Three CA-Related Covariance Issues and their Solutions

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Reporting on the work of B. Bowman and S. Casali

Covariance Realism in Prediction: ASW Dynamic Consider Parameter

Covariance Accuracy in Prediction

- **Orbit determination process generates a state estimate (position and velocity) for an object at a particular time (the epoch time)**
  - Usually a time within the fit-span for a batch differential correction
- **The associated covariance is a formal estimate of the error of that fit**
  - Based on the observations used, their times, and their expected uncertainties
  - Gives estimation variances of the fitted parameters and their correlations
- **CA, however, is always performed in prediction**
  - Uses states and covariances propagated to a TCA
- **For propagated covariances to be representative, must "consider" sources of error encountered in prediction but not part of fit**
  - Implementation is called in ASW "Dynamic Consider Parameter," or DCP
- **Two major sources**
  - Atmospheric density forecast error
  - Satellite frontal area uncertainty

Atmospheric Density Forecast Error

- **Atmospheric density values needed for satellite drag solutions**
- **Neutral atmospheric density models used for this**
  - Number of different models available; ASW uses JBH09
  - Fed by space weather indices
    - EVU indices: F10, S10, M10, Y10
    - Joule heating indices: Ap, Dst
- **During fit, actual measured values of these indices available**
  - Density error is thus model error, which is generally not very large
- **During prediction, predicted values of these indices must be used**
  - Ability to predict these values very primitive
    - Weighted autoregressive fit of last five solar rotations' worth of data (135 days)
    - Perhaps acceptable for EVU indices; not so predictive for Joule heating / storms
- **Need to consider this additional error in predicted covariance**

Atmospheric Density Forecast Error Calculation Step I: Debias Density Model*

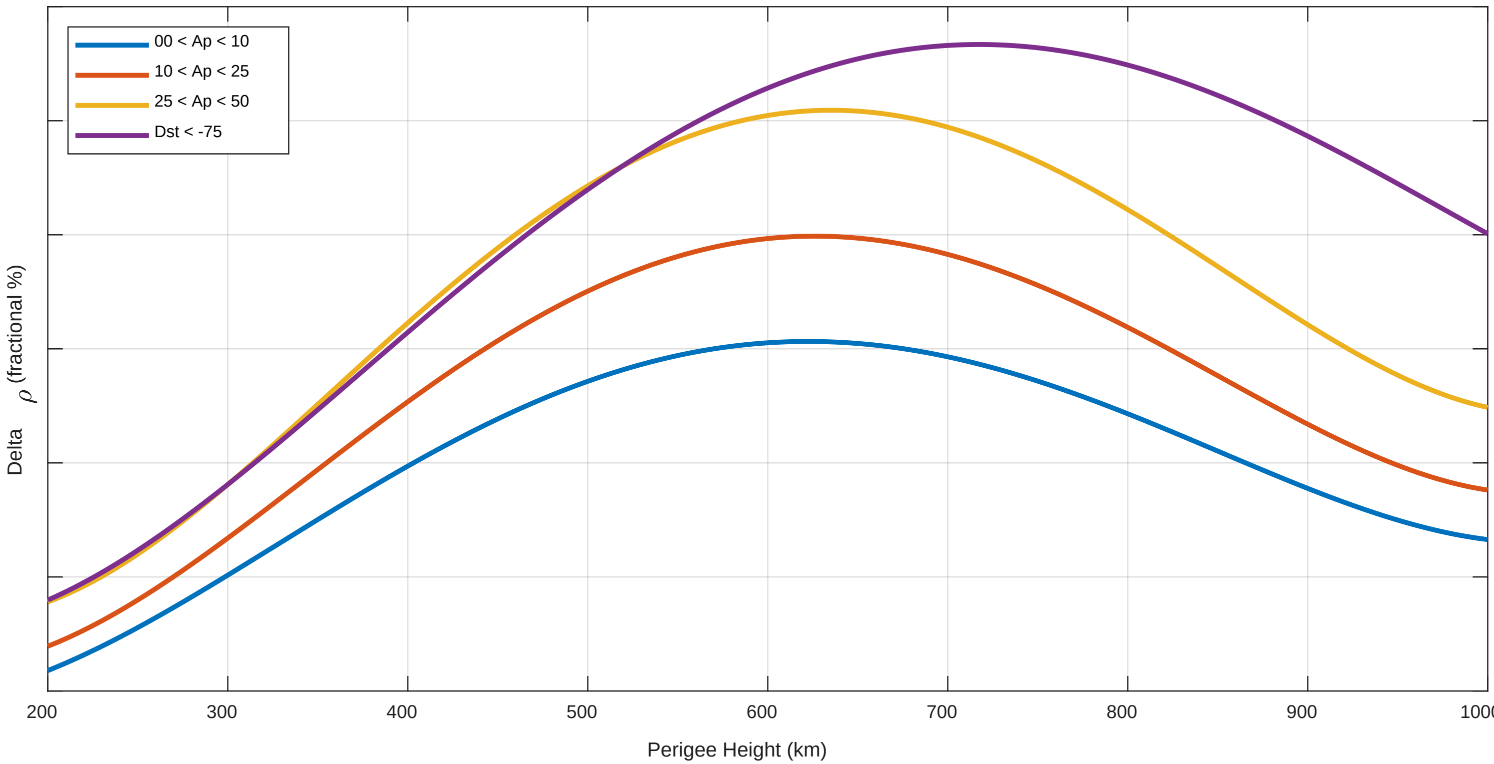
- **Obtain tracking data on disbursed group of LEO satellites that have very stable ballistic coefficients (B terms)**
  - About 100 such satellites used for present approach
- **Perform ensemble differential correction of all of these data**
  - Six-element correction for all of the (100) satellites
  - Constrained B-term correction for the (100) satellites also
    - Few percent correction allowed to enable adjustment for local density error
  - Global atmospheric model correction factors to temperature coefficients
- **Global correction factors ("DCA Coefficients") used to correct atmospheric model application to past data, employing definitive solar indices**
- **Correction factors can be projected into the future, to apply to satellite states propagated beyond the current time**
  - Low-order autoregression model used

* See Casali, S. and Barker, W. "Dynamic Calibration Atmosphere (DCA) for the High Accuracy Satellite Drag Model (HASDM)." AAS Astrodynamics Specialist Conference, Monterey CA, Paper 2002-4888, Aug. 2002.

Atmospheric Density Forecast Error Calculation Step II: Determine Forecast Errors

- **Calculate density field for time point in the past**
  - Use definitive space weather indices and associated DCA coefficients for particular time point
- **Calculate same density field, but with "predicted" values**
  - Predicted space weather indices that would have been used for that time point
  - Projected DCA coefficients that would have been applied at that time point
- **Set of density residuals obtained for this time point**
  - Averaged value of $\Delta\rho/\rho$ serves as statement of global density error
  - Standard deviation of all such $\Delta\rho/\rho$ serves as statement of error for time point
- **Results parameterized by satellite perigee height and solar activity (not EUV but Joule heating / geomagnetic activity)**
- **Weighted average used for single error value**
  - $\frac{5}{9}\sigma_{Day\ 1} + \frac{3}{9}\sigma_{Day\ 2} + \frac{1}{9}\sigma_{Day\ 3}$
  - Considers greater effect of error on first day, as it will grow quadratically with further days of prediction

Atmospheric Density Forecast Error: Function of Perigee Height and Solar Activity



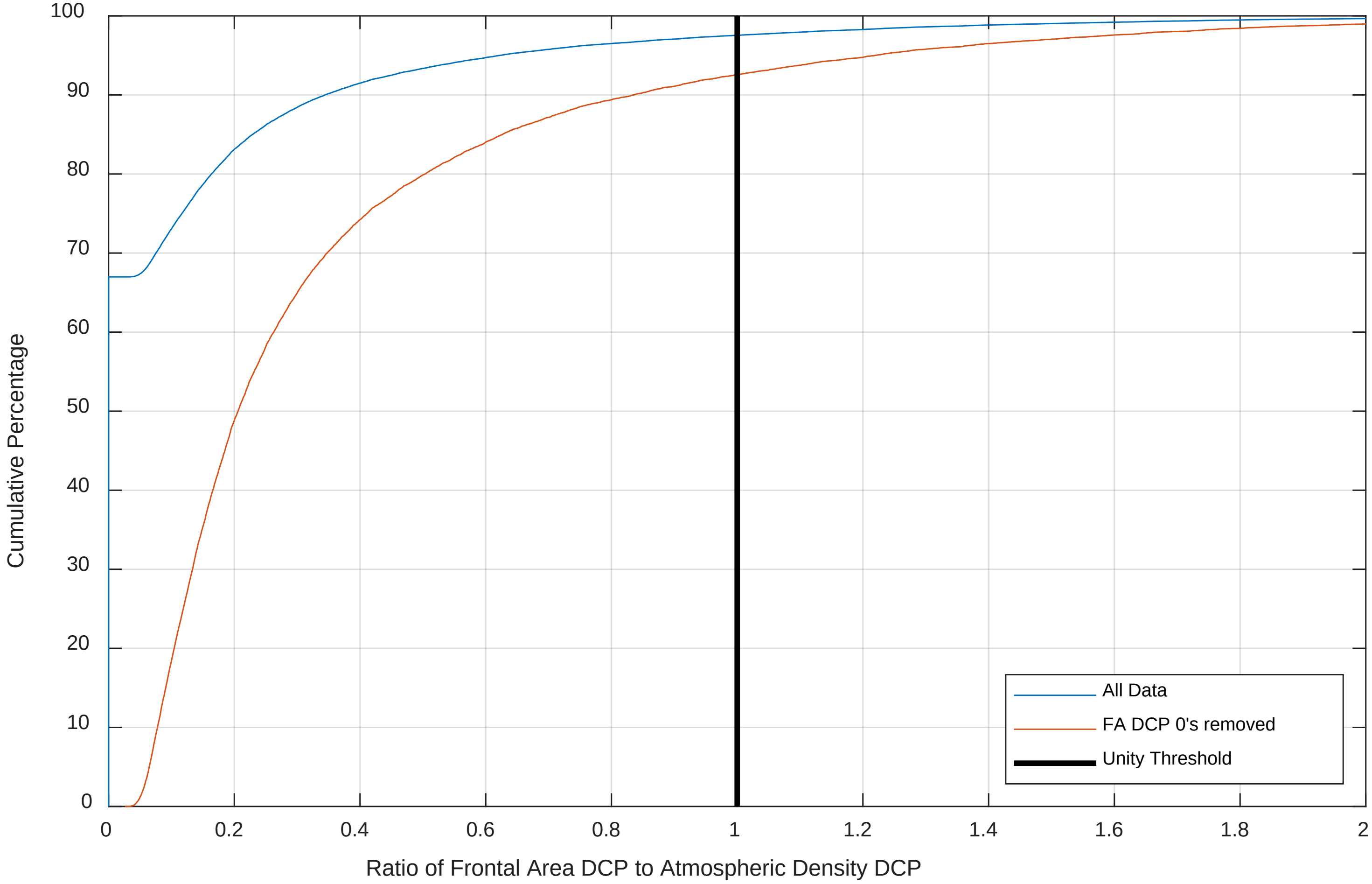
Satellite Frontal Area Uncertainty: Concept

- **HASDM technique gives confidence that density estimates with "definitive" solar indices are very accurate**
  - Solved-for ballistic coefficients should therefore very rarely be absorbing errors in atmospheric density estimate (possible due to aliasing, as shown below)
  - $\ddot{r} = -\frac{1}{2} B \rho v_r^2 e_v$
- **Ballistic coefficient (B) is given by combination of drag coefficient, satellite frontal area, and satellite mass**
  - $\ddot{r} = -\frac{1}{2} \left(C_D \frac{A}{M} \right) \rho v_r^2 e_v$
- **C_D and M essentially constant; and, as stated above, B not expected to be absorbing error in *ρ***
- **So changes in B over time should be attributable to changes in satellite frontal area**
  - Small amount of localized definitive density error aliased into B

Satellite Frontal Area Uncertainty: Calculation

- **Every satellite evaluated independently**
- **B histories easily available from update histories for each object**
- **However, number of data regularization activities needed**
  - B generation times moved to middle of OD interval
  - Outliers rejected
  - Datasets manifesting certain irregularities also rejected
  - Single value for each day determined, using averaging and interpolation
- **For datasets that survive above tests, assemble $\Delta B/B_{avg}$ at 1, 2, and 3 days of offset, and calculate standard deviations**
    - Bavg the B value at the center of the fit-span
    - Offset values that number of days from this averaged value
  - Same weighted sum to produce single value:
    - $\frac{5}{9}\sigma_{Day\ 1} + \frac{3}{9}\sigma_{Day\ 2} + \frac{1}{9}\sigma_{Day\ 3}$
  - As stated previously, contains small amount of localized definitive density error

Density vs Frontal Area Errors: Comparison



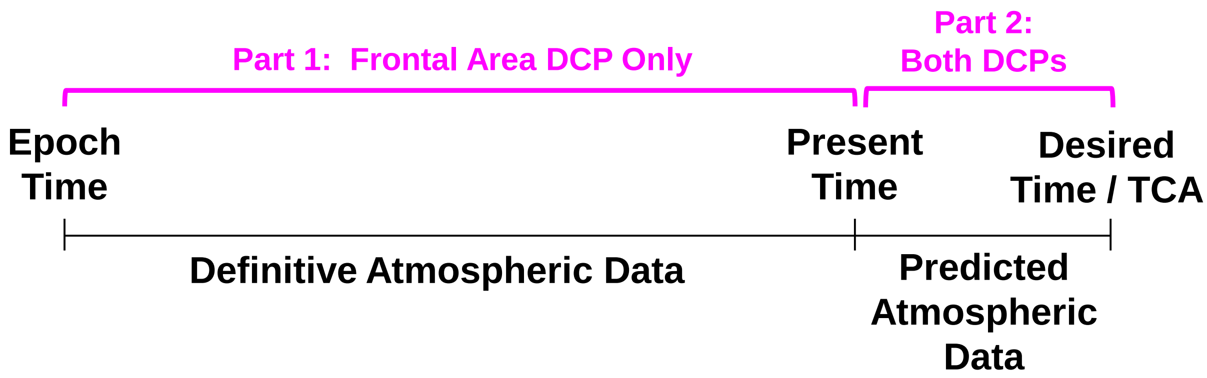
Density and Frontal Area Consider Parameters: Application

- **Consider parameters are standard deviations; squares are variances**
- **Both consider parameters can be considered additional B errors**
  - Frontal area variation is B error directly
  - Atmospheric density forecast error is error in ρ but is aliased with B
- **Method of application is to add consider parameter variances to B variance**
  - $\sigma^2_{Btotal} = \sigma^2_B + \sigma^2_{Atm} + \sigma^2_{FA}$
  - But consider subtlety outlined on next slide
- **Will be placed in orange box in example at right**

	U (m)	V (m)	W (m)	Udot (m/s)	Vdot (m/s)	Wdot (m/s)	B (m2/kg)	AGOM (m2/kg)
U	6.84E+01	-2.73E+02	6.38E+00	2.76E-01	-7.14E-02	8.75E-03	-3.83E-02	-3.83E-02
V	-2.73E+02	1.10E+05	3.23E+01	-1.17E+02	-8.99E-02	2.51E-02	-1.28E-01	-1.28E-01
W	6.38E+00	3.23E+01	4.47E+00	-3.26E-02	-6.83E-03	1.81E-03	-3.73E-03	-3.73E-03
Udot	2.76E-01	-1.17E+02	-3.26E-02	1.24E-01	1.10E-04	-2.47E-05	1.46E-04	1.46E-04
Vdot	-7.14E-02	-8.99E-02	-6.83E-03	1.10E-04	7.57E-05	-9.39E-06	4.10E-05	4.10E-05
Wdot	8.75E-03	2.51E-02	1.81E-03	-2.47E-05	-9.39E-06	2.06E-05	-4.39E-06	-4.39E-06
B	-5.07E-03	1.30E+00	4.34E-05	-1.38E-03	7.97E-07	7.26E-07	1.64E-05	-6.28E-07
AGOM	-3.83E-02	-1.28E-01	-3.73E-03	1.46E-04	4.10E-05	-4.39E-06	-6.28E-07	2.31E-05

Temporal Placement of Consider Parameters

- **Part 1: propagation from epoch time to present time**
  - Uses definitive (issued) space weather indices and HASDM coefficients
  - No atmospheric density forecast error, since all in the past
  - Drag error combination of model error (small) and satellite frontal area fluctuation (object-dependent), so frontal area consider parameter applied
- **Part 2: propagation from present time to desired future time**
  - Atmospheric density model error (small), satellite frontal area fluctuation, and atmospheric density forecast error all in play
  - Both consider parameters applied for this portion
- **For CA, long propagation times are almost always in Part 1**



Summary/Significance

- **Covariances provided in CDMs have been pre-adjusted to compensate for expected drag-related prediction errors**
  - Satellite frontal area uncertainty (usually smaller)
  - Atmospheric density forecast error (usually larger)
  - Modification to B variance; affects all covariance terms during propagation
- **Adjustments include compensation for solar storms**
  - *Anemomilos* solar event prediction module part of JBH09 density model
  - High predicted Ap and (actually low predicted) Dst values will increase covariance size, reflecting additional prediction error
    - If storm predictions are inaccurate, covariances will be improperly sized
  - Covariances thus change with each 8-hour upload of space weather indices
- **Frontal area DCP in play for entire propagation interval, density forecasting DCP applies only from present time to future propagated state**

Reporting on the work of S. Casali *et al.**

Decorrelating Joint Covariance for 2-D Pc Calculation

*Casali, S., Hall, D., Snow, D., Hejduk, M., Johnson, L., Skrehart, B., and Baars, L. "Effect of Cross-Correlation of Orbital Error on Probability of Collision Determination." 2018 AAS/AIAA Astrodynamics Specialist Conference (paper # AAS 18-272), Snowbird UT, August 2018.

Joint Covariances

- **2-D Pc calculation requires formation of joint position covariance**
  - Comparison of joint error to HBR circle placed at relative position vector at TCA
- **If primary and secondary satellite position errors are uncorrelated, joint covariance formation is straightforward**
  - $P_J = P_P + P_S$
- **If covariances contain correlated error, then this error must be removed in order to form joint covariance**
  - Otherwise joint covariance too large; affects Pc
- **Questions**
  - Is there significant common error in covariances?
  - How might such error be identified and removed?

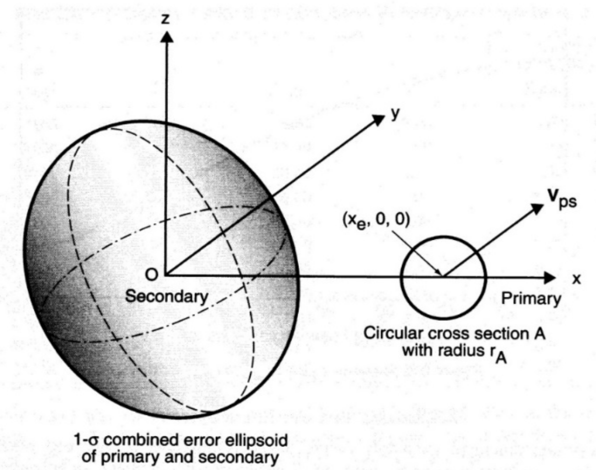


Figure taken from Chan (2008)

Correlated Error in Covariances

- **Fit error**
  - Dynamical model errors, manifested in satellite position error in the same way – small
  - Unmodeled sensor observation errors when data from a common sensor used in OD fit for both objects – also small
- **Prediction error**
  - B error due to frontal area uncertainty – uncorrelated between satellites
  - Global atmospheric density forecast error – can be large
    - Discussed in earlier part of presentation—the major contributor to the dynamic consider parameter
    - B-terms of both primary and secondary covariances modified to account for this error; spreads to entire covariance through propagation to TCA
- **Focus should be on removing shared portion of expected global atmospheric density forecast error**

Correlated Error: Theoretical Development

- **Joint covariance formation relationship**
  - $\mathbf{P}_J = \mathbf{P}_p + \mathbf{P}_s - E[(\mathbf{r}_p - \boldsymbol{\mu}_p)(\mathbf{r}_s - \boldsymbol{\mu}_s)^T] - E[(\mathbf{r}_s - \boldsymbol{\mu}_s)(\mathbf{r}_p - \boldsymbol{\mu}_p)^T]$
  - $\mathbf{r}$ is satellite actual state; $\boldsymbol{\mu}$ is mean state (OD state estimate)
  - Joint covariance is sum of primary and secondary covariances, with their cross-correlations subtracted
- **Satellite state error at TCA (here primary): $\mathbf{r}_p - \boldsymbol{\mu}_p = \boldsymbol{\Phi}_p \boldsymbol{\delta}_p + \mathbf{G}_p \Delta_g$**
  - $\boldsymbol{\Phi}$ is state transition vector, used to project epoch error to TCA
  - $\boldsymbol{\delta}$ is error vector at epoch
  - Δ_g is global atmospheric density parameter error
  - $\mathbf{G}$ is sensitivity vector mapping global density error into state error at TCA
- **Plugging in terms and simplifying, one obtains**
  - $\mathbf{P}_m = \mathbf{P}_p + \mathbf{P}_s - \sigma^2_g (\mathbf{G}_p \mathbf{G}^T{}_s + \mathbf{G}_s \mathbf{G}^T{}_p)$
    - $\sigma^2_g = E[\Delta^2_g]$
- **More elaborate form, allowing for different primary/secondary Δ_g**
  - $\mathbf{P}_m = \mathbf{P}_p + \mathbf{P}_s - \sigma_{s/g}\, \sigma_{p/g}\, \mathbf{G}_p \mathbf{G}^T{}_s - \sigma_{s/g}\, \sigma_{p/g}\, \mathbf{G}_s \mathbf{G}^T{}_p$

Covariance Decorrelation: Technical Implementation

- **Estimation techniques exist for G, based on information in CDM**
- **However, not necessary—CDM enhancement to provide G and σ^2_g values scheduled for ASW version release 19-2 (fall 2019)**
  - Drag sensitivity vectors (G) for primary and secondary objects
  - Atmospheric density forecast error consider parameters for primary and secondary objects
  - These data to be placed in CDM comments field in standardized way
  - Accompanying math specification will explain how to use these data to generate decorrelated joint covariance
- **Decorrelated joint covariance can be used to calculate more realistic Pc**

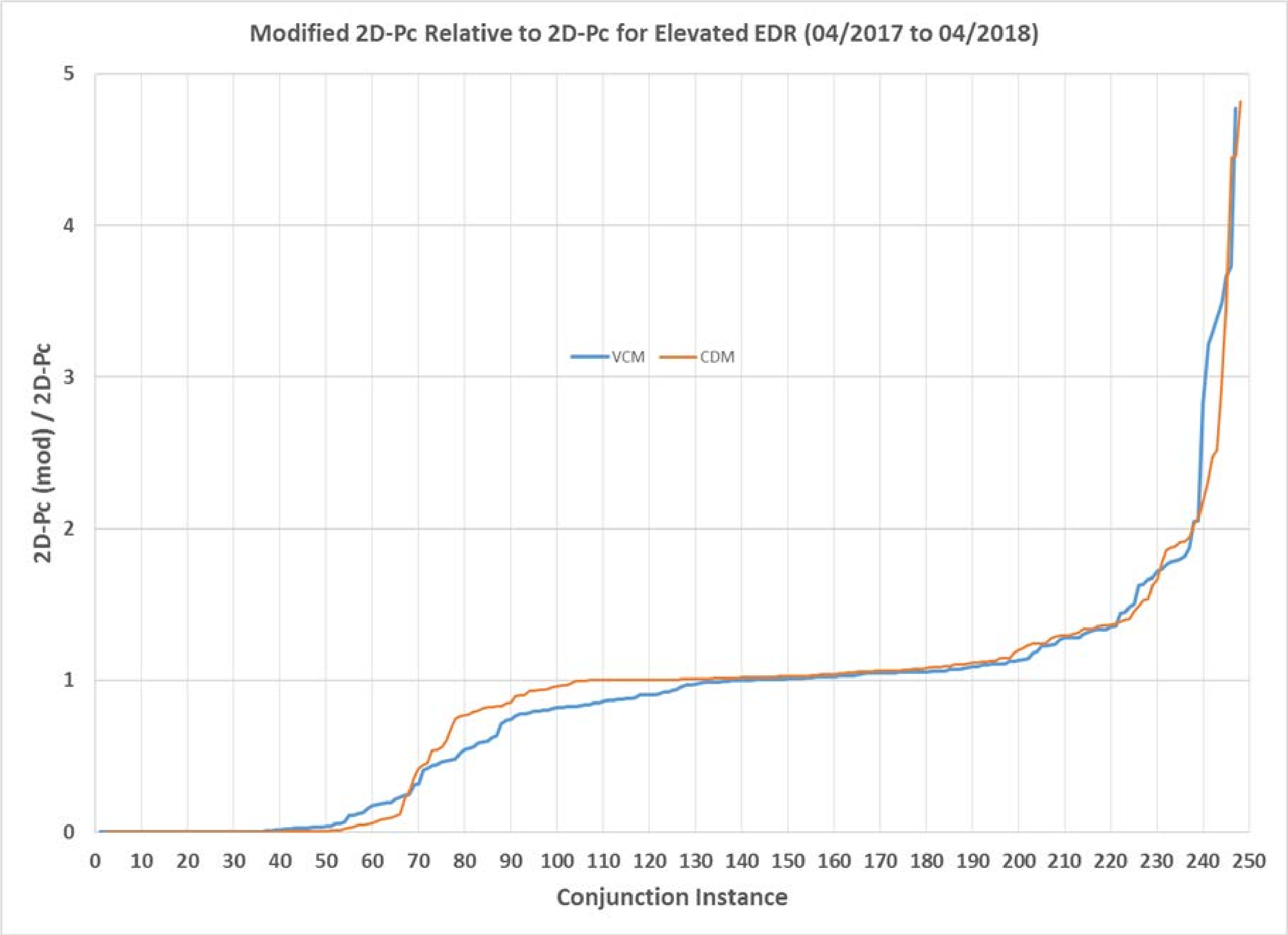
Covariance Cross-correlation: When does it Matter?

- **When drag uncertainty small, effect not significant**
  - Cross-correlation may be high, but absolute amount of correlated error low; removing a small value introduces little change
- **Head-on events not significantly affected**
  - Drag affects TCA uncertainty but not collision probability
- **Events with similar drag levels can negate effect of global error**
  - Could have effect of cancelling out entire forecast density error compensation
- **So higher-drag, crossing conjunctions will benefit the most from covariance decorrelation**

Covariance Cross-correlation: How much does it Matter?

- **250 higher-drag conjunctions profiled (APR 2017 – APR 2018)**
  - Energy dissipation rate (EDR) values > 0.0006 W/kg
- **2-D Pc ratio calculated (decorrelated Pc / uncorrected Pc)**
- **Two types of results shown**
  - Exact calculation, using numerically-calculated sensitivity vectors
  - Estimation method, using only information on current CDM
- **30% show significant reductions in Pc from decorrelation**
- **10% show considerable increases in Pc from decorrelation**
- **Effect will only increase as solar max is approached**

Cross-Correlation Effect on Pc: Profiling of Higher EDR Conjunctions



Summary/Significance

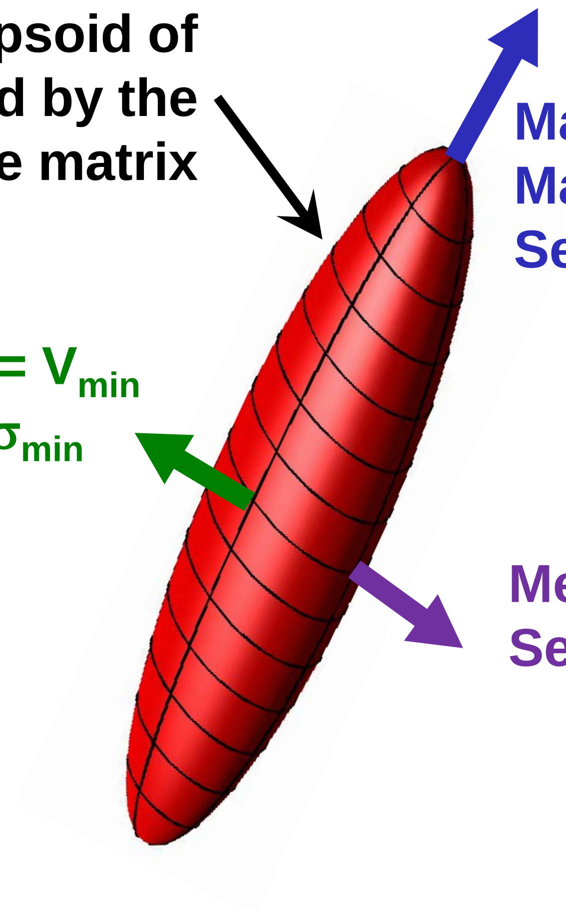
- **Covariance error cross-correlation especially problematic with global atmospheric density error in prediction**
  - Relevant only to higher-drag situations and crossing conjunctions
  - However, in such situations effect of decorrelation on calculated Pc can be significant
- **CDM to be modified for upcoming ASW release to provide information needed to perform covariance decorrelation**
  - Straightforward calculation
  - Can be used in tandem with certain covariance scaling techniques to investigate Pc sensitivity to covariance error

Reporting on the work of D. Hall *et al.**

Non-Positive-Definite Covariances: Issues and Repair

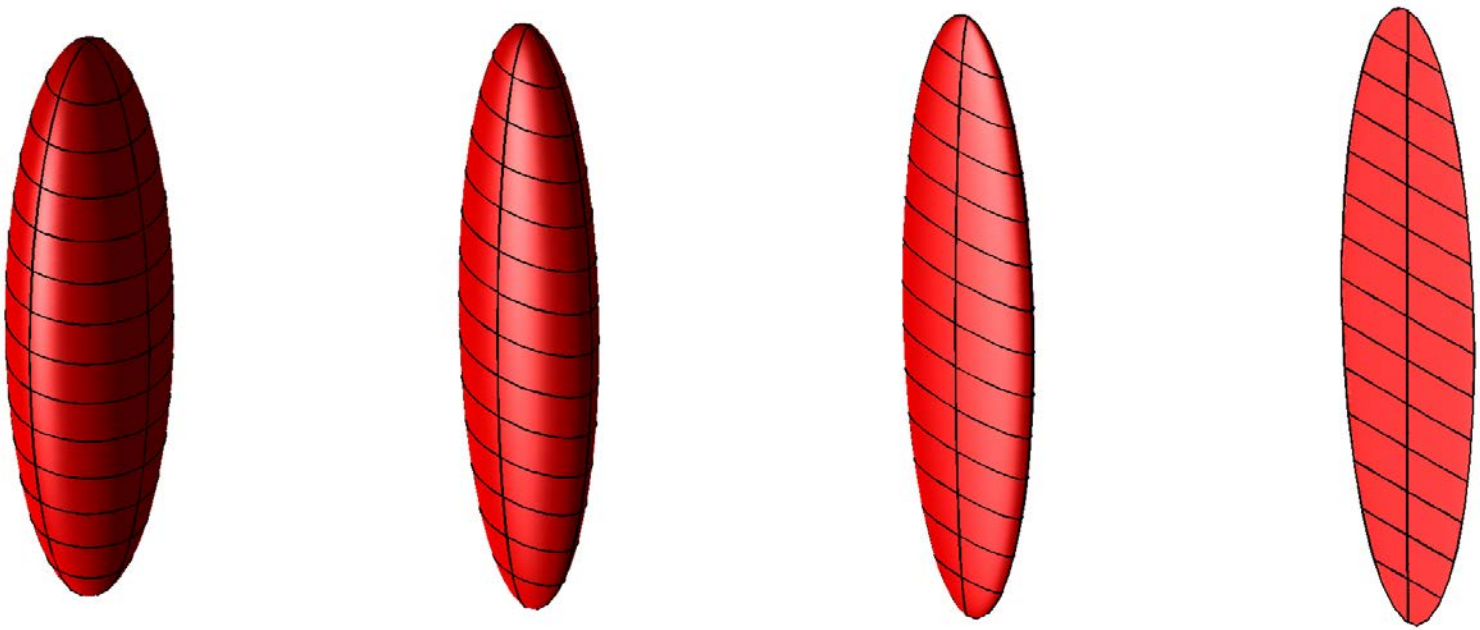
*Hall, D.T., Hejduk, M.D., and Johnson, L.C. "Remediating Non-Positive-Definite State Covariances for Collision Probability Estimation." 2017 AAS/AIAA Astrodynamics Specialist Conference, Stevenson WA, August 2017.

Geometrical Visualization of a 3x3 Covariance Matrix



Transition from Positive Definite to Semi-Positive Definite 3x3 Covariances

$\lambda_{min} = \lambda_{med}$ $\sigma_{min} = \sigma_{med}$	$\lambda_{min} = \lambda_{med} / 4$ $\sigma_{min} = \sigma_{med} / 2$	$\lambda_{min} = \lambda_{med} / 100$ $\sigma_{min} = \sigma_{med} / 10$	$\lambda_{min} = 0$ $\sigma_{min} = 0$



Positive definite (PD) covariances have positive σ_{min} values
Positive semi-definite (PSD) covariances have zero σ_{min} values
Non-positive definite (NPD) covariances have imaginary σ_{min} values

Origin/Causes of NPD Covariances

- **According to OD estimation theory, NPD should never arise**
  - Covariance is $(A^TWA)^{-1}$; A is partial derivative matrix and W is obs weighting
  - Linear algebra theorem that A^TWA must be at least positive semi-definite
  - However, numerical issues can create NPD condition
- **Insufficient numerical precision can alter covariance sufficiently to push small eigenvalue(s) negative**
  - Increase in numerical precision of CDM over OCM greatly reduced NPDs
- **Interpolation of covariances in propagation can produce NPD result**
  - Covariance not treated as a whole, but individual components interpolated
  - Some minor interpolation always necessary because desired time to propagation will not fall on a natural integration boundary
  - Analysis indicates that nearly all NDP covariances with adequate numerical precision appear to arise from propagation interpolation
- **Observability problems**
  - Due to very small OD datasets; never actually seen operationally

NPD Frequencies: CARA CDMs from OCT 2016 to AUG 2018

	Primary		Secondary		Joint	
	Pos - Vel 6 x 6	Pos 3 x 3	Pos - Vel 6 x 6	Pos 3 x 3	Pos 3 x 3	Projected 2 x 2
PD	855802	909623	905034	906836	909614	909623
NPD	53822	1	1811	9	3	1
Null	0	0	2772	2772	0	0
Default	0	0	7	7	7	0
Total	909624	909624	909624	909624	909624	909624
PD	94.0830	99.9999	99.4954	99.6935	99.9989	99.9999
NPD	5.9170	0.0001	0.1991	0.0010	0.0003	0.0001
Null	0	0	0.3047	0.3047	0	0
Default	0	0	0.0008	0.0008	0.0008	0

Should You Care about NPD? It Depends

- **If using the 2-D Pc calculation, unlikely to be a problem**
  - Only the position portion of the covariances are used—rarely NPD
  - Primary and secondary position covariances combined
    - Usually eliminates problem even if both are NPD
  - Combined covariance projected into conjunction plane
    - Often eliminates any NPD condition in combined covariance
  - In entire CARA history, only a single conjunction manifested NPD in conjunction plane
- **If using Monte Carlo, then situation is problematic**
  - Cannot sample from a distribution with a negative variance
- **If calculating Mahalanobis distances, then problematic also**
  - Useful for covariance realism, covariance scaling, and dilution region mapping
- **Prudent to identify some sort of repair technique**

Three Methods Studied for Remediating Non-Positive Definite Covariance Matrices

	Spectrum Shifting	Higham Remediation*	Eigenvalue Clipping
Description	Add an offset to all eigenvalues, and then reconstruct covariances using original eigenvectors	Find closest PSD covariance and/or correlation matrix in terms of Frobenius norm	Clip the eigenvalues at a minimum limit, and then reconstruct covariances using original eigenvectors
Advantages	-Relatively simple to implement	-Mathematically well defined, as used by the financial industry -Algorithms and codes posted on-line	-Simplest to implement -Constrained by physics-based considerations - Produces fully PD position covariances
Dis-advantages	-Assumes original eigenvectors can be used for matrix reconstruction -Offset applied to all eigenvalues, even if they don't need remediation -Not constrained using any physics-based considerations	-Most complicated to implement in software -Only designed to produce PSD matrices -The "closest Frobenius norm" criterion, while well mathematically defined, is not a physics-based criterion	-Assumes original eigenvectors can be used for matrix reconstruction

*N.J. Higham, "Computing the Nearest Correlation Matrix—A Problem from Finance," *IMA Journal of Numerical Analysis* 22, 329-343, 2002.

Pc Estimates for Conjunctions with Remediated NPD Covariances

29499_conj_38481_20160727_045045_20160815_125105*

TCA 6x6 ECI covariance status: C_p = NPD C_s = NPD C_p+C_s = NPD

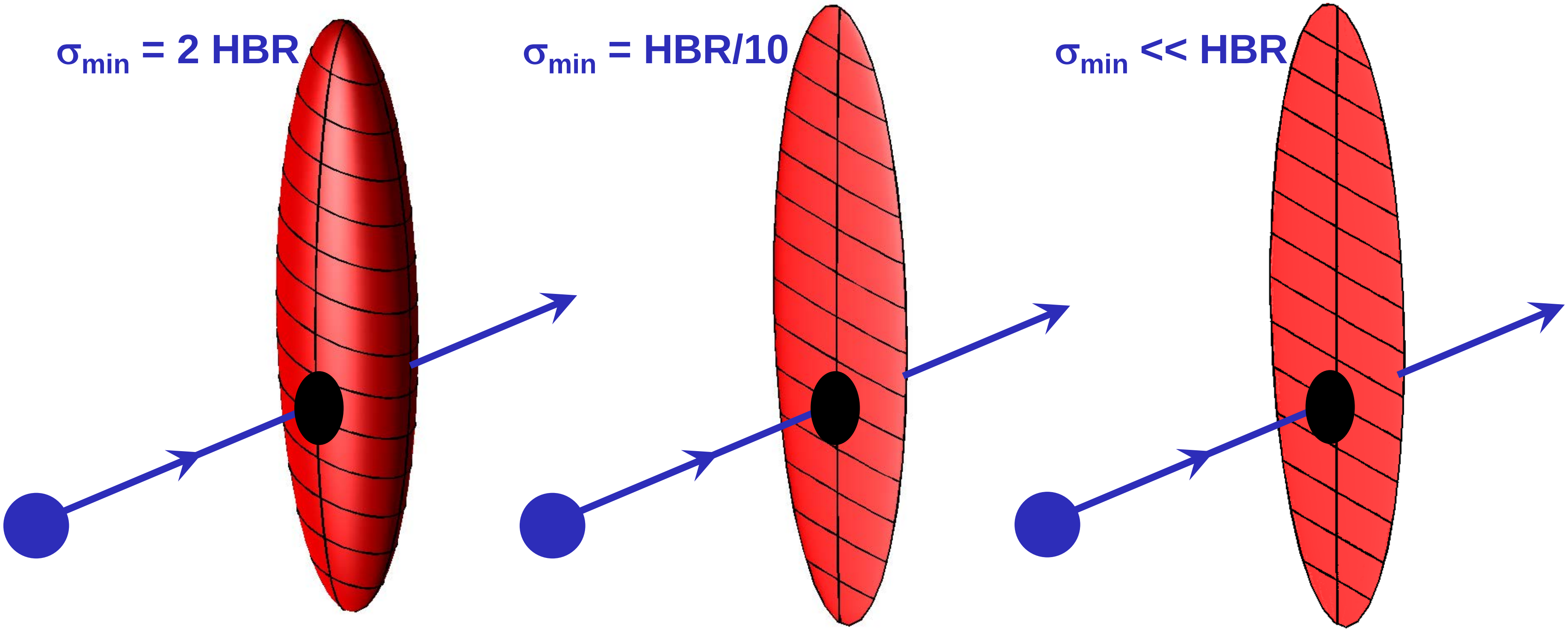
Remediation Method	2D Pc
No covariance remediation	1.3217×10^{-3}
Spectrum shifting remediation	1.3217×10^{-3}
Higham remediation	1.3464×10^{-3}
Eigenvalue clipping	1.3217×10^{-3}

This ~1.8% Pc difference was the largest seen among 430,000 events

***Primary: METOP-A Secondary: COSMOS 2251 DEB**

Collision Probability as a Function of σ_{min}

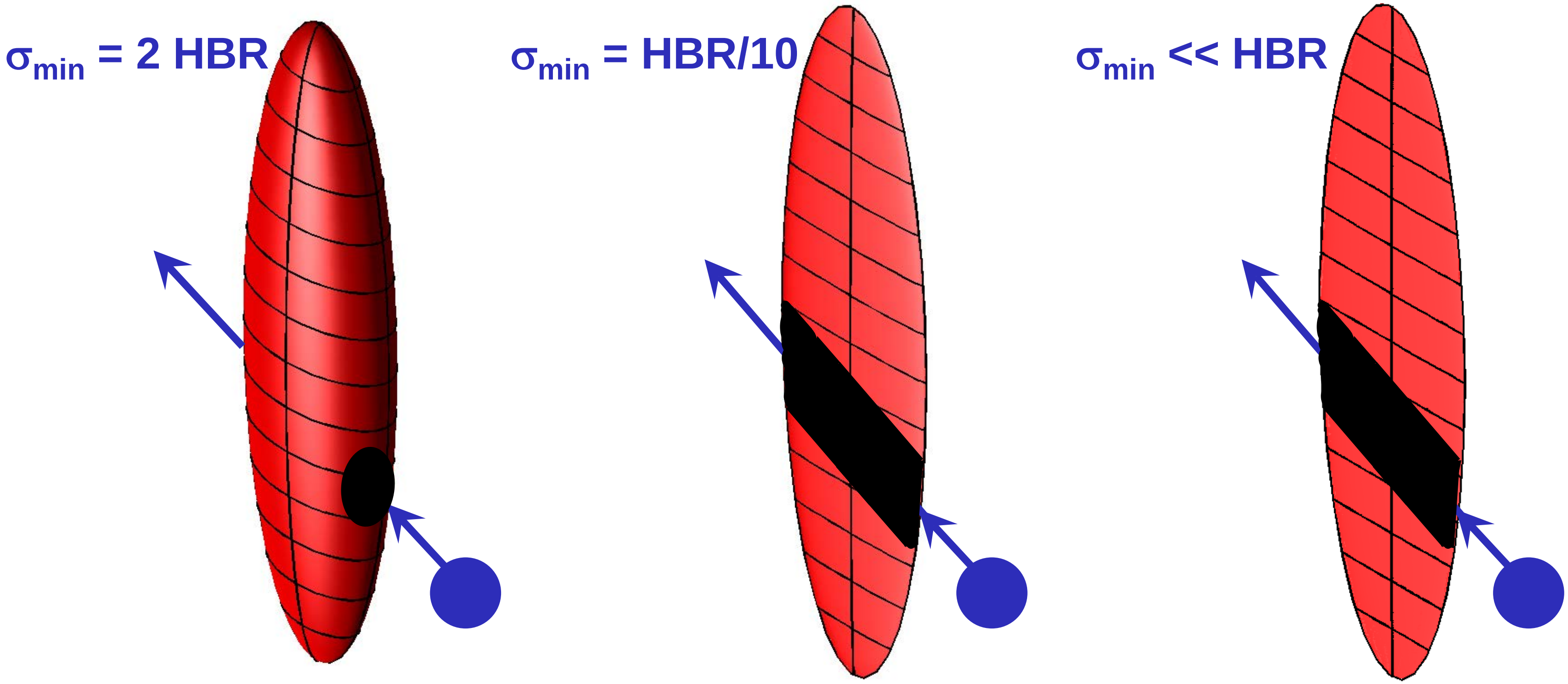
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



These three conjunctions produce similar Pc values (because they've carved out similar fractions of the PDF)

Collision Probability Visualization: Pc as a Function of σ_{min}

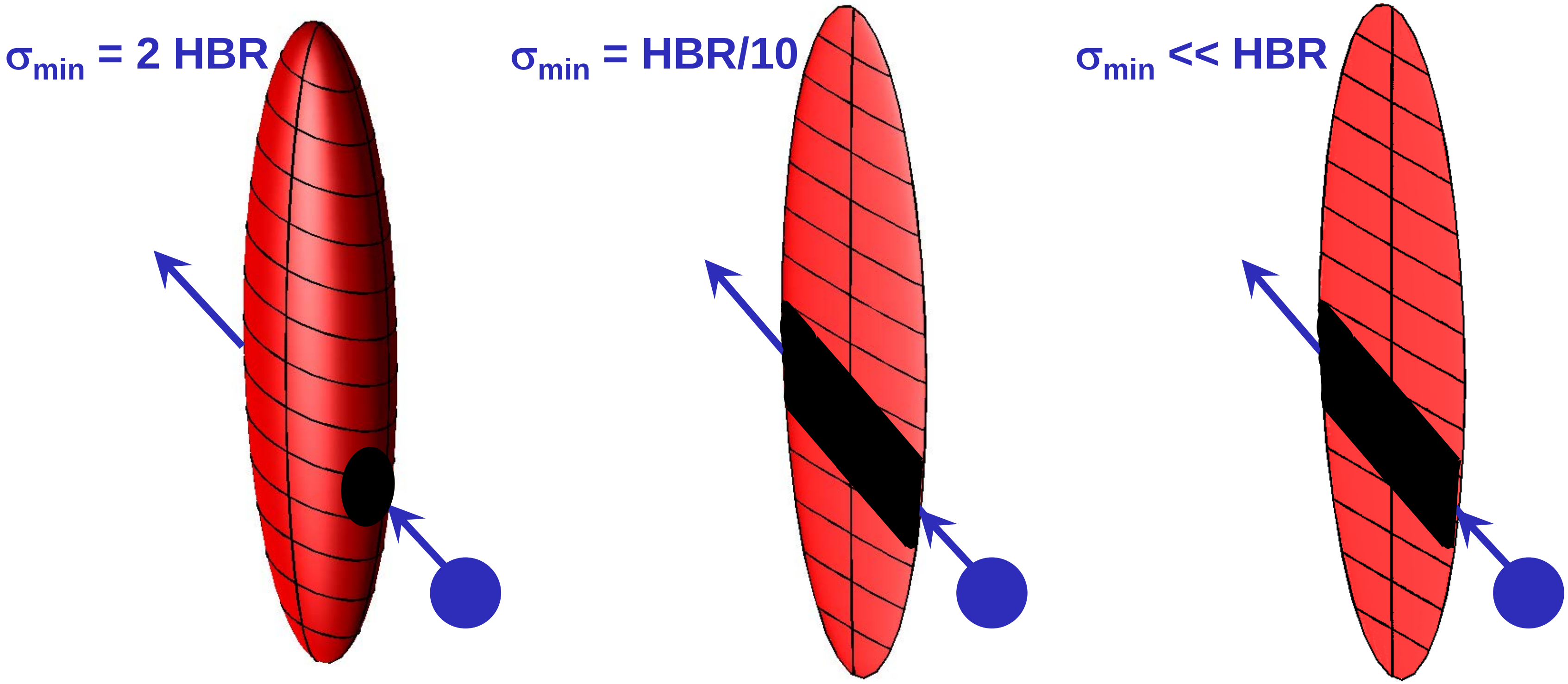
The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



The two conjunctions on the right will produce similar Pc values
The one on the left will produce a smaller Pc value

Collision Probability Visualization: Pc as a Function of σ_{min}

The collision probability represents an integral of the relative position PDF over the volume carved out along the path of the collision sphere



Pc values are insensitive to the σ_{min} value whenever $0 < \sigma_{min} << \text{HBR}$
This can be used to set a sensible eigenvalue clipping level

Conclusions

- **NPD covariances do arise from time to time in CDMs**
  - Relatively infrequent due to CDM's improvements in precision
  - Causes are believed to be fully understood
- **Unlikely to be an issue for 2-D Pc calculation**
- **For Monte Carlo and other applications, remediation is required**
  - Number of methods work well
  - Eigenvalue clipping is recommended as simplest approach
  - Result does not appear to be affected significantly by choice of method
- **On the whole, not an issue for CA if handled properly**