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On the Calculation of Exact Cumulative Distribution Statistics for Burgers Equation

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Abstract

A mathematical procedure is presented for the calculation of exact cumulative distribution statistics for a viscosity-free variant of Burgers nonlinear partial differential equation (PDE) in one space dimension and time subject to sinusoidal initial data with uncertain (random variable) amplitude or phase shift. Analytical solutions of nonlinear PDEs with uncertain initial and/or boundary data are invaluable benchmarks in assessing approximate uncertainty quantification techniques. The Burgers equation solution with uncertain initial data results in nonsmooth solution behavior in both physical and random variable dimensions which provides a severe test for approximate uncertainty quantification techniques. Mathematical proofs are provided to verify that exact cumulative distribution statistics can be systematically and robustly obtained for all forward time.

1 Introduction

Exact analytical solutions for deterministic nonlinear PDE models are invaluable benchmarks in assessing the accuracy of numerical approximations. Unfortunately, it is often difficult or impossible to obtain these exact solutions in a closed form. The difficulty is compounded when sources of uncertainty (e.g. random variable parameters or fields) are introduced into the PDE model so that the solution is a random variable function and uncertainty statistics (e.g. moment statistics or probability distributions) of output quantities of interest are sought.

Analytical solutions of the deterministic Burgers equation model, with or without a second-order differential viscosity term, are often used in evaluating the accuracy of numerical methods for conservation laws. In the present work, a viscosity-free variant of Burgers equation with sinusoidal initial data in a periodic spatial domain is considered. Even though the initial data is smooth, the solution becomes discontinuous in finite time. The exact piecewise smooth solution to this problem can be obtained using the method of characteristics in each smooth region. The boundary location between smooth regions is determined from the Rankine-Hugoniot jump conditions and an entropy selection principle [4].

A single source of uncertainty is then introduced into the deterministic Burgers equation initial data via a random variable with prescribed probability measure, $\mathbb{X} \sim P$. The Burgers equation solution is then a random variable function for which uncertainty statistics may be calculated. A notable feature of this random variable solution is the discontinuous behavior with respect to both physical independent variables and the random variable $\mathbb{X}$. This solution behavior degrades the accuracy of many numerical methods in uncertainty quantification that rely on high solution regularity with respect to random variable dimensions. The purpose of this paper is to show that given random variable inputs, the exact[1] random variable solution for Burgers equation $\mathbb{Y}(\mathbb{X})$ can be readily constructed from which the cumulative distribution function (CDF)

$$CDF_{\mathbb{Y}}(y) = Prob[\mathbb{Y} < y] \tag{1}$$

can be calculated. Given exact $\mathbb{Y}$ and/or $CDF_{\mathbb{Y}}(y)$, other uncertainty statistics are easily obtained, i.e.,

- expectation

$$E[\mathbb{Y}] = \int \mathbb{Y} dP \ , \tag{2}$$

[1] modulo implicit function root finding

- variance

$$V[\mathbb{Y}] = \int (\mathbb{Y} - E[\mathbb{Y}])^2 dP \ , \tag{3}$$

- probability density function (PDF)

$$PDF_{\mathbb{Y}}(y) = \frac{dCDF_{\mathbb{Y}}(y)}{dy} \ . \tag{4}$$

Calculation of these quantities serve as important benchmarks in uncertainty quantification for first-order conservation laws.

2 Background

2.1 A deterministic Burgers equation model

Our starting point is a viscosity-free spatially periodic form of Burgers equation with sinusoidal initial data, i.e.,

$$\partial_t u + \partial_x u^2/2 = 0 \tag{5a}$$
$$u(x,0) = A \sin(2\pi x) \tag{5b}$$

where $u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ denotes the dependent solution variable, $u^2/2$ is a quadratically nonlinear flux function, and $A > 0$ is the amplitude of the sinusoidal initial data. The evolution of this equation, as depicted in Figure 1, shows a pronounced steepening of the sinusoidal initial data which eventually becomes discontinuous at $x = \frac{1}{2}$ for $t > \frac{1}{2\pi A}$.

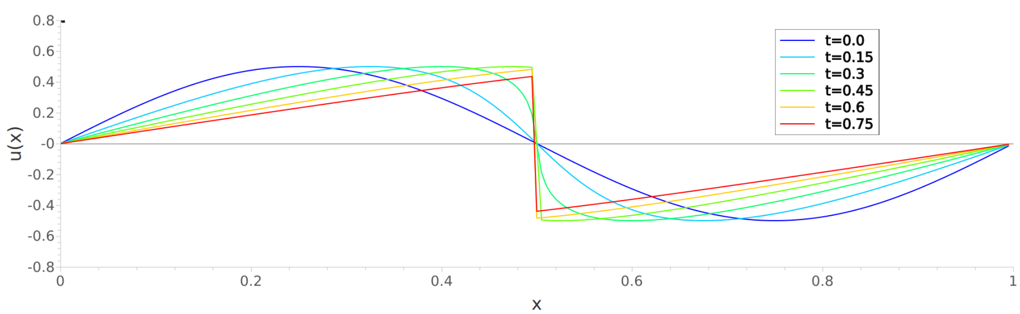

Figure 1: Burgers equation solutions $u(x,t)$ for fixed $t = \{0.0, 0.15, 0.3, 0.45, 0.6, 0.75\}$ and $A = 1/2$.

2.2 Burgers equation with uncertain initial data

Let (Ω, Σ, P) denote a probability space with event outcomes in Ω, a σ-algebra Σ, and probability measure P. Our interest lies in an random variable form of Burgers equation with uncertain sinusoidal initial data depending on a random variable $\mathbb{X}(\omega)$, $\omega \in \Omega$. Two forms of uncertain initial data are considered corresponding to (1) phase uncertainty and (2) amplitude uncertainty as described next.

[**Burgers equation with phase uncertainty**] Let $\mathbb{X}(\omega)$ denote a random variable associated with phase shift in the sinusoidal initial data. As a first test problem, we pose the following Burgers equation problem with phase uncertain initial data:

$$\partial_t u_{\mathbb{X}} + \partial_x u_{\mathbb{X}}^2/2 = 0 \tag{6a}$$

$$u_{\mathbb{X}}(x, 0, \omega) = A \sin(2\pi(x + \mathbb{X}(\omega))) \tag{6b}$$

where $u_{\mathbb{X}}(x, t, \omega) : [0, 1] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ and $A > 0$. The spatially periodic solution $u_{\mathbb{X}}(x, 4/10, \mathbb{X}(\omega))$ is shown in Figure 2 for $-\frac{1}{10} \leq \mathbb{X}(\omega) \leq \frac{1}{10}$. The effect of phase uncertainty is to shift in x the location of the stationary discontinuity that develops in Burgers equation solution realizations.

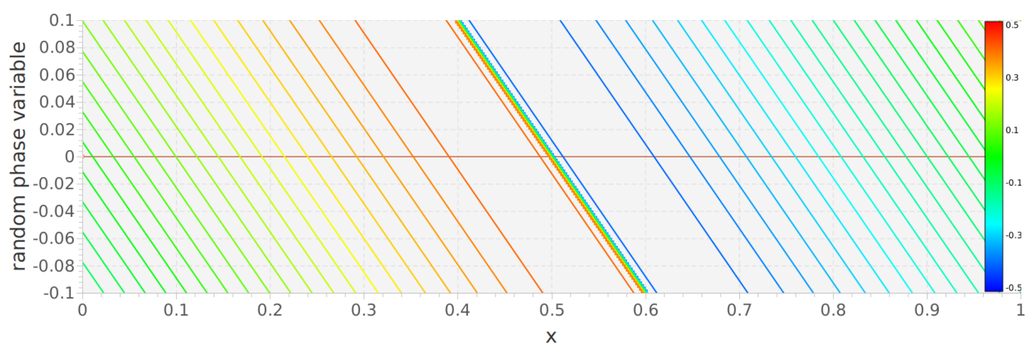

Figure 2: Contours of Burgers equation exact solution, $u_{\mathbb{X}}(x, 4/10, \mathbb{X}(\omega))$, with phase uncertain initial data and $A = 1/2$.

As mentioned previously, in the interval $x \in [4/10, 6/10]$ the random variable solution is only piecewise smooth in the random variable dimension. Numerical methods that require global smoothness in random variable dimensions (e.g. polynomial chaos and stochastic collocation) suffer a significant deterioration in accuracy in this region.

[**Burgers equation with amplitude uncertainty**] Let $\mathbb{X}(\omega)$ denote a positive random variable associated with amplitude of the sinusoidal initial data. As a second test problem, we pose the following Burgers equation problem with amplitude uncertain initial data

$$\partial_t u_{\mathbb{X}} + \partial_x u_{\mathbb{X}}^2/2 = 0 \tag{7a}$$
$$u_{\mathbb{X}}(x,0,\omega) = \mathbb{X}(\omega)\sin(2\pi x) \tag{7b}$$

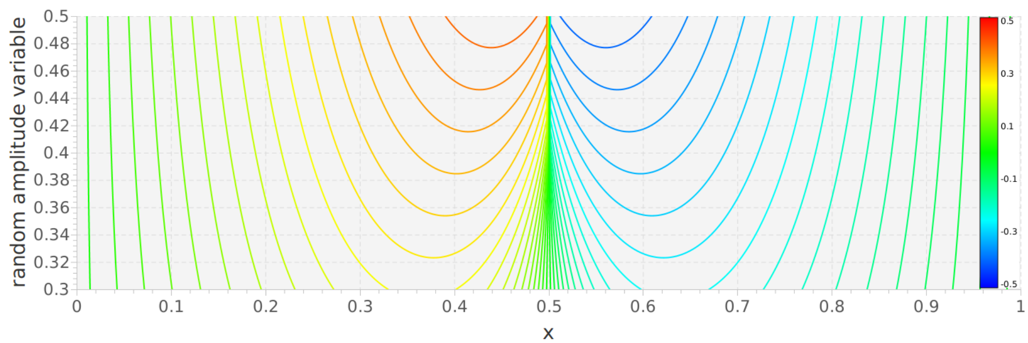

Figure 3: Contours of Burgers equation exact solution, $u_{\mathbb{X}}(x, 4/10, \mathbb{X}(\omega))$, with amplitude uncertain initial data.

where $u_{\mathbb{X}}(x,t,\omega) : [0,1] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$ and $0 < A_{min} \leq \mathbb{X}(\omega) \leq A_{max}$. The spatially periodic solution $u_{\mathbb{X}}(x, 4/10, \mathbb{X}(\omega))$ is shown in Figure 3 for $\frac{3}{10} \leq \mathbb{X}(\omega) \leq \frac{1}{2}$. Note the formation of discontinuity at $x = 1/2$ for values of the amplitude random variable greater than $\frac{5}{4\pi}$.

3 Calculating exact solutions of Burgers equation

The Burgers equation solution associated with (5a)-(5b) consists of two smooth regions, $(x,t) \in (0,1/2) \times \mathbb{R}^+$ and $(x,t) \in (1/2,1) \times \mathbb{R}^+$, separated by a entropy-satisfying stationary discontinuity at $x = 1/2$. The fixed location of the discontinuity greatly simplifies the task of constructing an exact piecewise solution. Specifically, it avoids the complication associated with finding a time-evolving discontinuity location that satisfies the proper jump conditions and an entropy selection principle. Rather, in each fixed smooth region, the solution is “classical” (see below) and can be straightforwardly calculated using the method of characteristics.

3.1 Classical solutions and the method of characteristics [4, 2]

As a prototype scalar conservation law, consider a function f depending only on u that satisfies the Cauchy initial value problem

$$\partial_t u + \partial_x f(u) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+ \tag{8a}$$
$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R} \tag{8b}$$

where $u(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ denotes the dependent solution variable, $f \in C^1(\mathbb{R})$ denotes the flux function, and $u_0(x) : \mathbb{R} \to \mathbb{R}$ the initial data. The solution u is a classical solution of (8a)-(8b) if $u \in C^1(\mathbb{R}, \mathbb{R}^+)$ satisfies this system pointwise. For classical solutions of the scalar conservation law, equation (8a) may be equivalently written in quasilinear form

$$\partial_t u + f'(u)\partial_x u = 0 \tag{9}$$

Next, let $(x(s), t(s))$ denote a curve in the (x,t) plane. Along this curve

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} \tag{10}$$

Clearly, if

$$\frac{dt}{ds} = 1 \quad \text{and} \quad \frac{dx}{ds} = f'(u), \tag{11}$$

then

$$\frac{du}{ds} = 0 \tag{12}$$

This latter equation implies that $u(x(s), t(s))$ is constant along the constrained curve. These constrained curves are referred to as characteristic curves for the quasilinear form (9). It follows from (11) that

$$\frac{d\,t}{d\,x} = \frac{1}{f'(u)} \tag{13}$$

from which it follows that characteristic curves have constant slope and therefore are straight lines. Consequently, a characteristic curve passing through the coordinate pairs $(x_0, 0)$ and (x, t) for $t > 0$ satisfies

$$\frac{t-0}{x-x_0} = \frac{1}{f'(u(x,t))} = \frac{1}{f'(u_0(x_0))} \tag{14}$$

from which the solution is obtained

$$u(x,t) = u_0(x_0(x,t)) \tag{15}$$

The function $x_0(x,t)$ will be referred to as the "pullback map" that satisfies the implicit relation

$$x - x_0(x,t) = t\, f'(u_0(x_0(x,t))) \tag{16}$$

Finding the pullback map for specific (x,t) pairs is the basis for the "method of characteristics" when applied to (8a)-(8b).

4 Exact solution of Burgers equation with sinusoidal initial data

The global structure of Burgers equation solution with sinusoidal initial data (5a)-(5b) is shown in Figure 4. It is convenient to denote left and right subdomains, $Q^L = (0,1/2) \times \mathbb{R}^+$ and $Q^R = (1/2,1) \times \mathbb{R}^+$, for which the lemmas given below will verify that the solution is classical and computable using the method of characteristics.

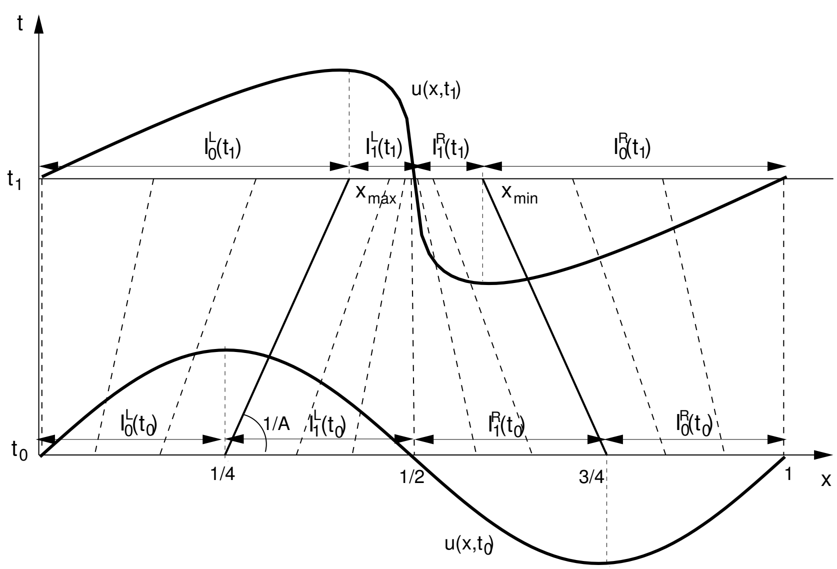

Figure 4: Burgers solutions $u(x,t_0)$ and $u(x,t_1)$ showing characteristics in the x-t plane, solution maximum location x_{max}, and solution minimum location x_{min}.

The solution is time invariant at $x = 0$ and $x = 1$, i.e., $u(0,t) = u(1,t) = 0$. As graphically depicted in this figure, the pullback map at these points reduces to $x_0(0,t) = 0$ and $x_0(1,t) = 1$. For a sufficiently large time, the solution at the isolated point $x = 1/2$ is multivalued with left and right limit values obtained from Q^L and Q^R, respectively.

The evolution of the solution maximum and minimum can be directly calculated for the Burgers equation problem (5a)-(5b), i.e.,

$$x_{max}(t;A) = \min(1/2, 1/4 + A\,t), \quad x_{min}(t;A) = \max(1/2, 3/4 - A\,t) \tag{17}$$

and used to define the left domain subintervals

$$I_0^L(t;A) = [0, x_{max}(t;A)], \quad I_1^L(t;A) = [x_{max}(t;A), 1/2] \tag{18}$$

and the right domain subintervals

$$I_0^R(t;A) = [x_{min}(t;A), 1], \quad I_1^R(t;A) = [1/2, x_{min}(t;A)] \tag{19}$$

that are later used in constructing bracketing intervals for root finding methods.

4.1 Characteristic pullback map iterations

In the specific case of Burgers equation with sinusoidal initial data (5a) - (5b), the characteristic pullback map relation (16) reduces to

$$x - x_0(x,t;A) = t\,A\,\sin(2\pi x_0(x,t;A)) \tag{20}$$

Values of the characteristic pullback map for specific (x,t) pairs are zeros of the function

$$F(\xi)(x,t;A) := x - \xi - t\,A\,\sin(2\pi\xi) \tag{21}$$

Given $\xi^*(x,t;A)$ satisfying $F(\xi^*)(x,t;A) = 0$, one obtains the desired solution

$$u(x,t;A) = u_0(\xi^*(x,t;A)) = A\sin(2\pi\xi^*(x,t;A)) \tag{22}$$

As a root finding problem, it is not clear how many roots (21) possesses and whether the root(s) can be robustly approximated to any desired level of accuracy. These questions are addressed below.

4.1.1 Bracketing characteristic pullback map iterations

There is a keen interest in constructing intervals that bracket isolated roots of the characteristic pullback map iteration function $F(\xi)$. The bisection method [3] applied to an initial bracketing interval $[\xi^{(0)}, \xi^{(1)}]$ produces a convergent sequence of root approximations such that the n-th member of the sequence approximates the root ξ^* with bound

$$|\xi^{(n)} - \xi^*| \leq \frac{\xi^{(1)} - \xi^{(0)}}{2^n} \tag{23}$$

Thus, the bisection method applied to intervals that bracket and isolate roots of $F(\xi)$ provides both a guarantee of convergence and a reliable error estimate. In practice, other bracket preserving methods such as the *regula-falsi* method, that employs bracketed secants, may offer significantly improved convergence rates but usually require minor modifications [1] to prevent a slowdown when one bracket limit is repeatedly retained in iterations.

Definition 1 (Isolated bracketing interval) *Given an interval $I = [x_0, x_1]$ and the continuous function $f(x) : C^0(I) \to \mathbb{R}$ which brackets roots of $f(x)$, $f(x_0)f(x_1) < 0$. Interval I is an isolated bracketing interval for $f(x)$ if and only if $f(x^*) = 0$ occurs exactly once for $x^* \in I$.*

The following lemma gives sufficient conditions for a bracketing interval to contain a single root.

Lemma 1 (Isolated bracketing interval) *Given an interval $I = [x_0, x_1]$ and a continuous function $f(x) : C^0(I) \to \mathbb{R}$ which brackets roots of $f(x)$, $f(x_0)f(x_1) < 0$. A sufficient condition for I to be an isolated bracketing interval is that either*

(a) $f(x)$ is strictly increasing or decreasing on I,

or

(b) $f(x)$ is strictly convex or concave on I.

Proof: From the given bracketing assumption $f(x_0)f(x_1) < 0$ and continuity of $f(x)$, at least one root $f(x^*) = 0$ must exist for $x^* \in I$. Strictly increasing or decreasing functions are injective. Assume the existence of a second distinct root, $f(y) = 0,\ y \in I$. From the injective property, $f(y) = f(x^*)$, implies $y = x^*$ which contradicts the assumption of a second distinct root and proves condition (a). Next, assume that $f(x)$ is a strictly convex function. The secant line passing through $f(x_0)$ and $f(x_1)$ crosses the zero axis from the given bracketing assumption $f(x_0)f(x_1) < 0$. A strictly convex function lies entirely below this secant line with exactly one minimum, $f(x_m)$, occurring either in the interval interior or on the the interval boundary. Therefore, x_m partitions the curve into at most two subcurves that are each strictly increasing or decreasing. When a single subcurve is present, it is strictly increasing or decreasing and satisfies $f(x_0)f(x_1) < 0$ so condition (a) directly applies. When two subcurves are present, one of these subcurves must lie entirely below the zero axis because the local secant line lies entirely below the zero axis. Thus, it can be concluded that the remaining strictly increasing or decreasing subcurve must cross the zero axis, i.e., either $f(x_0)f(x_m) < 0$ or $f(x_1)f(x_m) < 0$. Thus, condition (a) applies to this subcurve and condition (b) is proved. The proof for concave functions follows a similar path and is omitted. ■

Using convexity and bracketing properties of the characteristic pullback map iteration function, the next lemma proves that isolated bracketing intervals exist when applied to the Burgers equation problem (5a) - (5b) in subdomains Q^L and Q^R.

Lemma 2 (Characteristic pullback map iteration function bracketing) *Let $F(\xi)(x,t;A)$ denote the characteristic pullback iteration function*

$$F(\xi)(x,t;A) := x - \xi - t\,A\sin(2\pi\xi)$$

for the Burgers equation problem (5a) - (5b). The intervals

- $[0, 1/2]$ *for* $(x,t) \in Q^L$
- $[1/2, 1]$ *for* $(x,t) \in Q^R$

are isolated bracketing intervals for the characteristic pullback map iteration function.

Proof: Consider the characteristic pullback map iteration function $F(\xi)(x,t;A)$ in the subdomain Q^L. The interval $[0, 1/2]$ satisfies the bracketing property for $(x,t) \in Q^L$, i.e.,

$$\begin{aligned} F(0) &= x > 0 \\ F(1/2) &= x - 1/2 < 0 \end{aligned}$$

Twice differentiation of the function $F(\xi)$ yields

$$F''(\xi)(x,t;A) = t\,A\,(2\pi)^2 \sin(2\pi\xi) \tag{24}$$

It follows that $F(\xi)(x,t;A)$ satisfies the necessary and sufficient condition for strict convexity of a twice differentiable function that $\{\xi \in [0,1/2] : F''(\xi) > 0\}$ is a dense set. Using Lemma 1, the stated lemma is proved for the bracketing interval $[0,1/2]$ and $(x,t) \in Q^L$. The proof for the bracketing interval $[1/2,1]$ and $(x,t) \in Q^R$ with $F''(\xi)$ concave follows a similar path that is omitted. ■

This lemma is useful in devising robust numerical methods for calculating the characteristic pullback map for (x,t) pairs. Even so, it is possible to construct improved (reduced) isolated bracketing intervals. These refined intervals are then used to determine domain-codomain properties of the pullback map.

Lemma 3 (Improved characteristic pullback map iteration isolated bracketing) *Let $x_{max}(t;A)$ and $x_{min}(t;A)$ denote the solution maximum and minimum locations for the Burgers equation problem (5a) - (5b) as defined in (17). Further, let $F(\xi)(x,t;A)$ denote the characteristic pullback iteration function*

$$F(\xi)(x,t;A) := x - \xi - t\,A\sin(2\pi\xi)$$

for this Burgers equation problem. The intervals

- $[0,1/4]$ *for* $(x,t) \in (0, x_{max}(t;A)) \times \mathbb{R}^+$
- $[1/4,1/2]$ *for* $(x,t) \in (x_{max}(t;A), 1/2) \times \mathbb{R}^+$
- $[3/4,1]$ *for* $(x,t) \in (x_{min}(t;A), 1) \times \mathbb{R}^+$
- $[1/2,3/4]$ *for* $(x,t) \in (1/2, x_{min}(t;A)) \times \mathbb{R}^+$

are isolated bracketing intervals for the characteristic pullback map iteration function.

Proof: The proof is identical to Lemma 2 but with the following time dependent bracket limits

$$\begin{aligned}
F(0) &= x > 0, \quad (x,t) \in (0, x_{max}(t;A)) \times \mathbb{R}^+ \\
F(1/4) &= x - 1/4 - t\,A \leq x - x_{max}(t;A) < 0, \quad (x,t) \in (0, x_{max}(t;A)) \times \mathbb{R}^+ \\
F(3/4) &= x - 3/4 + t\,A \geq x - x_{min}(t;A) > 0, \quad (x,t) \in (x_{min}(t;A), 1) \times \mathbb{R}^+ \\
F(1) &= x - 1 < 0, \quad (x,t) \in (x_{min}(t;A), 1) \times \mathbb{R}^+
\end{aligned}$$

the following bracket limits when $x_{max}(t;A) < 1/2$

$$\begin{aligned}
F(1/4) &= x - 1/4 - t\,A = x - x_{max}(t;A) > 0, \quad (x,t) \in (x_{max}(t;A), 1/2) \times \mathbb{R}^+ \\
F(1/2) &= x - 1/2 < 0, \quad (x,t) \in (x_{max}(t;A), 1/2) \times \mathbb{R}^+
\end{aligned}$$

and the following bracket limits when $x_{min}(t;A) > 1/2$

$$\begin{aligned}
F(1/2) &= x - 1/2 > 0, \quad (x,t) \in (1/2, x_{min}(t;A)) \times \mathbb{R}^+ \\
F(3/4) &= x - 3/4 + t\,A = x - x_{min}(t;A) < 0, \quad (x,t) \in (1/2, x_{min}(t;A)) \times \mathbb{R}^+
\end{aligned}$$

F is strictly convex in Q^L and strictly concave in Q^R so that the stated lemma is proved. ■

4.2 Burgers equation problem (5a)-(5b) computability

The isolated bracketing property when combined with a root finding method such as the bisection method, described earlier, is the basis for a robust algorithm for constructing exact solutions of the Burger equation problem (5a) - (5b).

Theorem 1 (Burgers equation problem (5a) - (5b) computability) *Given the Burgers equation problem (5a) - (5b), the solution $u(x,t;A)$ for any $(x,t) \in Q^L \cup Q^R$ can be computed, assuming exact arithmetic, with guaranteed reliability to a specified precision ϵ using at most $\log_2 \frac{1/2}{\epsilon}$ steps of the bisection bracketing method.*

Proof: The theorem follows immediately from Lemma 2 together with the error convergence estimate (23) for the bisection root finding method. ■

4.3 Further properties of the characteristic pullback map

4.3.1 Characteristic pullback map domain-codomain relationships

From Lemma 3 and guaranteed convergence of the bisection root finding method (23), the following domain-codomain relationships can be deduced for any given time $t \geq 0$. These relationships are used later in Sect. 5.

Lemma 4 (Pullback map domain-codomain relationships) *Given the Burgers equation problem (5a) - (5b), the characteristic pullback map relation*

$$x - x_0(x,t;A) = t\,A\sin(2\pi x_0(x,t;A))$$

exhibits the following domain-codomain relationships

$$\begin{aligned} x_0(x,t;A) &: Q^L \to (0,1/2) \\ x_0(x,t;A) &: Q^R \to (1/2,1) \end{aligned}$$

and using the refined subintervals

$$\begin{aligned} x_0(x,t;A) &: (0, x_{max}(t;A)) \times \mathbb{R}^+ \to (0,1/4) \\ x_0(x,t;A) &: (x_{max}(t;A), 1/2) \times \mathbb{R}^+ \to (1/4,1/2) \\ x_0(x,t;A) &: (1/2, x_{min}(t;A)) \times \mathbb{R}^+ \to (1/2,3/4) \\ x_0(x,t;A) &: (x_{min}(t;A), 1) \times \mathbb{R}^+ \to (3/4,1) \end{aligned}$$

Proof: The proof follows from Lemmas 2-3 and convergence of the bisection bracketing method in Theorem 1. ■

4.3.2 Monotonicity of the characteristic pullback map

It is instructive and useful later on to algebraically verify that the solution in Q^L and Q^R remains classical for all time $t \geq 0$. Solutions obtained from the method of characteristics are of the form

$$u(x,t) = u_0(x_0(x,t))$$

and are classical if the solution gradients

$$\frac{\partial u}{\partial x} = u_0' \frac{\partial x_0}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial t} = u_0' \frac{\partial x_0}{\partial t}$$

remain bounded and hold pointwise. For the Burgers equation problem (5a) - (5b), u_0' is trivially bounded and a direct calculation yields

$$\frac{\partial x_0}{\partial t} = -A \sin(2\pi x_0(x,t;A)) \frac{\partial x_0}{\partial x}$$

so that boundedness of solution gradients reduces to the problem of proving boundedness of spatial derivatives of the characteristic pullback map. This question is addressed in the next lemma.

Lemma 5 (Pullback map monotonicity) *The Burgers equation problem (5a) - (5b) characteristic pullback map $x_0(x,t)$ satisfying the following implicit relation*

$$x - x_0(x,t;A) = t\, A \sin(2\pi x_0(x,t;A)) \tag{25}$$

is a bounded strictly increasing function for a fixed time t and $(x,t) \in Q^L \cup Q^R$.

Proof: Given the characteristic pullback map relation (25), the spatial partial derivative is readily obtained

$$\frac{\partial x_0}{\partial x} = \frac{1}{1 + 2\pi\, t\, A \cos(2\pi x_0(x,t;A))} \tag{26}$$

This partial derivative is positive at $x = 0$

$$\frac{\partial x_0}{\partial x}(x=0,t) = \frac{1}{1+2\pi t\, A} > 0$$

and can only change sign by the denominator in (26) passing through zero at some critical space-time (x^*, t^*) that satisfies

$$t^* = -\frac{1}{2\pi A \cos(2\pi x_0(x^*,t^*;A))} \tag{27}$$

and

$$x^* = x_0(x^*,t^*;A) - \frac{1}{2\pi} \tan(2\pi x_0(x^*,t^*;A)) \tag{28}$$

To prove positivity and boundedness of the derivative (26), it must be shown that no $(x^*,t^*) \in Q^L \cup Q^R$ exists that satisfies these equations. Assume that a critical space-time $(x^*,t^*) \in Q^L$ does exist (similarly for $(x^*,t^*) \in Q^R$).

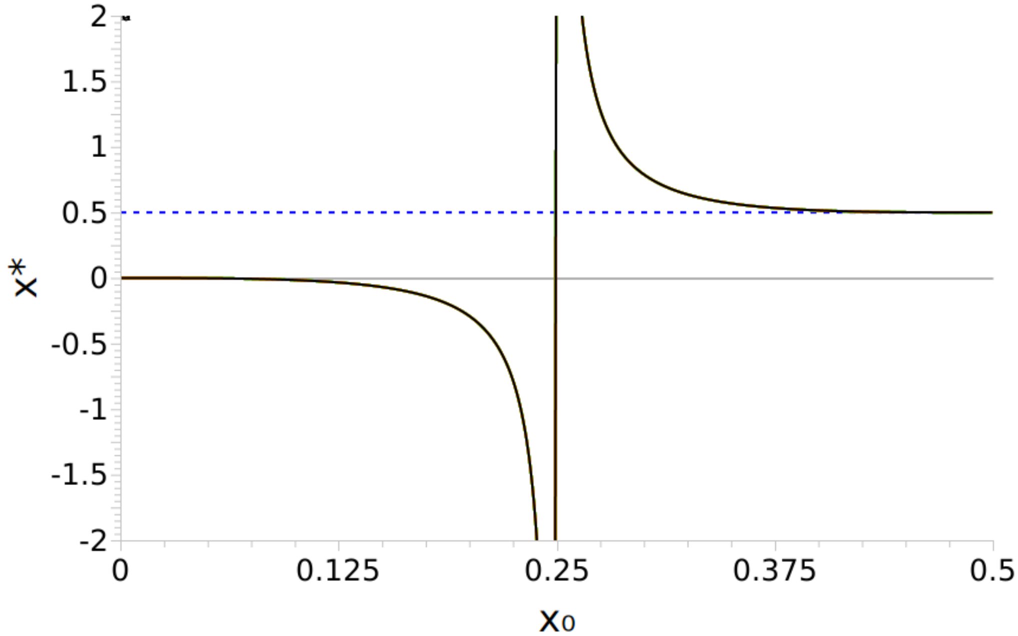

Figure 5: Graph of x^* versus x_0, $x^* = x_0 - \frac{1}{2\pi}\tan(2\pi x_0)$.

From Lemma 4, $(x^*, t^*) \in Q^L$ implies $x_0 \in (0, 1/2)$. The graph of x^* versus x_0 (see Figure 5) reveals that for $x_0 \in (0, 1/2)$, the function $x^* \notin (0, 1/2)$. More specifically,

$$x^* < 0 \text{ for } x_0 \in (0, 1/4)$$

$$x^* > 1/2 \text{ for } x_0 \in (1/4, 1/2)$$

and

$$|x^*| = \infty \text{ for } x_0 = 1/4$$

This shows that $(x^*, t^*) \notin Q^L$ and the stated assumption is contradicted. Thus, the denominator in (26) never vanishes which implies that

$$\frac{\partial x_0}{\partial x}(x, t) > 0, \quad (x, t) \in Q^L$$

A similar analysis in Q^R yields

$$\frac{\partial x_0}{\partial x}(x, t) > 0, \quad (x, t) \in Q^R$$

and the stated lemma is proved. ■

5 Calculating an output cumulative probability distribution for Burgers equation with uncertainty

5.1 Calculating the cumulative probability distribution for an output random variable

Let $\mathbb{X} \sim P$ denote a random variable with prescribed probability measure and $\mathbb{Y}$ an output random variable that satisfies

$$\mathbb{Y} = g(\mathbb{X}) \tag{29}$$

The cumulative distribution associated with $\mathbb{Y}$ can be directly calculated

$$\begin{aligned} CDF_{\mathbb{Y}}(y) &= Prob[\mathbb{Y} < y] \\ &= Prob[g(\mathbb{X}) < y] \\ &= Prob[\{\mathbb{X} < g^{-1}(y)\}] \end{aligned}$$

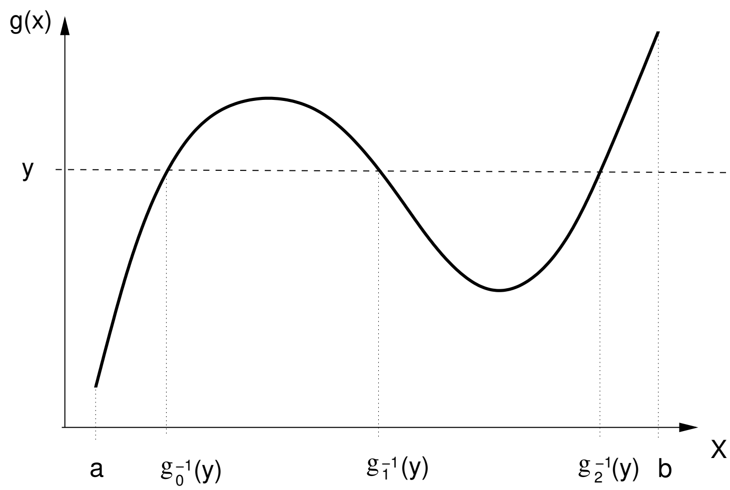

Figure 6: Non-monotone $g(x)$ versus x.

This latter equation is complicated by the fact that the inverse of $g(y)$ may not be unique when $g(y)$ is non-monotone as depicted in Figure 6. These multiple inverses form a set of cardinality $N(y)$ with this set denoted here by $\{g_0^{-1}(y), g_1^{-1}(y), \ldots, g_{N-1}^{-1}(y)\}$ with the convention $g_i^{-1}(y) \geq g_j^{-1}(y)$ if $i > j$. Again referring to Figure 6, for convenience define $g_{-1}^{-1}(y) \equiv a$ and $g_N^{-1}(y) \equiv b$

with $CDF_{\mathbb{X}}(g_{-1}^{-1}(y)) = 0$ and $CDF_{\mathbb{X}}(g_N^{-1}(y)) = 1$. Using these added definitions, the cumulative distribution associated with the random variable $\mathbb{Y}$ is then canonically given by

$$CDF_{\mathbb{Y}}(y) = \begin{cases} \sum_{i=0}^{\lfloor N(y)/2 \rfloor} CDF_{\mathbb{X}}(g_{2i}^{-1}(y)) - CDF_{\mathbb{X}}(g_{2i-1}^{-1}(y)), & \text{if } g(g_0^{-1}(y)) \text{ increasing} \\ 1 - \sum_{i=0}^{\lfloor N(y)/2 \rfloor} CDF_{\mathbb{X}}(g_{2i}^{-1}(y)) - CDF_{\mathbb{X}}(g_{2i-1}^{-1}(y)), & \text{otherwise} \end{cases} \tag{30}$$

The following example gives a concrete application of this formula for $N(y) = 2$.

5.1.1 Example: calculation of an output probability distribution

Let $\mathbb{X} \sim P$ denote a random variable. Assume P has a uniform probability distribution

$$PDF_{\mathbb{X}}(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}, \quad CDF_{\mathbb{X}}(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{otherwise} \end{cases}$$

Next, let $g(x)$ denote the parabolic function

$$g(x) = 4x(1-x)$$

Over the nonzero support of $PDF_{\mathbb{X}}(x)$, this function is the mapping $g : [0,1] \to [0,1]$ with values of x satisfying $g(x) = y$ given by $\{\frac{1}{2}(1-\sqrt{1-y}), \frac{1}{2}(1+\sqrt{1-y})\}$ such that $g(x)$ is a locally increasing function at the first root and a locally decreasing function at the second root. Using equation (30), a random variable $\mathbb{Y}$ satisfying the random variable equation

$$\mathbb{Y} = g(\mathbb{X})$$

has the following cumulative distribution function for $y \in [0,1]$

$$\begin{aligned} CDF_{\mathbb{Y}}(y) &= 1 + CDF_{\mathbb{X}}(g_0^{-1}(y)) - CDF_{\mathbb{X}}(g_1^{-1}(y)) \\ &= 1 + g_0^{-1}(y) - g_1^{-1}(y) \\ &= 1 - \sqrt{1-y} \end{aligned}$$

Using (4), $PDF_{\mathbb{Y}}(y)$ is obtained from $CDF_{\mathbb{Y}}(y)$ by differentiation.

5.2 Calculating the output cumulative probability distribution for Burgers equation with phase uncertainty

Recall the phase uncertain initial data problem (6a) and (6b) repeated here

$$\begin{aligned} \partial_t u_{\mathbb{X}} + \partial_x u_{\mathbb{X}}^2/2 &= 0 \text{ in } [0,1] \times \mathbb{R}^+ \times \Omega \\ u_{\mathbb{X}}(x,0,\omega) &= A \sin(2\pi(x + \mathbb{X}(\omega))) \end{aligned}$$

Let $g(\xi)(x,t;A)$ denote a solution of the deterministic Burgers equation problem with ξ phase-shifted initial data. This function is related to the unshifted initial data problem by the relation

$$g(\xi)(x,t;A) = u(x+\xi,t;A) \tag{32}$$

Further, define the phase uncertainty iteration function

$$G(\xi)(x,t;\tilde{u},A) := g(\xi)(x,t;A) - \tilde{u} \tag{33}$$

that effectively inverts $g(\cdot)$, i.e., $\xi(\tilde{u})(x,t;A) = g_i^{-1}(\tilde{u})(x,t;A)$. In implementations, it is preferable to use the unshifted reference problem

$$G_0(\eta)(t;\tilde{u},A) := u(\eta,t;A) - \tilde{u} \tag{34}$$

to calculate unshifted roots and then calculate the shifted roots via (modulo periodicity)

$$\xi(\tilde{u})(x,t;A) = \eta(\tilde{u})(t;A) - x \tag{35}$$

The following results prove that roots of $G_0(\eta)$ can be reliably computed using bracketed iteration.

Lemma 6 (Phase uncertainty iteration function convexity/concavity) *Let $u(x,t;A)$ denote a solution of the Burgers equation problem (5a)-(5b). The phase uncertainty iteration function*

$$G_0(\eta)(t;\tilde{u},A) := u(\eta,t;A) - \tilde{u} \tag{36}$$

for a given fixed $\tilde{u}$ is strictly

- *concave for $(\eta,t) \in [0,1/2] \times \mathbb{R}^+$*

- *convex for* $(\eta, t) \in [1/2, 1] \times \mathbb{R}^+$

Proof: The phase uncertainty iteration function simplifies to

$$G_0(\eta)(t; \tilde{u}, A) = A \sin(2\pi x_0(\eta, t; A)) - \tilde{u} \tag{37}$$

and after twice differenting

$$G_0''(\eta)(t; \tilde{u}, A) = -\frac{(2\pi)^2 A \sin(2\pi x_0(\eta, t; A))}{(1 + 2\pi t\, A \cos(2\pi x_0(\eta, t; A)))^3} \tag{38}$$

In the proof of Lemma 5, the denominator in this formula is proven strictly positive for $(\eta, t) \in Q^L \cup Q^R$. From Lemma 4, $(\eta, t) \in Q^L$ implies $x_0(\eta, t; A) \in (0, 1/2)$ and $\sin(2\pi x_0(\eta, t; A)) > 0$. Similarly, $(\eta, t) \in Q^R$ implies $x_0(\eta, t; A) \in (1/2, 1)$ and $\sin(2\pi x_0(\eta, t; A)) < 0$. Thus, the sets $\{(\eta, t) \in [0, 1/2] \times \mathbb{R}^+ : G_0''(\eta)(t; \tilde{u}, A) < 0\}$ and $\{(\eta, t) \in [1/2, 1] \times \mathbb{R}^+ : G_0''(\eta)(t; \tilde{u}, A) > 0\}$ are dense and the stated lemma is proved. ■

Recall that explicit formulas for the location of the solution minimum and maximum for the Burgers equation problem (5a)-(5b) are given in (17) from which the solution minimum

$$u_{min}(t; A) = A \sin(2\pi x_0(x_{min}(t; A), t; A)) \tag{39}$$

and solution maximum

$$u_{max}(t; A) = A \sin(2\pi x_0(x_{max}(t; A), t; A)) \tag{40}$$

are obtained. The phase uncertainty random variable solution at a time t is bounded between these limits, i.e.,

$$u_{min}(t; A) \leq u_{\mathbb{X}}(x, t, \omega) \leq u_{max}(t; A), \quad (x, t) \in Q^L \cup Q^R \tag{41}$$

The next lemma proves that two roots of $G_0(\eta)(x, t; \tilde{u})$ exist with isolated bracketing intervals for $u_{min}(t; A) < \tilde{u} < u_{max}(t; A)$. To insure that two roots are always obtained (rather than just one), it is convenient to supplant the multivalued solution at $x = 1/2$ with the single value $u(x = 1/2, t) = 0$ on the closure boundary of Q^L and Q^R.

Lemma 7 (Phase uncertainty iteration function isolated bracketing) *Let $u(x,t;A)$ denote a solution of the Burgers equation problem (5a)-(5b) and $G_0(\eta)(t;\tilde{u},A)$ the phase uncertainty iteration function*

$$G_0(\eta)(t;\tilde{u},A) := u(\eta,t;A) - \tilde{u} \tag{42}$$

for $u_{min}(t;A) < \tilde{u} < u_{max}(t;A)$. The phase uncertainty iteration function possesses exactly two isolated bracketing intervals for $0 < \tilde{u} < u_{max}(t;A)$

- $I_0^L(t;A) = [0, x_{max}(t;A)]$
- $I_1^L(t;A) = [x_{max}(t;A), 1/2]$

and exactly two isolated bracketing intervals for $u_{min}(t;A) < \tilde{u} < 0$

- $I_1^R(t;A) = [1/2, x_{min}(t;A)]$
- $I_0^R(t;A) = [x_{min}(t;A), 1]$

as depicted in Figure 7.

Proof: Assume $0 < \tilde{u} < u_{max}(t;A)$, a direct evaluation of the bracket limits verifies the bracketing properties

$$\begin{aligned} G_0(0) &= \tilde{u} > 0 \\ G_0(x_{max}(t;A)) &= \tilde{u} - u_{max}(t;A) < 0 \\ G_0(1/2) &= \tilde{u} > 0 \end{aligned}$$

When combined with the concavity result of Lemma 6, $[0, x_{max}(t;A)]$ and $[x_{max}(t;A), 1/2]$ are isolated bracketing intervals. Since $[0, x_{max}(t;A)] \cup [x_{max}(t;A), 1/2]$ completely covers $[0, 1/2]$, no additional isolated bracketing intervals are possible for $\eta \in [0, 1/2]$. The proof of isolated bracketing intervals when $u_{min}(t;A) < \tilde{u} < 0$ follows a similar path and is omitted. ■

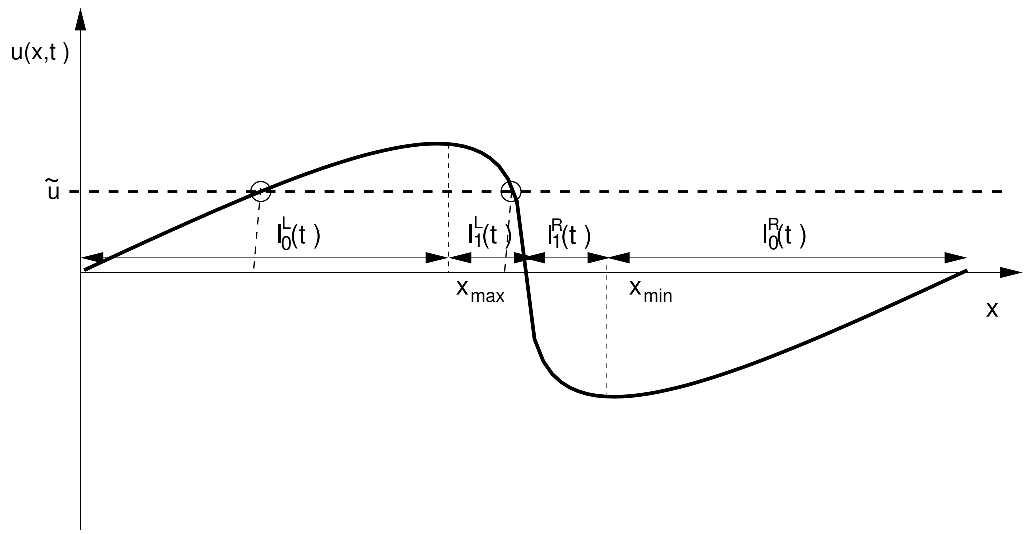

Figure 7: Burgers equation solution identifying the two roots (circles) of $G(\xi)(x,t;\tilde{u},A) = 0$ for $0 < \tilde{u} < u_{max}(t;A)$.

Theorem 2 (Phase Uncertainty Computability) *Let $u(x,t;A)$ denote a solution of the Burgers equation problem (5a)-(5b) and $G_0(\eta)(t;\tilde{u},A)$ the phase uncertainty iteration function*

$$G_0(\eta)(t;\tilde{u},A) := u(\eta,t;A) - \tilde{u} \tag{43}$$

for $u_{min}(t;A) < \tilde{u} < u_{max}(t;A)$. The two roots of the phase uncertainty iteration function can be reliably computed, assuming exact arithmetic, with guaranteed reliability to a specified precision ϵ using at most $\log_2 \frac{1/2}{\epsilon}$ steps of the bisection root finding method.

Proof: The theorem follows immediately from Lemma 7 together with the error convergence estimate (23) for the bisection root finding method. ■

Theorem 2 proves that the two roots of the phase uncertainty iteration function, $G_0(\eta)(t;\tilde{u},A)$, can be reliably computed. Equation (35) then provides the transformation of these roots to roots of $G(\xi)(x,t;\tilde{u};A)$, namely, $\{\xi_0(\tilde{u})(x,t;A), \xi_1(\tilde{u})(x,t;A)\}$ with the ordering convention $\xi_0(\tilde{u})(x,t;A) \leq \xi_1(\tilde{u})(x,t;A)$. Using (30), the cumulative probability distribution formula for the phase uncertain Burgers problem (6a)-(6b) solution is then given by

$$CDF_{u_{\mathbb{X}}}(\tilde{u})(x,t;A) \;=\; \begin{cases} 0 & \tilde{u} \le u_{min}(t;A) \\ CDF_{\mathbb{X}}(\xi_1(\tilde{u}))(x,t;A) - CDF_{\mathbb{X}}(\xi_0(\tilde{u}))(x,t;A), & u_{min}(t;A) < \tilde{u} < 0 \\ 1 + CDF_{\mathbb{X}}(\xi_0(\tilde{u}))(x,t;A) - CDF_{\mathbb{X}}(\xi_1(\tilde{u}))(x,t;A), & 0 < \tilde{u} < u_{max}(t;A) \\ 1 & \tilde{u} \ge u_{max}(t;A) \end{cases}$$

5.2.1 Example: Burgers equation phase uncertainty output statistics, $\mathbb{X}(\omega) \sim \mathcal{U}[-.1,.1]$

Output statistics for the phase uncertain Burgers equation problem (6a)-(6b) with uniform probability measure, $\mathbb{X}(\omega) \sim \mathcal{U}[-.1,.1]$ are presented in Figure 8.

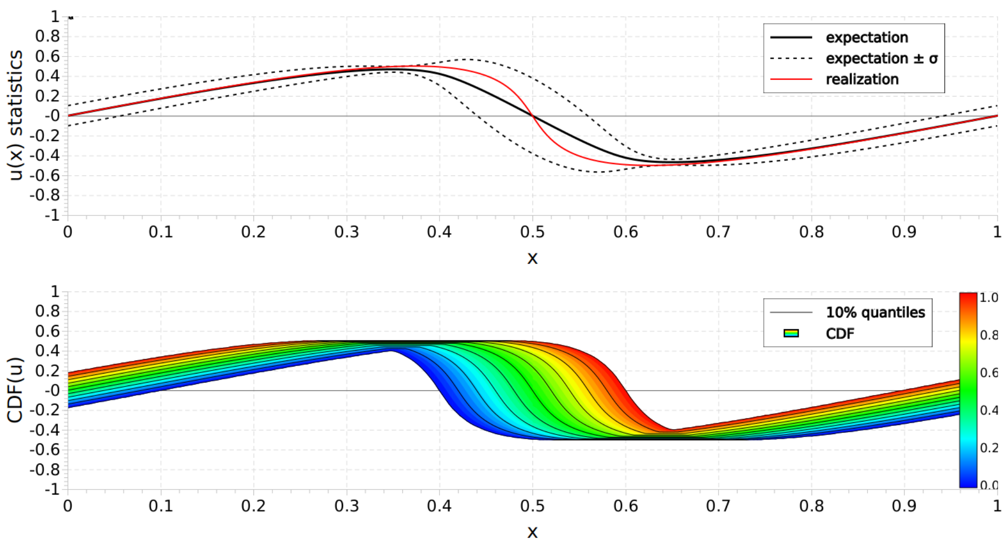

Figure 8: Burgers equation with phase uncertainty, $\mathbb{X}(\omega) \sim \mathcal{U}[-.1,.1]$. Moment statistics and representative realization (top) and shaded cumulative distribution with 10% quantile lines (bottom) at time $t = 0.25$

The cumulative distribution function, $CDF_{u_{\mathbb{X}}}(u)(x, t = 1/4; 1/2)$, (shaded region) together with quantiles of 10% probability are shown in Figure 8 (bottom). Moment statistics and a representative realization have been graphed in Figure 8 (top) for reference. Figure 9 shows graphs of the solution cumulative distribution function at $x = 0.2$ (left) and $x = 0.46$ (right) with the latter figure showing significant nonlinear distortion of the output cumulative distribution due to uncertainty in the discontinuity location resulting from phase uncertainty.

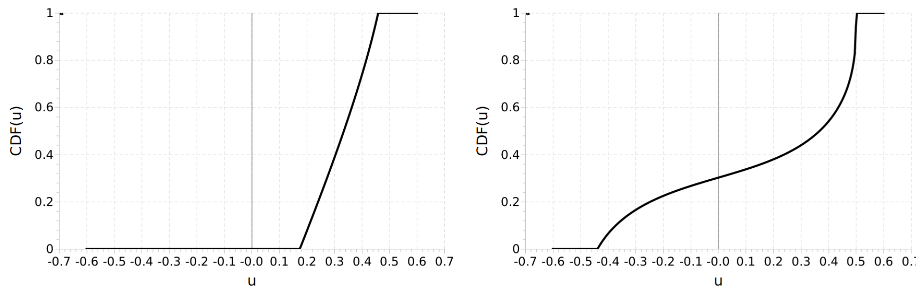

Figure 9: Burgers equation with phase uncertainty, $\mathbb{X}(\omega) \sim \mathcal{U}[-.1, .1]$. Graphs of the cumulative distribution at $x = 0.2$ (left) and $x = 0.46$ (right) at time $t = 0.25$

5.2.2 Example: Burgers equation phase uncertainty output statistics, $\mathbb{X}(\omega) \sim \mathcal{N}_3(m = 0, \sigma = 0.05)$

Output statistics for the phase uncertain Burgers equation problem (6a)-(6b) with normal distribution probability measure truncated at 3σ, $\mathbb{X}(\omega) \sim \mathcal{N}_3[m = 0, 0.05]$ are presented in Figure 10.

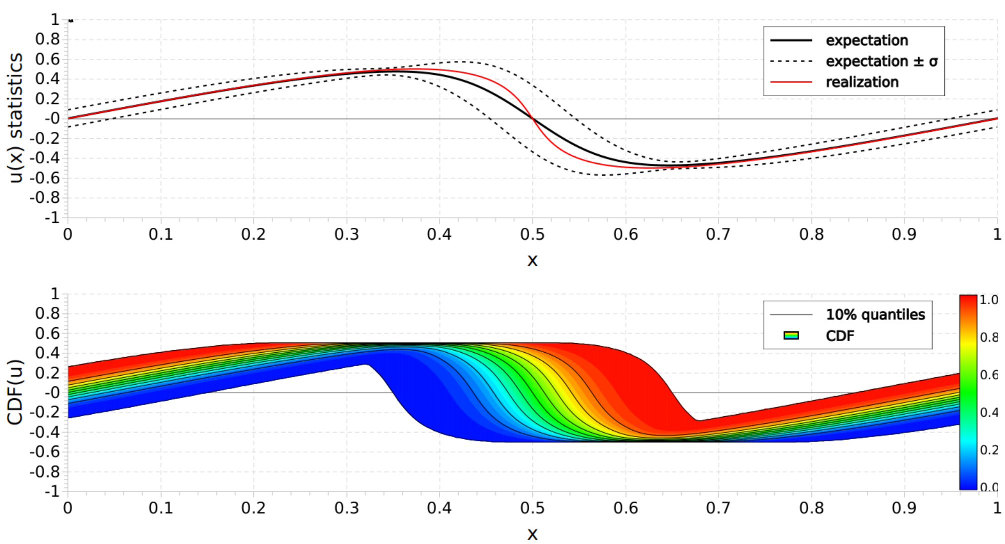

Figure 10: Burgers equation with phase uncertainty, $\mathbb{X}(\omega) \sim \mathcal{N}_3(m = 0, \sigma = 0.05)$. Moment statistics and representative realization (top) and shaded cumulative distribution with 10% quantiles (bottom) at time $t = 0.25$

The cumulative distribution function, $CDF_{u_{\mathbb{X}}}(u)(x, t = 1/4; 1/2)$, (shaded region) together with quantiles of 10% probability are shown in Figure 10 (bottom). Moment statistics and a representative realization have been graphed in Figure 10 (top) for reference. Figure 11 shows graphs of the solution cumulative distribution function at $x = 0.2$ (left) and $x = 0.46$ (right). These results appear similar to the previous uniform distribution results with the most pronounced differences in the discontinuity region apparently due to the distribution tails in the normal distribution.

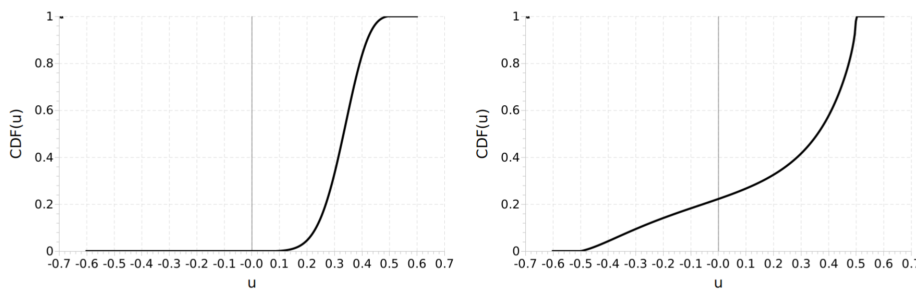

Figure 11: Burgers equation with phase uncertainty, $\mathbb{X}(\omega) \sim \mathcal{N}_3(m = 0, \sigma = 0.05)$. Graphs of the cumulative distribution at $x = 0.2$ (left) and $x = 0.46$ (right) at time $t = 0.4$

5.3 Calculating the output cumulative probability distribution for Burgers equation with amplitude uncertainty

Recall the amplitude uncertain initial data problem (7a) and (7b) repeated here

$$\begin{aligned} \partial_t u_{\mathbb{X}} + \partial_x u_{\mathbb{X}}^2/2 &= 0 \text{ in } [0,1] \times \mathbb{R}^+ \times \Omega \\ u_{\mathbb{X}}(x, 0, \omega) &= \mathbb{X}(\omega) \sin(2\pi x) \text{ for } 0 < A_{min} \leq \mathbb{X}(\omega) \leq A_{max} \end{aligned}$$

Let $h(\zeta)(x,t) : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ denote a solution of the deterministic Burgers equation problem with ζ amplitude initial data. This function is related to the Burgers solution by

$$h(\zeta)(x, t) = u(x, t; \zeta) \tag{45}$$

Further, define the amplitude uncertainty iteration function

$$H(\zeta)(x,t;\tilde{u}) := h(\zeta)(x,t) - \tilde{u} \tag{46}$$

that effectively inverts $h(\cdot)$, i.e., $\zeta(\tilde{u}) = h^{-1}(\tilde{u})(x,t)$. At this point, the number of roots of $H(\zeta)(x,t;\tilde{u})$ is unknown and a robust strategy for computing them is desired. The following lemmas prove that the amplitude uncertainty iteration function contains a single root that can be reliably computed using bracketed iteration. The next lemma proves that this function is a strictly monotone function.

Lemma 8 (Amplitude uncertainty iteration function monotonicity) *Let $u(x,t;A)$ denote a solution of the Burgers equation problem (5a)-(5b). The amplitude uncertainty iteration function*

$$H(\zeta)(x,t;\tilde{u}) := h(\zeta)(x,t) - \tilde{u} \tag{47}$$

for a given fixed $\tilde{u}$ is strictly

- *increasing for $\zeta > 0$ and $(x,t) \in Q^L$*
- *decreasing for $\zeta > 0$ and $(x,t) \in Q^R$*

Proof: The amplitude uncertainty iteration function simplifies to

$$H(\zeta)(x,t;\tilde{u}) \quad = \quad \zeta \sin(2\pi x_0(x,t;\zeta)) - \tilde{u}$$

with

$$x_0(x,t;\zeta) = x - t\,\zeta \sin(2\pi x_0(x,t;\zeta)) \tag{48}$$

and by differentiation

$$H'(\zeta)(x,t;\tilde{u}) = \frac{\sin(2\pi x_0(x,t;\zeta)}{1 + 2\pi\, t\, \zeta \cos(2\pi x_0(x,t;\zeta)} \tag{49}$$

In the proof of Lemma 5, the denominator in this formula is proven strictly positive for $(x,t) \in Q^L \cup Q^R$ and $\zeta > 0$. From Lemma 4, $(x,t) \in Q^L$ implies $x_0(x,t;\zeta) \in (0,1/2)$ and $\sin(2\pi x_0(x,t;\zeta)) > 0$. Similarly, $(x,t) \in Q^R$ implies $x_0(x,t;\zeta) \in (1/2,1)$ and $\sin(2\pi x_0(x,t;\zeta)) < 0$. Combining these results proves the stated lemma. ■

Lemma 8 implies that $H(\zeta)$ is injective and has at most one root. The next lemma proves that this single root with isolated bracketed interval always exists whenever $u(x,t;A_{min}) < \tilde{u} < u(x,t;A_{max})$.

Lemma 9 (Amplitude uncertainty iteration function isolated bracketing) *Let $u(x,t;A)$ denote a solution of the Burgers equation problem (5a)-(5b) and $H(\zeta)(x,t;\tilde{u})$ the amplitude uncertainty iteration function*

$$H(\zeta)(x,t;\tilde{u}) := h(\zeta)(x,t) - \tilde{u} \tag{50}$$

with $u(x,t;A_{min}) < \tilde{u} < u(x,t;A_{max})$. The interval $[A_{min}, A_{max}]$ is an isolated bracketing interval for $H(\zeta)(x,t;\tilde{u})$.

Proof: Evaluating the amplitude uncertainty iteration function at the bracket limits verifies the bracketing property for $u(x,t;A_{min}) < \tilde{u} < u(x,t;A_{max})$

$$\begin{aligned} H(A_{min}) &= \tilde{u} - u(x,t;A_{min}) > 0 \\ H(A_{max}) &= \tilde{u} - u(x,t;A_{max}) < 0 \end{aligned}$$

When combined with the strict monotonicity results of Lemma 8, the stated lemma is proved. ■

Theorem 3 (Amplitude Uncertainty Computability) *Let $u(x,t;A)$ denote a solution of the Burgers equation problem (5a)-(5b) and $H(\zeta)(x,t;\tilde{u})$ the amplitude uncertainty iteration function*

$$H(\zeta)(x,t;\tilde{u}) := u(x,t;\zeta) - \tilde{u} \tag{51}$$

for $u(x,t;A_{min}) < \tilde{u} < u(x,t;A_{max})$ The single root of the amplitude uncertainty iteration function can be reliably computed, assuming exact arithmetic, with guaranteed reliability to a specified precision ϵ using at most $\log_2 \frac{A_{max}-A_{min}}{\epsilon}$ steps of the bisection root finding method.

Proof: The theorem follows immediately from Lemma 9 together with the error convergence estimate (23) for the bisection root finding method. ■

Theorem 3 proves that the single isolated root of the amplitude uncertainty function can be reliably computed. Using (30), the cumulative probability distribution then reduces to

$$CDF_{u_{\mathbb{X}}}(\tilde{u})(x,t) = \begin{cases} 0, & \tilde{u} \le u_{min}(x,t;A_{min},A_{max}) \\ 1 - CDF_{\mathbb{X}}(\zeta(\tilde{u}))(x,t), & u_{min}(x,t;A_{min},A_{max}) < \tilde{u} < 0 \\ CDF_{\mathbb{X}}(\zeta(\tilde{u}))(x,t), & 0 < \tilde{u} < u_{max}(x,t;A_{min},A_{max}) \\ 1, & \tilde{u} \ge u_{max}(x,t;A_{min},A_{max}) \end{cases} \tag{52}$$

where

$$u_{min}(x,t;A_{min},A_{max}) = \begin{cases} u(x,t;A_{min}), & x < 1/2 \\ u(x,t;A_{max}), & \text{otherwise} \end{cases}$$

and

$$u_{max}(x,t;A_{min},A_{max}) = \begin{cases} u(x,t;A_{max}), & x < 1/2 \\ u(x,t;A_{min}), & \text{otherwise} \end{cases}$$

5.3.1 Example: Burgers equation amplitude uncertainty output statistics, $\mathbb{X}(\omega) \sim \mathcal{U}[.3,.5]$

Output statistics for the amplitude uncertain Burgers equation problem (7a)-(7b) with uniform probability measure, $\mathbb{X}(\omega) \sim \mathcal{U}[.3,.5]$ are presented in Figure 12.

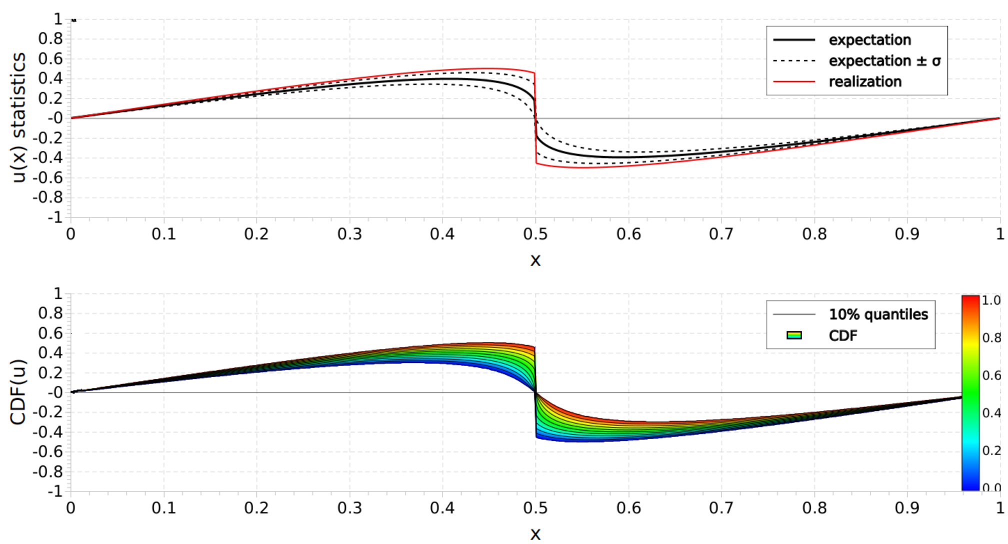

Figure 12: Burgers equation with amplitude uncertainty, $\mathbb{X}(\omega) \sim \mathcal{U}[.3,.5]$. Moment statistics and representative realization (top) and shaded cumulative distribution with 10% quantiles (bottom) at time $t = .4$

The cumulative distribution function, $CDF_{u_{\mathbb{X}}}(u)(x,t=4/10;1/2)$, (shaded region) together with quantiles of 10% probability are shown in Figure 12 (bottom). Moment statistics and a representative realization have been graphed in Figure 12 (top) for reference. Figure 13 shows graphs of the solution cumulative distribution function at $x = 0.2$ (left) and $x = 0.46$

(right). In sharp contrast to phase uncertainty, the present results show only relatively small deviation from a uniform distribution.

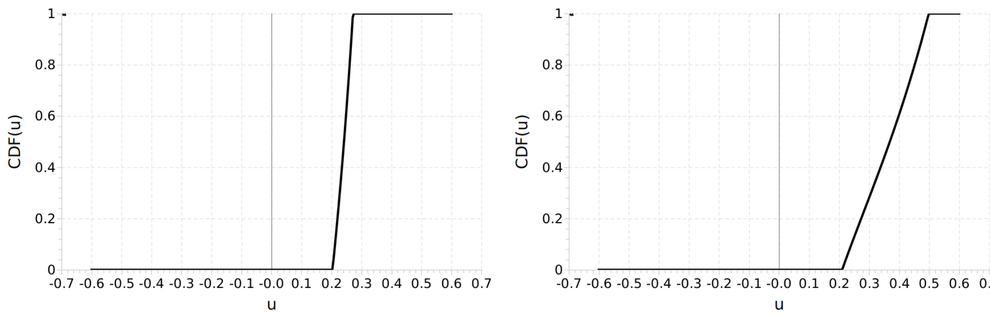

Figure 13: Burgers equation with amplitude uncertainty, $\mathbb{X}(\omega) \sim \mathcal{U}[.3, .5]$. Graphs of the cumulative distribution at $x = 0.2$ (left) and $x = 0.46$ (right) at time $t = 0.4$

5.3.2 Example: Burgers equation amplitude uncertainty output statistics, $\mathbb{X}(\omega) \sim \mathcal{N}_3(m = 0.35, \sigma = 0.05)$

Output statistics for the amplitude uncertain Burgers equation problem (7a)-(7b) with normal distribution probability measure truncated at 3σ, $\mathbb{X}(\omega) \sim \mathcal{N}_3[m = 0.35, 0.05]$ are presented in Figure 14.

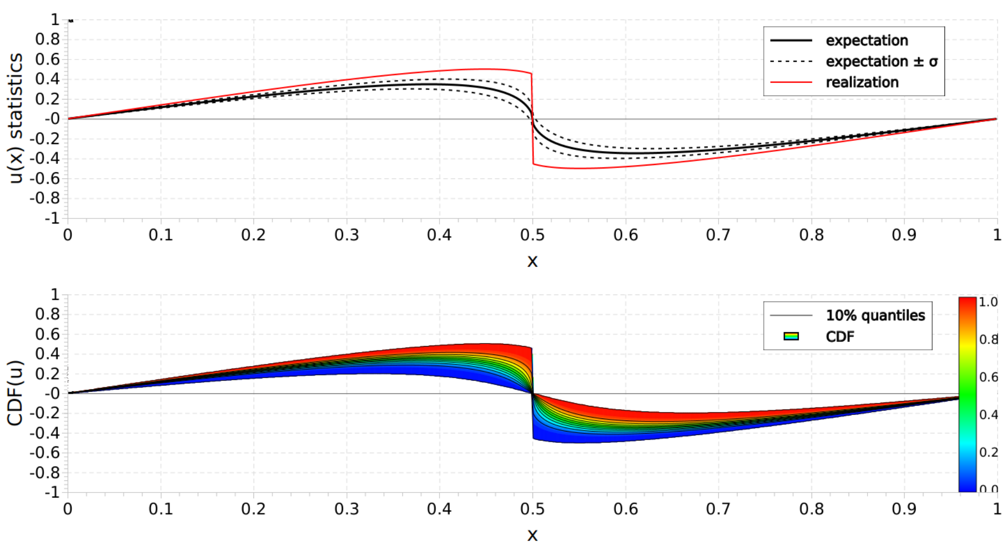

Figure 14: Burgers equation with amplitude uncertainty, $\mathbb{X}(\omega) \sim \mathcal{N}_3(m = 0.35, \sigma = 0.05)$. Moment statistics and representative realization (top) and shaded cumulative distribution with 10% quantiles (bottom) at time $t = .4$

The cumulative distribution function, $CDF_{u_{\mathbb{X}}}(u)(x, t = 1/4; 1/2)$, (shaded region) together with quantiles of 10% probability are shown in Figure 14 (bottom). Moment statistics and a representative realization have been graphed in Figure 14 (top) for reference. Figure 15 shows graphs of the solution cumulative distribution function at $x = 0.2$ (left) and $x = 0.46$ (right). Again in sharp contrast with the phase uncertainty results, the present graphs results show only relatively small deviation from a normal distributions.

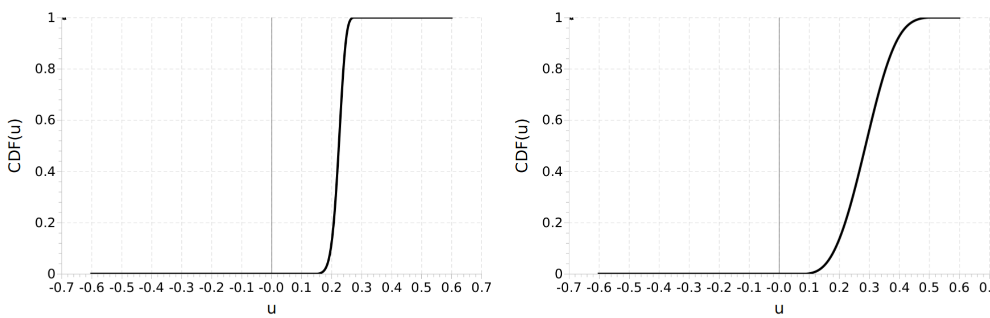

Figure 15: Burgers equation with amplitude uncertainty, $\mathbb{X}(\omega) \sim \mathcal{N}_3(m = 0.35, \sigma = 0.05)$. Graphs of the cumulative distribution at $x = 0.2$ (left) and $x = 0.46$ (right) at time $t = 0.4$

6 Concluding Remarks

A robust procedure and underlying theory have been presented for calculating exact uncertainty statistics for Burgers equation with uncertain sinusoidal initial data. This model problem together with exact uncertainty statistics provides a benchmark for assessing numerical methods in uncertainty quantification.

The exact solution to Burgers equation problem with uncertain sinusoidal initial data also provides insight into difficulties encountered by many numerical methods. In particular, the exact solution exhibits a piecewise smooth behavior in random variable dimensions that can greatly degrade the accuracy of numerical methods that rely on global smoothness in random variable dimensions.

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