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Determining the Transport of Magnetic Helicity and Free Energy in the Sun's Atmosphere

Peter W. Schuck and Spiro K. Antiochos
Heliophysics Science Division, NASA Goddard Space Flight Center, 8800 Greenbelt Rd., Greenbelt, MD 20771, USA; peter.schuck@nasa.gov
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Abstract

The most important factors determining solar coronal activity are believed to be the availability of magnetic free energy and the constraint of magnetic helicity conservation. Direct measurements of the helicity and magnetic free energy in the coronal volume are difficult, but their values may be estimated from measurements of the helicity and free energy transport rates through the photosphere. We examine these transport rates for a topologically open system such as the corona, in which the magnetic fields have a nonzero normal component at the boundaries, and derive a new formula for the helicity transport rate at the boundaries. In addition, we derive new expressions for helicity transport due to flux emergence/submergence versus photospheric horizontal motions. The key feature of our formulas is that they are manifestly gauge invariant. Our results are somewhat counterintuitive in that only the lamellar electric field produced by the surface potential transports helicity across boundaries, and the solenoidal electric field produced by a surface stream function does not contribute to the helicity transport. We discuss the physical interpretation of this result. Furthermore, we derive an expression for the free energy transport rate and show that a necessary condition for free energy transport across a boundary is the presence of a closed magnetic field at the surface, indicating that there are current systems within the volume. We discuss the implications of these results for using photospheric vector magnetic and velocity field measurements to derive the solar coronal helicity and magnetic free energy, which can then be used to constrain and drive models for coronal activity.

Key words: magnetic fields – magnetic reconnection – Sun: coronal mass ejections (CMEs) – Sun: flares – Sun: magnetic fields – Sun: photosphere

1. Introduction

The Sun's atmosphere, the chromosphere–corona, is characterized by ubiquitous bursts of energy release ranging from sporadic giant coronal mass ejections (CMEs) and eruptive flares with energies up to 10^{33} erg (e.g., Forbes 2000; Aulanier et al. 2013) to ever-present nanoflares with energies of the order of 10^{24} erg or less (Parker 1988; Klimchuk 2006). All these forms of solar activity share a common underlying scenario. First, magnetic energy is injected into the corona either directly by the convective motions of the high-β photosphere acting on the low-β coronal magnetic field, or by the emergence of pre-stressed magnetic fields through the photosphere and into the corona. Second, the energy builds up until some fast process such as an ideal or resistive instability releases the magnetic energy, transferring it to the plasma in the form of heating, mass motion, and/or energetic particle acceleration. The amount of energy released (the size of the coronal event) depends on the dynamical constraints of the system. For the corona, there are two general dynamical constraints due primarily to the timescales for coronal evolution. Because the Alfvén speed V_A is three orders of magnitude or more faster in the corona than in the photosphere, the magnetic flux through the photosphere is expected to be approximately constant during an energy release event. Accordingly, the minimum energy state for the coronal field is the unique potential (current-free) field with that photospheric normal flux distribution. This result implies that the maximum available energy for release is only the "free" energy, defined as the difference between the energy of the coronal field and the energy of this potential state.

However, even this free energy overestimates the available energy for a coronal event, because the system is further constrained by the magnetic topology. In the corona, the timescale for resistive dissipation $\tau_{\rm d} \sim L^2/\eta$ is very large compared to observed evolutionary timescales $\tau_{\rm t} = V_{\rm A}/L$, where L is the scale of the system and η is the magnetic diffusivity. In other words, the corona has a very large Lundquist number $\tau_{\rm d}/\tau_{\rm t} = L\, V_{\rm A}/\eta \simeq 10^9$, which means that magnetic disturbances can cross the system long before they can be diffused. In this situation, the coronal evolution must be primarily ideal, which imposes additional constraints on the final state. If the evolution were perfectly ideal, then every field line would be a constant of the motion, leading to an infinite number of constraints, and the final state along with the energy available for release through an ideal evolution would be highly constrained by the initial magnetic topology. Fortunately, this is not the case; because the corona does have finite resistivity, magnetic reconnection can occur wherever current sheets form. For this reason, reconnection is widely believed to be the dominant energy release mechanism in most forms of solar activity, including CMEs, flares (e.g., Karpen et al. 2012), jets (e.g., Wyper et al. 2017), and coronal heating (Klimchuk 2006). Reconnection, however, preserves the topological constraint of total magnetic helicity, which is described in detail directly below. The implication is that the only relevant constraints for determining the energy available for explosive coronal activity are the magnetic free energy and the magnetic helicity. Accurate measurement of these quantities would greatly enable both the understanding and prediction of solar activity, but this would require measuring the full vector magnetic field in the coronal volume, which is not experimentally feasible at present. Because the free energy and helicity are injected into the corona through the photosphere, a more observationally feasible approach to constraining coronal energy release would be to measure the time-integrated transport rate of free energy and helicity through the photosphere. The goal of this paper is to derive rigorous expressions that are straightforward to understand physically and can be used effectively for measuring the free energy and helicity transport with photospheric observations.

1.1. Magnetic Helicity

While energy is undoubtedly the best known and most widely used quantity throughout physics, magnetic helicity is not generally well known or understood; therefore, we start with a discussion of its salient properties. Magnetic helicity is simply the quantification of magnetic flux linkages. The topological concept of linking numbers (Gauss 1867; Călugăreanu 1959; White 1969) and their connections with helicity introduced by Lord Kelvin (then Sir William Thomson 1868), plays a fundamental role in science from astrophysics (Woltjer 1958) to fluid dynamics (Moffatt 1969), and biochemistry (Fuller 1971; Crick 1976). The Gauss[1] linking number $\mathcal{L}_{ij}$ represents the number of times a closed curve i encircles a second closed curve j in space ($i \neq j$). The Călugăreanu–White linking number $\mathcal{L}_i$ represents the number of crossings of the edges of a ribbon curve, which may be further decomposed, $\mathcal{L}_i \equiv \mathrm{Tw}_i + \mathrm{Wr}_i$, where Tw_i is the twist and Wr_i is the writhe (Călugăreanu 1959; Pohl 1968; White 1969). Twist is a measure of how much a ribbon is twisted about its own axis and writhe is a measure of the nonplanarity of the ribbon axis itself (Dennis & Hannay 2005). Correspondingly, helicity measures the total linkage of magnetic field lines in a plasma or vortex lines in a fluid, and is given by the simple expression (see review by Moffatt 2014)

$$H = \int_{\mathcal{V}} d^3x\, \boldsymbol{A} \cdot \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{1a}$$

where $\mathcal{V}$ is a connected volume, $\boldsymbol{A}$ is the vector potential, and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ relates the vector potential to the observable magnetic field $\boldsymbol{B}$. For fluid flow, $\boldsymbol{A}$ is replaced by the fluid velocity $\boldsymbol{v}$ and $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ is the vorticity. Note that in contrast with energy, H is not positive definite. Helicity is a pseudo-scalar that changes sign under a transformation from a right-handed to a left-handed frame of reference, i.e., $H \neq \reflectbox{$H$} = -H$, where H and $\reflectbox{$H$}$ represent the helicity in the right-handed and left-handed coordinate systems.

For a set of N closed flux tubes or vortices, the expression for helicity, 1(a), can be shown to be formally related to the linking numbers (Moffatt 1969; Berger & Field 1984; Moffatt & Ricca 1992; Ricca 2002)

$$H = \sum_i^N \mathcal{L}_i\, \Psi_i + 2 \sum_{i \neq j}^{N,N} \mathcal{L}_{ij}\, \Psi_i\, \Psi_j, \tag{1b}$$

where Ψ_i is the toroidal flux of each elemental flux tube or the circulation of each vortex. The first term is identified as the "self-helicity" and the second term as the "mutual helicity." Despite the broad relevance of helicity, the concept is fraught with practical challenges in application to astrophysical plasmas. The magnetic helicity is only defined in terms of the integral over a closed volume $\mathcal{V}$ bounded by the surface $\mathcal{S}$. Thus, a complete knowledge of $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathcal{V}$ is required to compute H. Furthermore, the field must satisfy strong constraints in order for H, as defined by 1(a), to be physically meaningful. Eichinvarianz or gauge invariance, a fundamental principle of modern physics, is a manifestation of the unobservability of the electric and magnetic potentials $(\psi/c, \boldsymbol{A})$ (Weyl 1919; Jackson & Okun 2001). Gauge-transforming 1(a) with $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda$ and using (98) and the Gauss–Ostrogradsky theorem, 122(a), produces

$$H \to H - \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot (\Lambda\, \boldsymbol{B}), \tag{2}$$

where $\hat{\boldsymbol{n}}$ is the inwardly[2] directed normal to the volume $\mathcal{V}$. Therefore, H is uniquely defined only for a magnetically isolated plasma, where the normal component of $\boldsymbol{B}$ vanishes on all bounding surfaces $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = 0$. (Hereafter, the term "isolated" is equivalent to "magnetically isolated.")

For ideal plasma motions in an isolated system, the quadratic invariant H is conserved, notwithstanding complex dynamical evolution of the system. Just over 60 yr ago, Woltjer (1958) demonstrated that helicity is preserved for the evolution of an isolated ideal low-β plasma and that linear force-free fields are the minimum energy state of this system with a prescribed value of H. If the evolution were truly ideal, there would be an infinite number of constraints, in which case the helicity would not be very useful. Taylor (1974), however, conjectured that even for a weakly dissipative isolated plasma, the magnetic helicity H is a robust invariant, meaning that H is approximately conserved even in the presence of dissipation and is insensitive to the details of the nonequilibrium mechanisms involved in the relaxation to a lower energy state. If correct, Taylor's (1974) bold conjecture constrains the final state of the field and plasma even under large-scale reconfiguration, due to magnetic reconnection. Taylor's hypothesis has been largely verified in laboratory devices, such as reverse field pinches, where conducting walls with $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = 0$ isolate the plasma (Butt et al. 1976). Furthermore, helicity conservation has been observed in numerous MHD simulations with high Lundquist number (e.g., MacNeice et al. 2004; Knizhnik et al. 2017). The magnetic helicity H is also a robust invariant in periodic geometries in the absence of a mean magnetic field. However, the presence of a mean field in a periodic system destroys the topological concept of linkage and H is no longer a robust invariant (Berger 1997; Watson & Craig 2001).

1.2. Application to the Corona

Astrophysical plasmas, and in particular the solar corona, can rarely be considered isolated or periodic. To address this situation, Berger & Field (1984) and Finn & Antonsen (1985) contemporaneously introduced the concept of relative helicity $\mathcal{H}$ for systems with magnetic fields having $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} \neq 0$ at one or more system boundaries. Berger & Field (1984) and Finn & Antonsen (1985) defined relative helicity $\mathcal{H}$ in terms of an

[1] See note dated 1833 January 22 in Gauss (1867) as discussed extensively in Ricca & Nipoti (2011).

[2] This is simply so that $\hat{\boldsymbol{n}}$ points in the radial direction in the photosphere and into the coronal volume, which is conceptually convenient for practical solar calculations.

arbitrary reference magnetic field $\boldsymbol{B}_{\mathrm{R}}$ that satisfies $\boldsymbol{B}_{\mathrm{R}} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = \boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}}$ on all surfaces, and with no constraint on the vector potentials. The two definitions are equivalent.

The magnetic field lines that thread the boundaries are defined as "open" fields. For the corona, a natural boundary is the photosphere, in which case the normal flux certainly does not vanish there; on the contrary, every field line in the corona is expected to connect to the photosphere. Note that the term "open" is used very differently here than in the usual solar physics terminology, where "open" denotes coronal flux that connects to the photosphere at only one end, as in a coronal hole. In the terminology of this paper and of the standard helicity literature, all coronal flux is open, because it intersects the photosphere, at least, once.

An important point that must be emphasized is that both the helicity and the helicity transport rate are defined rigorously only for a finite closed volume. This implies that measuring the magnetic field at a single surface, the photosphere, is not sufficient to determine the coronal helicity or its evolution; however, it is not possible with present instrumentation to measure the field accurately in the hot corona. As a result, the approximation that is generally made is to consider a coronal volume that is finite but sufficiently large to contain an active region, for example, and then to assume that the helicity within this volume is due only to injection or loss through the photosphere. If a CME/eruptive flare occurs in the active region, the assumption clearly breaks down, but even without eruptions, reconnection may transport helicity into or out of a particular coronal volume (Antiochos 2013). On the other hand, given the available solar observations, the assumption to ignore all boundaries but the photosphere is virtually unavoidable.

With the results of Berger & Field (1984), Berger (1984) extended the Taylor conjecture to a coronal system and argued that relative helicity $\mathcal{H}$ is a robust invariant for a flare or other form of fast solar activity. Of course, solar eruptions do change the relative helicity in the lower corona by ejecting twisted field out into the heliosphere. The relative helicity is also changed by the transport of twisted field across the photospheric surface $\mathcal{S}$ and the twisting and tangling of footpoints by motion in the surface which can link coronal field lines. Berger & Field (1984) derived a simple expression for the relative helicity transport rate for a closed volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$

$$\dot{\mathcal{H}} \equiv \frac{\partial \mathcal{H}}{\partial t} = \overbrace{-2\,c \int_{\mathcal{V}} d^3x\, \boldsymbol{E} \cdot \boldsymbol{B}}^{\dot{\mathcal{H}}_{\mathcal{V}}} \overbrace{-\,2\,c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \left(\boldsymbol{A}_{\mathrm{PC}} \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \phi_{\mathrm{P}}}{\partial t}\boldsymbol{A}_{\mathrm{PC}}\right)}^{\dot{\mathcal{H}}_{\mathcal{S}}}, \tag{3}$$

where $\boldsymbol{E}$ is the electric field, ϕ_{P} is the magnetic scalar potential,

$$\boldsymbol{P} = -\boldsymbol{\nabla}\phi_{\mathrm{P}}, \tag{4}$$

determined from

$$\nabla^2\phi_{\mathrm{P}} = 0 \quad \boldsymbol{x} \in \mathcal{V}, \tag{5a}$$

$$\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = -\left.\frac{\partial \phi_{\mathrm{P}}}{\partial n}\right|_{\mathcal{S}}, \tag{5b}$$

and $\boldsymbol{A}_{\mathrm{PC}}$ is the unique vector potential of the potential magnetic field determined by

$$\boldsymbol{P} = \boldsymbol{\nabla} \times \boldsymbol{A}_{\mathrm{PC}} \tag{6a}$$

in the Coulomb gauge

$$\boldsymbol{\nabla} \cdot \boldsymbol{A}_{\mathrm{PC}} = 0, \tag{6b}$$

with the boundary condition that the normal component of the vector potential vanishes at the surface $\mathcal{S}$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{A}_{\mathrm{PC}}|_{\mathcal{S}} = 0. \tag{6c}$$

This boundary condition can always be satisfied by adding the appropriate gradient of a harmonic function to any $\boldsymbol{A}_{\mathrm{PC}}$ (Cantarella et al. 2002). Relationships 6(b) and 6(c) define an intrinsically solenoidal vector $\boldsymbol{A}_{\mathrm{PC}} = \boldsymbol{\nabla} \times \boldsymbol{f}$ according to Helmholtz's theorem (see Appendix E and Kemmer 1977).

The first integral $\dot{\mathcal{H}}_{\mathcal{V}}$ in (3) represents the generation and dissipation of helicity in the volume $\mathcal{V}$, and the second integral $\dot{\mathcal{H}}_{\mathcal{S}}$ represents the helicity transport rate across the boundary $\mathcal{S}$. Generally, the electric field $\boldsymbol{E}$ may comprise both ideal and nonideal terms through a generalized Ohm's law. Indeed, $\boldsymbol{E}$ may be nonideal in highly localized regions of the volume permitting reconnection but ideal throughout most of the volume. For this scenario, the generation and dissipation of helicity in the volume $\dot{\mathcal{H}}_{\mathcal{V}}$ can be ignored, and $\dot{\mathcal{H}} \simeq \dot{\mathcal{H}}_{\mathcal{S}}$ (Taylor 1974; Berger 1984). Nonetheless, if $\boldsymbol{E}$ can be measured directly on the surface, then $\dot{\mathcal{H}}_{\mathcal{S}}$ is the surface helicity transport rate regardless of the nonideal properties of the evolution on the surface or in the volume. However, if the evolution in the volume is substantially nonideal, then $\dot{\mathcal{H}} \neq \dot{\mathcal{H}}_{\mathcal{S}}$. Using the ideal magnetohydrodynamics (MHD) Ohm's law,

$$\boldsymbol{E} = -\frac{1}{c}\,\boldsymbol{v} \times \boldsymbol{B}, \tag{7}$$

where $\boldsymbol{v}$ is the single fluid bulk plasma velocity, the surface helicity transport rate simplifies to

$$\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} = 2 \oint_{\mathcal{S}} dS\, \underbrace{(\boldsymbol{A}_{\mathrm{PC}} \cdot \boldsymbol{B})v_n}_{\text{emergence}} - 2 \oint_{\mathcal{S}} dS\, \underbrace{(\boldsymbol{A}_{\mathrm{PC}} \cdot \boldsymbol{v})B_n}_{\text{shearing}}, \tag{8}$$

where the subscript "n" indicates the normal component of a vector. Berger (1984, 1999) and Berger & Ruzmaikin (2000) have identified these two terms as the emergence and shearing terms (see also Kusano et al. 2002, 2003; Yamamoto & Sakurai 2009). The emergence term is, in principle, due to the emergence through the photosphere of a twisted field from the solar interior, and the shearing term is presumed to represent the helicity generated by the shearing and twisting of field lines by tangential motions on the photospheric surface. There is a long history of estimating each term independently (Chae 2001; Nindos et al. 2003; Pevtsov et al. 2003; Pariat et al. 2005; Démoulin 2007; Liu et al. 2014). However, there are several concerns with the interpretation and use of this equation:

1. The terms in (8) are neither in part, nor in whole, manifestly gauge invariant. Indeed, (8) is expressed explicitly in the Coulomb gauge, 6(a)–6(c). While the combination of the emergence and shearing terms with other terms implicitly contained in (8) is gauge invariant (see (28) in Section 2), neither term is independently gauge invariant, and their physical interpretation is

ambiguous. If the separation of helicity transport into emergence and shearing is physically valid, there should be a gauge-invariant expression with that interpretation. Consequently, without further justification, the two surface integrals generally cannot be considered independently as observables.

2. Central to the interpretation of relative helicity is the role of flux threading the bounding surface. If the system is both isolated, $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = 0$, and ideal, then $\mathcal{H}$ should be a robust invariant. In other words, helicity transport across the surface $\mathcal{S}$ must require a finite $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}}$; however, (8) is only indirectly related to $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}}$ through the tangential components of $\boldsymbol{A}_{\rm PC}$.
3. Prior & Yeates (2014, p. 2) note that when "the boundary conditions $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}}$ are changing in time, ... the evolution of the relative helicity will mix up both real topological changes in $\boldsymbol{B}$ and those simply due to the change of $\boldsymbol{P}$."
4. While the combination of the emergence and shearing terms is independent of the flow parallel to the magnetic field because they result from (7), each term, by itself, is dependent on the value of the parallel flow $v_{\|}$. Thus, to have any hope of estimating each term independently, one has to formally subtract the flow parallel to the magnetic field $\boldsymbol{v}_{\perp} \rightarrow \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}/(\boldsymbol{B} \cdot \boldsymbol{B})$. However, if one subtracts this flow then, for example, $v_{\perp n}$ explicitly depends on the values of the flow tangent to the surface, complicating the interpretation of (8)!

We conclude from this discussion that while (8) is a fully accurate expression for the helicity transport rate through a closed boundary, it does not afford a clear physical interpretation. Consequently, we undertake below to derive new expressions for both the helicity and free energy transport into the corona that allow for physically intuitive interpretation and straightforward calculation from data. The organization of the paper is as follows. In Section 2, we first discuss key features of the Finn & Antonsen (1985) and Berger & Field (1984) relative helicity formulas that are needed for our derivations. In Section 3, we revisit the expression for the relative helicity transport rate derived by Berger & Field (1984) and propose an alternative expression that is manifestly gauge invariant on the surface. Section 4 discusses the implications of the new expression: its gauge invariance, its equivalence with the Berger & Field (1984) results, and its relationship with the lamellar electric field on the boundary. Section 5 develops the gauge-invariant emerging and shearing terms and briefly discusses nonideal effects. In Section 6, we turn to the free energy and derive an expression for the free energy transport rate, which shows that the transport of free energy across the surface $\mathcal{S}$ requires electric currents in the enclosed volume—in other words, the coronal field must be nonpotential. Appendix A derives a general expression for the reference electric field. The other Appendices provide some vector relationships in a volume (Appendix B), a brief introduction to vector calculus on a surface (Appendix C), some integral relationships (Appendix D), and the Helmholtz theorem in volumes and the Helmholtz–Hodge theorem on surfaces (Appendix E)—all of which are necessary for this paper. Given the detailed nature of some of the expressions derived in this paper, we tried to include all the material required for the reader to understand the derivations without having to consult outside sources.

2. The Relative Helicity

As discussed in Section 1, astrophysical plasmas can rarely be considered isolated. To address the field lines threading bounding surfaces of a plasma, Finn & Antonsen (1985) introduced the relative helicity formula (equivalent to Berger & Field 1984)

$$\mathcal{H} \equiv \int_{\mathcal{V}} d^3x\, \hbar, \tag{9a}$$

where

$$\hbar \equiv (\boldsymbol{A} + \boldsymbol{A}_{\rm R}) \cdot (\boldsymbol{B} - \boldsymbol{B}_{\rm R}) \tag{9b}$$

is the helicity integrand, and $\boldsymbol{B}_{\rm R}$ and $\boldsymbol{A}_{\rm R}$ are arbitrary reference fields. While using the term "density" for the helicity integrand $\hbar$ is tempting, doing so would be misleading. Unlike a true density, $\hbar$ is not a physically meaningful quantity, because helicity is a topological property that cannot be localized. The integrand $\hbar$ is not an observable; its value is at the discretion of the observer through the gauge freedom. Any observable with respect to $\hbar$ requires an integral over a spatial volume $\mathcal{V}$ with appropriate boundary conditions on the vector potentials and magnetic fields.

Magnetic fields are assumed to be solenoidal, which implies that

$$\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0 \tag{10}$$

for every and any closed surface, or equivalently, that there are no magnetic monopoles anywhere,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla} \cdot \boldsymbol{B}_{\rm R} = 0 \quad \text{(everywhere)}. \tag{11a}$$

The absence of monopoles permits the magnetic fields $\boldsymbol{B}$ and $\boldsymbol{B}_{\rm R}$ to be expressed in terms of vector potentials

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \quad \text{and} \quad \boldsymbol{B}_{\rm R} = \boldsymbol{\nabla} \times \boldsymbol{A}_{\rm R}. \tag{11b}$$

The gauge invariance of the relative helicity is proven by gauge-transforming 9(a)–(b) with $\boldsymbol{A} \rightarrow \boldsymbol{A} + \boldsymbol{\nabla}\Lambda'$ and $\boldsymbol{A}_{\rm R} \rightarrow \boldsymbol{A}_{\rm R} + \boldsymbol{\nabla}\Lambda''$. Defining $\Lambda = \Lambda' + \Lambda''$

$$\mathcal{H}' = \int_{\mathcal{V}} d^3x (\boldsymbol{A} + \boldsymbol{A}_{\rm R} + \boldsymbol{\nabla}\Lambda) \cdot (\boldsymbol{B} - \boldsymbol{B}_{\rm R}), \tag{12}$$

and using the original definition, 9(a)–(b), this becomes

$$\mathcal{H}' = \mathcal{H} + \int_{\mathcal{V}} d^3x\, \boldsymbol{\nabla}\Lambda \cdot (\boldsymbol{B} - \boldsymbol{B}_{\rm R}). \tag{13}$$

Applying (98) produces

$$\begin{aligned} \mathcal{H}' = \mathcal{H} &+ \int_{\mathcal{V}} d^3x\, \boldsymbol{\nabla} \cdot [\Lambda(\boldsymbol{B} - \boldsymbol{B}_{\rm R})] \\ &- \int_{\mathcal{V}} d^3x\, \Lambda\, \boldsymbol{\nabla} \cdot (\boldsymbol{B} - \boldsymbol{B}_{\rm R}). \end{aligned} \tag{14}$$

The second term can be converted to an integral of the surface bounding $\mathcal{V}$ with 122(a)

$$\begin{aligned} \mathcal{H}' = \mathcal{H} &- \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [\Lambda(\boldsymbol{B} - \boldsymbol{B}_{\rm R})] \\ &- \int_{\mathcal{V}} d^3x\, \Lambda\, \boldsymbol{\nabla} \cdot (\boldsymbol{B} - \boldsymbol{B}_{\rm R}). \end{aligned} \tag{15}$$

If there are no monopoles, 11(a), and the normal component of $\boldsymbol{B}$ and $\boldsymbol{B}_{\rm R}$ match on the boundary,

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{B} - \boldsymbol{B}_{\rm R})|_{\mathcal{S}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_{\rm R})|_{\mathcal{S}} = 0, \tag{16a}$$

then 9(a)–(b) is gauge invariant, $\mathcal{H}' = \mathcal{H}$, with respect to independent gauge transformations of $\boldsymbol{A}$ and $\boldsymbol{A}_{\rm R}$. Note, however, that 16(a) does imply some restrictions on the relative values of $\boldsymbol{A}$ and $\boldsymbol{A}_{\rm R}$ at the boundary. The null space of $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{f}$ is $\boldsymbol{f} = \boldsymbol{\nabla}_{\mathcal{S}}\Lambda + \tau\,\hat{\boldsymbol{n}}$ on $\mathcal{S}$ (see Appendix E.2 Equations (138)–(140)). Here, the subscript "$\mathcal{S}$" indicates the gradient (or vector) is tangent to the surface $\mathcal{S}$ (see Appendix C and in particular (108)), and $\boldsymbol{\nabla}_{\mathcal{S}}$ is the surface gradient operator defined in 109(c). Consequently, the boundary condition 16(a) implies that the tangential components of $\boldsymbol{A}$ and $\boldsymbol{A}_{\rm R}$ must be equivalent to within the gradient of a scalar on the surface $\mathcal{S}$,

$$\boldsymbol{A} - \boldsymbol{A}_{\rm R} = \tau\,\hat{\boldsymbol{n}} + \boldsymbol{\nabla}_{\mathcal{S}}\Lambda \quad \in \mathcal{S}. \tag{16b}$$

Note that gauge invariance for the Finn & Antonsen formula 9(a)–(b) is a direct result of the fact that there is no relative flux $\boldsymbol{B} - \boldsymbol{B}_{\rm R}$ threading the bounding surface 16(a). Thus, we again see that the normal component of the magnetic flux at the bounding surface plays a crucial role for defining a rigorous helicity observable. Finn & Antonsen's insight into defining relative helicity in terms of $\boldsymbol{B} - \boldsymbol{B}_{\rm R}$ is fundamentally related to the Helmholtz theorem (see Appendix E.1) and an isolated system. The vector field $\boldsymbol{B}$, while solenoidal ($\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$), is not intrinsically solenoidal for an arbitrary volume $\mathcal{V}$ bounded by $\mathcal{S}$, because $\boldsymbol{B}$ generally admits a mixed description $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{f} - \boldsymbol{\nabla}\xi$ in $\mathcal{V}$ according to the Helmholtz theorem. However, the "closed field" (Kusano et al. 1995; Berger 1999), $\boldsymbol{B}_{\rm cl} = \boldsymbol{B} - \boldsymbol{B}_{\rm R} = \boldsymbol{\nabla} \times \boldsymbol{f}$ in $\mathcal{V}$ with $\hat{\boldsymbol{n}} \cdot \boldsymbol{B}_{\rm cl}|_{\mathcal{S}} = 0$, is intrinsically solenoidal by construction and, therefore, magnetically isolated.

3. The Relative Helicity Transport Rate

Because the Finn & Antonsen (1985) formula is fully rigorous and consistent with Berger & Field (1984), its time dependence can be explored to determine the helicity transport rate. Expanding 9(b) and using 11(b) followed by (100), the integrand $\mathscr{h}$ becomes

$$\mathscr{h} = \boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{A}_{\rm R} \cdot \boldsymbol{B}_{\rm R} + \boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{A}_{\rm R}). \tag{17}$$

Taking the time derivative,

$$\frac{\partial \mathscr{h}}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{A}_{\rm R} \cdot \boldsymbol{B}_{\rm R}) + \boldsymbol{\nabla} \cdot \frac{\partial}{\partial t}(\boldsymbol{A} \times \boldsymbol{A}_{\rm R}), \tag{18}$$

and using Faraday's Law of induction,

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}, \tag{19}$$

and the relationship between the electric field, the magnetic vector potential $\boldsymbol{A}$, and electric scalar potential ψ

$$\boldsymbol{E} \equiv -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\psi, \tag{20}$$

yields

$$\begin{aligned}\frac{\partial \mathscr{h}}{\partial t} = &-c(\boldsymbol{E} \cdot \boldsymbol{B} - \boldsymbol{E}_{\rm R} \cdot \boldsymbol{B}_{\rm R}) - c(\boldsymbol{B} \cdot \boldsymbol{\nabla}\psi - \boldsymbol{B}_{\rm R} \cdot \boldsymbol{\nabla}\psi_{\rm R})\\ &- c\,\boldsymbol{A} \cdot \boldsymbol{\nabla} \times \boldsymbol{E} + c\,\boldsymbol{A}_{\rm R} \cdot \boldsymbol{\nabla} \times \boldsymbol{E}_{\rm R}\\ &+ c\boldsymbol{\nabla} \cdot (\boldsymbol{A}_{\rm R} \times \boldsymbol{E} - \boldsymbol{A} \times \boldsymbol{E}_{\rm R})\\ &- c\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{\nabla}\psi_{\rm R} - \boldsymbol{A}_{\rm R} \times \boldsymbol{\nabla}\psi),\end{aligned} \tag{21}$$

where the reference electric field $\boldsymbol{E}_{\rm R}$ satisfies Faraday's law (19) for $\boldsymbol{B}_{\rm R}$ and the corresponding relationship (20) between the reference electric field $\boldsymbol{E}_{\rm R}$, the magnetic vector potential $\boldsymbol{A}_{\rm R}$, and the electric scalar potential $\psi_{\rm R}$. Applying (100) to the third, fourth, and fifth groups of terms, using 11(b), and the absence of monopoles 11(a), and regrouping the results, the rate of change of the helicity integrand becomes

$$\begin{aligned}\frac{\partial \mathscr{h}}{\partial t} = &-2\,c(\boldsymbol{E} \cdot \boldsymbol{B} - \boldsymbol{E}_{\rm R} \cdot \boldsymbol{B}_{\rm R}) + c\boldsymbol{\nabla} \cdot [(\boldsymbol{A} + \boldsymbol{A}_{\rm R})\\ &\times (\boldsymbol{E} - \boldsymbol{E}_{\rm R})] - c\boldsymbol{\nabla} \cdot [(\boldsymbol{B} - \boldsymbol{B}_{\rm R})(\psi + \psi_{\rm R})].\end{aligned} \tag{22}$$

Using (100) and (19), Berger & Field (1984) express the volumetric rate of change of the self-helicity of the reference field as

$$\begin{aligned}\boldsymbol{E}_{\rm R} \cdot \boldsymbol{B}_{\rm R} = \boldsymbol{E}_{\rm R} \cdot \boldsymbol{\nabla} \times \boldsymbol{A}_{\rm R} &= \boldsymbol{\nabla} \cdot (\boldsymbol{A}_{\rm R} \times \boldsymbol{E}_{\rm R}) + \boldsymbol{A}_{\rm R} \cdot \boldsymbol{\nabla} \times \boldsymbol{E}_{\rm R},\\ &= \boldsymbol{\nabla} \cdot (\boldsymbol{A}_{\rm R} \times \boldsymbol{E}_{\rm R}) - \frac{1}{c}\boldsymbol{A}_{\rm R} \cdot \frac{\partial \boldsymbol{B}_{\rm R}}{\partial t}.\end{aligned} \tag{23a}$$

The choice of a potential reference field (4) permits further simplification,

$$\begin{aligned}\boldsymbol{E}_{\rm P} \cdot \boldsymbol{P} &= \boldsymbol{\nabla} \cdot (\boldsymbol{A}_{\rm P} \times \boldsymbol{E}_{\rm P}) + \frac{1}{c}\boldsymbol{A}_{\rm P} \cdot \frac{\partial \boldsymbol{\nabla}\phi_{\rm P}}{\partial t},\\ &= \boldsymbol{\nabla} \cdot (\boldsymbol{A}_{\rm P} \times \boldsymbol{E}_{\rm P}) + \boldsymbol{\nabla} \cdot \left(\frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{A}_{\rm P}\right) - \frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{\nabla} \cdot \boldsymbol{A}_{\rm P},\end{aligned} \tag{23b}$$

producing

$$\begin{aligned}\frac{\partial \mathscr{h}}{\partial t} = &-2\,c\,\boldsymbol{E} \cdot \boldsymbol{B} - 2\,\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{\nabla} \cdot \boldsymbol{A}_{\rm P} + 2\,\boldsymbol{\nabla} \cdot \left(\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{A}_{\rm P}\right)\\ &+ c\boldsymbol{\nabla} \cdot [(\boldsymbol{A} + \boldsymbol{A}_{\rm P}) \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P}) + 2\,\boldsymbol{A}_{\rm P} \times \boldsymbol{E}_{\rm P}]\\ &- c\boldsymbol{\nabla} \cdot [(\boldsymbol{B} - \boldsymbol{P})(\psi + \psi_{\rm P})].\end{aligned} \tag{24}$$

Integrating over the volume $\mathcal{V}$ and applying 122(a) produces

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial t} = &-2\,c\int_{\mathcal{V}} d^3x\left(\boldsymbol{E} \cdot \boldsymbol{B} + \frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{\nabla} \cdot \boldsymbol{A}_{\rm P}\right)\\ &- 2\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}} \cdot \left(\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{A}_{\rm P}\right)\\ &- c\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}} \cdot [(\boldsymbol{A} + \boldsymbol{A}_{\rm P})\boldsymbol{E} - \boldsymbol{E}_{\rm P}\\ &+ 2\,\boldsymbol{A}_{\rm P} \times \boldsymbol{E}_{\rm P}] + c\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}} \cdot [(\boldsymbol{B} - \boldsymbol{P})(\psi + \psi_{\rm P})].\end{aligned} \tag{25}$$

Expressing the volumetric rate of change of self-helicity of the reference field according to 23(b) is a critical choice in the derivation of the helicity transport equation. The left-hand side of 23(b) is, in principle, composed of electric and magnetic field observables, albeit of potential reference fields. In contrast, the right-hand side involves the vector potential and consequently, as we shall see below, constraints on a gauge-dependent quantity. Applying the magnetic field boundary conditions 16(a) and constraints on $\boldsymbol{A}$ implied by boundary conditions 16(b) results in

$$\frac{\partial \mathcal{H}}{\partial t} = -2\,c\int_{\mathcal{V}} d^3x\left(\boldsymbol{E}\cdot\boldsymbol{B} + \frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\rm P}\right) - 2\,c\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot\left(\boldsymbol{A}_{\rm P}\times\boldsymbol{E} + \frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{A}_{\rm P}\right). \quad (26)$$

The normal component of $\boldsymbol{A}_{\rm P}$ can always be eliminated on any surface,

$$\boldsymbol{A}_{\rm P}\cdot\hat{\boldsymbol{n}}|_{\mathcal{S}} = 0, \quad (27a)$$

through a gauge transformation with Neumann boundary conditions (see also Section III.B in Clegg et al. 2000),

$$\boldsymbol{A}_{\rm P}^{'} \rightarrow \boldsymbol{A}_{\rm P} + \boldsymbol{\nabla}\Lambda,\ \ \nabla^2\Lambda = 0,\ \ \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\Lambda|_{\mathcal{S}} = -\hat{\boldsymbol{n}}\cdot\boldsymbol{A}_{\rm P}|_{\mathcal{S}}, \implies \hat{\boldsymbol{n}}\cdot\boldsymbol{A}_{\rm P}^{'}|_{\mathcal{S}} = 0. \quad (27b)$$

Using the ideal MHD relationship (7) for the surface electric field produces

$$\frac{\partial \mathcal{H}}{\partial t} = -2\,c\int_{\mathcal{V}} d^3x\left(\overbrace{\boldsymbol{E}\cdot\boldsymbol{B}}^{(7)} + \frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\overbrace{\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\rm P}}^{6(b)}\right) - 2\oint_{\mathcal{S}} dS\,\frac{\partial \phi_{\rm P}}{\partial t}\overbrace{\hat{\boldsymbol{n}}\cdot\boldsymbol{A}_{\rm P}}^{27(a)} + 2\oint_{\mathcal{S}} dS\,\underbrace{(\boldsymbol{A}_{\rm P}\cdot\boldsymbol{B})v_n}_{\text{emergence}} - 2\oint_{\mathcal{S}} dS\,\underbrace{(\boldsymbol{A}_{\rm P}\cdot\boldsymbol{v})B_n}_{\text{shearing}}, \quad (28)$$

where because of the referenced equations in the overbraces the first three terms vanish, resulting finally in the expression in (8). The simplicity of (8) is a consequence of the choice of gauge, which eliminates several terms from the more general expression (28) and the ideal Ohm's law. Note that setting the volume term $\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\rm P} = 0$ and the surface term $\hat{\boldsymbol{n}}\cdot\boldsymbol{A}_{\rm P}| = 0$ leaves the surface term $2\,\boldsymbol{A}_{\rm P}\times\boldsymbol{E}_{\rm P}$ originating from the volumetric rate of change $\boldsymbol{E}_{\rm P}\cdot\boldsymbol{P}$ in 23(b). This term is used to cancel another surface term of the same form originating from $(\boldsymbol{A}+\boldsymbol{A}_{\rm P})\times(\boldsymbol{E}-\boldsymbol{E}_{\rm P})$ in (22). Thus, the volumetric rate of change of the self-helicity of the potential reference field is not guaranteed to be zero in the Coulomb gauge unless the surface integral of $\boldsymbol{A}_{\rm P}\times\boldsymbol{E}_{\rm P}$ is zero! We address this issue directly in Section 4.2.

As noted by Prior & Yeates (2014), the remaining emergence and shearing surface terms can entangle the transport of the self-helicity of the potential reference field across the boundary with changes in the self-helicity of the potential reference field in the volume $\mathcal{V}$. Additionally, while (28) is gauge invariant, (8) is not, because it is written explicitly in a particular gauge. Without further justification, the emergence and shearing terms cannot be considered observables as they are not independently gauge invariant. On the other hand, emergence and shearing are clearly physical processes that, in principle, are observable as long as all components of the velocity and magnetic fields can be measured at the boundary; consequently, there must exist manifestly gauge-invariant expressions for these quantities.

Given the concerns with (28) and by extension (8), we endeavor to derive an alternate expression for the helicity transport across a surface. Restarting from (22) but with a potential reference field (4), the rate of change of the helicity integrand becomes

$$\frac{\partial \mathscr{h}}{\partial t} = -2\,c(\boldsymbol{E}\cdot\boldsymbol{B} - \boldsymbol{E}_{\rm P}\cdot\boldsymbol{P}) + c\boldsymbol{\nabla}\cdot[(\boldsymbol{A}+\boldsymbol{A}_{\rm P})\times(\boldsymbol{E}-\boldsymbol{E}_{\rm P})] - c\boldsymbol{\nabla}\cdot[(\boldsymbol{B}-\boldsymbol{P})(\psi+\psi_{\rm P})]. \quad (29)$$

Integrating over the volume and applying the Gauss–Ostrogradsky theorem 122(a) produces

$$\frac{\partial H}{\partial t} = -2\,c\int_{V} d^3x(\boldsymbol{E}\cdot\boldsymbol{B} - \boldsymbol{E}_{\rm P}\cdot\boldsymbol{P}) - c\oint dS\,\hat{\boldsymbol{n}}\cdot [(\boldsymbol{A}+\boldsymbol{A}_{\rm P})](\boldsymbol{E}-\boldsymbol{E}_{\rm P}) + c\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot [(\boldsymbol{B}-\boldsymbol{P})(\psi+\psi_{\rm P})]. \quad (30)$$

The second term in the volume integral represents the generation of helicity in the volume caused by changes in the potential reference field. Instead of employing 23(a)–(b), we require that the evolution of the potential reference field does not generate any relative helicity in the volume

$$\int_{\mathcal{V}} d^3x\,\boldsymbol{E}_{\rm P}\cdot\boldsymbol{P} \equiv 0. \quad (31)$$

This critically important ansatz merits some discussion because it elevates the potential magnetic field to a special status, which in fact, the potential field $\boldsymbol{P}$ does possess. For a given distribution of normal flux at a closed boundary, the potential field is physically the unique ground state of the system. It is the only field that has no sources i.e., electric currents, anywhere in the volume. Note that because MHD ignores the displacement current, then all currents are due to actual material sources. Consequently, the potential field $\boldsymbol{P}$ is not simply some convenient reference field, but a well-defined state for a physical system. Any deviation from the potential field anywhere in the volume increases the energy of the system. Furthermore, if a system evolves through a sequence of potential states, then no free energy can be generated by this evolution. These unique properties of the potential field also apply to the relative helicity $\mathcal{H}$ of the system. As long as the system evolves through potential states, at least in a quasi-static sense, then no relative helicity can be generated. By construction, in the Berger & Field (1984) or Finn & Antonsen (1985) formalisms, there is no self-helicity of the reference field $\boldsymbol{B}_{\rm R}$ and in our formulation, none can be generated purely by changes in the potential reference field $\boldsymbol{B}_{\rm R} = \boldsymbol{P}$. This choice directly addresses the concern raised by Prior & Yeates (2014) in the introduction.

The reference electric field $\boldsymbol{E}_{\rm P}$ that satisfies Faraday's law for the changing reference magnetic field $\boldsymbol{P}$ and the ansatz (31) above can now be determined. In general, $\boldsymbol{E}_{\rm P}$ can be decomposed into an solenoidal (inductive) piece $\boldsymbol{\Sigma}_{\rm P} = \boldsymbol{\nabla}\times\boldsymbol{f}$ and an irrotational (electrostatic) component $\boldsymbol{\nabla}\Lambda_{\rm P}$:

$$\boldsymbol{E}_{\rm P} = \boldsymbol{\Sigma}_{\rm P} + \boldsymbol{\nabla}\Lambda_{\rm P}, \quad (32a)$$

which must satisfy the following conditions for compatibility with the ansatz (31):

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Sigma}_{\rm P} = 0 \quad \in \mathcal{V}, \quad (32b)$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\Sigma}_{\rm P}|_{\mathcal{S}} = 0, \tag{32c}$$

$$\nabla^2 \Lambda_{\rm P} = \varrho_{\rm P} \quad \in \mathcal{V}, \tag{32d}$$

$$\Lambda_{\rm P}|_{\mathcal{S}} = \text{constant}, \tag{32e}$$

and $\boldsymbol{\nabla} \cdot \boldsymbol{E}_{\rm P} = \varrho_{\rm P}$ is arbitrary. These conditions guarantee that the $\boldsymbol{E}_{\rm P}$ does not generate any change in the relative helicity and, as noted in 133(a) of Appendix E, the boundary conditions ensure orthogonality between $\boldsymbol{\Sigma}_{\rm P}$ and $\boldsymbol{\nabla}\Lambda_{\rm P}$. In particular, conditions 32(b) and (c) ensure that $\boldsymbol{\Sigma}_{\rm P}$ is intrinsically solenoidal and does not change the self-helicity of the potential reference field,

$$\begin{aligned} \int_{\mathcal{V}} d^3x\, \boldsymbol{\Sigma}_{\rm P} \cdot \boldsymbol{P} &= -\int_{\mathcal{V}} d^3x\, \boldsymbol{\Sigma}_{\rm P} \cdot \boldsymbol{\nabla}\phi_{\rm P} \\ &= \int_{\mathcal{V}} d^3x\, \phi_{\rm P} (\overbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}_{\rm P}}^{32(b)}) + \oint_{\mathcal{S}} dS\, \overbrace{\hat{\boldsymbol{n}} \cdot (\phi_{\rm P}\, \boldsymbol{\Sigma}_{\rm P})}^{32(c)} \equiv 0, \end{aligned} \tag{33a}$$

and Laplace's equation, 5(a), and condition 32(e) combined with the solenoidal property of $\boldsymbol{B}$ in (10) ensure that $\Lambda_{\rm P}$ does not change the self-helicity of the potential reference field,

$$\begin{aligned} \int_{\mathcal{V}} d^3x\, \boldsymbol{\nabla}\Lambda_{\rm P} \cdot \boldsymbol{P} &= -\int_{\mathcal{V}} d^3x\, \boldsymbol{\nabla}\Lambda_{\rm P} \cdot \boldsymbol{\nabla}\phi_{\rm P} \\ &= \int_{\mathcal{V}} d^3x\, \Lambda_{\rm P}\, \overbrace{\nabla^2 \phi_{\rm P}}^{5(a)} + \oint_{\mathcal{S}} dS\, \overbrace{\Lambda_{\rm P}\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}\phi_{\rm P}}^{(10)32(e)} \equiv 0. \end{aligned} \tag{33b}$$

This surface integral on the right is zero by the solenoidal property of magnetic fields when $\Lambda_{\rm P}$ is a constant on $\mathcal{S}$, and the volume integral on the right is zero by 5(a). Note that $\boldsymbol{\nabla}_{\mathcal{S}}\Lambda_{\rm P} = 0$ and $\nabla_{\mathcal{S}}^2 \Lambda_{\rm P} = 0$ on $\mathcal{S}$, and $\boldsymbol{\nabla}\Lambda_{\rm P}$ does not contribute to $\boldsymbol{\nabla} \times \boldsymbol{E}_{\rm P}$ in $\mathcal{V}$ or on $\mathcal{S}$ because it is a complete gradient.

Additionally, $\boldsymbol{E}_{\rm P}$ must satisfy Faraday's law (19) for $\partial \boldsymbol{P}/\partial t$, which specifies $\boldsymbol{\nabla} \times \boldsymbol{E}_{\rm P}$ and by extension $\boldsymbol{\nabla} \times \boldsymbol{\Sigma}_{\rm P}$. Temporarily deferring the calculation of $\boldsymbol{\Sigma}_{\rm P}$ to Section 4.3.1 and Appendix A, we note that the existence and uniqueness of $\boldsymbol{\Sigma}_{\rm P}$ by (19), 32(b), and 32(c) have been firmly established by numerous authors (see Girault & Raviart 1986; Jiang et al. 1994; Amrouche et al. 1998; Amrouche & Seloula 2013; Cheng & Shkoller 2017, and references therein). Thus, while $\boldsymbol{E}_{\rm P}$ exhibits some arbitrariness through $\boldsymbol{\nabla}\Lambda_{\rm P}$ and $\varrho_{\rm P}$, the existence and uniqueness of $\boldsymbol{\Sigma}_{\rm P}$ throughout $\mathcal{V}$ permit the expression of (30) with boundary condition 16(a) and ansatz (31) as

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial t} = &-2\, c \int_{\mathcal{V}} d^3x [\boldsymbol{E} \cdot \boldsymbol{B} - \phi_{\rm P}\, \overbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}_{\rm P}}^{32(b)} - \Lambda_{\rm P}\, \overbrace{\nabla^2 \phi_{\rm P}}^{5(a)}] \\ &+ c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [2\, \overbrace{\phi_{\rm P}\, \boldsymbol{\Sigma}_{\rm P}}^{32(c)} + 2\, \overbrace{\Lambda_{\rm P}\, \boldsymbol{\nabla}\phi_{\rm P}}^{(10)32(e)} \\ &- (\boldsymbol{A} + \boldsymbol{A}_{\rm P}) \times (\boldsymbol{E} - \boldsymbol{\Sigma}_{\rm P} - \overbrace{\boldsymbol{\nabla}\Lambda_{\rm P}}^{32(e)})]. \end{aligned} \tag{34}$$

The terms with overbraces are zero because of the referenced equations resulting in

$$\begin{aligned} \dot{\mathcal{H}} \equiv \frac{\partial \mathcal{H}}{\partial t} = &\overbrace{-2\, c \int_{\mathcal{V}} d^3x\, \boldsymbol{E} \cdot \boldsymbol{B}}^{\dot{\mathcal{H}}_{\mathcal{V}}} \\ &\overbrace{-\, c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [(\boldsymbol{A} + \boldsymbol{A}_{\rm P}) \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P})]}^{\dot{\mathcal{H}}_{\mathcal{S}}}. \end{aligned} \tag{35}$$

As expected, $\Lambda_{\rm P}$, the irrotational constituent of $\boldsymbol{E}_{\rm P}$, plays no role in helicity transport. The final surface integral involves only the tangential components of $\boldsymbol{E}_{\rm P}$, which are unique on $\mathcal{S}$, because $\hat{\boldsymbol{n}} \times \boldsymbol{E}_{\rm P} \equiv \hat{\boldsymbol{n}} \times \boldsymbol{\Sigma}_{\rm P}$, i.e., $\boldsymbol{E}_{{\rm P}\mathcal{S}} = \boldsymbol{\Sigma}_{\rm P}$. We show in the next section that the representation above for the surface helicity transport $\dot{\mathcal{H}}_{\mathcal{S}}$ has major advantages over the Berger & Field formula (8).

4. Implications

4.1. Gauge Invariance

The most important advantage of Equation (35) for determining the helicity transport across the boundary is that the surface helicity transport rate $\dot{\mathcal{H}}_{\mathcal{S}}$ is independently and manifestly gauge invariant. Because $\dot{\mathcal{H}}_{\mathcal{V}}$ is comprised entirely of observables $\boldsymbol{E}$ and $\boldsymbol{B}$, the gauge invariance of $\dot{\mathcal{H}}$ depends on the gauge invariance of $\dot{\mathcal{H}}_{\mathcal{S}}$. Gauge-transforming $\dot{\mathcal{H}}_{\mathcal{S}}$ with $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Lambda'$ and $\boldsymbol{A}_{\rm P} \to \boldsymbol{A}_{\rm P} + \boldsymbol{\nabla}\Lambda''$, where $\Lambda = \Lambda' + \Lambda''$ produces

$$\begin{aligned} \frac{\partial \mathcal{H}_{\mathcal{S}}'}{\partial t} = &-c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{A} + \boldsymbol{A}_{\rm P}) \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P}) \\ &- c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}\Lambda \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P}), \end{aligned} \tag{36}$$

this can be rewritten with (99) and (19):

$$\begin{aligned} \frac{\partial \mathcal{H}_{\mathcal{S}}'}{\partial t} = \frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} &- 2\, c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times [\Lambda(\boldsymbol{E} - \boldsymbol{E}_{\rm P})] \\ &- 2 \oint_{\mathcal{S}} dS\, \Lambda\, \hat{\boldsymbol{n}} \cdot \frac{\partial}{\partial t}(\boldsymbol{B} - \boldsymbol{P}), \end{aligned} \tag{37}$$

which proves $\partial \mathcal{H}_{\mathcal{S}}'/\partial t \equiv \partial \mathcal{H}_{\mathcal{S}}/\partial t$ and by extension the gauge invariance of $\partial \mathcal{H}/\partial t$, because the second term is zero by Stokes' theorem on a closed surface 124(a) and the third term is zero by the boundary condition 16(a). This new formula directly addresses our primary concern in using the Berger & Field (1984) formula for helicity transport—namely, the lack of manifest gauge invariance of its surface helicity transport rate. This demonstrates that the volumentric rate of change of helicity $\dot{\mathcal{H}}_{\mathcal{V}}$ and the surface helicity transport $\dot{\mathcal{H}}_{\mathcal{S}}$ are independently observables inasmuch as the surface $\mathcal{S}$ is closed.

4.2. Equivalence with Berger & Field (1984)

The Berger & Field (1984) result (3) and the new expression (35) are equivalent for a closed surface. First, we note that by using the boundary relation 16(b), which relates the tangential components of the vector potential, our surface helicity transport expression $\dot{\mathcal{H}}_{\mathcal{S}}$ above can be written entirely in terms of $\boldsymbol{A}$ or $\boldsymbol{A}_{\rm P}$ as

$$\begin{aligned} \frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} &= -2\, c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [\boldsymbol{A} \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P})] \\ &= -2\, c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [\boldsymbol{A}_{\rm P} \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P})]. \end{aligned} \tag{38}$$

Comparing the expression on the right with the Berger & Field (1984) formula, which uses the Coulomb gauge, $\boldsymbol{A}_{\rm PC}$, we see that they are equivalent if and only if

$$\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{A}_{\rm PC} \times \boldsymbol{E}_{\rm P}) = 0. \quad (39)$$

This result follows directly from our ansatz (31), which can be written with 23(b) as

$$\int_{\mathcal{V}} d^3x\, \boldsymbol{E}_{\rm P} \cdot \boldsymbol{P} = -\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{A}_{\rm P} \times \boldsymbol{E}_{\rm P}) - \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \left(\frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\boldsymbol{A}_{\rm P}\right) - \int_{\mathcal{V}} d^3x\, \frac{1}{c}\frac{\partial \phi_{\rm P}}{\partial t}\, \boldsymbol{\nabla} \cdot \boldsymbol{A}_{\rm P} \equiv 0. \quad (40)$$

Substituting $\boldsymbol{A}_{\rm PC}$ in the expression above, we note that the last two terms on the right vanish due to the properties 6(b) and (c) that define the Coulomb gauge vector potential, and with the properties of $\boldsymbol{E}_{\rm P}$ given by 32(a)–(33b), this becomes

$$\int_{\mathcal{V}} d^3x\, \boldsymbol{E}_{\rm P} \cdot \boldsymbol{P} = -\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{A}_{\rm PC} \times \boldsymbol{E}_{\rm P}) = -\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot (\boldsymbol{A}_{\rm PC} \times \boldsymbol{\Sigma}_{\rm P}) \equiv 0, \quad (41)$$

and the Berger & Field (1984) result is, indeed, equivalent to the new expression when integrated over a closed surface; consequently, one is free to use the old result if it is computationally more advantageous.

4.3. Electric Field Determining Helicity Transport

Another important implication of our expression is that the transport across the boundary is due solely to the lamellar part of the electric field on the boundary $\boldsymbol{E}_{\mathcal{S}} \sim \boldsymbol{\nabla}_{\mathcal{S}}\zeta$ (Scharstein 1991). To prove this, we must determine the solenoidal and lamellar components of $\boldsymbol{E}_{\rm P}$ and $\boldsymbol{E}$ on the bounding surface $\mathcal{S}$. The Helmholtz–Hodge theorem (see Appendix E.2) provides the mathematical framework for this decomposition given the surface components of $\boldsymbol{E}_{\rm P}$ and $\boldsymbol{E}$. The electric field $\boldsymbol{E}$ is a specified field in the helicity transport problem, but $\boldsymbol{E}_{\rm P\mathcal{S}} = \boldsymbol{\Sigma}_{\rm P}$ is related to changes in the normal component of the surface magnetic field. The existence and uniqueness of solutions of the div-curl system represented by $\boldsymbol{\Sigma}_{\rm P}$, e.g., (19), 32(b), and 32(c), are well established (Girault & Raviart 1986; Jiang et al. 1994; Amrouche et al. 1998; Amrouche & Seloula 2013; Cheng & Shkoller 2017). Thus, $\boldsymbol{\Sigma}_{\rm P}$ can always be computed from changes in the normal component of the magnetic field $\partial B_n/\partial t$. However, the details of determining $\boldsymbol{\Sigma}_{\rm P}$ in general geometries adds unnecessary complexity, given the simple geometries that are typically of interest to solar physicists (Cartesian, which is flat; spherical with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$, which has the same curvature along the orthogonal principal directions, etc.). In these simple geometries, the surface components of $\boldsymbol{E}_{\rm P\mathcal{S}} = \boldsymbol{\Sigma}_{\rm P}$ can be determined from $\partial B_n/\partial t$ and directly measured using a $\dot{\boldsymbol{B}}$ loop embedded in the surface. Thus, in Sections 4.3.1–4.3.3 below, we assume that $\boldsymbol{\Sigma}_{\rm P}$ has a priori been determined and develop the general formalism for computing helicity transport. We defer the discussion of determining $\boldsymbol{\Sigma}_{\rm P}$ from $\partial B_n/\partial t$ in general curvilinear geometries to Appendix A.

Up to this point, the detailed properties of the surface $\mathcal{S}$ have largely been ignored. However, moving forward, the geometry of the differentiable surface $\mathcal{S}$ bounding the volume $\mathcal{V}$ becomes intertwined with the definition of the differential operators (see Appendix C). For the remainder of Section 4, we assume that the bounding surface $\mathcal{S}$ is at least a $\mathcal{C}^3$, closed (compact and without boundaries), and simply connected hypersurface, where $\mathcal{C}^k$ indicates continuity up to the kth derivative. This includes the $\mathcal{C}^\infty$ spherical surface, which is the most relevant for solar physics. Reusken (2018) notes that the $\mathcal{C}^2$ continuity of $\mathcal{S}$ is sufficient but not necessary for the definition of the surface differential operators in the Helmholtz–Hodge decomposition on $\mathcal{S}$. Indeed, Chapter 7 of Morrey (1966) extends the Helmholtz–Hodge decomposition to $\mathcal{C}^1$ surfaces with and without boundaries using differential forms. However, $\mathcal{C}^3$ is sufficient for the existence of $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} \propto \partial \boldsymbol{J}/\partial t$ on the surface $\mathcal{S}$, which is necessary for the analysis of relative helicity transport. Our results can be extended mutatis mutandis to multiply connected domains[3] and surfaces that are $\mathcal{C}^3$ almost everywhere in a set of measure zero sense (Halmos 1974).

4.3.1. The Reference Electric Field $\boldsymbol{E}_{\rm P}$ on the Bounding Surface $\mathcal{S}$

As shown in Section 3, the reference electric field $\boldsymbol{E}_{\rm P}$ that changes the potential field $\boldsymbol{P}$ can be decomposed into an instrinsically solenoidal part $\boldsymbol{\Sigma}_{\rm P}$ and irrotational part $\boldsymbol{\nabla}\Lambda_{\rm P}$. The intrinsically solenoidal constituent $\boldsymbol{\Sigma}_{\rm P}$ is unique, whereas the irrotational constituent $\boldsymbol{\nabla}\Lambda_{\rm P}$ is arbitrary, subject to Dirichlet boundary conditions 32(e) on $\Lambda_{\rm P}$ such that $\boldsymbol{E}_{\rm P\mathcal{S}} = \boldsymbol{\Sigma}_{\rm P}$. The irrotational part plays no role in helicity generation and transport, and thus helicity transport through the boundary only requires determining the unique tangential components of $\boldsymbol{\Sigma}_{\rm P}$. On the surface $\mathcal{S}$, the intrinsically solenoidal component $\boldsymbol{\Sigma}_{\rm P}$ may be decomposed with the Helmholtz–Hodge theorem outlined in Appendix E.2 (see also Hodge 1959; Scharstein 1991; Van Bladel 1993; Bhatia et al. 2013; O'Neil 2018; Reusken 2018) in the Dupin surface coordinate system (u_1, u_2, n) discussed in Appendix C (see also Weatherburn 1955; Tai 1992; Van Bladel 2007)

$$\boldsymbol{\Sigma}_{\rm P} = \tau_{\rm P}\,\hat{\boldsymbol{n}} - \frac{1}{c}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\frac{\partial \chi_{\rm P}}{\partial t} - \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\rm P}. \quad (42)$$

Here, the variable $\boldsymbol{\Sigma}_{\rm P}$ is understood to be differentiable on $\mathcal{S}$ and in some neighborhood of $\mathcal{S}$. In other words, $\boldsymbol{E}_{\rm P}$ exists not just on the surface $\mathcal{S}$ but also in the volume $\mathcal{V}$ and consequently is a function of the parameterization of the surface by u_1 and u_2 in the Dupin surface coordinate system, as well as n, the coordinate normal to the surface $\mathcal{S}$. The terms in (42) are mutually orthogonal when integrated over a closed surface $\mathcal{S}$. In the case of a multiply connected volume, a mutually orthogonal harmonic term $\boldsymbol{\Omega}_{\rm P}$ can be added to the decomposition on the surface or the volume can be converted to a simply connected one with the appropriate cuts and auxiliary surfaces as mentioned in footnote 3. However, for a simply connected volume bounded by a $\mathcal{C}^k$ surface with $k \geqslant 2$, such as a sphere with $k = \infty$, the dimension of the harmonic space is zero, $\boldsymbol{\Omega} = 0$, and the harmonic field can thus be ignored in the decomposition (see Lemma 4.3 in Reusken 2018).

The second term in (42) is purely solenoidal as $\boldsymbol{\nabla} \cdot [\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}(\partial \chi_{\rm P}/\partial t)] = 0$, and the third term is irrotational with respect to the normal component of the curl

[3] The simply connected assumption may be relaxed by noting that any multiply connected volume can be transformed into a simply connected volume by $\mathfrak{g}$ cuts, where $\mathfrak{g}$ is the genus of the bounding surface $\mathcal{S}$, which is equivalent to the number of holes (see Appendix E.1).

$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\rm P} = 0$, but it is not perfectly irrotational. In particular, $\boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\rm P}$ is not irrotational with respect to the three-dimensional curl operator because it is an incomplete gradient of a scalar (see discussion at the end of Appendix C). Thus, $\boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\rm P}$ contains a solenoidal component in general three-dimensional curvilinear coordinates. Nonetheless, this term is commonly referred to as the "lamellar term" in the Helmholtz–Hodge decomposition (Scharstein 1991) because its so-called "surface curl" $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathcal{S}} \times \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\rm P}$ is identically zero.

The lamellar and solenoidal components of $\boldsymbol{E}_{\rm P}$ on $\mathcal{S}$ are determined by

$$\nabla^2_{\mathcal{S}}\zeta_{\rm P} = -\boldsymbol{\nabla}_{\mathcal{S}} \cdot [(\hat{\boldsymbol{n}} \times \boldsymbol{E}_{\rm P}) \times \hat{\boldsymbol{n}}] = -\boldsymbol{\nabla}_{\mathcal{S}} \cdot \boldsymbol{\Sigma}_{\rm P} \quad \in \mathcal{S}, \tag{43a}$$

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathcal{S}} \times \left(\frac{1}{c}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\frac{\partial \chi_{\rm P}}{\partial t}\right) = -\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathcal{S}} \times \boldsymbol{\Sigma}_{\rm P}$$
$$= \frac{1}{c}\frac{\partial P_n}{\partial t} \Longrightarrow \nabla^2_{\mathcal{S}}\chi_{\rm P} = P_n \quad \in \mathcal{S}, \tag{43b}$$

where (120) has been used and

$$\tau_{\rm P}|_{\mathcal{S}} = 0, \tag{43c}$$

to satisfy 32(c). Here, 32(b) with 43(b), 43(c), and 111(c) implies

$$\nabla^2_{\mathcal{S}}\zeta_{\rm P} = \frac{\partial \tau_{\rm P}}{\partial n}. \tag{43d}$$

Note that the only finite homogeneous solutions for 43(a) or 43(b) ($\nabla^2_{\mathcal{S}}\zeta_{\rm P}$ and $\nabla^2_{\mathcal{S}}\chi_{\rm P} = 0$, on $\mathcal{S}$) are $\chi_{\rm P} =$ constant and $\zeta_{\rm P} =$ constant, i.e., there is no nonzero harmonic vector on $\mathcal{S}$ which is at least $\mathcal{C}^2$ everywhere.[4] Thus, $\chi_{\rm P}$ and $\zeta_{\rm P}$ are unique on a closed, simply connected, $\mathcal{C}^{k\geqslant 2}$ surface $\mathcal{S}$ to within a constant. For simple, smooth, unbounded geometries such as a plane or a sphere, 43(b) and 43(c) are sufficient to uniquely determine $\boldsymbol{\Sigma}_{\rm P}$ on $\mathcal{S}$ because $\zeta_{\rm P}$ is constant on $\mathcal{S}$ as discussed below.

However, 43(a) implies that in general curvilinear coordinates $\chi_{\rm P}$ and $\zeta_{\rm P}$ are coupled. Substituting $\boldsymbol{E}_{\rm P}$ into Faraday's Law (19) for $\boldsymbol{P}$ and using (99) and (117) on the first term and rearranging,

$$\frac{\partial \boldsymbol{P}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}_{\rm P} = \boldsymbol{\nabla} \times \left(\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\frac{\partial \chi_{\rm P}}{\partial t}\right)$$
$$- c\left(\tau_{\rm P} + \frac{\partial \zeta_{\rm P}}{\partial n}\right)\boldsymbol{\nabla} \times \hat{\boldsymbol{n}} + c\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\left(\tau_{\rm P} + \frac{\partial \zeta_{\rm P}}{\partial n}\right) \quad \in \mathcal{S}. \tag{44}$$

Here, $\zeta_{\rm P}$ only appears in the form $\partial\zeta_{\rm P}/\partial n$, a consequence of its manifestation as an incomplete gradient. Taking the curl of (44), and using 114(d), (99), and (103),

$$\frac{\partial \boldsymbol{\nabla} \times \boldsymbol{P}}{\partial t} = -\boldsymbol{\nabla} \times \left\{\boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \times \left(\frac{\partial \chi_{\rm P}}{\partial t}\,\hat{\boldsymbol{n}}\right)\right]\right.$$
$$\left. + c\boldsymbol{\nabla} \times \left[\left(\tau_{\rm P} + \frac{\partial \zeta_{\rm P}}{\partial n}\right)\hat{\boldsymbol{n}}\right]\right\} \quad \in \mathcal{S}. \tag{45}$$

By definition, the potential magnetic field cannot have any associated currents in the volume, thus the normal component provides a constraint on $\partial\zeta_{\rm P}/\partial n$ with $\tau_{\rm P} = 0$ on $\mathcal{S}$:

$$\nabla^2_{\mathcal{S}}\left(\tau_{\rm P} + \frac{\partial \zeta_{\rm P}}{\partial n}\right) = \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \left\{\boldsymbol{\nabla} \times \left[\left(\tau_{\rm P} + \frac{\partial \zeta_{\rm P}}{\partial n}\right)\hat{\boldsymbol{n}}\right]\right\}$$
$$= -\frac{1}{c}\,\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \left[\nabla^2\left(\frac{\partial \chi_{\rm P}}{\partial t}\,\hat{\boldsymbol{n}}\right)\right] \quad \in \mathcal{S}. \tag{46}$$

Generally, $\boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\rm P}$ produces a solenoidal component when the total curl is applied and this solenoidal component is dependent on the missing part of the gradient: $\partial\zeta_{\rm P}/\partial n$. This constraint requires the existence of $\nabla^2_{\mathcal{S}}(\partial\zeta_{\rm P}/\partial n)$, which our assumption of $\mathcal{C}^3$ of $\mathcal{S}$ ensures. For several geometries, such as planes and spheres, the right-hand side of (46) vanishes and $\nabla^2_{\mathcal{S}}(\partial\zeta_{\rm P}/\partial n) = 0$, $\zeta_{\rm P} =$ constant on $\mathcal{S}$, and $\zeta_{\rm P}$ can essentially be ignored in the subsequent analysis.

4.3.2. The Electric Field $\boldsymbol{E}$

The electric field $\boldsymbol{E}$ may also be decomposed with the Helmholtz–Hodge theorem:

$$\boldsymbol{E} = \tau\,\hat{\boldsymbol{n}} - \frac{1}{c}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\frac{\partial \chi}{\partial t} - \boldsymbol{\nabla}_{\mathcal{S}}\zeta \quad \in \mathcal{S}. \tag{47}$$

Again, the solenoidal and lamellar components are determined by

$$\nabla^2_{\mathcal{S}}\zeta = -\boldsymbol{\nabla} \cdot [(\hat{\boldsymbol{n}} \times \boldsymbol{E}) \times \hat{\boldsymbol{n}}] = -\boldsymbol{\nabla}_{\mathcal{S}} \cdot \boldsymbol{E}_{\mathcal{S}} \quad \in \mathcal{S}, \tag{48a}$$

and

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathcal{S}} \times \left(\frac{1}{c}\,\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\frac{\partial \chi}{\partial t}\right) = -\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathcal{S}} \times \boldsymbol{E}$$
$$= \frac{1}{c}\frac{\partial P_n}{\partial t} \Longrightarrow \nabla^2_{\mathcal{S}}\chi = P_n \quad \in \mathcal{S}, \tag{48b}$$

where (120) was used. The electric field $\boldsymbol{E}$ must satisfy the same boundary conditions on the normal component of the curl as $\boldsymbol{\Sigma}_{\rm P}$,[5] and thus,

$$\frac{\partial \chi}{\partial t} \equiv \frac{\partial \chi_{\rm P}}{\partial t} \quad \in \mathcal{S} \tag{49}$$

to within a constant. For any geometry, $\partial\chi/\partial t$ and $\partial\chi_{\rm P}/\partial t$ can be determined directly from the change in the normal component of the magnetic field at the surface.

[4] This is related to the hairy ball theorem, which can be stated as a continuous tangent vector field on a sphere must have at least one zero (Brouwer 1912). Thus, a constant tangent vector (other than zero) is forbidden on a sphere and shapes continuously deformable to a sphere. See Chapter II, Section 22 in Shubin & Andersson (2001) and the Laplace equation on a sphere discussed by Esparza-López et al. (2016).

[5] Practically, for a numerical simulation, the instantaneous electric field data corresponding to the left-hand side of 48(b) may not be equal to the magnetic data corresponding to the right-hand side of 48(b) because of discretization, dissipation, etc. Indeed, deviations from equality for $\partial B_n/\partial t = -c\,\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{E}$ is indication of how well the induction equation is satisfied by the numerical data from the simulation.

4.3.3. The Surface Helicity Transport $\dot{\mathcal{H}}_{\mathcal{S}}$

Substituting (42) and (47) into (35) produces the surface helicity transport rate

$$\begin{aligned}\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} &= -c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [(\boldsymbol{A} + \boldsymbol{A}_{\rm P}) \times (\boldsymbol{E} - \boldsymbol{E}_{\rm P})] \\ &= -c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot [\boldsymbol{\nabla}(\zeta - \zeta_{\rm P}) \times (\boldsymbol{A} + \boldsymbol{A}_{\rm P})]. \end{aligned} \tag{50}$$

Using (99), the general surface helicity transport rate takes the simple form

$$\begin{aligned}\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} &= -c \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times [(\zeta - \zeta_{\rm P})(\boldsymbol{A} + \boldsymbol{A}_{\rm P})] \\ &\quad + c \oint_{\mathcal{S}} dS (\zeta - \zeta_{\rm P}) \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times (\boldsymbol{A} + \boldsymbol{A}_{\rm P}) \\ &\equiv 2\, c \oint_{\mathcal{S}} dS (\zeta - \zeta_{\rm P}) B_n, \end{aligned} \tag{51}$$

where the first surface integral vanishes by 124(a).

This simple and somewhat surprising expression (51) provides key insight into helicity transport. First, it shows that only the lamellar ($\sim$irrotational) part of $\boldsymbol{E}$ is involved in the helicity transport across $\mathcal{S}$. In simple geometries, $\nabla_{\mathcal{S}}\zeta$, is the electrostatic component, whereas the twisting and tangling of field lines might be expected to be due to an inductive electric field. Resolution to this apparent paradox can be achieved by noting that for ideal motions,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} \propto \boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{B} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}, \tag{52}$$

which shows that the irrotational part of $\boldsymbol{E}$ is related to the plasma vorticity and the transport of electric current, as expected for helicity injection. Second, the difference $\zeta - \zeta_{\rm P}$ can be interpreted as that part of $\boldsymbol{E}$ not involved in changing the normal component of $\boldsymbol{B}$ at the boundary, implying that the instantaneous change of the potential reference magnetic field at the boundary does not contribute to helicity transport.

Although the mean value of $\zeta - \zeta_{\rm P} \equiv \zeta_0$ is subject to the choice of the observer, (51) is manifestly gauge invariant. The solenoidal property of the magnetic field $\oint_{\mathcal{S}} dS\, \boldsymbol{B}_n = 0$ ensures that $\partial \mathcal{H}_{\mathcal{S}}/\partial t$ is independent of this constant ζ_0. Nonetheless, the integrand is only determined to within a constant ζ_0 times the normal component of $\boldsymbol{B}$, i.e., $\zeta_0\, B_n$. Consequently, the value of the "helicity flux density" can be adjusted at an arbitrary location to be positive, negative, or zero by fiat through the choice of ζ_0 without changing the observable $\partial \mathcal{H}_{\mathcal{S}}/\partial t$. Given that the sign of $(\zeta - \zeta_{\rm P})B_n$ cannot be determined uniquely at any location, helicity transport cannot be assigned a unique local interpretation, thereby demonstrating again that the concept of "helicity flux density" is not meaningful. This point is proven explicitly in the example discussed in Section 5.2 below.

As noted above, in curvilinear coordinates, $\chi_{\rm P}$ and $\zeta_{\rm P}$ are generally coupled. However, in several geometries (Cartesian, which is flat; spherical with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$, which has the same curvature along the orthogonal principal directions, etc.) $\partial\zeta_{\rm P}/\partial n$ and $\partial\chi_{\rm P}/\partial t$ completely decouple—the right-hand side of (45) has no projection along the normal component and $\nabla_{\mathcal{S}}^2 \partial\zeta_{\rm P}/\partial n = 0$ on $\mathcal{S}$. Indeed, for spherical boundaries, the intrinsically solenoidal reference electric field for the potential magnetic field may be represented everywhere in $\mathcal{V}$ by $\boldsymbol{\Sigma}_{\rm P} = -c^{-1}\, \hat{\boldsymbol{r}} \times \boldsymbol{\nabla}_{\mathcal{S}}(\partial\chi_{\rm P}/\partial t)$ with $\hat{\boldsymbol{r}} \cdot \boldsymbol{\Sigma}_{\rm P} = 0$ and $\nabla^2(\partial\chi_{\rm P}/\partial t) = 0$ (Backus 1986). Thus, for the volume between two spherical boundaries, there is no radial component of $\boldsymbol{\Sigma}_{\rm P}$ anywhere! In this case, $\boldsymbol{\nabla}_{\mathcal{S}} \cdot \boldsymbol{\Sigma}_{\rm P} = \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma}_{\rm P} - \partial\hat{\boldsymbol{r}} \cdot \boldsymbol{\Sigma}_{\rm P}/\partial r \equiv 0$ on every radial surface in $\mathcal{V}$ and in particular $\zeta_{\rm P} =$ constant on the boundaries $\mathcal{S}$.[6] We emphasize that this is not the case in general curvilinear coordinates. However, for simple, smooth, unbounded, geometries such as a plane or sphere, the helicity transport equation admits a particularly simple form

$$\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} \equiv 2\, c \oint_{\mathcal{S}} dS\, \zeta\, B_n. \tag{53}$$

This form has been noted by Berger (1999) for application to laboratory plasmas with flux-conserving boundary conditions such that $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = 0$ and $\boldsymbol{\mathcal{E}} \equiv -\boldsymbol{\nabla}_{\mathcal{S}}\zeta$. We emphasize that we have made no such boundary assumptions in deriving (53). Indeed, we argue below in the conclusions that the expression (53) above is, in fact, the form that should generally be used for measuring the helicity transport through the photosphere and into the corona.

5. Surface Helicity Transport: Emerging, Shearing, and Nonideal Effects

For ideal boundary motions, the surface helicity transport $\partial\mathcal{H}_{\mathcal{S}}/\partial t$ can be changed by two MHD processes: (1) emergence—the transport of linked flux across the surface $\mathcal{S}$, and (2) shearing—the twisting and tangling of footpoints by motions in the surface. Using the helicity transport expression (51), we can now derive the helicity transport terms due to emergence and shearing. For comparison with previous work, recall that for ideal surface motions Berger (1984) decomposed the integrand of the surface helicity transport (8) into two terms: one involving the motion of magnetic flux through $\mathcal{S}$ (emergence) and a second which transports helicity through the surface by motions tangent to $\mathcal{S}$ (shearing)

$$\begin{aligned}\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} &= -2 \oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{A}_{\rm PC} \times (\boldsymbol{v} \times \boldsymbol{B}) \\ &= 2 \oint_{\mathcal{S}} dS\, \underbrace{(\boldsymbol{A}_{\rm PC} \cdot \boldsymbol{B}) v_n}_{\text{Emergence}} \\ &\quad - 2 \oint_{\mathcal{S}} dS\, \underbrace{(\boldsymbol{A}_{\rm PC} \cdot \boldsymbol{v}) B_n}_{\text{Shearing}}. \end{aligned} \tag{54}$$

The first expression involves $\boldsymbol{v} \times \boldsymbol{B}$, which ensures that velocity parallel $v_{\parallel}$ to the magnetic field plays no role in helicity transport through $\mathcal{S}$. However, once this integrand is decomposed into emerging and shearing constituents, in the second expression, each term can individually depend on $v_{\parallel}$. To clarify this point, the decomposition (108) adapted to the magnetic field is

$$\boldsymbol{v}_{\parallel} = (\boldsymbol{v} \cdot \boldsymbol{B})\boldsymbol{B}/B^2, \tag{55a}$$

$$\boldsymbol{v}_{\perp} = (\boldsymbol{B} \times \boldsymbol{v}) \times \boldsymbol{B}/B^2 = \boldsymbol{v} - v_{\parallel}\, \boldsymbol{B}/B, \tag{55b}$$

which separates plasma flows parallel to the magnetic field ($\boldsymbol{v}_{\parallel}$) and perpendicular to the magnetic field ($\boldsymbol{v}_{\perp}$) by construction.

[6] This special geometry also leads to the analogous representation for $\boldsymbol{A}_{\rm PC}$ for the potential magnetic field leading to $\boldsymbol{\nabla}_{\mathcal{S}} \cdot \boldsymbol{A}_{\rm PC} = 0$ on $\mathcal{S}$ (e.g., Berger & Ruzmaikin 2000).

The total velocity in terms of the magnetic field vector is

$$\boldsymbol{v} = \boldsymbol{v}_\perp + v_\parallel \boldsymbol{B}/B, \tag{56}$$

producing the emerging and shearing helicity transport terms

$$\left(\frac{\partial \mathcal{H}_\mathcal{S}}{\partial t}\right)_{\rm Em} = 2 \oint_\mathcal{S} dS (\boldsymbol{A}_{\rm PC} \cdot \boldsymbol{B}_\mathcal{S})(v_{\perp n} + v_\parallel B_n/B), \tag{57a}$$

$$\left(\frac{\partial \mathcal{H}_\mathcal{S}}{\partial t}\right)_{\rm Sh} = -2 \oint_\mathcal{S} dS [\boldsymbol{A}_{\rm PC} \cdot (\boldsymbol{v}_{\perp\mathcal{S}} + v_\parallel \boldsymbol{B}_\mathcal{S}/B)]\, B_n. \tag{57b}$$

The $\boldsymbol{v}_\parallel$ terms locally cancel when combined in (8) and thus lead to no net pointwise helicity transport. However, these parallel velocities, which play no role in the magnetic evolution, bias the individual emergence and shearing terms. To correct this bias, only the components of the perpendicular velocity 55(b) are usually used to compute the emerging and shearing helicity transport:

$$\left(\frac{\partial \mathcal{H}_\mathcal{S}}{\partial t}\right)_{\rm Em} = 2 \oint_\mathcal{S} dS (\boldsymbol{A}_{\rm PC} \cdot \boldsymbol{B}_\mathcal{S}) \boldsymbol{v}_{\perp n}, \tag{58a}$$

$$\left(\frac{\partial \mathcal{H}_\mathcal{S}}{\partial t}\right)_{\rm Sh} = -2 \oint_\mathcal{S} dS (\boldsymbol{A}_{\rm PC} \cdot \boldsymbol{v}_{\perp\mathcal{S}}) B_n, \tag{58b}$$

but these expressions are still explicitly gauge dependent.

A more rigorous procedure is to split the contributions to the $\boldsymbol{v} \times \boldsymbol{B}$ electric field into emergence and shearing terms by constructing electric fields corresponding to the two MHD surface processes:

$$\text{Emerging } \boldsymbol{E}_{\rm Em} = -v_{\perp n} \hat{\boldsymbol{n}} \times \boldsymbol{B}_\mathcal{S}/c, \tag{59a}$$

$$\text{Shearing } \boldsymbol{E}_{\rm Sh} = -(\boldsymbol{v}_{\perp\mathcal{S}} \times B_n \hat{\boldsymbol{n}} + \boldsymbol{v}_{\perp\mathcal{S}} \times \boldsymbol{B}_\mathcal{S})/c, \tag{59b}$$

each of which individually satisfies $\boldsymbol{E}_{\rm Em} \cdot \boldsymbol{B} = \boldsymbol{E}_{\rm Sh} \cdot \boldsymbol{B} = 0$. Note that the shearing term supports an electric field in the normal direction except when $\boldsymbol{B}_\mathcal{S} = 0$. For ideal motions, 48(a) separates into two terms for emerging and shearing:

$$\nabla^2_\mathcal{S} \zeta_{\rm Em} = \frac{1}{c} \boldsymbol{\nabla}_\mathcal{S} \cdot (\boldsymbol{v}_{\perp n} \hat{\boldsymbol{n}} \times \boldsymbol{B}_\mathcal{S}) \quad \in \mathcal{S} \tag{60a}$$

$$\nabla^2_\mathcal{S} \zeta_{\rm Sh} = \frac{1}{c} \boldsymbol{\nabla}_\mathcal{S} \cdot (\boldsymbol{v}_{\perp\mathcal{S}} \times \hat{\boldsymbol{n}} B_n) \quad \in \mathcal{S}, \tag{60b}$$

where by superposition $\zeta = \zeta_{\rm Em} + \zeta_{\rm Sh}$. Note that $(\hat{\boldsymbol{n}} \times \boldsymbol{E}_{\rm Sh}) \times \hat{\boldsymbol{n}}$ in 48(a) under the divergence kills the normal electric field $\boldsymbol{v}_\mathcal{S} \times \boldsymbol{B}_\mathcal{S}$. In many unbounded geometries of interest (spheres with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$, planes, etc.), $\zeta_{{\rm P}i} =$ constant, and $\zeta_{{\rm P}i}$ has no effect on the emergence or shearing helicity transport and can simply be ignored. The surface helicity transport equation is then

$$\frac{\partial \mathcal{H}_\mathcal{S}}{\partial t} = 2\, c \sum_i \oint_\mathcal{S} dS\; \zeta_i\, B_n, \tag{61}$$

where, here and below, the sum over $i \in$ (Em, Sh) represents the emerging and shearing terms. Each term is individually gauge invariant in this formalism by the solenoidal property of $\boldsymbol{B}$ and the uniqueness, to within arbitrary constants, of the ζ_i's. Consequently, each term in the sum can be considered independently as an observable.

In general curvilinear coordinates, the helicity transport equation involves $\zeta_{{\rm P}i}$

$$\frac{\partial \mathcal{H}_\mathcal{S}}{\partial t} = 2\, c \sum_i \oint_\mathcal{S} dS (\zeta_i - \zeta_{{\rm P}i}) B_n, \tag{62}$$

and the individual contributions to $\boldsymbol{\Sigma}_{\rm P}$ from emergence and shearing must be determined. Because $\boldsymbol{\Sigma}_{\rm P}$ is determined directly from the changes in the normal component of the magnetic field at the boundary, then as with the electric field above, we can maintain gauge invariance by splitting these changes into the contributions from emerging and shearing velocities:

$$\left(\frac{\partial P_n}{\partial t}\right)_{\rm Em} = \nabla^2_\mathcal{S} \frac{\partial \chi_{\rm PEm}}{\partial t} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_\mathcal{S} \times (v_{\perp n} \hat{\boldsymbol{n}} \times \boldsymbol{B}_\mathcal{S}) \quad \in \mathcal{S}, \tag{63a}$$

$$\left(\frac{\partial P_n}{\partial t}\right)_{\rm Sh} = \nabla^2_\mathcal{S} \frac{\partial \chi_{\rm PSh}}{\partial t} = \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_\mathcal{S} \times (\boldsymbol{v}_{\perp\mathcal{S}} \times \hat{\boldsymbol{n}} B_n) \quad \in \mathcal{S}. \tag{63b}$$

Each of these processes individually can produce a change in $\boldsymbol{P}$ at the surface $\mathcal{S}$ and the $\partial\chi_{{\rm P}i}/\partial t$'s can be established directly from these changes. Again, the normal electric field $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times (\boldsymbol{v}_\mathcal{S} \times \boldsymbol{B}_\mathcal{S}) = 0$ does not affect the value $\partial\chi_{\rm PEm}/\partial t$ or $\partial\chi_{\rm PSh}/\partial t$ and by superposition, $\partial\chi_{\rm P}/\partial t = \partial\chi_{\rm PEm}/\partial t + \partial\chi_{\rm PSh}/\partial t$. The emergence and shearing terms for $\boldsymbol{\Sigma}_{\rm P}$ can now be expressed in terms of the relevant constituents of $\partial B_n/\partial t$ using (94):

$$\boldsymbol{\Sigma}_{\rm PEm}(\boldsymbol{x}, t) = c^{-1} \int_\mathcal{V} d^3x' \, \mathbb{K}_\mathcal{V}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{\nabla}' \oint_\mathcal{S} dS'' \, \mathcal{G}_{\rm N}(\boldsymbol{x}', \boldsymbol{x}'') \left[\frac{B_n(\boldsymbol{x}'', t)}{\partial t}\right]_{\rm Em}, \tag{64a}$$

$$\boldsymbol{\Sigma}_{\rm PSh}(\boldsymbol{x}, t) = c^{-1} \int_\mathcal{V} d^3x' \, \mathbb{K}_\mathcal{V}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{\nabla}' \oint_\mathcal{S} dS'' \, \mathcal{G}_{\rm N}(\boldsymbol{x}', \boldsymbol{x}'') \left[\frac{B_n(\boldsymbol{x}'', t)}{\partial t}\right]_{\rm Sh}. \tag{64b}$$

Taking the surface divergence of the potential reference electric fields produces expressions for the emerging and shearing lamellar potentials in terms of the motions corresponding to the two ideal MHD surface processes:

$$\nabla^2_\mathcal{S} \zeta_{\rm PEm}(\boldsymbol{x}, t) = -c^{-1} \boldsymbol{\nabla}_\mathcal{S} \cdot \int_\mathcal{V} d^3x' \, \mathbb{K}_\mathcal{V}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{\nabla}' \oint_\mathcal{S} dS'' \, \mathcal{G}_{\rm N}(\boldsymbol{x}', \boldsymbol{x}'') \hat{\boldsymbol{n}}'' \cdot \boldsymbol{\nabla}_\mathcal{S}'' \times [v_{\perp n}(\boldsymbol{x}'', t) \hat{\boldsymbol{n}}'' \times \boldsymbol{B}_\mathcal{S}(\boldsymbol{x}'', t)], \tag{65a}$$

$$\nabla^2_\mathcal{S} \zeta_{\rm PSh}(\boldsymbol{x}, t) = -c^{-1} \boldsymbol{\nabla}_\mathcal{S} \cdot \int_\mathcal{V} d^3x' \, \mathbb{K}_\mathcal{V}(\boldsymbol{x}, \boldsymbol{x}') \cdot \boldsymbol{\nabla}' \times \oint_\mathcal{S} dS'' \, \mathcal{G}_{\rm N}(\boldsymbol{x}', \boldsymbol{x}'') \hat{\boldsymbol{n}}'' \cdot \boldsymbol{\nabla}_\mathcal{S}''[\boldsymbol{v}_{\perp\mathcal{S}}(\boldsymbol{x}'', t) \times \hat{\boldsymbol{n}}'' B_n(\boldsymbol{x}'', t)], \tag{65b}$$

where again, by superposition, $\zeta_{\rm P} = \zeta_{\rm PEm} + \zeta_{\rm PSh}$. Note that 63(a)–(b) and 65(a)–(b) directly couple the $\chi_{{\rm P}i}$ and $\zeta_{{\rm P}i}$. The $\zeta_{{\rm P}i}$ are unique to within an arbitary constant and dependent only on observables $\boldsymbol{v}$ and $\boldsymbol{B}$.

For application to observations, it is useful to re-express the ideal plasma velocity in terms of the constituents that produce the changes in the normal component of the magnetic field $\boldsymbol{v}_{B_n}$ and transport helicity $\boldsymbol{v}_{\mathcal{H}}$ across the boundary. For an ideal electric field, the Helmholtz–Hodge decomposition on $\mathcal{S}$ has

$$\tau = \frac{1}{B_n}\boldsymbol{B}_{\mathcal{S}} \cdot \left[\frac{1}{c}\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\frac{\partial\chi}{\partial t} + \boldsymbol{\nabla}_{\mathcal{S}}\zeta\right] \tag{66}$$

in (47) to enforce $\boldsymbol{E} \cdot \boldsymbol{B} = 0$. The ideal plasma velocity can be expressed on $\mathcal{S}$ as

$$\boldsymbol{v} = \boldsymbol{v}_{\parallel}\boldsymbol{B}/B + \sum_i(\boldsymbol{v}_{B_n,i} + \boldsymbol{v}_{\mathcal{H},i}), \tag{67a}$$

where

$$\begin{aligned}\boldsymbol{v}_{B_n,i} &= \frac{c}{B^2}\left\{\boldsymbol{\Sigma}_{\mathrm{P}i} - \frac{\hat{\boldsymbol{n}}}{B_n}\boldsymbol{B}_{\mathcal{S}} \cdot \boldsymbol{\Sigma}_{\mathrm{P}i}\right\} \times \boldsymbol{B},\\ &= \frac{c}{B^2}\boldsymbol{B} \times \left\{\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\left(\frac{1}{c}\frac{\partial\chi_{\mathrm{P}i}}{\partial t}\right) + \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\mathrm{P}i}\right.\\ &\quad\left. - \frac{\hat{\boldsymbol{n}}}{B_n}\boldsymbol{B}_{\mathcal{S}} \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\left(\frac{1}{c}\frac{\partial\chi_{\mathrm{P}i}}{\partial t}\right) + \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\mathrm{P}i}\right]\right\},\end{aligned} \tag{67b}$$

$$\begin{aligned}\boldsymbol{v}_{\mathcal{H},i} &= \frac{c}{B^2}\left\{(\boldsymbol{E}_i - \boldsymbol{\Sigma}_{\mathrm{P}i})_{\mathcal{S}} - \frac{\hat{\boldsymbol{n}}}{B_n}\boldsymbol{B}_{\mathcal{S}} \cdot (\boldsymbol{E}_i - \boldsymbol{\Sigma}_{\mathrm{P}i})\right\} \times \boldsymbol{B},\\ &= \frac{c}{B^2}\boldsymbol{B} \times \left[\boldsymbol{\nabla}_{\mathcal{S}}(\zeta_i - \zeta_{\mathrm{P}i}) - \frac{\hat{\boldsymbol{n}}}{B_n}\boldsymbol{B}_{\mathcal{S}} \cdot \boldsymbol{\nabla}_{\mathcal{S}}(\zeta_i - \zeta_{\mathrm{P}i})\right]\end{aligned} \tag{67c}$$

are the ideal MHD constituent velocities that change the normal component of the magnetic field and that transport helicity, respectively. Note that all three components of the ideal plasma velocity can be reconstructed from just the surface components of the ideal electric field consituents produced by the emerging $\boldsymbol{E}_{\mathrm{Em}}$ and shearing $\boldsymbol{E}_{\mathrm{Sh}}$ processes, due to the ideal MHD constraints $\boldsymbol{v}_{B_n,i} \cdot \boldsymbol{B} = 0$ and $\boldsymbol{v}_{\mathcal{H},i} \cdot \boldsymbol{B} = 0$. The total ideal electric field is reconstructed exactly by the sum of the $\boldsymbol{v} \times \boldsymbol{B}$ produced by the constituent velocities $\sum_i \boldsymbol{E}_i = -\sum_i(\boldsymbol{v}_{\mathcal{H},i} + \boldsymbol{v}_{B_n,i}) \times \boldsymbol{B}/c$. If we interpret $\boldsymbol{v}_{B_n}$ as reconstructing the reference electric field that produces changes in $\boldsymbol{P}$ on $\mathcal{S}$, then

$$\begin{aligned}\boldsymbol{E}_{\mathrm{P}i} &= -\boldsymbol{v}_{B_n,i} \times \boldsymbol{B}/c + \boldsymbol{\nabla}\Lambda'_{\mathrm{P}_i},\\ &= -\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\left(\frac{1}{c}\frac{\partial\chi_{\mathrm{P}i}}{\partial t}\right) - \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\mathrm{P}i} + \frac{\hat{\boldsymbol{n}}}{B_n}\boldsymbol{B}_{\mathcal{S}}\\ &\quad \cdot\left[\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\left(\frac{1}{c}\frac{\partial\chi_{\mathrm{P}i}}{\partial t}\right) + \boldsymbol{\nabla}_{\mathcal{S}}\zeta_{\mathrm{P}i}\right] + \boldsymbol{\nabla}\Lambda'_{\mathrm{P}_i},\\ &= \boldsymbol{\Sigma}_{\mathrm{P}i} + \boldsymbol{\nabla}\Lambda_{\mathrm{P}i} \in \mathcal{S}.\end{aligned} \tag{68a}$$

In other words, $\boldsymbol{v}_{B_n,i}$ is the ideal MHD plasma velocity consistent with the change in the normal component of the magnetic field. Here, $\boldsymbol{v}_{B_n,i}$ reconstructs the surface components of $\boldsymbol{E}_{\mathrm{P}}$ exactly, and the freedom in Λ_{P} can account for the normal component of $\boldsymbol{E}_{\mathrm{P}}$ produced by $\boldsymbol{v} \times \boldsymbol{B}$:

$$\begin{aligned}&\frac{\partial\Lambda_{\mathrm{P}i}}{\partial n}\\ &= \frac{\partial{\Lambda_{\mathrm{P}i}}'}{\partial n} + \frac{1}{B_n}\boldsymbol{B}_S \cdot \left[\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_S\left(\frac{1}{c}\frac{\partial\chi_{\mathrm{P}i}}{\partial t}\right) + \boldsymbol{\nabla}_S\zeta_{\mathrm{P}i}\right] \in S,\end{aligned} \tag{68b}$$

which, with 119(c), requires that

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_{\mathrm{P}i} \equiv \varrho_{\mathrm{P}} = \overbrace{\nabla^2_{\mathcal{S}}\Lambda_{\mathrm{P}i}}^{32(\mathrm{e})} - \mathcal{J}\frac{\partial\Lambda_{\mathrm{P}i}}{\partial n} + \frac{\partial^2\Lambda_{\mathrm{P}i}}{\partial n^2} \quad \in \mathcal{S}, \tag{68c}$$

where the term with the overbrace is zero by the referenced equation. As an example of the considerable freedom in Λ_{P} relevant to solar physics, consider the space between two spherical shells at r_1 and r_2. Writing

$$\begin{aligned}&\Lambda_{\mathrm{P}}\\ &= \frac{(r - r_1)(r - r_2)^2 E_{n1}(\vartheta, \varphi) + (r - r_1)^2(r - r_2)E_{n2}(\vartheta, \varphi)}{(r_2 - r_1)^2},\end{aligned} \tag{68d}$$

which satisfies $\Lambda_{\mathrm{P}} = 0$ at both $r = r_1$ and $r = r_2$ while leaving $\partial\Lambda_{\mathrm{P}}/\partial n$ arbitrary at those two surfaces, demonstrates that $\boldsymbol{E}_{\mathrm{P}}$ can accomodate the normal component of $\boldsymbol{E}$ produced by $\boldsymbol{v} \times \boldsymbol{B}$.

While ζ_{PEm} and ζ_{PSh} are not observables, $\boldsymbol{v}_{\mathcal{H},\mathrm{Em}}$ and $\boldsymbol{v}_{\mathcal{H},\mathrm{Sh}}$ depend on the gradients of the corresponding lamellar potentials and thus can be used to determine where the two MHD surface processes are operating. Note, however, that a finite $\boldsymbol{v}_{\mathcal{H},\mathrm{Em}}$ or $\boldsymbol{v}_{\mathcal{H},\mathrm{Sh}}$ does not necessarily imply a finite helicity transport due to the nonlocal nature of helicity. This is demonstrated directly below in Section 5.2.

5.1. Nonideal Effects

Although we have focused on ideal transport effects, we emphasize that (35) and (51) do not inherently require ideal evolution to estimate $\partial\mathcal{H}_{\mathcal{S}}/\partial t$, which is itself a gauge-invariant observable. If we have independent measurements of $\boldsymbol{E}$, $\boldsymbol{v}$, and $\boldsymbol{B}$, then the nonideal electric field E_{NI} can be estimated from

$$E_{\mathrm{NI}} = \boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}, \tag{69a}$$

which produces a nonideal change in the normal component of $\boldsymbol{B}$ given by

$$\left(\frac{\partial B_n}{\partial t}\right)_{\mathrm{NI}} = -c\,\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times E_{\mathrm{NI}}. \tag{69b}$$

Once E_{NI} and $(\partial B_n/\partial t)_{\mathrm{NI}}$ are established, then the corresponding $\partial\chi_{\mathrm{NI}}/\partial t$ and ζ_{NI} can be computed directly from the Helmholtz–Hodge decomposition of E_{NI}. Similarly, the nonideal $\boldsymbol{\Sigma}_{\mathrm{PNI}}$ can in principle be determined as in Appendix A and then ζ_{PNI} can be computed from the Helmholtz–Hodge decomposition of $\boldsymbol{\Sigma}_{\mathrm{PNI}}$. The nonideal surface helicity transport

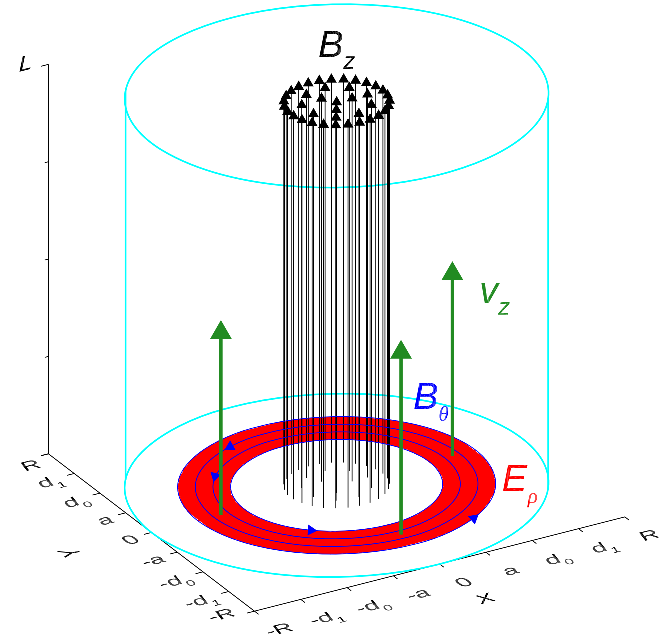

Figure 1. Emergence of the disconnected linked field. The bounding cylindrical surface (cyan) contains a uniform column of the vertical field B_z for $\rho \leqslant a$ (black arrows). Closed rings of the azimuthal field B_θ are injected at the bottom boundary $d_0 \leqslant \rho \leqslant d_1$ (blue arrows) by a uniform vertical flow v_z (green arrows), which produces a ring of radial electric field $E_\rho = v_z\, B_\theta / c$ (red).

takes the form

$$\left(\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t}\right)_{\rm NI} = 2\, c \sum_i \oint_{\mathcal{S}} dS (\zeta_{\rm NI} - \zeta_{\rm PNI}) B_n. \tag{70}$$

Of course, generally, a nonideal electric field will produce a volumetric contribution in addition to the surface term above, leading to a nonideal term in $\dot{\mathcal{H}}_{\mathcal{V}}$ and an overall nonideal contribution to the helicity transport rate of $\dot{\mathcal{H}}$.

5.2. The Emergence of Disconnected Linked Flux

To clarify the nonlocal nature of our expressions for helicity transport, consider the simple case of the emergence of a horizontal ring of the closed field (zero self-helicity) so that it links with a vertical column of the untwisted field, Figure 1. In this case all the helicity generated is due to the linkage of the two systems. The key point is that this case demonstrates clearly that the concept of a "helicity flux density" is not valid. Let the domain be the cylinder $\rho \in [0, R]$ and $z \in [0, L]$ shown in Figure 1. The initial magnetic field is given by

$$\begin{aligned} \boldsymbol{B}(\rho, z) = {} & B_1\, U(-z)[U(\rho - d_0) - U(\rho - d_1)]\, \hat{\boldsymbol{\theta}} \\ & + B_0[1 - U(\rho - a)]\hat{\boldsymbol{z}}, \end{aligned} \tag{71a}$$

where $R \gg d_1 > d_0 > a$, and the velocity is simply a constant vertical flow,

$$\boldsymbol{v}(\rho, z) = v_0\, \hat{\boldsymbol{z}}, \tag{71b}$$

with the unit step function defined as

$$U(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0. \end{cases} \tag{72}$$

Neither cap of the cylinder is flux balanced, but the entire surface satisfies $\oint_{\mathcal{S}} dS\, B_n = 0$. The potential due to emergence is

$$\begin{aligned} \zeta_{\rm Em}(\rho, 0) = \zeta_0 - \frac{v_0\, B_1}{c}[&(\rho - d_0)U(\rho - d_0) \\ & - (\rho - d_1)U(\rho - d_1)], \end{aligned} \tag{73a}$$

where we redefine the arbitrary constant ζ_0 in terms of the value of $\zeta_{\rm Em}$ as $\rho \to R$, namely ζ_R,

$$\zeta_0 = \zeta_R + \frac{v_0\, B_1}{c}(d_1 - d_0), \tag{73b}$$

then by continuity,

$$\zeta_{\rm Em}(\rho \geqslant d_1, 0) = \zeta_{\rm Em}(R, z) = \zeta_{\rm Em}(\rho, L) = \zeta_R. \tag{73c}$$

Because no magnetic field penetrates the sides for $\rho = R$ and $z \in [0, L]$ or the caps for $\rho \geqslant a$, the helicity transport rate is then

$$\begin{aligned} \frac{d\mathcal{H}}{dt} = {} & \overbrace{2\, c\, B_0\, \pi\, a^2\, \zeta_R + 2\, B_0\, \pi\, a^2\, B_1(d_1 - d_0)v_0}^{\text{bottom surface}} \\ & - \overbrace{2\, c\, B_0\, \pi\, a^2\, \zeta_R}^{\text{top surface}} \\ = {} & 2\, B_0\, \pi\, a^2\, B_1(d_1 - d_0)v_0. \end{aligned} \tag{74}$$

The choice of constant ζ_R has no influence on the net helicity transport rate.

Intuitively, we would say that the helicity transport in this situation is "emerging" at the bottom boundary between $\rho = d_0$ and $\rho = d_1$, where the closed field is being transported across the boundary. However, the choice of constants influences this interpretation. First, we choose $\zeta_{\rm Em}(0, 0) = 0$ and $\zeta_R = (v_0\, B_1/c)(d_1 - d_0)$. This implies counterintuitively that there is no helicity transport through the bottom boundary, and the helicity enters through the top boundary. Alternatively, if we choose $\zeta_R = 0$, then there is no helicity transport through the top boundary and the helicity appears to enter through the bottom boundary. Note that in both cases, the integrand is only nonzero at boundary locations where $B_n \neq 0$, i.e., where the normal field penetrates the bounding surface, but there are no footpoint motions there, because $\boldsymbol{v} \times \boldsymbol{B} = 0$! Even knowing the field line connectivity for this situation does not resolve the nonlocality of where helicity is injected on the surface $\mathcal{S}$. We conclude, therefore, that helicity injection, just like helicity itself, is inherently nonlocal. Associating a local interpretation to the helicity transport is incorrect; consequently, observational quantities such as "helicity flux maps" are misleading (Pariat et al. 2005, 2007; Chandra et al. 2010; Romano et al. 2011; Vemareddy et al. 2012; Vemareddy 2015).

6. Free Energy Flux

As with relative helicity, free energy is defined only with respect to a reference potential field. To determine its transport, let us consider a closed volume $\mathcal{V}$ with boundary $\mathcal{S}$ and magnetic field $\boldsymbol{B}$. There is a unique potential field $\boldsymbol{P}$ that satisfies $\boldsymbol{\nabla} \times \boldsymbol{P} = 0$ in $\mathcal{V}$ and $\boldsymbol{P} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = \boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}}$. Following Berger (1999), the closed field in the volume may be defined as

$$\boldsymbol{B}_{\rm cl} \equiv \boldsymbol{B} - \boldsymbol{P} \quad \text{with} \quad \boldsymbol{B}_{\rm cl} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = 0. \tag{75}$$

The closed field carries all the current in the volume. Note, however, that $\boldsymbol{B}_{\rm cl}$ may be potential with $\boldsymbol{\nabla} \times \boldsymbol{B}_{\rm cl} = 0$ in some parts of the volume, or even on some or all of the surface. At the surface $\mathcal{S}$, $\boldsymbol{B}_{\rm cl}$ represents the tangential field produced by current systems in the volume or normal to the surface.

Because the corona is low β, the energy transfer into the corona is almost completely via the magnetic field. In this case, the rate of energy transfer through the surface $\mathcal{S}$ into $\mathcal{V}$ is given by the Poynting flux:

$$\frac{d\mathcal{E}}{dt} = \frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot(\boldsymbol{E}\times\boldsymbol{B}). \tag{76}$$

The total potential magnetic energy in the volume $\mathcal{V}$ is

$$\mathcal{E}_{\rm P} = \frac{1}{8\,\pi}\int_{\mathcal{V}} d^3x(\boldsymbol{P}\cdot\boldsymbol{P}). \tag{77}$$

The goal is to derive the equivalent equation for the rate of potential energy transfer across the surface $\mathcal{S}$ and subtract this from (76) to obtain the rate of free energy transfer across the surface $\mathcal{S}$:

$$\frac{d\mathcal{E}_{\rm F}}{dt} = \frac{d\mathcal{E}}{dt} - \frac{d\mathcal{E}_{\rm P}}{dt}. \tag{78}$$

Because $\boldsymbol{P}$ is the potential field, it may be determined from (4) in the volume $\mathcal{V}$, and the last term in (78) above may be recast as

$$\begin{aligned}\frac{dE_{\rm P}}{dt} &= \frac{1}{4\,\pi}\int_V d^3x\,\boldsymbol{P}\cdot\frac{\partial\boldsymbol{P}}{\partial t}\\ &= -\frac{1}{4\,\pi}\int_V d^3x\,\nabla\phi_{\rm P}\cdot\frac{\partial\boldsymbol{P}}{\partial t}\\ &= -\frac{1}{4\,\pi}\int_V d^3x\left[\nabla\cdot\left(\phi_{\rm P}\frac{\partial\boldsymbol{P}}{\partial t}\right) - \phi_{\rm P}\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{P})\right],\\ &= \frac{1}{4\,\pi}\oint_S dS\,\phi_{\rm P}\frac{\partial P_n}{\partial t}.\end{aligned} \tag{79}$$

(Recall that we are using the convention that $\hat{\boldsymbol{n}}$ on the surface $\mathcal{S}$ points into the coronal volume $\mathcal{V}$).

Using Faraday's law of induction (19),[7]

$$\frac{d\mathcal{E}_{\rm P}}{dt} = -\frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot(\phi_{\rm P}\,\nabla\times\boldsymbol{E}); \tag{80}$$

with (99) and 124(a), this becomes

$$\begin{aligned}\frac{d\mathcal{E}_{\rm P}}{dt} &= \frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot[\boldsymbol{E}\times\boldsymbol{P} - \nabla\times(\phi_{\rm P}\,\boldsymbol{E})]\\ &= \frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot(\boldsymbol{E}\times\boldsymbol{P}).\end{aligned} \tag{81}$$

Substituting (76) and (81) into (78), the rate of change in the free energy is

$$\begin{aligned}\frac{d\mathcal{E}_{\rm F}}{dt} &= \frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot[\boldsymbol{E}\times(\boldsymbol{B}-\boldsymbol{P})]\\ &= \frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot(\boldsymbol{E}\times\boldsymbol{B}_{\rm cl}).\end{aligned} \tag{82}$$

The result that the free energy transport depends on the presence of a finite $\boldsymbol{B}_{\rm cl}$ at the boundary may seen somewhat obscure, but in fact, it can be understood from straightforward force arguments. If the field is purely potential at the boundary, then it exerts no stress there, and any instantaneous dynamics of the boundary do no work on the field. The presence of a $\boldsymbol{B}_{\rm cl}$ is required for the boundary to add/subtract free energy to/from the field. Furthermore, while the Poynting vector $\boldsymbol{E}\times\boldsymbol{B}$ defines a physically valid flux density, the free energy transport depends on the details of the boundary, which determines the potential field $\boldsymbol{P}$. Consequently, unlike energy transport (see Vol. II, p. 27-6–27-8, Feynman et al. 1989), free energy (and helicity) transport has physical significance only in the context of a specified volume.

We also note that free energy transport across the boundary requires currents in the volume $\mathcal{V}$ or at the surface $\mathcal{S}$. This can be seen by substituting (47) with 48(a)–48(b) for $\boldsymbol{E}$, producing

$$\begin{aligned}\frac{d\mathcal{E}_{\rm F}}{dt} = -\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot\Bigg\{&\frac{1}{4\,\pi}\left[\hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\left(\frac{\partial\chi_{\rm P}}{\partial t}\right)\right]\\ &\times\boldsymbol{B}_{\rm cl} + \frac{c}{4\,\pi}\nabla_{\mathcal{S}}\zeta\times\boldsymbol{B}_{\rm cl}\Bigg\}.\end{aligned} \tag{83}$$

Using (95) on the first term with $\boldsymbol{B}_{\rm cl}\cdot\hat{\boldsymbol{n}}$ followed by (112) and using (116) on the second term produces

$$\begin{aligned}\frac{d\mathcal{E}_{\rm F}}{dt} = \oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot\Bigg\{&\frac{\hat{\boldsymbol{n}}}{4\,\pi}\left[\nabla_{\mathcal{S}}\cdot\left(\frac{\partial\chi_{\rm P}}{\partial t}\boldsymbol{B}_{\rm cl}\right) - \left(\frac{\partial\chi_{\rm P}}{\partial t}\right)\nabla_{\mathcal{S}}\cdot\boldsymbol{B}_{\rm cl}\right]\\ &- \frac{c}{4\,\pi}\nabla_{\mathcal{S}}\times(\zeta\,\boldsymbol{B}_{\rm cl}) + \frac{c}{4\,\pi}\zeta\,\nabla_{\mathcal{S}}\times\boldsymbol{B}_{\rm cl}\Bigg\}.\end{aligned} \tag{84}$$

Applying 124(a) and 125(b) and noting that any current though the surface must be produced by the curl of $\boldsymbol{B}_{\rm cl}$,

$$\hat{\boldsymbol{n}}\cdot\nabla\times\boldsymbol{B}_{\rm cl} = \hat{\boldsymbol{n}}\cdot\nabla\times\boldsymbol{B} = \frac{4\,\pi}{c}J_n \tag{85}$$

leads to

$$\frac{d\mathcal{E}_{\rm F}}{dt} = \oint_{\mathcal{S}} dS\left[\zeta\,J_n - \frac{1}{4\,\pi}\frac{\partial\chi_{\rm P}}{\partial t}\nabla_{\mathcal{S}}\cdot\boldsymbol{B}_{\rm cl}\right]. \tag{86}$$

The first term involves local currents normal to the surface and the lameller electric field, whereas the second term describes purely inductive changes in the surface fields due to currents in the volume and involves the solenoidal (inductive) electric field. We conclude that as with helicity, a net free energy transport requires the presence of electric currents. Furthermore, the free energy and free energy transport are nonlocal quantities that depend on the surface $\mathcal{S}$ bounding the volume $\mathcal{V}$ that determines the potential field. Whereas the Poynting flux $\boldsymbol{E}\times\boldsymbol{B}$ can be calculated locally and is invariant to changes in the shape of the volume away from the local point of interest, the local free energy transport $\boldsymbol{E}\times\boldsymbol{B}_{\rm cl}$ can change in response to nonlocal modifications in the shape of the volume.

7. Discussion and Conclusions

The most important conclusion from the results above is that there does, indeed, exist a gauge-invariant expression for the surface helicity transport, namely, (35) which we repeat here for completeness:

$$\frac{\partial\mathcal{H}_{\mathcal{S}}}{\partial t} = -c\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot[(\boldsymbol{A}+\boldsymbol{A}_{\rm P})\times(\boldsymbol{E}-\boldsymbol{E}_{\rm P})].$$

[7] Note that $\boldsymbol{E}_{\rm P}$ from the previous discussion can be substituted for $\boldsymbol{E}$ here.

Although this expression may not always have computational advantages over the Berger & Field (1984) expression for the helicity transport, it has major theoretical advantages. It resolves the long-standing concern that, while the volumetric helicity itself could be readily expressed in a fully gauge-invariant form using the Berger & Field (1984) or Finn & Antonsen (1985) formulas, the surface helicity transport apparently could not. The Berger & Field (1984) expression explicitly requires the Coulomb gauge. Our expression above is valid in any gauge for either $\boldsymbol{A}$ or $\boldsymbol{A}_{\rm P}$ and with arbitrary $\boldsymbol{\nabla} \cdot \boldsymbol{E}_{\rm P} = \varrho_{\rm P}$.

The key physical insight that is used to derive this gauge-invariant expression is the requirement that no self-helicity is generated in the potential field by its evolution—in other words, ansatz (31). In fact, this ansatz is physically no different than the standard assumption used in every discussion of helicity—that a potential magnetic field has zero helicity. Given this assumption, then (31) inevitably follows. Mathematically, this ansatz leads to the decomposition of the reference electric field,

$$\boldsymbol{E}_{\rm P} = \boldsymbol{\Sigma}_{\rm P} + \boldsymbol{\nabla}\Lambda_{\rm P} = -\frac{1}{c}\,\frac{\partial \boldsymbol{A}_{\rm P}}{\partial t} - \boldsymbol{\nabla}\psi_{\rm P}, \tag{87}$$

into a unique intrinsically solenoidal reference electric field $\boldsymbol{\Sigma}_{\rm P}$ and the gradient of an arbitrary function $\Lambda_{\rm P}$ which satisfies Dirichlet boundary conditions on $\mathcal{S}$. For a closed domain, the potential magnetic field is determined uniquely by its boundary values, and its temporal evolution is also uniquely specified by the evolution of these boundary values; consequently, there should exist a unique intrinsically solenoidal electric field $\boldsymbol{\Sigma}_{\rm P}$ corresponding to the changing $\boldsymbol{P}$. The conditions on $\boldsymbol{\Sigma}_{\rm P}$ and $\Lambda_{\rm P}$ are summarized by Faraday's law (19) which relates $\boldsymbol{\nabla} \times \boldsymbol{\Sigma}_{\rm P}$ and $\partial \boldsymbol{P}/\partial t$ and conditions 32(b)–32(e). Note, however, that $\boldsymbol{E}_{\rm P}$ itself is not unique because its divergence is arbitrary, and thus, the vector potential in (87) admits a gauge transformation.

From the general surface helicity transport expression above, it is possible to derive a somewhat simpler, but still fully general, expression (51), included here for convenience:

$$\frac{\partial \mathcal{H}_{\mathcal{S}}}{\partial t} \equiv 2\,c \oint_{\mathcal{S}} dS(\zeta - \zeta_{\rm P})B_n.$$

1. This expression is manifestly gauge invariant as it is only dependent on the potentials ζ and $\zeta_{\rm P}$ and the observable B_n. While ζ and $\zeta_{\rm P}$ are still arbitrary to within a constant, e.g., $\zeta \rightarrow \zeta + \zeta_0$, this constant ζ_0 has no effect on the rate of change of relative helicity by virtue of the solenoidal property of the magnetic field $\left(\oint_{\mathcal{S}} dS\ \zeta_0\ B_n = 0\right)$.
2. This expression is explicitly dependent on the flux threading the surface.
   (a) The helicity $\mathcal{H}$ can only be changed by boundary motions if there is flux threading the bounding surface. Isolation means $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} \equiv 0$.
   (b) The helicity $\mathcal{H}$ in an isolated system evolving according to ideal motions is a robust invariant. If $\boldsymbol{B} \cdot \hat{\boldsymbol{n}}|_{\mathcal{S}} = 0$, then $\partial\mathcal{H}/\partial t \equiv 0$.
3. This expression is only dependent on the lamellar electric field. The instantaneous solenoidal (inductive) electric field, which changes the normal component of the magnetic field on the boundary, does not contribute to helicity transport across the boundary. Thus, Prior & Yeates's (2014, p. 2) criticism of Berger & Field (1984) that when "the boundary conditions $B_n|_{\mathcal{S}}$ are changing in time, ... the evolution of the relative helicity will mix up both real topological changes in $\boldsymbol{B}$ and those simply due to the change of $\boldsymbol{P}$" does not hold for (51).
4. This expression can be unambiguously decomposed into independent gauge-invariant expressions for the helicity transport produced by the emergence of the magnetic field represented by $\zeta_{\rm Em}$, the shearing of the magnetic field represented by $\zeta_{\rm Sh}$, and nonideal effects represented by $\zeta_{\rm NI}$, where $\zeta \equiv \zeta_{\rm Em} + \zeta_{\rm Sh} + \zeta_{\rm NI}$.

This final conclusion is highly important for studies of the energy buildup leading to solar eruptions. Given accurate vector magnetograph data, the expressions derived in Section 5 can be applied to measure the different contributions to the helicity injection. For the fully general case that includes $\zeta_{\rm P}$, the calculations would be somewhat tedious, but for the special case of a spherical domain (a coronal domain consisting of the volume between two spherical shells), then the $\zeta_{\rm P}$ drops out, and the transport reduces to the simple expression (53). While this simple expression is valid only for special domains, the spherical domain should always be used, if possible, for the solar atmosphere.

As discussed in the introduction, the concept of helicity is valid only for a closed system. While the photosphere constitutes a true boundary to the corona, there are no other physically meaningful boundaries. Given this fact, the typical assumptions are to take as the closing boundary some spherical surface sufficiently far above the corona so that the radial field can be approximated as vanishing there. In that case, (53) implies that only the photosphere contributes to the helicity transport. Under this assumption, our result above, (53), can be used to determine the gauge-invariant helicity transport through a spherical photospheric surface.

Many authors, however, have attempted to measure coronal helicity transport using box-like rather than spherical domains (e.g., Chae 2001; Kusano et al. 2002; Nindos et al. 2003; Pevtsov et al. 2003; Pariat et al. 2005; Démoulin 2007; Vemareddy et al. 2012; Liu et al. 2014). The usual assumption in using such a domain is that there is minimal helicity transport through the side boundaries in the corona, because the velocities are small there. The problem with using such a domain is that the fully general helicity transport expressions, including calculation of the intrinsically solenoidal part of the reference electric field, must be used. Given the complexities of the calculations involved, we defer to a subsequent paper a discussion and demonstration of the detailed application of our results to finite Cartesian domains. Another problem with having side walls in the corona is that even with minimal photospheric velocities, coronal reconnection may efficiently transport helicity to the walls via the process of helicity condensation (Antiochos 2013).

Our first-principles approach to the surface helicity transport rate can be connected with the transport discussed by Mackay et al. (2014) in the context of helicity condensation (Antiochos 2013). The rate of change in the vector potential produced by the electric field transporting helicity can be written generally for emergence or shearing in spherical coordinates as

$$\left[\frac{\partial \boldsymbol{A}}{\partial t}\right]_{\mathcal{H}} = c\,\boldsymbol{\nabla}_{\mathcal{S}}\zeta.$$

This leads to the same form for the evolution of the surface components of the magnetic field proposed by Mackay et al. (2014),[8]

$$\left[\frac{\partial \boldsymbol{B}_{\mathcal{S}}}{\partial t}\right]_{\mathcal{H}} = \boldsymbol{\nabla}\times\left[\frac{\partial \boldsymbol{A}}{\partial t}\right]_{\mathcal{H}} = -c\,\boldsymbol{\nabla}\times\left(\hat{\boldsymbol{r}}\,\frac{\partial \zeta}{\partial r}\right), \left[\frac{\partial B_n}{\partial t}\right]_{\mathcal{H}} = 0.$$

A solution to this system is $\boldsymbol{v}_{\mathcal{S}} = (c/B_r)\hat{\boldsymbol{r}}\times\boldsymbol{\nabla}\zeta$ for any ζ. If we choose ζ to be constant on flux surfaces, $\boldsymbol{v}_{\mathcal{S}} = (c/B_r)\hat{\boldsymbol{r}}\times\boldsymbol{\nabla}_{\mathcal{S}}\zeta(B_r)$, then the surface helicity transport can be written $\partial\mathcal{H}_{\mathcal{S}}/\partial t \equiv 2\,c\oint_{\mathcal{S}} dS\,\zeta(B_r)B_r$ as purely a function of B_r. If we consider a statistical ensemble of vorticies of scale l and average rotation rate ω_l, the lamellar potential can be written $\zeta(B_r) = B_r\,l^2\,\omega_l/(2\,c)$ and the result $\partial\mathcal{H}_{\mathcal{S}}/\partial t \equiv 2\oint_{\mathcal{S}} dS(l^2\,\omega_l/2)B_r^2$ from Mackay et al. (2014) is recovered.

The other essential quantity determining solar coronal activity is magnetic free energy. As discussed in the introduction, energy and energy transport are generally much more familiar to most researchers than their helicity counterparts; however, for completeness, we have derived the expression for free energy transport across the surface $\mathcal{S}$ of a closed volume:

$$\frac{dE_{\mathrm{F}}}{dt} = \frac{c}{4\,\pi}\oint_{\mathcal{S}} dS\,\hat{\boldsymbol{n}}\cdot(\boldsymbol{E}\times\boldsymbol{B}_{\mathrm{cl}}).$$

Note that the free energy transport is manifestly dependent on the closed field $\boldsymbol{B}_{\mathrm{cl}} = \boldsymbol{B} - \boldsymbol{P}$ on the boundary. This result emphasizes that the field at the boundary must be nonpotential for free energy to be injected into the corona. Unlike energy flux, there can be no free energy transport into the volume in the absence of electric currents within the volume or at the surface $\mathcal{S}$.

Although the free energy transport expression may seem simpler than the expressions for helicity transport, it is, in fact, much less amenable to measurement by observations. The problem is that the free energy transport depends primarily on the tangential components of the magnetic field at the boundary. These components are more difficult to measure than the normal component near the center of the solar disk where active regions are observed at the highest resolution by present instruments such as *SDO*/HMI. Hopefully, progress in instrumentation will result in broader coverage of magnetic and velocity field measurements over the solar surface with sufficient accuracy that the expressions derived in this paper can be applied to yield major new insights into the mechanisms of solar coronal activity.

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Appendix A
General Expression for the Intrinsically Solenoidal Reference Electric Field $\boldsymbol{\Sigma}_{\mathrm{P}}$

Returning to the problem of computing $\boldsymbol{\Sigma}_{\mathrm{P}}$ in general geometries, the Helmholtz decomposition in Appendix E ensures $\boldsymbol{\Sigma}_{\mathrm{P}}$ is completely determined when the tangential components of $\boldsymbol{\Sigma}_{\mathrm{P}}$ on $\mathcal{S}$ are known; however, this is an awkward boundary condition for (19), 32(b), and 32(c), as $\boldsymbol{\Sigma}_{\mathrm{P}}$ is not known a priori anywhere, and only $\boldsymbol{\nabla}\times\boldsymbol{\Sigma}_{\mathrm{P}} = -c^{-1}\,\partial\boldsymbol{P}/\partial t$ is known everywhere.[9] Thus, we seek to determine the inverse curl $\boldsymbol{\nabla}^{-1}\times(\boldsymbol{\nabla}\times\boldsymbol{\Sigma}_{\mathrm{P}})$ knowing $\partial\boldsymbol{P}/\partial t$ everywhere. In free space, the inverse curl has been known for roughly 200 yr; it is the Biot–Savart law (Jackson 1975), which plays a fundamental role in fluid mechanics and electromagnetism by connecting the vorticity, magnetic field, or current with the fluid velocity, vector potential, or magnetic field respectively. For a closed volume $\mathcal{V}$, the Biot–Savart operator

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathrm{P}}(\boldsymbol{x},\,t) &= -\frac{1}{4\,\pi\,c}\int_{\mathcal{V}} d^3x'\,\frac{\partial\boldsymbol{P}(\boldsymbol{x}',\,t)}{\partial t}\times\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\\ &= -\frac{1}{4\,\pi\,c}\,\boldsymbol{\nabla}\times\int_{\mathcal{V}} d^3x'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\frac{\partial\boldsymbol{P}(\boldsymbol{x}',\,t)}{\partial t}\end{aligned} \tag{88}$$

satisfies Faraday's law (19) if and only if $\boldsymbol{\nabla}\cdot\partial\boldsymbol{P}/\partial t = 0$ and $\partial\hat{\boldsymbol{n}}\cdot\boldsymbol{P}/\partial t|_{\mathcal{S}} = 0$ (Cantarella et al. 2001). This is straightforward to see from the application of the curl to the right-hand side of (88), which leads to

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{\Sigma}_{\mathrm{P}}(\boldsymbol{x},\,t) = -c^{-1}\,&\frac{\partial\boldsymbol{P}(\boldsymbol{x}',\,t)}{\partial t}\\ &- \frac{1}{4\,\pi\,c}\,\boldsymbol{\nabla}\Bigg[\int_{\mathcal{V}} d^3x'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\boldsymbol{\nabla}'\cdot\frac{\partial\boldsymbol{P}(\boldsymbol{x}',\,t}{\partial t}\\ &+ \oint_{\mathcal{S}} dS'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\frac{\partial\hat{\boldsymbol{n}}'\cdot\boldsymbol{P}(\boldsymbol{x}',\,t)}{\partial t}\Bigg],\end{aligned} \tag{89}$$

with (102), (91), 126(e), (98), and 122(a). The first condition is satisfied by any magnetic field, but the second condition is impossible to satisfy for any time-dependent potential magnetic field. While a great deal of effort has been expended in the literature proving the existence and uniqueness of solutions of the div-curl system represented by (19), 32(b), and 32(c) (Girault & Raviart 1986; Jiang et al. 1994; Amrouche et al. 1998; Amrouche & Seloula 2013; Cheng & Shkoller 2017), surprisingly, only recently has a general closed-form solution been developed for this system in bounded domains of Riemannian three-manifolds (locally Euclidean three-space). Enciso et al. (2018) constructed an integral solution to (19),

[8] Note the difference in notation between Mackay et al. (2014) and the present work. The former denotes the cyclonic parameter by $\zeta = l^2\,\omega_l/2$, and the present work denotes the lamellar potential by ζ.

[9] The difficulty in applying the Helmholtz decomposition here is that 126(b) is written in terms of the free-space Green's function $\mathcal{G}_{\mathrm{F}}$. All the information about the geometry is encoded in the details of the integrals. In contrast, consider the potential magnetic field, where the Green's function $\mathcal{G}_{\mathrm{N}}$ is solved for an impulse response with homogeneous Neumann boundary conditions (91)–92(b). This encodes the geometry into the Green's function, permitting direct knowledge of the ϕ_{P} everywhere from its normal gradient, namely B_n, on the bounding surface $\mathcal{S}$ (Sakurai 1982; Nemenman & Silbergleit 1999).

32(b), and 32(c) of the form

$$\boldsymbol{\Sigma}_{\rm P}(\boldsymbol{x}, t) = -c^{-1}\int_{\mathcal{V}} d^3x'\, \mathbb{K}_{\mathcal{V}}(\boldsymbol{x}, \boldsymbol{x}')\cdot\frac{\partial \boldsymbol{P}(\boldsymbol{x}', t)}{\partial t}, \tag{90}$$

where $\mathbb{K}_{\mathcal{V}}(\boldsymbol{x}, \boldsymbol{x}')$ is a matrix-valued integral kernel. This solution is unique up to a harmonic field $\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} = 0$, $\boldsymbol{\nabla}\times\boldsymbol{\Omega} = 0$, $\hat{\boldsymbol{n}}\cdot\boldsymbol{\Omega}|_{\mathcal{S}} = 0$, which cannot be supported in a simply connected volume $\mathcal{V}$ (see footnote 3). The details of the kernel $\mathbb{K}_{\mathcal{V}}(\boldsymbol{x}, \boldsymbol{x}')$ are beyond the scope of this paper. However, it permits the expression of a formal relationship between $\boldsymbol{\Sigma}_{\rm P}$ and changes in the normal component of the magnetic field.

The potential reference electric field (90) can be re-expressed entirely in terms of surface values by noting that a solution for $\boldsymbol{P}$ in (4), 5(a)–(b) may be constructed as the convolution of a Green's function and the normal component of $\boldsymbol{P}$ on the surface $\mathcal{S}$:

$$\phi_{\rm P}(\boldsymbol{x}, t) = \langle\phi_{\rm P}(t)\rangle + \oint_{\mathcal{S}} dS'\, \mathcal{G}_{\rm N}(\boldsymbol{x}, \boldsymbol{x}')P_n(\boldsymbol{x}', t), \tag{91}$$

where $\langle\phi_{\rm P}(t)\rangle$ is arbitrary often set to zero or set to the average of $\phi_{\rm P}$ in the volume. The Green's function $\mathcal{G}_{\rm N}$ is developed from the Poisson equation

$$\nabla^2\mathcal{G}_{\rm N}(\boldsymbol{x}, \boldsymbol{x}') = \nabla'^2\mathcal{G}_{\rm N}(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}'),\quad \text{for } \boldsymbol{x} \text{ and } \boldsymbol{x}' \in \mathcal{V}, \tag{92a}$$

where ∇'^2 operates on the primed coordinates with homogeneous Neumann boundary conditions at $\mathcal{S}$,

$$\hat{\boldsymbol{n}}'\cdot\boldsymbol{\nabla}'\mathcal{G}_{\rm N}(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{\mathcal{S}} \text{ for } \boldsymbol{x}' \text{ on } \mathcal{S} \text{ and } \boldsymbol{x} \text{ within } \mathcal{V}, \tag{92b}$$

and $\mathcal{S}$ represents the total surface area bounding $\mathcal{V}$. A solution to 5(a)–(b) exists if only if the compatibility condition

$$\int_{\mathcal{V}} d^3x\, \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi_{\rm P} = -\oint_{\mathcal{S}} dS\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_{\rm P} = \oint_{\mathcal{S}} dS\, P_n \equiv 0,\quad \text{for } \boldsymbol{x} \in \mathcal{V} \tag{93a}$$

is satisfied, which follows from the solenoidal properties of the magnetic field and the boundary condition on the scalar magnetic potential,

$$-\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\phi_{\rm P} = P_n \quad \text{for } \boldsymbol{x} \in \mathcal{S}. \tag{93b}$$

Using (91) in (90) permits the formal expression of $\boldsymbol{\Sigma}_{\rm P}$ entirely in terms of surface values of $\partial P_n/\partial t$ as a double convolution:

$$\boldsymbol{\Sigma}_{\rm P}(\boldsymbol{x}, t) = c^{-1}\int_{\mathcal{V}} d^3x'\, \mathbb{K}_{\mathcal{V}}(\boldsymbol{x}, \boldsymbol{x}') \cdot\boldsymbol{\nabla}'\oint_{\mathcal{S}} dS''\, \mathcal{G}_{\rm N}(\boldsymbol{x}', \boldsymbol{x}'')\frac{\partial P_n(\boldsymbol{x}'', t)}{\partial t}. \tag{94}$$

Appendix B Vector Identities

For the convenience of the reader, we include some vector identities and surface operators used throughout this paper. Van Bladel (2007) contains a fairly comprehensive inventory of vector relations.

B.1. Triple Products

$$\boldsymbol{f}\times(\boldsymbol{g}\times\boldsymbol{h}) = (\boldsymbol{f}\cdot\boldsymbol{h})\boldsymbol{g} - (\boldsymbol{f}\cdot\boldsymbol{g})\boldsymbol{h}, \tag{95}$$

$$\boldsymbol{f}\cdot(\boldsymbol{g}\times\boldsymbol{h}) = \boldsymbol{g}\cdot(\boldsymbol{h}\times\boldsymbol{f}) = \boldsymbol{h}\cdot(\boldsymbol{f}\times\boldsymbol{g}). \tag{96}$$

B.2. Integration by Parts

Below, ξ, $\boldsymbol{f}$, and $\boldsymbol{g}$ are suitably continuously differentiable functions:

$$\boldsymbol{\nabla}(\boldsymbol{f}\cdot\boldsymbol{g}) = \boldsymbol{f}\times(\boldsymbol{\nabla}\times\boldsymbol{g}) + \boldsymbol{g}\times(\boldsymbol{\nabla}\times\boldsymbol{f}) + (\boldsymbol{g}\cdot\boldsymbol{\nabla})\boldsymbol{f} + (\boldsymbol{f}\cdot\boldsymbol{\nabla})\boldsymbol{g}, \tag{97}$$

$$\boldsymbol{\nabla}\cdot(\xi\boldsymbol{f}) = \xi\boldsymbol{\nabla}\cdot\boldsymbol{f} + \boldsymbol{f}\cdot\boldsymbol{\nabla}\xi, \tag{98}$$

$$\boldsymbol{\nabla}\times(\xi\boldsymbol{f}) = \xi\boldsymbol{\nabla}\times\boldsymbol{f} + \boldsymbol{\nabla}\xi\times\boldsymbol{f}, \tag{99}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{f}\times\boldsymbol{g}) = \boldsymbol{g}\cdot(\boldsymbol{\nabla}\times\boldsymbol{f}) - \boldsymbol{f}\cdot(\boldsymbol{\nabla}\times\boldsymbol{g}). \tag{100}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{f}\times\boldsymbol{g}) = \boldsymbol{f}\,\boldsymbol{\nabla}\cdot\boldsymbol{g} - \boldsymbol{g}\,\boldsymbol{\nabla}\cdot\boldsymbol{f} + (\boldsymbol{g}\cdot\boldsymbol{\nabla})\boldsymbol{f} - (\boldsymbol{f}\cdot\boldsymbol{\nabla})\boldsymbol{g}. \tag{101}$$

B.3. Laplacian

The vector Laplacian is

$$\nabla^2\boldsymbol{f} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{f}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{f}). \tag{102}$$

The curl of this equation leads to

$$\boldsymbol{\nabla}\times(\nabla^2\boldsymbol{f}) = -\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{f})]. \tag{103}$$

Appendix C Dupin Surface Coordinates, Surface Vectors, and Operators

Key to performing vector calculus on a surface $\mathcal{S}$ is Dupin surface coordinates. A brief overview is provided below for convenience, and the reader is referred to seminal texts by Weatherburn (1955), Tai (1992), and Van Bladel (2007). Below, the expressions from Weatherburn (1955) and Van Bladel (2007) are used, but Tai (1992) is an excellent reference for connecting the different definitions and occasionally ambiguous notations employed for surface operators. For clarity, the surface operators used in this paper will be explicitly defined and connected with their familiar volumetric counterparts. As Tai & Fang (1991) note, several familiar coordinate systems have natural surfaces corresponding to the Dupin system: Cartesian (x, y, z) with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, or $\hat{\boldsymbol{z}}$; spherical (r, ϑ, φ) with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{r}}$; and cylindrical (ρ, θ, z) with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}}$ or $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$. In contrast, the toroidal system does not have a natural Dupin surface. Chapter 2 in Tai (1992) provides an example of deriving the Dupin surface for a conical section in cylindrical coordinates.

C.1. Dupin Surface Coordinates

A diagram of the Dupin surface coordinate system is shown in Figure 2. A point $\boldsymbol{x}$ on a regular, but open or closed, surface may be labeled with the coordinates u_1 and u_2 with unit vectors $(\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \hat{\boldsymbol{n}})$, where $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$ are tangent to the surface and $\hat{\boldsymbol{n}} = \hat{\boldsymbol{u}}_1\times\hat{\boldsymbol{u}}_2$ is normal to the surface. The coordinates u_1 and u_2 can be chosen concomitant with the principal directions, which are orthogonal $\hat{\boldsymbol{u}}_1\cdot\hat{\boldsymbol{u}}_2 = 0$, and the coordinate n denotes the normal distance measured linearly from the surface. Along the principal directions $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$, the normals $\hat{\boldsymbol{n}}$ at contiguous

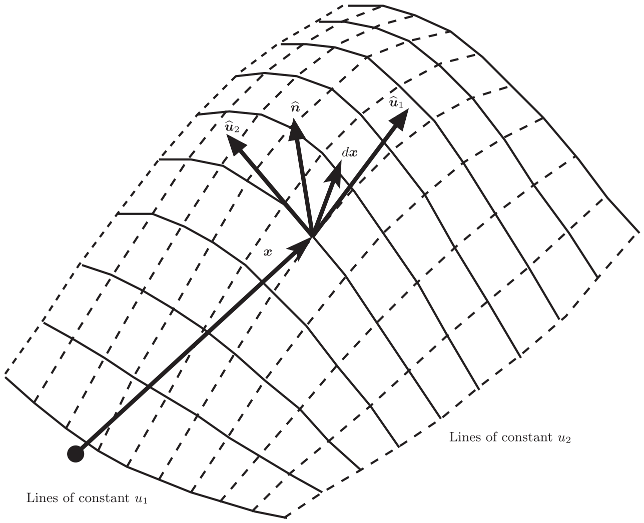

Figure 2. Dupin surface coordinate system. Here, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{u}}_1 \times \hat{\boldsymbol{u}}_2$, and the total differential of the position vector $\boldsymbol{x}$ from a point on the surface to a neighboring point in space $\boldsymbol{x} + d\boldsymbol{x}$ is $d\boldsymbol{x} = h_1\, du_1\, \hat{\boldsymbol{u}}_1 + h_2\, du_2\, \hat{\boldsymbol{u}}_2 + dn\, \hat{\boldsymbol{n}}$. After Tai (1992).

points intersect the normal at point $\boldsymbol{x}$. The distance from point $\boldsymbol{x}$ to this point of intersection is a principal radius of curvature at $\boldsymbol{x}$ correspondingly denoted R_1 and R_2 for $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$, respectively. The total differential of the position vector $\boldsymbol{x}$ from a point on the surface to a neighboring point in space $\boldsymbol{x} + d\boldsymbol{x}$ is $d\boldsymbol{x} = h_1\, du_1\, \hat{\boldsymbol{u}}_1 + h_2\, du_2\, \hat{\boldsymbol{u}}_2 + dn\, \hat{\boldsymbol{n}}$, where

$$\frac{\partial \boldsymbol{x}}{\partial u_1} = h_1\, \hat{\boldsymbol{u}}_1, \quad \frac{\partial \boldsymbol{x}}{\partial u_2} = h_2\, \hat{\boldsymbol{u}}_2, \quad \frac{\partial \boldsymbol{x}}{\partial n} = h_n\, \hat{\boldsymbol{n}}, \tag{104a}$$

with

$$h_n \equiv 1, \tag{104b}$$

where h_1, h_2, and h_n are the scale factors that form the metric tensor $h_i = \sqrt{g_{ii}}$, and $i = 1, 2, n$ with $g_{ij} = 0$ for $i \neq j$. The continuity $\mathcal{C}^k$ of the surface implies that g_{ii} is $\mathcal{C}^{k-1}$ (see p. 296 in Morrey 1966). From 104(a), a $\mathcal{C}^1$ surface is sufficient to define a proper normal $\hat{\boldsymbol{n}}$ and the tangent vectors $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$. The scale factors are not completely independent as (see p. 26 in Morse 1953)[10]

$$\begin{aligned} &\frac{\partial \hat{\boldsymbol{u}}_1}{\partial u_1} = -\frac{\hat{\boldsymbol{u}}_2}{h_2}\frac{\partial h_1}{\partial u_2} - \frac{\hat{\boldsymbol{n}}}{h_n}\frac{\partial h_1}{\partial n}, \\ &\frac{\partial \hat{\boldsymbol{u}}_1}{\partial u_2} = \frac{\hat{\boldsymbol{u}}_2}{h_1}\frac{\partial h_2}{\partial u_1}, \quad \frac{\partial \hat{\boldsymbol{u}}_1}{\partial n} = \frac{\hat{\boldsymbol{n}}}{h_1}\frac{\partial h_n}{\partial u_1} = 0, \end{aligned} \tag{105a}$$

$$\begin{aligned} &\frac{\partial \hat{\boldsymbol{u}}_2}{\partial u_2} = -\frac{\hat{\boldsymbol{u}}_1}{h_1}\frac{\partial h_2}{\partial u_1} - \frac{\hat{\boldsymbol{n}}}{h_n}\frac{\partial h_2}{\partial n}, \\ &\frac{\partial \hat{\boldsymbol{u}}_2}{\partial u_1} = \frac{\hat{\boldsymbol{u}}_1}{h_2}\frac{\partial h_1}{\partial u_2}, \quad \frac{\partial \hat{\boldsymbol{u}}_2}{\partial n} = \frac{\hat{\boldsymbol{n}}}{h_2}\frac{\partial h_n}{\partial u_2} = 0, \end{aligned} \tag{105b}$$

$$\begin{aligned} &\frac{\partial \hat{\boldsymbol{n}}}{\partial n} = -\frac{\hat{\boldsymbol{u}}_1}{h_1}\frac{\partial h_n}{\partial u_1} - \frac{\hat{\boldsymbol{u}}_2}{h_2}\frac{\partial h_n}{\partial u_2} = 0, \\ &\frac{\partial \hat{\boldsymbol{n}}}{\partial u_1} = \frac{\hat{\boldsymbol{u}}_1}{h_n}\frac{\partial h_1}{\partial n}, \quad \frac{\partial \hat{\boldsymbol{n}}}{\partial u_2} = \frac{\hat{\boldsymbol{u}}_2}{h_n}\frac{\partial h_2}{\partial n}. \end{aligned} \tag{105c}$$

[10] For pedagogical purposes, the scale factor h_n is carried through, understanding that 104(b) holds.

A $\mathcal{C}^2$ surface $\mathcal{S}$ is sufficient for 105(a)–(c) to be well defined, which in turn are sufficient for a well-defined surface divergence, surface curl, and Laplace–Beltrami operator (surface Laplacian) on a scalar described below. However, the vector fields in physical problems are not only defined on the surface but also in some neighborhood of it. For example, the potential field constraint (46) requires that $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E}_{\mathrm{P}} = 0$ be well defined on the surface $\mathcal{S}$. A surface with $\mathcal{C}^3$ continuity almost everywhere is sufficient to satisfy this physical requirement.[11] However, a $\mathcal{C}^3$ surface implies additional relationships for the h_i's. Differentiating the first and last equations in 105(a) by $\partial/\partial n$ and $\partial/\partial u_1$, respectively, produces

$$\frac{\partial^2 \hat{\boldsymbol{u}}_1}{\partial n \partial u_1} = -\hat{\boldsymbol{u}}_2\left(\frac{\partial h_1}{\partial u_2}\frac{\partial}{\partial n}\frac{1}{h_2} + \frac{1}{h_2}\frac{\partial^2 h_1}{\partial u_2\, \partial n}\right) - \frac{\hat{\boldsymbol{n}}}{h_n}\frac{\partial^2 h_1}{\partial n^2}, \tag{106a}$$

$$\frac{\partial^2 \hat{\boldsymbol{u}}_1}{\partial u_1 \partial n} = \frac{\partial h_n}{\partial u_1}\frac{\partial}{\partial u_1}\left(\frac{\hat{\boldsymbol{n}}}{h_1}\right) + \frac{\hat{\boldsymbol{n}}}{h_1}\frac{\partial^2 h_n}{\partial u_1^2} = 0. \tag{106b}$$

Equating these leads to

$$\frac{\partial^2 h_1}{\partial u_2 \partial n} = \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\frac{\partial h_2}{\partial n} \quad \text{and} \quad \frac{\partial^2 h_1}{\partial n^2} = 0. \tag{106c}$$

Cross-differentiating the first two equations in 105(a) and substituting the appropriate terms from 105(b) and (c) produce

$$\begin{aligned} \frac{\partial^2 \hat{\boldsymbol{u}}_1}{\partial u_2 \partial u_1} = &\frac{\hat{\boldsymbol{u}}_1}{h_2}\frac{\partial h_1}{\partial u_2}\frac{1}{h_1}\frac{\partial h_2}{\partial u_1} - \hat{\boldsymbol{u}}_2\left(\frac{1}{h_2}\frac{\partial^2 h_1}{\partial u_2^2}\right. \\ &\left. - \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\frac{1}{h_2}\frac{\partial h_2}{\partial u_2} + \frac{1}{h_n}\frac{\partial h_1}{\partial n}\frac{1}{h_n}\frac{\partial h_2}{\partial n}\right) \\ &+ \frac{\hat{\boldsymbol{n}}}{h_n}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\frac{\partial h_2}{\partial n} - \frac{\partial^2 h_1}{\partial u_2 \partial n}\right), \end{aligned} \tag{107a}$$

$$\begin{aligned} \frac{\partial^2 \hat{\boldsymbol{u}}_1}{\partial u_1 \partial u_2} = &\frac{\hat{\boldsymbol{u}}_1}{h_1}\frac{\partial h_2}{\partial u_1}\frac{1}{h_2}\frac{\partial h_1}{\partial u_2} \\ &+ \hat{\boldsymbol{u}}_2\left(\frac{1}{h_1}\frac{\partial^2 h_2}{\partial u_1^2} - \frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\frac{1}{h_1}\frac{\partial h_1}{\partial u_1}\right). \end{aligned} \tag{107b}$$

Equating the $\hat{\boldsymbol{u}}_2$ component leads to a new relationship

$$\begin{aligned} &\frac{1}{h_1}\frac{\partial^2 h_2}{\partial u_1^2} + \frac{1}{h_2}\frac{\partial^2 h_1}{\partial u_2^2} - \frac{1}{h_1}\frac{\partial h_2}{\partial u_1}\frac{1}{h_1}\frac{\partial h_1}{\partial u_1} \\ &- \frac{1}{h_2}\frac{\partial h_1}{\partial u_2}\frac{1}{h_2}\frac{\partial h_2}{\partial u_2} + \frac{1}{h_n}\frac{\partial h_1}{\partial n}\frac{1}{h_n}\frac{\partial h_2}{\partial n} = 0. \end{aligned} \tag{107c}$$

Equations 105(a)–(c), 106(c), and 107(c) ensure that (103) is satisfied when $\boldsymbol{f} = \psi\, \hat{\boldsymbol{n}}$.

C.2. Surface Vectors and Operators

A surface $\mathcal{S}$ of at least $\mathcal{C}^1$ is sufficient to define the differential operators and relationships below. A surface vector may be decomposed into a normal vector and a tangent vector

[11] Schulz & Schulz (2016) note on p. 44 that a $\mathcal{C}^3$ surface is usually sufficient in physics, but mathematicians assume $\mathcal{C}^\infty$ so that they can avoid the "boring" details of the surface.

denoted with subscripts "n" and "$\mathcal{S}$," respectively,

$$\boldsymbol{f} = \boldsymbol{f}_n + \boldsymbol{f}_{\mathcal{S}} = \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{f}) - \hat{\boldsymbol{n}}\times(\hat{\boldsymbol{n}}\times\boldsymbol{f}). \tag{108}$$

The familiar gradient operator written in Dupin surface coordinates is

$$\boldsymbol{\nabla} = \frac{\hat{\boldsymbol{u}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\boldsymbol{u}}_2}{h_2}\frac{\partial}{\partial u_2} + \hat{\boldsymbol{n}}\frac{\partial}{\partial n}. \tag{109a}$$

The surface gradient operator is defined as the projection of the gradient operator on the surface $\mathcal{S}$

$$\boldsymbol{\nabla}_{\mathcal{S}} = \frac{\hat{\boldsymbol{u}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\boldsymbol{u}}_2}{h_2}\frac{\partial}{\partial u_2}. \tag{109b}$$

Symbolically, the relationship between the surface gradient operator and the familiar gradient operator is determined from (108) as

$$\boldsymbol{\nabla}_{\mathcal{S}}\xi \equiv (\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}\xi)\times\hat{\boldsymbol{n}} = \boldsymbol{\nabla}\xi - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\xi. \tag{109c}$$

A surface $\mathcal{S}$ of at least $\mathcal{C}^2$ is sufficient to define the differential operators and relationships below. The "first curvature" (see p. 78 in Weatherburn 1955) is the sum of the principal curvatures

$$\mathcal{J} \equiv \frac{1}{R_1} + \frac{1}{R_2} = -\frac{\partial \log(h_1 h_2)}{\partial n}. \tag{110}$$

The familiar divergence operator in Dupin surface coordinates is

$$\boldsymbol{\nabla}\cdot\boldsymbol{f} = \frac{1}{h_1 h_2}\left[\frac{\partial(h_2 f_1)}{\partial u_1} + \frac{\partial(h_1 f_2)}{\partial u_2}\right] - \mathcal{J}f_n + \frac{\partial f_n}{\partial n}. \tag{111a}$$

The surface divergence is defined

$$\boldsymbol{\nabla}_{\mathcal{S}}\cdot\boldsymbol{f} = \frac{1}{h_1 h_2}\left[\frac{\partial(h_2 f_1)}{\partial u_1} + \frac{\partial(h_1 f_2)}{\partial u_2}\right] - \mathcal{J}f_n. \tag{111b}$$

Symbolically, the relationship between the surface divergence operator and the familiar divergence operator is then

$$\boldsymbol{\nabla}_{\mathcal{S}}\cdot\boldsymbol{f} \equiv \boldsymbol{\nabla}\cdot\boldsymbol{f} - \hat{\boldsymbol{n}}\cdot\frac{\partial\boldsymbol{f}}{\partial n} = \boldsymbol{\nabla}_{\mathcal{S}}\cdot\boldsymbol{f}_{\mathcal{S}} - \mathcal{J}f_n. \tag{111c}$$

Note that when the vector $\boldsymbol{f}$ is tangent to the surface $\mathcal{S}$ with $f_n = 0$, $\partial f_n/\partial n = 0$, then

$$\boldsymbol{\nabla}\cdot\boldsymbol{f}_{\mathcal{S}} = \boldsymbol{\nabla}_{\mathcal{S}}\cdot\boldsymbol{f}_{\mathcal{S}}. \tag{111d}$$

The surface divergence satisfies a relationship formally equivalent to (98)

$$\boldsymbol{\nabla}_{\mathcal{S}}\cdot(\xi\boldsymbol{f}) = \boldsymbol{f}\cdot\boldsymbol{\nabla}_{\mathcal{S}}\xi + \xi\boldsymbol{\nabla}_{\mathcal{S}}\cdot\boldsymbol{f}. \tag{112}$$

The familiar curl operator in Dupin surface coordinates takes the form

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{f} = \frac{1}{h_1 h_2}\Bigg\{ & h_1\left[\frac{\partial f_n}{\partial u_2} - \frac{\partial(h_2 f_2)}{\partial n}\right]\hat{\boldsymbol{u}}_1 \\ & + h_2\left[\frac{\partial(h_1 f_1)}{\partial n} - \frac{\partial f_n}{\partial u_1}\right]\hat{\boldsymbol{u}}_2 \\ & + \left[\frac{\partial(h_2 f_2)}{\partial u_1} - \frac{\partial(h_1 f_1)}{\partial u_2}\right]\hat{\boldsymbol{n}}\Bigg\}.\end{aligned} \tag{113a}$$

The surface curl is defined

$$\begin{aligned}\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{f} = \frac{1}{h_1 h_2}\Bigg\{ & \left[h_1\frac{\partial f_n}{\partial u_2} - h_1 f_2\frac{\partial h_2}{\partial n}\right]\hat{\boldsymbol{u}}_1 \\ & - \left[h_2\frac{\partial f_n}{\partial u_1} - h_2 f_1\frac{\partial h_1}{\partial n}\right]\hat{\boldsymbol{u}}_2 \\ & + \left[\frac{\partial(h_2 f_2)}{\partial u_1} - \frac{\partial(h_1 f_1)}{\partial u_2}\right]\hat{\boldsymbol{n}}\Bigg\}.\end{aligned} \tag{113b}$$

The normal component of the curl and surface curl are equivalent,

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{f} = \hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{f}, \tag{113c}$$

but the relationship between the tangential components is not immediately apparent from 113(a) and 113(b). However, symbolically, the familiar curl and the surface curl are related by

$$\boldsymbol{\nabla}\times\boldsymbol{f} = \boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{f} + \hat{\boldsymbol{n}}\times\frac{\partial\boldsymbol{f}}{\partial n}. \tag{113d}$$

The normal $\hat{\boldsymbol{n}}$ satisfies

$$\boldsymbol{\nabla}_{\mathcal{S}}\cdot\hat{\boldsymbol{n}} = -\mathcal{J}, \tag{114a}$$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{n}} = -\mathcal{J}, \tag{114b}$$

$$\boldsymbol{\nabla}_{\mathcal{S}}\times\hat{\boldsymbol{n}} = 0, \tag{114c}$$

$$\boldsymbol{\nabla}\times\hat{\boldsymbol{n}} = 0, \tag{114d}$$

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\xi \equiv \frac{\partial\xi}{\partial n}, \tag{114e}$$

$$\hat{\boldsymbol{n}}\cdot\frac{\partial\hat{\boldsymbol{n}}}{\partial n} = 0. \tag{114f}$$

The surface divergence satisfies a relationship formally equivalent to (100),

$$\boldsymbol{\nabla}_{\mathcal{S}}\cdot(\boldsymbol{f}\times\boldsymbol{g}) = \boldsymbol{g}\cdot\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{f} - \boldsymbol{f}\cdot\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{g}, \tag{115a}$$

which for $\boldsymbol{f} = \hat{\boldsymbol{n}}$ with 114(c) becomes

$$\boldsymbol{\nabla}_{\mathcal{S}}\cdot(\hat{\boldsymbol{n}}\times\boldsymbol{g}) = -\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{g}. \tag{115b}$$

The surface curl satisfies a relationship formally equivalent to (99),

$$\boldsymbol{\nabla}_{\mathcal{S}}\times(\xi\boldsymbol{f}) = \xi\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{f} + \boldsymbol{\nabla}_{\mathcal{S}}\xi\times\boldsymbol{f}. \tag{116}$$

The tangent derivative operator may be written as (see p. 712 in Wu et al. 2007)

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla} = \hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_{\mathcal{S}}. \tag{117}$$

For any tangent vectors $\boldsymbol{f}_{\mathcal{S}}$ and $\boldsymbol{g}_{\mathcal{S}}$ satisfying $\hat{\boldsymbol{n}}\cdot\boldsymbol{f}_{\mathcal{S}} = 0$ and $\boldsymbol{g}_{\mathcal{S}} = \hat{\boldsymbol{n}}\times\boldsymbol{f}_{\mathcal{S}}$ (see p. 712 in Wu et al. 2007),

$$\begin{aligned}\boldsymbol{\nabla}_{\mathcal{S}}\cdot\boldsymbol{f}_{\mathcal{S}} &= (\hat{\boldsymbol{n}}\times\boldsymbol{\nabla})\cdot\boldsymbol{g}_{\mathcal{S}} = \hat{\boldsymbol{n}}\cdot(\boldsymbol{\nabla}\times\boldsymbol{g}_{\mathcal{S}}) \\ &= \hat{\boldsymbol{n}}\cdot(\boldsymbol{\nabla}_{\mathcal{S}}\times\boldsymbol{g}_{\mathcal{S}}).\end{aligned} \tag{118}$$

The familiar Laplacian operator in Dupin surface coordinates takes the form

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\xi = \frac{1}{h_1 h_2}\sum_{j=1,2}\frac{\partial}{\partial u_j}\left(\frac{h_1 h_2}{h_j^2}\frac{\partial\xi}{\partial u_j}\right) \\ - \mathcal{J}\frac{\partial\xi}{\partial n} + \frac{\partial^2\xi}{\partial n^2}.\end{aligned} \tag{119a}$$

The surface Laplacian or Laplace–Beltrami operator may be expressed as

$$\nabla_{\mathcal{S}} \cdot \nabla_{\mathcal{S}} \xi = \nabla_{\mathcal{S}}^2 \xi = \frac{1}{h_1 h_2} \sum_{j=1,2} \frac{\partial}{\partial u_j} \left(\frac{h_1 h_2}{h_j^2} \frac{\partial \xi}{\partial u_j} \right). \tag{119b}$$

Symbolically, these are related by

$$\nabla_{\mathcal{S}}^2 \xi = \nabla^2 \xi + \mathcal{J} \frac{\partial \xi}{\partial n} - \frac{\partial^2 \xi}{\partial n^2}. \tag{119c}$$

Note that (118) with $\boldsymbol{f}_{\mathcal{S}} = \nabla_{\mathcal{S}} \xi$ and $\boldsymbol{g}_{\mathcal{S}} = \hat{\boldsymbol{n}} \times \boldsymbol{f}_{\mathcal{S}}$ implies that

$$\hat{\boldsymbol{n}} \cdot \nabla \times (\hat{\boldsymbol{n}} \times \nabla_{\mathcal{S}} \xi) = \hat{\boldsymbol{n}} \cdot \nabla_{\mathcal{S}} \times (\hat{\boldsymbol{n}} \times \nabla_{\mathcal{S}} \xi) = \nabla_{\mathcal{S}}^2 \xi. \tag{120}$$

Unlike the volumetric operator which satisfies $\nabla \times \nabla \xi = 0$, only the normal component vanishes identically for the surface operators (see p. 233 in Weatherburn 1955)

$$\hat{\boldsymbol{n}} \cdot (\nabla_{\mathcal{S}} \times \nabla_{\mathcal{S}} \xi) = 0, \tag{121a}$$

but with 113(c), there is also

$$\hat{\boldsymbol{n}} \cdot (\nabla \times \nabla_{\mathcal{S}} \xi) = 0. \tag{121b}$$

Using this with 115(b) implies

$$\nabla_{\mathcal{S}} \cdot (\hat{\boldsymbol{n}} \times \nabla_{\mathcal{S}} \xi) = 0. \tag{121c}$$

More specifically, $\nabla_{\mathcal{S}} \times \nabla_{\mathcal{S}} \xi = 0$ only for surfaces with zero Gaussian curvature (see p. 122 in Ludu 2012).

Appendix D Integral Relationships

Arapura (2016) provides a introductory treatment of the Gauss–Ostrogradsky and Stokes' theorems. Below, $\boldsymbol{f}$ is continuously differentiable ($\mathcal{C}^1$), $\mathcal{S}$ is a closed $\mathcal{C}^1$ surface bounding $\mathcal{V}$, and $\hat{\boldsymbol{n}}$ is the inwardly directed normal. The Gauss–Ostrogradsky theorems are

$$\int_{\mathcal{V}} d^3x \, \nabla \cdot \boldsymbol{f} = - \oint_{\mathcal{S}} dS \, \hat{\boldsymbol{n}} \cdot \boldsymbol{f}, \tag{122a}$$

$$\int_{\mathcal{V}} d^3x \, \nabla \times \boldsymbol{f} = - \oint_{\mathcal{S}} dS \, \hat{\boldsymbol{n}} \times \boldsymbol{f}. \tag{122b}$$

An inwardly directed normal to the coronal volume corresponds to radially outward vector in the photosphere, and so our convention is to measure positive flux into the coronal volume.

Below, ℓ is the $\mathcal{C}^1$ contour bounding the $\mathcal{C}^1$ open surface $\mathcal{S}$ with the line element $\boldsymbol{dl}$. The normal $\hat{\boldsymbol{n}}$ is oriented according to the right-hand rule in relation to the contour ℓ with $\boldsymbol{dl} \cdot \hat{\boldsymbol{n}} = 0$ locally because $\boldsymbol{dl}$ lies in the surface $\mathcal{S}$. Stokes' theorems for an open surface take the form

$$\int_{\mathcal{S}} dS \, \hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{f}) = \oint_{\ell} \boldsymbol{dl} \cdot \boldsymbol{f}, \tag{123a}$$

$$\int_{\mathcal{S}} dS (\hat{\boldsymbol{n}} \times \nabla \xi) = \oint_{\ell} \boldsymbol{dl} \, \xi, \tag{123b}$$

and for a closed surface,

$$\oint_{\mathcal{S}} dS \, \hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{f}) = 0, \tag{124a}$$

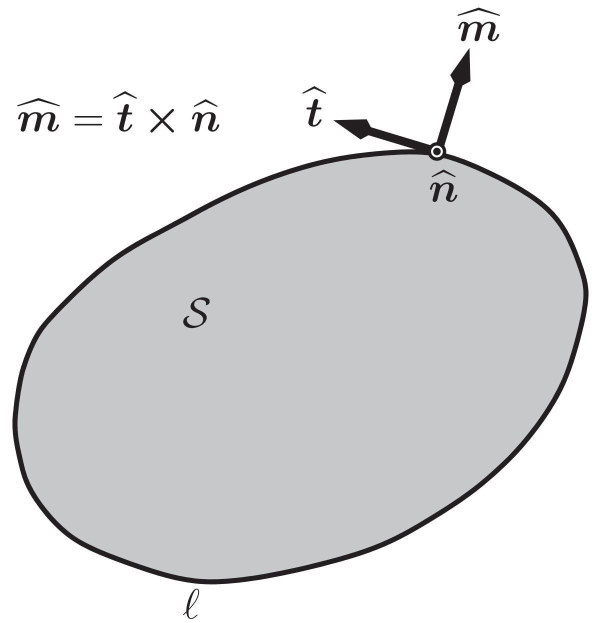

Figure 3. Unit vectors on an open surface $\mathcal{S}$. After Van Bladel (2007).

$$\oint_{\mathcal{S}} dS (\hat{\boldsymbol{n}} \times \nabla \xi) = 0. \tag{124b}$$

Below, the unit vector $\widehat{\boldsymbol{m}}$ is in the tangent plane and perpendicular to contour ℓ; it points outward from the $\mathcal{C}^1$ surface $\mathcal{S}$ enclosed by contour ℓ. The unit vector $\hat{\boldsymbol{t}}$ is tangent to contour ℓ, $\hat{\boldsymbol{n}}$ is normal to the surface at the edge, and $\hat{\boldsymbol{t}} = \hat{\boldsymbol{n}} \times \widehat{\boldsymbol{m}}$ as shown in Figure 3. The surface divergence theorem is

$$\int_{\mathcal{S}} dS \, \nabla_{\mathcal{S}} \cdot \boldsymbol{f} + \int_{\mathcal{S}} dS \mathcal{J} (\boldsymbol{f} \cdot \hat{\boldsymbol{n}}) = \int_{\ell} dl \, \boldsymbol{f} \cdot \widehat{\boldsymbol{m}}. \tag{125a}$$

The first integral vanishes for a closed surface if $\boldsymbol{f}$ is everywhere tangent to the surface $\mathcal{S}$,

$$\oint_{\mathcal{S}} dS \, \nabla_{\mathcal{S}} \cdot \boldsymbol{f}_{\mathcal{S}} = 0. \tag{125b}$$

The integrals 122(a)–125(b) have been generalized to rough surfaces and even fractal surfaces that have no proper unit normal vector $\hat{\boldsymbol{n}}$ within the framework of differential forms (Harrison 1999). Mathematicians have noticed that one side of these integrals can be used to define the other side under more general conditions. Indeed, Harrison (1993) notes that Whitney (1957) used the left-hand side of 123(a) to define the right-hand side for a rough boundary. Generally, the smoother $\boldsymbol{f}$ is, the rougher $\mathcal{S}$ or ℓ can be. Furthermore, these integrals 122(a)–125(b) can be extended to the piecewise $\mathcal{C}^1$ surfaces $\mathcal{S}$ and piecewise $\mathcal{C}^1$ contours ℓ by considering $\mathcal{C}^1$ surface patches $\mathcal{S}_i$ that comprise $\mathcal{S}$ and dividing the corresponding bounding contours ℓ_i into $\mathcal{C}^1$ segments.

Appendix E Vector Decompositions

Fundamental to the new formula (51) for helicity transport across a surface $\mathcal{S}$ is determining $\boldsymbol{\Sigma}_{\rm P}$ uniquely in the enclosed volume $\mathcal{V}$ and on the enclosing surface $\mathcal{S}$. To accomplish this, two major theorems of vector calculus are employed: the Helmholtz decomposition in a finite volume $\mathcal{V}$ and Helmholtz–

Hodge decomposition on a surface $\mathcal{S}$. Zhou (2006) provides a recent rigorous treatment of the former and Van Bladel (1993) discusses the latter. Van Bladel (1958) is a comprehensive reference that describes the splitting of the Helmholtz decomposition in a volume for various combinations of boundary conditions $\boldsymbol{F}$ (see the table on p. 22 of Van Bladel 1958). Cantarella et al. (2002) provides a comprehensive introduction to the connection between topology and the Helmholtz–Hodge decompositions for divergence-free vectors in multiply connected volumes. These theorems are stated below and the reader is referred to the cited references for the proofs.

E.1. Helmholtz Decomposition in a Volume

Zhou (2006, p. 95) states the Helmholtz decomposition theorem as (see also Gui & Dou 2007, Morse 1953, pp. 52–54):

Theorem 1. *Any finite, integrable, and continuously differentiable vector function $\boldsymbol{F}(\boldsymbol{x})$ given in a simply connected volume $\mathcal{V}$ enclosed by $\mathcal{S}$ can be completely and uniquely decomposed into a sum of an irrotational part and a solenoidal part. The two parts are independent. Mathematically, it is the identity*

$$\boldsymbol{F}(\boldsymbol{x}) = -\overbrace{\boldsymbol{\nabla}\xi}^{\text{Irr}} + \overbrace{\boldsymbol{\nabla}\times\boldsymbol{f}}^{\text{Sol}}; \quad \boldsymbol{\nabla}\cdot\boldsymbol{f} = 0 \quad \in \quad \mathcal{V}, \tag{126a}$$

where

$$\boldsymbol{f}(\boldsymbol{x}) = \int_{\mathcal{V}} d^3x'\, \mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{\nabla}'\times\boldsymbol{F}(\boldsymbol{x}') + \oint_{\mathcal{S}} dS'\, \mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}')\hat{\boldsymbol{n}}(\boldsymbol{x}')\times\boldsymbol{F}(\boldsymbol{x}'), \tag{126b}$$

$$\xi(\boldsymbol{x}) = \int_{\mathcal{V}} d^3x'\, \mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{\nabla}'\cdot\boldsymbol{F}(\boldsymbol{x}') - \oint_{\mathcal{S}} dS'\, \mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}')\hat{\boldsymbol{n}}(\boldsymbol{x}')\cdot\boldsymbol{F}(\boldsymbol{x}'), \tag{126c}$$

where $\mathcal{G}_{\rm F}$ is the free-space Green's function,

$$\mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}') = -\frac{1}{4\,\pi}\,\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{126d}$$

which satisfies

$$\nabla\mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{126e}$$

The exact statement in Zhou (2006) is more general than what is stated above, being applicable to piecewise continuous functions in a multiply connected volume, which is beyond the scope of this paper. Strictly speaking, 126(a)–(d) applies to a single boundary surface, but disconnected bounding surfaces, such as a volume between spherical shells $R_< < r < R_>$, can be incorporated directly with the appropriate auxiliary surfaces. Furthermore, a multiply connected region can be transformed into a simply connected region by $\mathfrak{g}$ cuts and $b_1 = 2\,\mathfrak{g}$ auxiliary surfaces, where b_1 is the first Betti number and $\mathfrak{g}$ is the genus of the bounding surface $\mathscr{S}$, which is equivalent to the number of holes (Yoshida 1998; Boulmezaoud et al. 1999). Each cut generates two new surfaces in the multiply connected volume, denoted $\mathscr{S}_i$, bracketing the cut. The fluxes at each of these surfaces must then be specified, namely $\int_{\mathscr{S}_i} dS\, \hat{\boldsymbol{n}}_i\cdot\boldsymbol{F}$, where $i = 1 \ldots b_1$ and $\hat{\boldsymbol{n}}_i$ is the normal to the surface $\mathscr{S}_i$, which in the convention of this paper points into the volume $\mathcal{V}$. Essentially, the closed harmonic field lines in the original multiply connected volume become open field lines in a new simply connected volume $\mathcal{V}$. By combining the two terms in 126(b) with 122(b) and (99) and $\boldsymbol{\nabla}'\mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}') = -\boldsymbol{\nabla}\mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}')$, it becomes clear that the condition $\boldsymbol{\nabla}\cdot\boldsymbol{f} = 0$ asserted in 126(a) holds:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{\nabla}\times\int_{\mathcal{V}} d^3x'\, \mathcal{G}_{\rm F}(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{F}(\boldsymbol{x}'). \tag{127}$$

Zhou (2006, p. 97) states the general uniqueness theorem as:

Theorem 2. *A vector function $\boldsymbol{F}(\boldsymbol{x})$ in $\mathcal{V}$ bounded by the surface $\mathcal{S}$ can be uniquely determined by its divergence, curl, and boundary values (both normal and tangential components) over the boundary $\mathcal{S}$, i.e., the solution to the system*

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{s}_{\rm curl}, \tag{128a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = s_{\rm div}, \tag{128b}$$

$$\boldsymbol{F}|_{\mathcal{S}} = \boldsymbol{F}_0|_{\mathcal{S}}, \tag{128c}$$

is unique.

Here $\boldsymbol{F}_0 = \boldsymbol{F}|_{\mathcal{S}}$ is the vector boundary condition on $\boldsymbol{F}$, and $\boldsymbol{s}_{\rm curl}$ and $s_{\rm div}$ are the sources of the curl and divergence in $\mathcal{V}$, respectively. In "Corollary 2" of the general uniqueness theorem of a vector function in Zhou (2006, p. 99), the situation for $\boldsymbol{\Sigma}_{\rm P}$ is stated concisely:

Corollary 1. *An intrinsically solenoidal vector function $\boldsymbol{F}(\boldsymbol{x})$ can be uniquely determined by its curl and tangential components over the boundary $\mathcal{S}$ (because the normal components are zero). That is, the solution to the mathematical problem*

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{s}_{\rm curl}, \tag{129a}$$

$$\hat{\boldsymbol{n}}\times\boldsymbol{F}|_{\mathcal{S}} = \hat{\boldsymbol{n}}\times\boldsymbol{F}_0|_{\mathcal{S}}, \tag{129b}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{F} = 0, \tag{129c}$$

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{F}|_{\mathcal{S}} = 0, \tag{129d}$$

is unique.

When 129(c) and (d) hold, this corollary has the following consequences for $\boldsymbol{f}(\boldsymbol{x})$ and $\xi(\boldsymbol{x})$ in 126(a)–(c):

$$\boldsymbol{\nabla}\times\boldsymbol{f} = \boldsymbol{F}, \tag{130a}$$

$$\hat{\boldsymbol{n}}\times(\boldsymbol{\nabla}\times\boldsymbol{f})|_{\mathcal{S}} = \hat{\boldsymbol{n}}\times\boldsymbol{F}_0|_{\mathcal{S}}, \tag{130b}$$

$$\xi = \xi_0 \quad \text{(meaningless constant)}, \tag{130c}$$

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{f}|_{\mathcal{S}} = 0. \tag{130d}$$

This representation can be achieved, more intuitively, albeit less rigorously, by noting that a goal of representing a vector $\boldsymbol{F}(\boldsymbol{x})$ in $\mathcal{V}$ by the Helmholtz decomposition 126(a) is to decompose it into linear independent orthogonal representations, which requires that

$$\int_{\mathcal{V}} d^3x\, \boldsymbol{\nabla}\times\boldsymbol{f}\cdot\boldsymbol{\nabla}\xi = 0. \tag{131}$$

Integrating by parts with (100) or (98) and using 122(a) and (96) produce

$$\int_{\mathcal{V}} d^3x\, \nabla\cdot(\boldsymbol{f}\times\nabla\xi) = -\oint_{\mathcal{S}} dS\, \boldsymbol{f}\cdot(\nabla\xi\times\hat{\boldsymbol{n}}) = -\oint_{\mathcal{S}} dS\, \nabla\xi\cdot(\hat{\boldsymbol{n}}\times\boldsymbol{f}) = 0, \tag{132a}$$

$$\int_{\mathcal{V}} d^3x\, \nabla\cdot(\xi\nabla\times\boldsymbol{f}) = -\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}}\cdot(\xi\nabla\times\boldsymbol{f}) = 0. \tag{132b}$$

Noting 124(a) for the last relationship implies that any of the boundary conditions

$$\xi|_{\mathcal{S}} = \text{constant}, \tag{133a}$$

$$\hat{\boldsymbol{n}}\times\boldsymbol{f}|_{\mathcal{S}} = 0, \tag{133b}$$

$$\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{f})|_{\mathcal{S}} = 0 \tag{133c}$$

are sufficient for orthogonality. Note that 133(b) automatically implies 133(c). All conditions holding simultaneously are sufficient for orthogonality but are not necessary (Maria Denaro 2003). The first boundary condition corresponds to homogeneous Dirichlet conditions $\xi|_{\mathcal{S}} = \xi_0$, and the second and third boundary conditions correspond to homogeneous Neumann conditions $\partial\xi/\partial n|_{\mathcal{S}} = -\hat{\boldsymbol{n}}\cdot\boldsymbol{F}$.

For the intrinsically solenoidal reference electric field $\boldsymbol{F} = \boldsymbol{\Sigma}_{\mathrm{P}}$ discussed in Section 4.3, with $\nabla\cdot\boldsymbol{F} = 0$ and $\hat{\boldsymbol{n}}\cdot\boldsymbol{F}|_{\mathcal{S}} = 0$, either choice leads to the same result for ξ. The function ξ is the solution to

$$\nabla^2\xi = 0 \quad \in \mathcal{V}, \tag{134a}$$

with

$$\xi|_{\mathcal{S}} = \xi_0, \tag{134b}$$

or

$$\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{f})|_{\mathcal{S}} = 0 \text{ leading to } \hat{\boldsymbol{n}}\cdot\nabla\xi|_{\mathcal{S}} = \hat{\boldsymbol{n}}\cdot\boldsymbol{F}|_{\mathcal{S}} = 0. \tag{134c}$$

Consequently, either choice has the solution $\xi = \xi_0 = $ constant and 130(a)–(d). In other words, $\boldsymbol{\Sigma}_{\mathrm{P}}$ is unique, but note that the complete reference electric field $\boldsymbol{E}_{\mathrm{P}} = \boldsymbol{\Sigma}_{\mathrm{P}} + \nabla\Lambda_{\mathrm{P}}$ does admit the gradient of a scalar potential Λ_{P}, which satisfies Dirichlet boundary conditions on $\mathcal{S}$.

E.2. Helmholtz–Hodge Decomposition on a Surface

The development of the Helmholtz–Hodge decomposition on a surface is more modern (Hodge 1959) and has several integral forms (Scharstein 1991; Van Bladel 1993; Backus et al. 1996; Imbert-Gerard & Greengard 2016; Kustepeli 2016). Reusken (2018) provides a comprehensive exposition of the Helmholtz–Hodge decomposition on a $\mathcal{C}^2$ surface, which is sufficient for our analysis

Theorem 3. *Any finite, square integrable, vector function $\boldsymbol{f}(\boldsymbol{x})$ on a $\mathcal{C}^2$ surface $\mathcal{S}$ may be represented by a normal component f_n and a tangent vector $\boldsymbol{f}_{\mathcal{S}}$ using (108). The tangent vector $\boldsymbol{f}_{\mathcal{S}}$ can be further uniquely decomposed into a solenoidal component (with no divergence) and lamellar component (with no normal component of the surface curl)*[12] and a harmonic field $\boldsymbol{\Omega}_{\mathcal{S}}$:

$$\boldsymbol{f} = \overbrace{\tau\,\hat{\boldsymbol{n}}}^{f_n} + \overbrace{\underbrace{\hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\phi}_{\text{solenoidal}} + \underbrace{\nabla_{\mathcal{S}}\psi}_{\text{lamellar}} + \boldsymbol{\Omega}_{\mathcal{S}}}^{f_{\mathcal{S}}}. \tag{135}$$

The scalars τ, ψ, and ϕ may be determined from

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{f} = f_n = \tau, \tag{136a}$$

$$\nabla^2_{\mathcal{S}}\psi = \nabla_{\mathcal{S}}\cdot[(\hat{\boldsymbol{n}}\times\boldsymbol{f}_{\mathcal{S}})\times\hat{\boldsymbol{n}}], \tag{136b}$$

and

$$\nabla^2_{\mathcal{S}}\phi = \hat{\boldsymbol{n}}\cdot(\nabla_{\mathcal{S}}\times\boldsymbol{f}_{\mathcal{S}}). \tag{136c}$$

The harmonic term can be determined by

$$\boldsymbol{\Omega}_{\mathcal{S}} \equiv \boldsymbol{f}_{\mathcal{S}} - \hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\phi - \nabla_{\mathcal{S}}\psi. \tag{137}$$

The second term in (135) is purely solenoidal as $\nabla\cdot(\hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\phi) = 0$. While the third term is irrotational with respect to the normal component of the surface curl $\hat{\boldsymbol{n}}\cdot\nabla_{\mathcal{S}}\times\nabla_{\mathcal{S}}\psi = 0$, it is not necessarily irrotational in three dimensions as discussed at the end of Appendix C. The harmonic term is both surface divergence free $\nabla_{\mathcal{S}}\cdot\boldsymbol{\Omega}_{\mathcal{S}} = 0$ and surface curl free $\hat{\boldsymbol{n}}\cdot(\nabla_{\mathcal{S}}\times\boldsymbol{\Omega}_{\mathcal{S}}) = 0$. For a simply connected $\mathcal{C}^k$ surface with $k \geqslant 2$, the harmonic term must be zero (see Lemma 4.3 in Reusken 2018).

The operator $\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{f})$ can be written in this representation with (99) and (118) as

$$\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{f}) = \hat{\boldsymbol{n}}\cdot[\nabla\times\nabla_{\mathcal{S}}\psi + \nabla\times(\hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\phi) + \tau\,\nabla\times\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\tau], \tag{138}$$

and using (120), 121(b), and 114(d),

$$\hat{\boldsymbol{n}}\cdot(\nabla\times\boldsymbol{f}) = \nabla^2_{\mathcal{S}}\phi. \tag{139}$$

The null space of this operator on $\mathcal{S}$ is

$$\boldsymbol{f}_{\mathcal{S}} = \tau\,\hat{\boldsymbol{n}} + \nabla_{\mathcal{S}}\psi + \boldsymbol{\Omega}_{\mathcal{S}}. \tag{140}$$

Here, τ is just as smooth as $\boldsymbol{f}$, and ψ and ϕ are somewhat smoother. The normal component is orthogonal to the surface component $\boldsymbol{f}_n\cdot\boldsymbol{f}_{\mathcal{S}} = 0$ in a pointwise sense, and the lamellar, solenoidal, and harmonic components are mutually orthogonal in an average sense over a closed surface:

$$\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}}\times\nabla_{\mathcal{S}}\phi\cdot\nabla_{\mathcal{S}}\psi = \oint_{\mathcal{S}} dS\, \nabla_{\mathcal{S}}\cdot[\hat{\boldsymbol{n}}\times(\psi\nabla\phi)] = 0, \tag{141a}$$

[12] In some literature, $\hat{\boldsymbol{n}}\cdot\nabla_{\mathcal{S}}\times\boldsymbol{f}_{\mathcal{S}}$, despite being a scalar, is denoted "the surface curl" of $\boldsymbol{f}_{\mathcal{S}}$ (Scharstein 1991).

where integration by parts (98) has been used and 125(b) has been invoked. Similarly,

$$\oint_{\mathcal{S}} dS\, \hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathcal{S}}\phi \cdot \boldsymbol{\Omega}_{\mathcal{S}} = \oint_{\mathcal{S}} dS\, \boldsymbol{\nabla}_{\mathcal{S}} \cdot (\boldsymbol{\Omega}_{\mathcal{S}} \times \phi\, \hat{\boldsymbol{n}}) - \oint_{\mathcal{S}} dS\, \phi\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathcal{S}} \times \boldsymbol{\Omega}_{\mathcal{S}} = 0, \tag{141b}$$

where integration by parts (100) has been used and 125(b) has been invoked, and

$$\oint_{\mathcal{S}} dS\, \boldsymbol{\nabla}_{\mathcal{S}}\psi \cdot \boldsymbol{\Omega}_{\mathcal{S}} = \oint_{\mathcal{S}} dS\, \boldsymbol{\nabla}_{\mathcal{S}} \cdot (\psi\, \boldsymbol{\Omega}_{\mathcal{S}}) - \oint_{\mathcal{S}} dS\, \psi\, \boldsymbol{\nabla}_{\mathcal{S}} \cdot \boldsymbol{\Omega}_{\mathcal{S}} = 0, \tag{141c}$$

where integration by parts (98) has been used and 125(b) has been invoked.

ORCID iDs

Peter W. Schuck https://orcid.org/0000-0003-1522-4632
Spiro K. Antiochos https://orcid.org/0000-0003-0176-4312

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