Analytic Spacecraft Attitude and Rate Estimation Performance During Attitude Sensor Outages

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Problem Background

- Interest in rapidly quantifying attitude filter performance
  - Star tracker, gyro component selection
  - Independent check on flight software implementation
- Analytic solutions to single axis case exist in literature
  - Farrenkopf (1974, 1978) for Rate Output Gyro (ROG)
    - Single-axis analog to Multiplicative Extended Kalman Filter (MEKF)
    - MEKF widely used (Space Shuttle, Orion, JWST, Hubble, etc)
  - Markley Reynolds (2000) for Rate Integrating Gyro (RIG)
    - Single-axis analog to MEKF extension to RIG (Crassidis Markley 2016)
- Attitude Sensor Outages
  - Attitude sensors' measurements can sporadically become unavailable
    - Sensor field-of-view occlusion
    - Sensor image smear during thruster maneuvers
    - Sensor reset due to radiation event
  - During attitude sensor outage, filter propagates with gyro
  - Is it possible to develop analytic expressions for filter performance during attitude sensor outages?

Single Axis Spacecraft with ROG

- Rate Output Gyro (ROG) Model

$$\omega_g(t) = \omega(t) + b(t) + \sigma_v n_v(t)$$

gyro meas | true rate | bias | angle random walk noise

$$\dot{b}(t) = \sigma_u n_u(t)$$

rate random walk noise

$n_v(t), n_u(t)$

independent
zero mean
unit variance
Gaussian noise

- Single Axis Spacecraft Model

$$\begin{aligned}\dot{\theta}(t) &= \omega(t) \\ &= \omega_g(t) - b(t) - \sigma_v n_v(t)\end{aligned}$$

$$\boldsymbol{x}(t) = \begin{bmatrix} \theta(t) \\ b(t) \end{bmatrix}$$

$$\boldsymbol{x}(t_{k+1}) = \Phi(\Delta t)\boldsymbol{x}(t_k) + \Gamma(\Delta t)\omega_g(t_k) + \boldsymbol{n}(t_k)$$

$$\Phi(\Delta t) = \begin{bmatrix} 1 & -\Delta t \\ 0 & 1 \end{bmatrix} \qquad \Gamma(\Delta t) = \begin{bmatrix} \Delta t \\ 0 \end{bmatrix}$$

$$\begin{aligned}Q(\Delta t) &= E\left[\boldsymbol{n}(t_k)\boldsymbol{n}^T(t_k)\right] \\ &= \begin{bmatrix} \sigma_v^2 \Delta t + \frac{1}{3}\sigma_u^2 \Delta t^3 & -\frac{1}{2}\sigma_u^2 \Delta t^2 \\ -\frac{1}{2}\sigma_u^2 \Delta t^2 & \sigma_u^2 \Delta t \end{bmatrix}\end{aligned}$$

Attitude Filter for ROG

$$\hat{\boldsymbol{x}}(t) = \begin{bmatrix} \hat{\theta}(t) \\ \hat{b}(t) \end{bmatrix}$$

- Use Gyro Measurements to Propagate

$$\hat{\boldsymbol{x}}(t_{k+1}^-) = \Phi(\Delta t)\hat{\boldsymbol{x}}(t_k^+) + \Gamma(\Delta t)\omega_g(t_k)$$

$$P(t_{k+1}^-) = \Phi(\Delta t)P(t_k^+)\Phi^T(\Delta t) + Q(\Delta t)$$

- Attitude Measurements (Star Tracker)

$$\begin{aligned} y(t_k) &= H\boldsymbol{x}(t_k) + \sigma_n n_n(t_k) \\ &= \theta(t_k) + \sigma_n n_n(t_k) \end{aligned}$$

$$R = E[\sigma_n^2 n_n^2(t)] = \sigma_n^2$$

- Attitude Measurement Update

$$K(t_k) = P(t_k^-)H^T\left(HP(t_k^-)H^T + R\right)^{-1}$$

$$P(t_k^+) = P(t_k^-) - K(t_k)HP(t_k^-)$$

$$\hat{\boldsymbol{x}}(t_k^+) = \hat{\boldsymbol{x}}(t_k^-) + K(t_k)\left(y(t_k) - H\hat{\boldsymbol{x}}(t_k^-)\right)$$

Farrenkopf's solution

- Steady State (SS) conditions:

$$P(-) = P(t_{k+1}^{-}) = P(t_k^{-}) = \begin{bmatrix} \sigma_{\theta\theta}^2(-) & \sigma_{\theta b}^2(-) \\ \sigma_{\theta b}^2(-) & \sigma_{bb}^2(-) \end{bmatrix}$$

$$P(+) = P(t_{k+1}^{+}) = P(t_k^{+}) = \begin{bmatrix} \sigma_{\theta\theta}^2(+) & \sigma_{\theta b}^2(+) \\ \sigma_{\theta b}^2(+) & \sigma_{bb}^2(+) \end{bmatrix}$$

- SS conditions and filter equations reduce to a single quartic (Farrenkopf 1974):

$$x^4 + S_u^2 x^3 + S_u^2 \left(\frac{1}{6} S_u^2 - S_v^2 - 2 \right) x^2 + S_u^4 x + S_u^4 = 0$$

- Farrenkopf (1978) solved the quartic analytically
  - Always only a single meaningful solution
  - Solution is analytic expressions for all elements of P(+), P(-)
- Farrenkopf's solution can be used to compute uncertainty of simple rate estimate

$$\hat{\omega}(t_k^+) = \omega_g(t) - \hat{b}(t_k^+)$$

$$\sigma_{\omega\omega}^2(+) = E\left[\left(\omega(t_k) - \hat{\omega}(t_k^+) \right)^2 \right] = \sigma_{bb}^2(+) + \sigma_v^2$$

ROG filter performance during outage

- Suppose filter is at steady state t_{ss}, and then attitude measurements no longer available
- Covariance grows according to

$$P(t_p^-) = \begin{bmatrix} \sigma_{\theta\theta}^2(t_p^-) & \sigma_{\theta b}^2(t_p^-) \\ \sigma_{\theta b}^2(t_p^-) & \sigma_{bb}^2(t_p^-) \end{bmatrix}$$

$$= \Phi(t_p - t_{ss}) P(t_{ss}^+) \Phi^T(t_p - t_{ss}) + Q(t_p - t_{ss})$$

- Possible to find analytic expressions for covariance

$$\sigma_{\theta\theta}^2(t_p^-) = \sigma_{\theta\theta}^2(+) - 2\Delta t_p \sigma_{\theta b}^2(+) + \Delta t_p^2 \sigma_{bb}^2(+) + \sigma_v^2 \Delta t_p + \frac{1}{3}\sigma_u^2 \Delta t_p^3$$

$$\sigma_{bb}^2(t_p^-) = \sigma_{bb}^2(+) + \sigma_u^2 \Delta t_p \qquad \Delta t_p = t_p - t_{ss}$$

- Rate estimate uncertainty is simply

$$\sigma_{\omega\omega}^2(t_p^-) = E\left[\left(\omega(t_p) - \hat{\omega}(t_p^-)\right)^2\right] = \sigma_{bb}^2(t_p^-) + \sigma_v^2$$

ROG filter performance

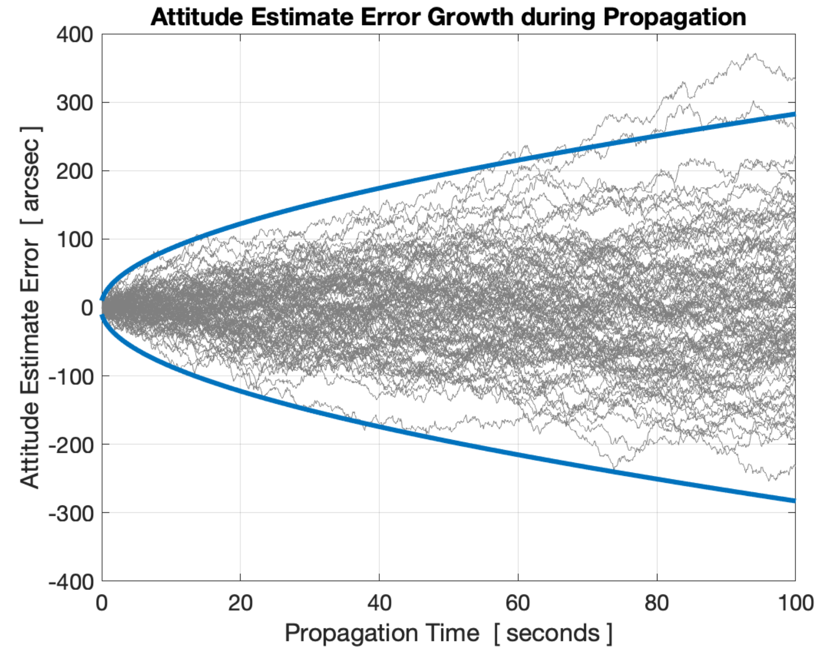
- 100 independent realizations
- high-performance MEMS ROG

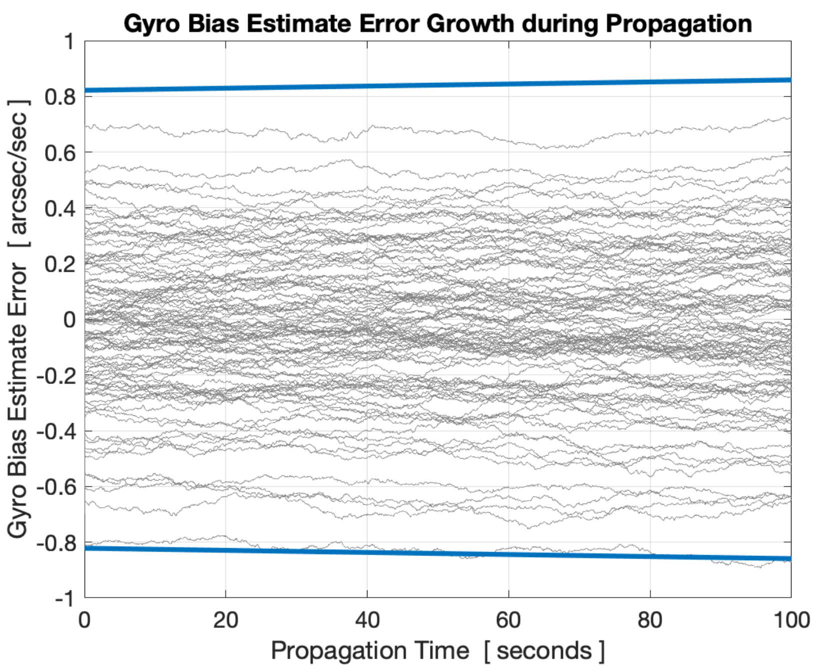
$$\sigma_v = 9.00\ \frac{arcsec}{\sqrt{s}}$$

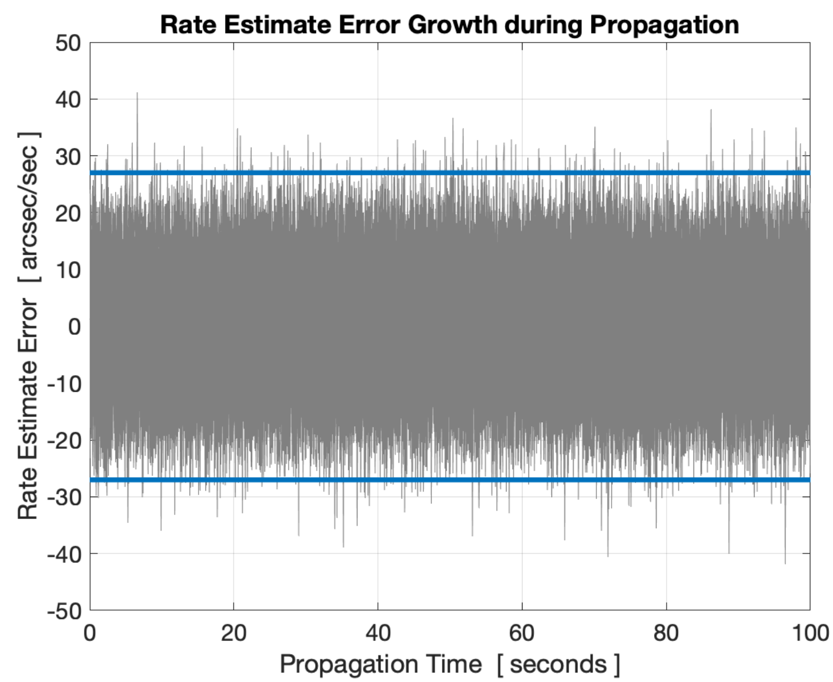
$$\sigma_u = 0.00833\ \frac{arcsec}{\sqrt{s^3}}$$

- high-performance CubeSat star tracker

$$\sigma_n = 5.00\ arcsec$$







Single Axis Spacecraft with RIG

- Rate Integrating Gyro (RIG)

RIG internal dynamics (true rate, bias, angle random walk, rate random walk):

$$\dot{\phi}(t) = \omega(t) + b(t) + \sigma_v n_v(t)$$

$$\dot{b}(t) = \sigma_u n_u(t)$$

RIG output (angle noise):

$$\phi_g(t_k) = \phi(t_k) + \sigma_e n_e(t_k)$$

- Single Axis Spacecraft Model

$$\boldsymbol{x}(t) = \begin{bmatrix} \theta(t) \\ b(t) \\ \phi(t) \end{bmatrix}$$

$$\boldsymbol{x}(t_{k+1}) = \Phi(\Delta t)\boldsymbol{x}(t_k) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \phi_g(t_{k+1}) + \boldsymbol{n}(t_{k+1})$$

$$\Phi(\Delta t) = \begin{bmatrix} 1 & -\Delta t & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} Q(\Delta t) &= E\left[\boldsymbol{n}(t_k)\boldsymbol{n}^T(t_k)\right] \\ &= \begin{bmatrix} \sigma_v^2 \Delta t + \frac{1}{3}\sigma_u^2 \Delta t^3 + \sigma_e^2 & -\frac{1}{2}\sigma_u^2 \Delta t^2 & \sigma_e^2 \\ -\frac{1}{2}\sigma_u^2 \Delta t^2 & \sigma_u^2 \Delta t & 0 \\ \sigma_e^2 & 0 & \sigma_e^2 \end{bmatrix} \end{aligned}$$

Attitude Filter for RIG

$$\hat{\boldsymbol{x}}(t) = \begin{bmatrix} \hat{\theta}(t) \\ \hat{b}(t) \\ \hat{\phi}(t) \end{bmatrix}$$

- Use Gyro Measurements to Propagate

$$\hat{\boldsymbol{x}}(t_{k+1}^-) = \Phi(\Delta t)\hat{\boldsymbol{x}}(t_k^+) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \phi_g(t_{k+1})$$

$$P(t_{k+1}^-) = \Phi(\Delta t)P(t_k^+)\Phi^T(\Delta t) + Q(\Delta t)$$

- Attitude Measurements (Star Tracker)

$$\begin{aligned} y(t_k) &= H\boldsymbol{x}(t_k) + \sigma_n n_n(t_k) \\ &= \theta(t_k) + \sigma_n n_n(t_k) \end{aligned}$$

$$R = E[\sigma_n^2 n_n^2(t)] = \sigma_n^2$$

- Attitude Measurement Update

$$K(t_k) = P(t_k^-)H^T\left(HP(t_k^-)H^T + R\right)^{-1}$$

$$P(t_k^+) = P(t_k^-) - K(t_k)HP(t_k^-)$$

$$\hat{\boldsymbol{x}}(t_k^+) = \hat{\boldsymbol{x}}(t_k^-) + K(t_k)\left(y(t_k) - H\hat{\boldsymbol{x}}(t_k^-)\right)$$

Markley Reynolds (2000) solution

- Steady State (SS) conditions:

$$P(-) = P(t_{k+1}^{-}) = P(t_k^{-}) = \begin{bmatrix} \sigma_{\theta\theta}^2(-) & \sigma_{\theta b}^2(-) & \sigma_{\theta\phi}^2(-) \\ \sigma_{\theta b}^2(-) & \sigma_{bb}^2(-) & \sigma_{b\phi}^2(-) \\ \sigma_{\theta\phi}^2(-) & \sigma_{b\phi}^2(-) & \sigma_{\phi\phi}^2(-) \end{bmatrix}$$

$$P(+) = P(t_{k+1}^{+}) = P(t_k^{+}) = \begin{bmatrix} \sigma_{\theta\theta}^2(+) & \sigma_{\theta b}^2(+) & \sigma_{\theta\phi}^2(+) \\ \sigma_{\theta b}^2(+) & \sigma_{bb}^2(+) & \sigma_{b\phi}^2(+) \\ \sigma_{\theta\phi}^2(+) & \sigma_{b\phi}^2(+) & \sigma_{\phi\phi}^2(+) \end{bmatrix}$$

- SS conditions and filter equations reduce to a single quartic (Markley Reynolds 2000):

$$\left(\zeta^2 - 2\left(\gamma + \frac{1}{4}S_u\right)\zeta + 1 + S_e^2\right)\left(\zeta^2 + 2\left(\gamma - \frac{1}{4}S_u\right)\zeta + 1 + S_e^2\right) = 0$$

- Markley and Reynolds found analytic solution
  - Always only a single physically-meaningful solution
  - Solution reduces to Farrenkopf's when $\sigma_e = 0$
- Solution can be used to find uncertainty of a rate estimate

$$\hat{\omega}(t_k^+) = \frac{\phi_g(t_k) - \phi_g(t_{k-1})}{\Delta t} - \hat{b}(t_{k-1}^+)$$

$$\begin{aligned} \sigma_{\omega\omega}^2(+) &= E\left[\tilde{\omega}^2(t_k^+)\right] \\ &= \sigma_{bb}^2(+) + \frac{1}{\Delta t}\sigma_v^2 + \frac{\Delta t}{3}\sigma_u^2 + \frac{2}{\Delta t}\sigma_e^2 \end{aligned}$$

RIG filter performance during outage

- Suppose filter is at steady state t_{ss}, and then attitude measurements no longer available
- Covariance grows according to

$$P(t_p^-) = \begin{bmatrix} \sigma^2_{\theta\theta}(t_p^-) & \sigma^2_{\theta b}(t_p^-) & \sigma^2_{\theta\phi}(t_p^-) \\ \sigma^2_{\theta b}(t_p^-) & \sigma^2_{bb}(t_p^-) & \sigma^2_{b\phi}(t_p^-) \\ \sigma^2_{\theta\phi}(t_p^-) & \sigma^2_{b\phi}(t_p^-) & \sigma^2_{\phi\phi}(t_p^-) \end{bmatrix}$$

$$\Delta t_p = t_p - t_{ss}$$

$$= \Phi(\Delta t_p) P(t_{ss}^+) \Phi^T(\Delta t_p) + Q(\Delta t_p)$$

- Possible to find analytic expressions for covariance

$$\sigma^2_{\theta\theta}(t_p^-) = \sigma^2_{\theta\theta}(+) + \Delta t_p^2 \sigma^2_{bb}(+) + \sigma^2_{\phi\phi}(+) - 2\Delta t_p \sigma^2_{\theta b}(+) - 2\sigma^2_{\theta\phi}(+)$$
$$+ 2\Delta t_p \sigma^2_{b\phi}(+) + \Delta t_p \sigma_v^2 + \frac{1}{3}\Delta t_p^3 \sigma_u^2 + \sigma_e^2$$

$$\sigma^2_{bb}(t_p^-) = \sigma^2_{bb}(+) + \Delta t_p \sigma_u^2$$

- Rate estimate uncertainty is

$$\sigma^2_{\omega\omega}(-) = E\left[\tilde{\omega}^2(t_j^-)\right]$$

$$\Delta t_j = t_j - t_p$$

$$= \sigma^2_{bb}(t_p^-) + \frac{1}{\Delta t_j}\sigma_v^2 + \frac{\Delta t_j}{3}\sigma_u^2 + \frac{2}{\Delta t_j}\sigma_e^2$$

$$t_j > t_p > t_{ss}$$

RIG performance during outage

- 100 independent realizations
- Ring Laser Gyro (RLG) - RIG

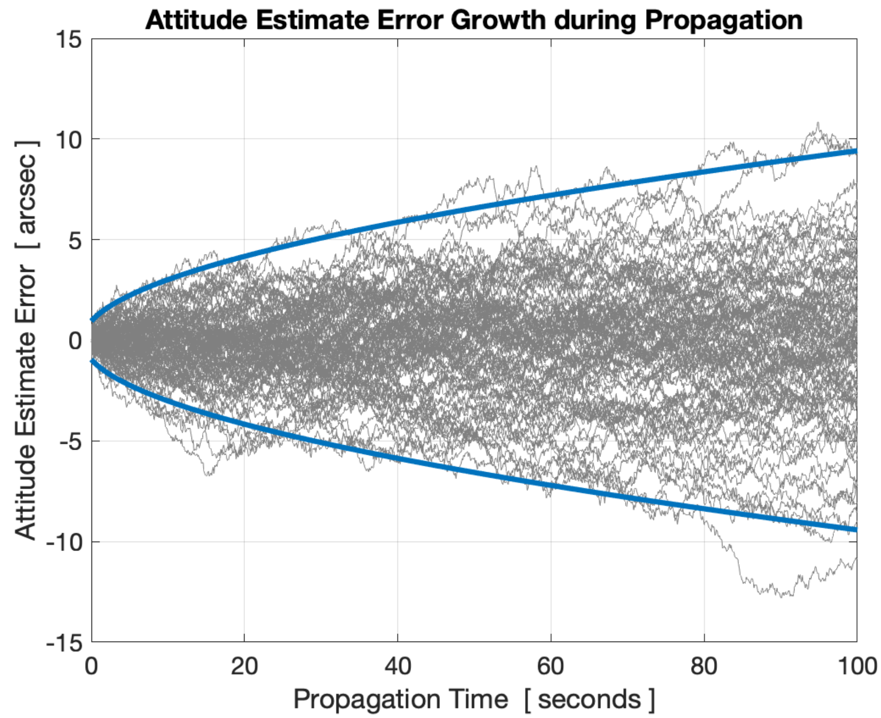
$$\sigma_v = 0.300 \ \frac{arcsec}{\sqrt{s}}$$

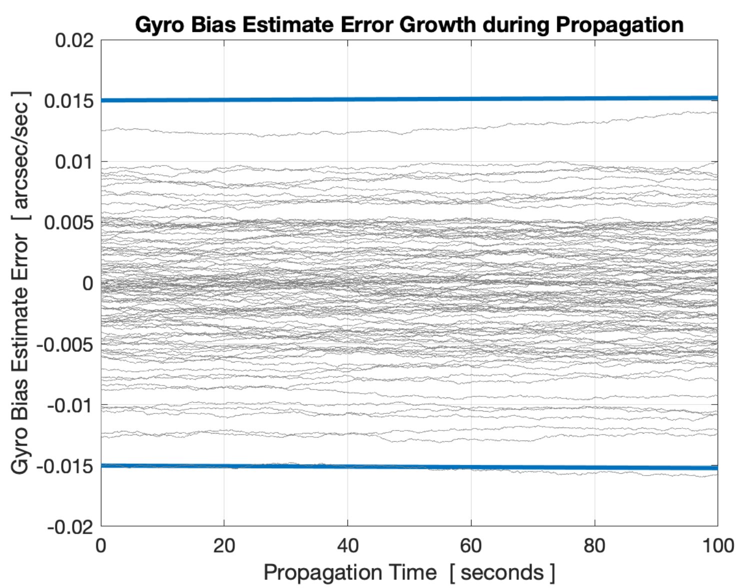
$$\sigma_u = 0.000083 \ \frac{arcsec}{\sqrt{s^3}}$$

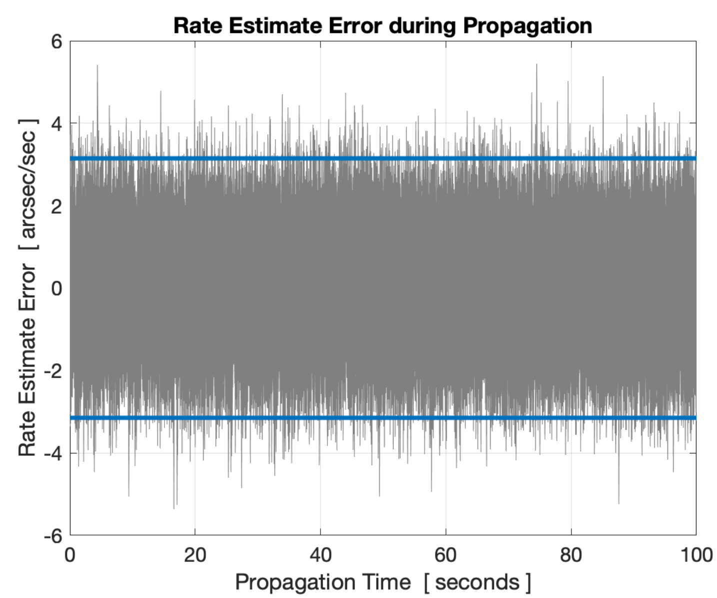
$$\sigma_e = 0.100 \ arcsec$$

- High-performance star tracker

$$\sigma_n = 3.09 \ arcsec$$







Conclusions

- Interest in quantifying attitude filter performance
  - Star tracker, gyro component selection
  - Independent check on flight implementation
- Analytic solutions exist in literature
  - Farrenkopf for Rate Output Gyro
  - Markley Reynolds for Rate Integrating Gyro
- Existing solutions can be used to analytic expressions for rate estimate uncertainty
- Analytic solutions can be extended to attitude sensor outage scenarios
  - New analytic results validated by Monte Carlo simulation