

Adaptive Control Law for PID

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Abstract—This paper is the final report for my Spring 2019 internship at the Kennedy Space Center in Cape Canaveral, Florida. The official title of my internship is ‘Launch Vehicle Control Study internship,’ and I spent the spring working in GMRO- Granular Mechanics and Regolith Operations. There are two components to my project involving developing an adaptive control law for a PID controller. The first component involves performing a deep study and analysis of adaptive control laws, with the goal of developing stability proofs about the control law and the gain and phase margins. The second component involves analyzing the data received after implementing the adaptive control law. Unfortunately, as of the time of writing this paper, the data is not considered ‘clean’ enough to analyze. Therefore, in this paper I will give an overview of adaptive control law stability proofs and will write about the data analysis separately. Section II includes several general definitions, and the later sections have additional definitions at the end of each section.

I. Introduction

A PID (proportional integral derivative) controller is a common control algorithm which is implemented in NASA’s rocket launch system. The idea behind the PID controller is to calculate the error (current position – desired position) and drive it to zero by using proportional, integral, and derivative influences on the controller. For example, when a sailor steers a ship which is heading towards location x but would like to turn the ship to reach location x^* , he/she would initially turn the wheel significantly, and as the ship proceeds towards x^* the sailor would slowly shift the wheel back to its original position, and thereby drive the error, $x^* - x$, to zero.

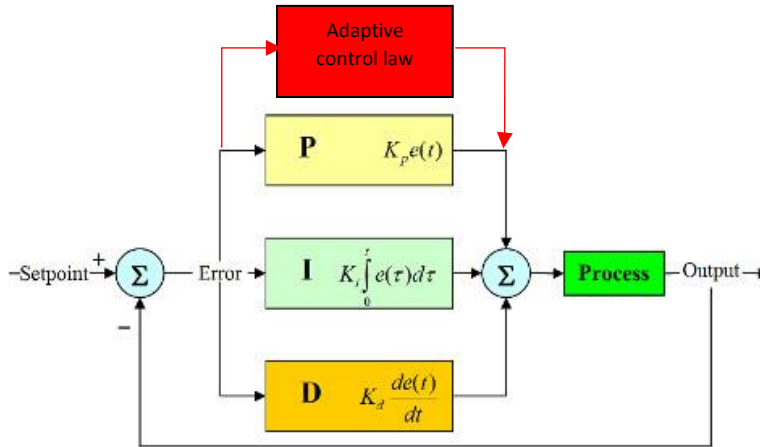
The mathematical formula for PID is:

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_d \frac{de}{dt}$$

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Where u is the control signal, e is the error (desired position – current position, or $x^* - x$) and k_p, k_i, k_d are constants (proportional gain, integral gain, derivative gain).
 A diagram of the PID controller with an adaptive control law in red is pictured below.



There are many mathematical proofs regarding the stability of PID-controlled systems and well-established robustness metrics, for example gain and phase margins. In general, we are trying to make the inherently linear infinite dimensional system asymptotically approach the output of a finite-dimensional reference model in a robust fashion. The difference between these two systems is the error which we are trying to drive to a neighborhood around zero.

There are two important theorems in [1] regarding stability of an infinite dimensional plant on a Hilbert space with disturbances of known waveform and unknown amplitude and phase. The first is Theorem 2, in which we assume almost strict dissipativity (ASD which we'll define later in this paper) and prove a robust stabilization result for linear dynamic systems on infinite-dimensional Hilbert spaces. Secondly, in Theorem 3 we show that adaptive model tracking is possible with very simple direct adaptive controller that knows very little specific information about the system it is controlling. Instead of having the error signals converging to 0, we will have them converge exponentially to a neighborhood of zero with a specific radius depending on the norm of the unknown disturbance.

II. Definitions

In order to begin going through the proofs, we must start out with several definitions.

General definitions of mathematical terms (closed linear operator, bounded linear operator, separable Hilbert space, dense topological space, marginally stable matrix, positive definite matrix, adjoint of an operator):

- **Linear**: an operator L is linear if for every pair of functions f, g and every scalar λ , the following conditions are satisfied:

- i. $L(f + g) = Lf + Lg$
- ii. $L(\lambda f) = \lambda Lf$

A linear operator A is **closed** where $A: D_A \rightarrow Y$ with X, Y being Banach spaces (complete³ normed vector space) over the same field of scalars, when we have:

$$\begin{aligned} x_n &\in D_A \\ x_n &\rightarrow x \\ Ax_n &\rightarrow y \end{aligned}$$

Then:

$$\begin{aligned} x &\in D_A \\ Ax &= y \end{aligned}$$

A **bounded** linear operator fulfills $\frac{\|Lv\|_Y}{\|v\|_X} \leq M < \infty$ for some $M \geq 0$. A bounded linear operator is closed.

- **Hilbert space** is a Banach space with inner product (\cdot, \cdot) which guarantees the norm: $\|u\| = (u, u)^{\frac{1}{2}}$

The norm must fulfill the following properties:

- (i) $(u, v) = (v, u) \quad \forall \frac{u}{v} \in H$
- (ii) (u, v) is linear $\leftrightarrow (u, \alpha v_1 + \beta v_2) = \alpha(u, v_1) + \beta(u, v_2)$
- (iii) $(u, u) \geq 0 \quad \forall u \in H$
- (iv) $(u, u) = 0 \leftrightarrow u = 0$

A Hilbert space is **separable** \leftrightarrow the space has a countable orthonormal⁴ basis. Any separable infinite dimensional Hilbert space is isometric to the space l^2 (square integrable functions).

- **Dense**: We have a topological space X with a subset $A \subseteq X$.

A is called dense in X if every point $x \in X$ we have either $x \in A$ or x is a limit point of A . In other words, $\text{closure}(A) = X$.

- Matrix F is a **marginally stable matrix** \leftrightarrow all the eigenvalues⁵ of F are zero or have negative real parts. The eigenvalues that have zero real parts are simple roots of the minimal polynomial of F . A minimal polynomial P of matrix F is a monic polynomial (polynomial with leading coefficient 1) of least degree such that $P(F) = 0$. Sometimes, the characteristic polynomial is the minimal polynomial.

³ Limit of every convergent sequence is in the space

⁴ Basis where all vectors are unit vectors and orthogonal to each other. Every vector in the infinite Hilbert space can be written as an infinite linear combination of the vectors in the basis.

⁵ To find the eigenvalues of matrix A , take the determinant of $(A - \lambda I)$, set it equal to 0 and solve for λ .

- **Positive definite matrix:** a symmetric $n \times n$ matrix A is considered positive definite when for all non-zero column vector x with n real numbers, $x^T Ax$ is strictly positive. This definition is equivalent to saying that the determinants of all upper-left submatrices (there will be n upper-left submatrices) of A are positive.
- **Adjoint of an operator:** Suppose we have $L: X \rightarrow Y$ with X, Y being Hilbert spaces. The adjoint operator is $L^*: Y \rightarrow X$ such that $\langle L(x), y \rangle = \langle x, L^*(y) \rangle, \forall x \in X, \forall y \in Y$. The adjoint L^* is unique.

III. Framework for proofs

X is an infinite dimensional separable Hilbert space with inner product (x, y) and corresponding norm $\|x\| = \sqrt{(x, x)}$. Let A be a closed linear operator with domain $D(A)$ dense in X .

Consider the linear infinite-dimensional plant with persistent disturbances:

$$(1) \begin{cases} \frac{\partial x(t)}{\partial t} = Ax(t) + Bu(t) + \Gamma u_D(t) + v, x(0) = x_0 \in D(A) \\ Bu = \sum_{i=1}^m b_i u_i \\ y(t) = Cx(t), y_i = (c_i, x(t)), i = 1, \dots, m \end{cases}$$

$x \in D(A)$ is the plant state

$b_i \in D(A)$ are actuator influence functions

$c_i \in D(A)$ are sensor influence functions

$u, y \in \mathbb{R}^m$: u is the control input, and y is the plant output

u_D is a disturbance with known basis functions ϕ_D

Assume v is bounded and has unknown disturbance such that $\|v\| \leq M_v < \infty$

Persistent disturbances: a disturbance vector $u_D \in \mathbb{R}^q$ is said to be persistent if it satisfies the disturbance generator equations:

$$(2) \begin{cases} u_D(t) = \theta z_D(t) \\ \dot{z}_D(t) = F_D(t) \text{ or } z_D(t) = L\phi_D(t) \end{cases}$$

F is a marginally stable matrix.

$\phi_D(t)$ is a vector of known functions forming a basis for all such possible disturbances, known as a ‘disturbance with known waveform but unknown amplitudes’.

We use the following linear finite-dimensional reference model:

$$(3) \begin{cases} \dot{x}_m = A_m x_m + B_m u_m \\ y_m = C_m x_m, x_m(0) = x_0^m \end{cases}$$

Where the reference model state x_m is an N_m dimensional vector with model output $y_m(t)$ having the same dimension as plant output $y(t)$. (In general the plant and reference models do not need to have the same dimension.)

The excitation of the model is accomplished via vector $u_m(t)$, generated by:

$$(4) \dot{u}_m = F_m u_m, u_m(0) = u_0^m$$

The model parameters (A_m, B_m, C_m, F_m) are assumed to be known.

The objective is to cause the output of the plant $y(t)$ to robustly asymptotically track the $y_m(t)$ reference model output defined above.

For this purpose, we will define the output error vector as:

$$(5) e_y \equiv y - y_m$$

As time goes to infinity, we will want $e_y \rightarrow N(0)$, where $N(0)$ is a neighborhood of the zero vector (of a specific radius which we will define later).

The direct adaptive control law will take the form of:

$$(6) u = G_m x_m + G_u u_m + G_e e_y + G_D \phi_D$$

The direct adaptive controller will have adaptive gains given by:

$$(7) \begin{cases} \dot{G}_u = -e_y u_m^* \gamma_u, \gamma_u > 0 \\ \dot{G}_m = -e_y x_m^* \gamma_m, \gamma_m > 0 \\ \dot{G}_e = -e_y e_y^* \gamma_e, \gamma_e > 0 \\ \dot{G}_D = -e_y \phi_D^* \gamma_D, \gamma_D > 0 \end{cases}$$

IV. SD, ASD, Theorem 2, Theorem 3

In this section, we will define ‘strict dissipativity’ and ‘almost strict dissipativity’, and state theorem 2 and theorem 3.

Firstly, we can combine two equations to create the following system regarding the error:

$$(8) \begin{cases} \frac{\partial e}{\partial t} = Ae + B\Delta u + v \\ e_y \equiv y - y_m = y - y_* = Ce \end{cases}$$

Strictly Dissipative (SD):

The triple (A_c, B, C) is said to be strictly dissipative (SD) if A_c is a densely defined, closed operator on $D(A_c) \subseteq X$. X is a complex Hilbert space with inner product (x, y) and corresponding norm $\|x\| \equiv \sqrt{(x, x)}$ and generates a C_0 semigroup of bounded operators $U(t)$, and (B, C) are bounded input/output operators with finite rank⁶ M .

$$\begin{aligned} B: \mathbb{R}^m &\rightarrow X \\ C: X &\rightarrow \mathbb{R}^m \end{aligned}$$

Additionally, there exist symmetric positive bounded operators P, Q on X such that they are bounded and coercive, meaning:

$$\begin{aligned} 0 \leq p_{\min} \|e\|^2 \leq (Pe, e) \leq p_{\max} \|e\|^2 \\ 0 \leq q_{\min} \|e\|^2 \leq (Qe, e) \leq q_{\max} \|e\|^2 \end{aligned}$$

And:

$$(9) \begin{cases} \operatorname{Re}(PA_c e, e) \equiv \frac{1}{2} [(PA_c e, e) + \overline{(PA_c e, e)}] = \\ \frac{1}{2} [(PA_c e, e) + (e, PA_c e)] = \\ -(Qe, e) \leq -q_{\min} \|e\|^2; e \in D(A_c) \\ PB = C^* \end{cases}$$

C^* is the adjoint of the operator C .

Almost Strictly Dissipative (ASD):

We say that (A, B, C) is almost strictly dissipative (ASD) when there exists a $G_* \in \mathbb{R}^{m \times m}$ such that (A_c, B, C) is strictly dissipative, where $A_c \equiv A + BG_*C$

Definitions for IV:

- **C_0 semigroup**: a semigroup is a set S together with a binary operation (for example, \cdot such that $S \times S \rightarrow S$) that satisfies the associative property: $\forall a, b, c \in S: (a \cdot b) \cdot c = a \cdot (b \cdot c)$
 C_0 semigroup is also known as a strongly continuous one-parameter semigroup, it is a representation of the semigroup $(\mathbb{R}_+, +)$ on a Banach space X that is continuous in the strong operator topology.
 Formally, a strongly continuous semigroup on Banach space X is a map $T: \mathbb{R}_+ \rightarrow L(X)$ such that the following conditions apply:
 - $T(0) = I$ (identity operator on X)
 - $\forall t, s \geq 0: T(t + s) = T(t)T(s)$
 - $\forall x_0 \in X: \|T(t)x_0 - x_0\| \rightarrow 0$ as $t \rightarrow 0$
- **Bounded operators**: Suppose we have two normed vector spaces X, Y and a linear transformation $L: X \rightarrow Y, v \in X$. If the ratio of $\|L(v)\|$ to $\|v\|$ is bounded above by the

⁶ Bounded operator such that the range is finite dimensional

same number over all nonzero vectors $v \in X$ then L is a bounded operator. More formally, $\exists M \geq 0$ such that $\forall v \in X, \frac{\|L(v)\|_Y}{\|v\|_X} \leq M < \infty$.

- **Symmetric operator** on Hilbert space: A linear operator A acting on Hilbert space H with dense domain $Dom(A)$ is symmetric if: $\langle Ax, y \rangle = \langle x, Ay \rangle \forall x, y \in Dom(A)$
- **Positive operator**: a symmetric operator A is called positive if $\langle Ax, y \rangle \geq 0 \forall x \in Dom(A)$

Hypothesis:

We have an ideal trajectory defined as:

$$(10) \quad \begin{cases} x_* = S_{11}^* x_m + S_{12}^* u_m + S_{13}^* z_D = S_1 z \\ u_* = S_{21}^* x_m + S_{22}^* u_m + S_{23}^* z_D = S_2 z \end{cases}$$

With: $z = [x_m \quad u_m \quad z_d]^T \in \mathbb{R}^L$

Where the ideal trajectory $x_*(t)$ is generated by the ideal control $u_*(t)$ from:

$$(11) \quad \begin{cases} \frac{\partial x_*}{\partial t} = Ax_* + Bu_* + \Gamma u_D \\ y_* = Cx_* = y_m \end{cases}$$

Assume:

1. (A, B, C) is ASD, meaning there exists a gain G_e^* such that the triple $(A_C \equiv A + BG_e^*G, B, C)$ is SD.
2. A is a densely defined, closed operator on $D(A) \subseteq X$ and generates a C_0 semigroup of bounded operators $U(t)$.
3. ϕ_D is bounded.

We can combine equations (6),(7),(10) to obtain:

$$\Delta u = u - u_* = \dots = G_e^* e_y + \Delta G \eta$$

Where:

$$\begin{aligned} \Delta G &\equiv G - G_* \\ G &= [G_e \quad G_m \quad G_u \quad G_D] \\ G_* &\equiv [G_e^* \quad S_{21}^* \quad S_{22}^* \quad S_{23}^* L] \\ \eta &\equiv [e_y \quad x_m \quad u_m \quad \phi_D]^T \end{aligned}$$

Additionally, we can combine equations (1),(6),(7),(8),(9) so that the error system becomes:

$$\left\{ \begin{array}{l} \frac{\partial e}{\partial t} = (A + BG_e^*C)e + B\Delta G\eta + v = A_c e + B\rho + v \\ \Delta \dot{G} = \dot{G} - \dot{G}_* = \dot{G} = -e_y \eta^* \gamma \\ \gamma \equiv \begin{bmatrix} \gamma_e & 0 & 0 & 0 \\ 0 & \gamma_m & 0 & 0 \\ 0 & 0 & \gamma_u & 0 \\ 0 & 0 & 0 & \gamma_D \end{bmatrix} > 0 \end{array} \right.$$

With: $e \in D(A)$, $\rho \equiv \Delta G\eta$, $e_y = ce$

Theorem 2 (Robust stabilization)

Statement:

Consider the coupled system of differential equations:

$$\left\{ \begin{array}{l} \dot{e} = A_c e + B\Delta Gz + v, \quad \Delta G = G(t) - G^* \\ e_y = Ce \\ \dot{G}(t) = -e_y z^T \gamma - aG(t) \end{array} \right.$$

$$\begin{array}{l} e \\ v \end{array} \in D(A_c)$$

$$z \in \mathbb{R}^m$$

$$\begin{bmatrix} e \\ G \end{bmatrix} \in \bar{X} \equiv X \times \mathbb{R}^{m \times m} \text{ is a Hilbert space,}$$

$$\text{with inner product } \left(\begin{bmatrix} e_1 \\ G_1 \end{bmatrix}, \begin{bmatrix} e_2 \\ G_2 \end{bmatrix} \right) \equiv (e_1, e_2) + tr^7(G_1 \gamma^{-1} G_2),$$

$$\text{and norm } \left\| \begin{bmatrix} e \\ G \end{bmatrix} \right\| \equiv (\|e\|^2 + tr(G_1 \gamma^{-1} G))^{1/2}$$

$G(t)$ is the $m \times m$ adaptive gain matrix

γ is any positive definite constant matrix of appropriate dimension.

Assume:

1. (A, B, C) is ASD with $A_c \equiv A + BG_*C$
2. There exists $M_G > 0$ such that $\sqrt{tr(G^*G^{*T})} \leq M_G$
3. There exists $M_v > 0$ such that $\sup_{t \geq 0} \|v(t)\| \leq M_v < \infty$

⁷ Tr = trace of a matrix = the sum of all the diagonal elements

⁸ Inverse of matrix A is the matrix that when multiplied with A will produce the identity matrix

⁹ Sup = supremum = least upper bound

4. There exists $\alpha > 0$ such that $\alpha \leq \frac{q_{\min}}{p_{\max}}$, where q_{\min}, p_{\max} are defined in the strictly dissipative definition
5. The positive definite matrix γ satisfies $tr(\gamma^{-1}) \leq \left(\frac{M_v}{\alpha M_G}\right)^2$

Then:

The gain matrix $G(t)$ is bounded, and state $e(t)$ approaches the ball of the radius exponentially at the rate e^{-at} :

$$\text{Radius: } R_* \equiv \frac{1+\sqrt{p_{\max}}}{\alpha\sqrt{p_{\min}}} M_v$$

Theorem 3:

Using the framework we set forth in Hypothesis 1 and applying Theorem 2, we have a robust state and output tracking of the reference model, meaning the error system approaches the zero vector with radius R_* as time goes to infinity:

$$\lim_{t \rightarrow \infty} \begin{bmatrix} e \\ \Delta G \end{bmatrix} = N(0, R_*)$$

Since C is a bounded linear operator, we have:

$$e_y = y - y_m = Ce$$

$$\lim_{t \rightarrow \infty} Ce = N(0, R_*)$$

With bounded adaptive gains:

$$G \equiv [G_e \quad G_m \quad G_u \quad G_D] = G_* + \Delta G$$

V. Conclusion

In theorem 2, we showed that if we have a linear infinite dimensional system which is almost strictly dissipative, we have a robust stabilization result of the error approaching a specific neighborhood around the zero vector. In theorem 3, we have a simple direct adaptive controller which knows very little information about the system and we show that the adaptive model tracking is still possible in this case. These theorems, together with the future data analysis, have many potential applications to improve PID systems which are implemented in a wide range of industries across the world.

VI. References

[1] Robust adaptive model tracking for distributed parameter control of linear infinite-dimensional systems in Hilbert space. (2014). *IEEE/CAA Journal of Automatica Sinica*, 1(3), 294-301. doi:10.1109/jas.2014.7004687