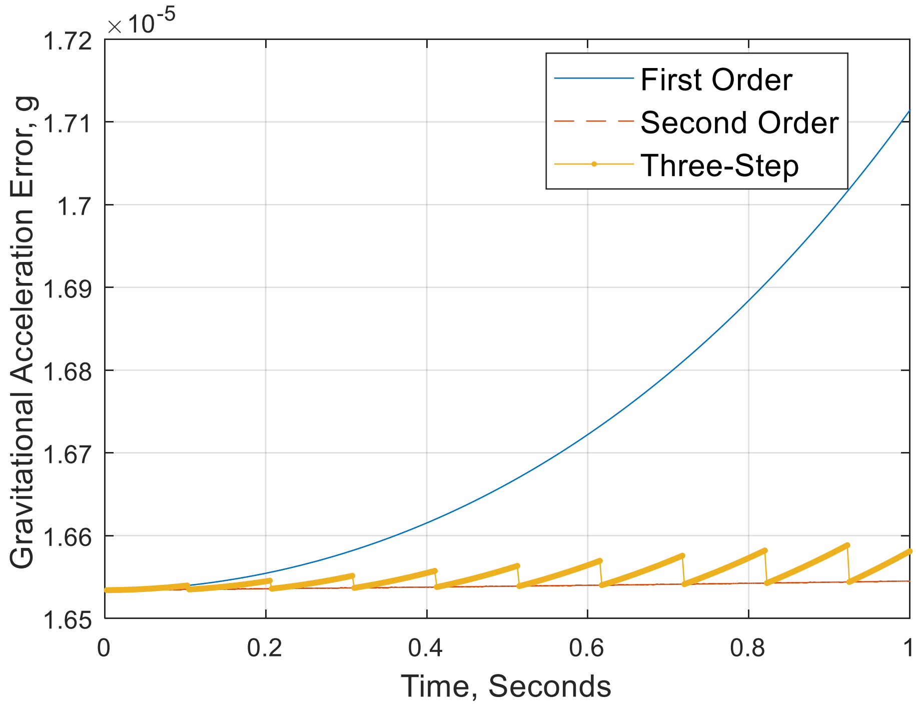

Fig. 4 Accuracy of the three step model for a sample 160km orbit. The full model was run at 1 Hz. For the three-step model, the second order update was run at a 0.1 Hz rate.

As shown in Figure 5, the error of the gravity model does increase over a time step between running full models. Using a second order model drastically decreases this increase in the error over the propagation interval. A three-step model results in approximately the same error level of the second order model, but with a lower computational requirement.

Accuracy at the Poles

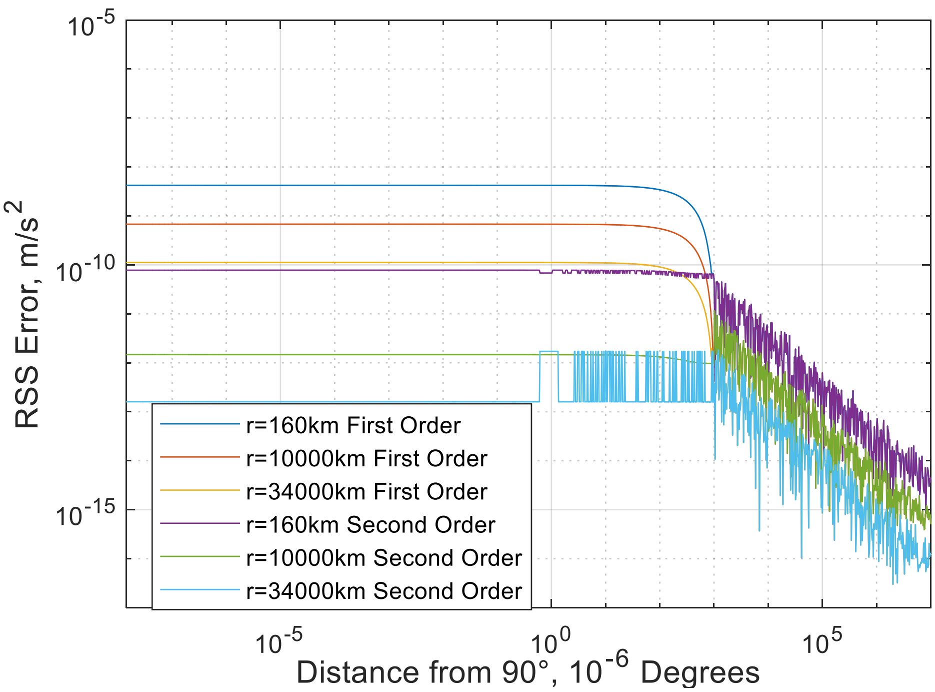

Fig. 5 Error caused by approaching the poles with delta=0.001 degrees using both first and second order expansion.

The propagation method was run with using a delta of 10^{-3} degrees. As this figure shows, using a first order expansion and delta=0.001 degrees results in completely avoiding any singularity, and the maximum error is less than 10^-8 m/s^2. If a second order expansion is used, this error can become less than 10^-10 m/s^2.

Propagation Timing Results

In order to test the effectiveness of the first and second order methods at decreasing the computation time, they were converted to C++ in order to be timed effectively. It was run at each propagation level, from running the full geopotential model each time to propagating 200 times between full models. This figure shows that there is diminishing returns when it comes to the number of times propagating; propagating 200 times is within 10% of the same run time as only propagating 20 times. This also shows that there is a large difference between the first and second order propagation methods. The three-step method is in between the first and second order methods in computation time, while maintaining accuracy close to the second order method.

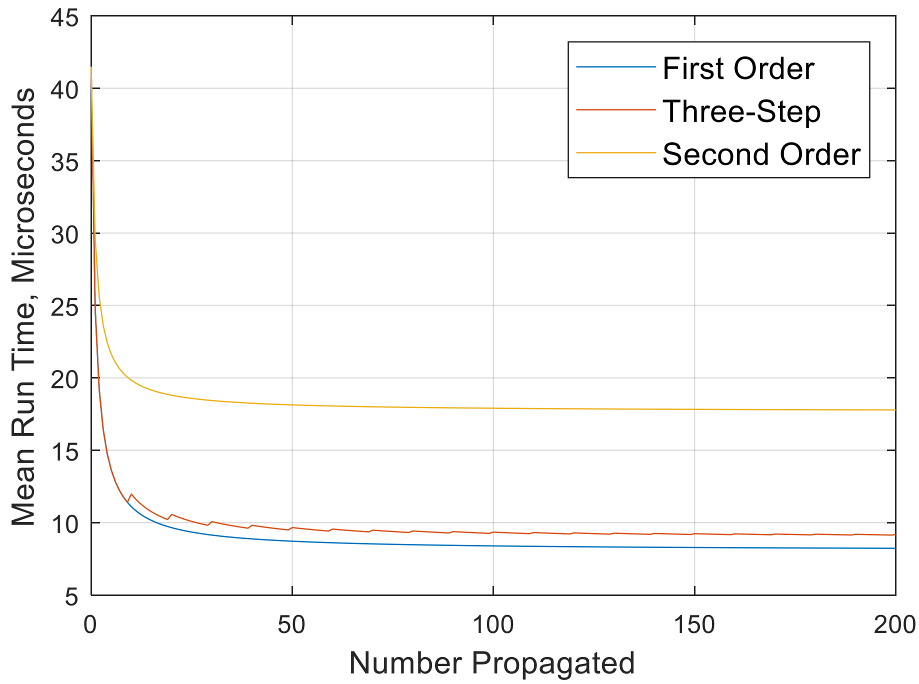

Fig. 6 Run time for various propagation intervals

APPLICATION TO THE EXPLORATION UPPER STAGE NAVIGATION SYSTEM

The geopotential implementation method that was originally planned for the Space Launch System (SLS) Exploration System Upper Stage (EUS) was the Cunningham method as described in Reference 10.[13] This is to be calculated at 100 Hz to match the output of the inertial navigation system used for SLS. The Cunningham method without propagation was determined to be too computationally intensive for the flight computers to process at 100 Hz. A trade study was accomplished to determine a more efficient method to estimate the geopotential; this paper is one of the products that came out of that trade study. The current SLS EUS navigation system still has the Cunningham method planned; however, it only calculates the full model at a 1 Hz rate. At the 100 Hz rate, a first order propagation is used to propagate the geopotential model to the current location. While this method is less efficient than using a forward column method and a second order or three-step propagation, it requires fewer changes to the navigation system design, while still decreasing the computational requirement to a level that is able to be handled by the flight computers.

FURTHER APPLICATIONS

Most of the above algorithms can be modified for any applications that require a similar equation to Eq. (1). One such application is the magnetic field of the Earth. The method described above, the forward column method, can be used after minor modification to calculate the magnetic field of the earth. This magnetic field can then be propagated in a very similar manner to either the first or second order methods.

CONCLUSION

This paper shows in detail the implementation of efficient geopotential estimation for use on satellites on-orbit. The methods for increasing the efficiency include using an efficient model for the full geopotential calculation, avoiding singularities, and utilizing a first and/or second order propagation method in order to reduce the needed frequency to compute the full model. These methods can increase the efficiency of geopotential estimation by an order of magnitude when included in flight software without introducing singularities. A properly implemented geopotential model can have the accuracy of an 8x8 model and approach the computational requirement of a J234 model.

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