

Meshless Local Petrov-Galerkin (MLPG) Method with Orthogonal Polynomials for Euler-Bernoulli Beam Problems

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ABSTRACT

In this paper, the feasibility of orthogonal polynomials in the meshless local Petrov Galerkin method (MLPG) method is studied. The orthogonal polynomials, Chebyshev and Legendre polynomials, are used in this MLPG method as trial functions. The test functions used were power functions with smooth derivatives at their ends. The performance of these methods is studied by applying these methods to Euler-Bernoulli beam problems.

The MLPG-Galerkin and Legendre methods passed all the patch tests for simple beam problems. Next the formulations are tested on complex beam problems such as beams with partial loadings and continuous beam problems. Problems with load discontinuities and additional supports require special attention. Near discontinuities, judicious choice of number of nodes and nodal placements are needed to obtain accurate deflections, slopes, moments and shear forces.

As polynomial functions are used, the large number of nodes can create a transformation matrix that is ill-conditioned, resulting in problems with the inversion of the matrix. The conditioning worsens as the number of nodes are increased beyond 20. Quadruple precision was needed for models to obtain accurate solutions. Even with quadruple precision the accuracy of the method suffers as the number of nodes is increased beyond 20. This appears to be a drawback of the MLPG-Chebyshev and MLPG-Legendre methods.

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INTRODUCTION

Nayroles et al.¹ proposed a diffused element method based on moving least square approximating functions. Since this paper in 1992, a considerable amount of research was invested in developing similar methods.²⁻⁴ These diffused element methods came to be known as element-free, mesh-free, or meshless methods and are increasingly being viewed as an alternative to the finite element method (FEM). As there are no elements in this method, there will be no locking of the meshes in both small and large deformation problems. The real advantages of the meshless methods will be realized when the method is applied to large deformation and progressive failure problems.

A meshless local Petrov-Galerkin (MLPG) formulation was introduced in Ref. 3. In the Galerkin formulations in Refs. 2 and 4, the trial and test functions in the weak form come from the same space, while in the Petrov-Galerkin formulations (Ref. 3) the trial and test functions come from different spaces. In Ref. 5, a Galerkin formulation was presented for beam (C1) problems using generalized moving least squares (GMLS) interpolants. In Ref. 6, a Petrov-Galerkin formulation was proposed with the GMLS interpolants as trial functions and spline functions as test functions. An alternative MLPG method with radial basis (RB) functions as trial functions that requires less computing effort was presented in Ref. 7.

Recent literature showed that orthogonal polynomials are being considered in the Rayleigh-Ritz (R-R) method [8-10]. Smith et al.⁹ considered Chebyshev, Legendre, Hermite, and Laguerre polynomials as R-R displacement functions. Singhvi and Kapania⁸ demonstrated the efficiency of using Chebyshev polynomials over simple polynomials in the R-R method in beam vibration problems. They found the convergence of Chebyshev polynomials to be superior to the simple polynomial functions.

In this paper, the feasibility of orthogonal polynomials as trial functions in the MLPG method is studied. These are referred to as the MLPG-Chebyshev and MLPG-Legendre methods. The performance of these methods is studied by applying these methods to several Euler-Bernoulli beam problems.

ANALYSIS

The governing equation for an Euler-Bernoulli beam is

$$EI \frac{d^4 w}{dx^4} = f \quad \text{in domain } \Omega \quad (0 \leq x \leq L) \quad \text{with boundaries } \Gamma \quad (x=0 \text{ and } x=L) \quad (1)$$

where w is the transverse displacement, L is the length, EI is the flexural rigidity of the beam, and f is the distributed load on the beam. The boundary conditions at $x=0$ and $x=L$ can have several combinations. The essential boundary conditions (EBCs) are of the form

$$\begin{aligned} w &= \tilde{w} & \text{on } \Gamma_w & \text{ and} \\ \frac{dw}{dx} &= \tilde{\theta} & \text{on } \Gamma_\theta, \end{aligned} \quad (2)$$

and the natural boundary conditions (NBCs) are of the form

$$\begin{aligned} V &= \tilde{V} & \text{on } \Gamma_V & \text{ and} \\ M &= \tilde{M} & \text{on } \Gamma_M \end{aligned} \quad (3)$$

where V and M are the shear force and bending moment, respectively, and are related to the deflection, w , as

$$V = -EI \frac{d^3 w}{dx^3} \quad \text{and} \quad M = EI \frac{d^2 w}{dx^2}, \quad (4)$$

and Γ_w , Γ_θ , Γ_V , and Γ_M denote the boundary points where deflection (w), slope (θ), shear (V), and moment (M) are prescribed, respectively. Note that the prescriptions of $\tilde{w}$ and $\tilde{V}$ and $\tilde{\theta}$ and $\tilde{M}$ are mutually disjoint, i.e., when $w = \tilde{w}$ is prescribed, the shear force V becomes the corresponding reaction, and when $\theta = \tilde{\theta}$ is prescribed, the moment M becomes the corresponding reaction.

The classical weighted residual form of the governing differential equation for fourth order problems is obtained by multiplying the residual by a weight function, v , integrating over the whole domain, and setting the integral to zero:

$$0 = \int_{\Omega} \left(EI \frac{d^4 w}{dx^4} - f \right) v \, dx. \quad (5)$$

The weak form of the weighted residual equation is set up by transferring the differentiation from the primary variable, w , to the weight function, v . This is achieved by integrating by parts twice, yielding

$$0 = EI \int_{\Omega} \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2} \, dx - \int_{\Omega} f v \, dx + n_x \left[EI \frac{d^3 w}{dx^3} v \right]_{\Gamma} - n_x \left[EI \frac{d^2 w}{dx^2} \frac{dv}{dx} \right]_{\Gamma} \quad (6)$$

where $n_x [v \cdot EI (d^3 w / dx^3)]_{\Gamma}$ and $n_x [(dv / dx) \cdot EI (d^2 w / dx^2)]_{\Gamma}$ are introduced as boundary terms and n_x is the direction cosine of the unit outward drawn normal to Ω with respect to the x -axis. The n_x thus takes values ± 1 in 1-D problems. In this paper, the essential boundary conditions are enforced by a penalty method (See Ref. 3 and 6). The penalty terms are written as

$$\alpha_w [(w - \tilde{w}) v]_{\Gamma_w} \quad (7a)$$

and

$$\alpha_\theta \left[\left(\frac{dw}{dx} - \tilde{\theta} \right) \frac{dv}{dx} \right]_{\Gamma_\theta} \quad (7b)$$

where α_w and α_θ are the penalty parameters to enforce the deflection and slope boundary conditions, respectively. Thus, including the penalty terms, Eq. (6) is written as

$$\begin{aligned} 0 = & EI \int_{\Omega} \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2} dx - \int_{\Omega} f v dx + \alpha_w [(w - \tilde{w})v]_{\Gamma_w} + \alpha_\theta \left[\left(\frac{dw}{dx} - \tilde{\theta} \right) \frac{dv}{dx} \right]_{\Gamma_\theta} \\ & + n_x \left[EI \frac{d^3 w}{dx^3} v \right]_{\Gamma} - n_x \left[EI \frac{d^2 w}{dx^2} \frac{dv}{dx} \right]_{\Gamma}. \end{aligned} \quad (8)$$

In Eq. (8), called the weak form of the governing differential equation, the chosen approximations for the variable w are called the trial functions, and the weight functions, v , are now called the test functions. In the current implementation, the test functions are chosen independently from the trial functions, and as such this is a Petrov-Galerkin method.

Consider an N -node model of a beam as shown in Figure 1a. The solution to the beam problem for a given loading and set of boundary conditions is sought as w and θ at each of the nodes in the model. To achieve this, first the trial functions are chosen for w and the test functions are chosen for v , and then the weak form in Eq. (8) is applied to each of the nodes in the model to yield a system of algebraic equations in terms of the unknown primary variables, w and θ , at each of the nodes in the model. Figure 1b illustrates the component of the trial function (the shape function) at node j and the component of the test function at node i . Unlike in the finite element method, the trial functions at node j are diffused and extend over a domain $(x_j - R_j)$ to $(x_j + R_j)$. Similarly, the test functions span over the domain $(x_i - R_o)$ to $(x_i + R_o)$. The variables R_o and R_j are user-defined parameters. As the test functions are defined over a local region, $(x_i - R_o)$ to $(x_i + R_o)$, i.e., as they are chosen to be zero for $x < (x_i - R_o)$ and $x > (x_i + R_o)$, the integrations over Ω in Eq. (8) reduce to integrations over a local sub-domain Ω_s , defined by $(x_i - R_o)$ to $(x_i + R_o)$, with boundary Γ_s (see Figure 1b), and the weak form reduces to

$$\begin{aligned} 0 = & EI \int_{\Omega_s} \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2} dx - \int_{\Omega_s} f v dx + \alpha_w [(w - \tilde{w})v]_{\Gamma_{sw}} + \alpha_\theta \left[\left(\frac{dw}{dx} - \tilde{\theta} \right) \frac{dv}{dx} \right]_{\Gamma_{s\theta}} \\ & + n_x \left[EI \frac{d^3 w}{dx^3} v \right]_{\Gamma_s} - n_x \left[EI \frac{d^2 w}{dx^2} \frac{dv}{dx} \right]_{\Gamma_s} \end{aligned} \quad (9)$$

where Γ_{sw} and $\Gamma_{s\theta}$ are the boundaries where w and θ are prescribed on the local boundary

$(\Gamma_s \cap \Gamma_w$ and $\Gamma_s \cap \Gamma_\theta)$. The penalty terms are considered only if the local sub-domain, Ω_s , intersects Γ_w or Γ_θ . In general, when a local boundary, Γ_s , intersects a global boundary, four boundary condition possibilities exist. These possibilities are $\Gamma_s \cap \Gamma_w, \Gamma_s \cap \Gamma_\theta, \Gamma_s \cap \Gamma_v$, and $\Gamma_s \cap \Gamma_M$ and are denoted $\Gamma_{sw}, \Gamma_{s\theta}, \Gamma_{sv}$, and Γ_{sM} , respectively. Additionally, when Γ_s coincides with an interior point, that point is denoted Γ_{sI} . Using these boundary condition possibilities, Eq. 9 is rewritten

$$\begin{aligned}
0 = & EI \int_{\Omega_s} \frac{d^2 w}{dx^2} \frac{d^2 v}{dx^2} dx - \int_{\Omega_s} f v dx + \alpha_w [(w - \tilde{w})v]_{\Gamma_{sw}} + \alpha_\theta \left[\left(\frac{dw}{dx} - \tilde{\theta} \right) \frac{dv}{dx} \right]_{\Gamma_{s\theta}} \\
& + n_x \left[EI \frac{d^3 w}{dx^3} v \right]_{\Gamma_{sI}} - n_x \left[EI \frac{d^2 w}{dx^2} \frac{dv}{dx} \right]_{\Gamma_{sI}} \\
& - n_x [\tilde{V} v]_{\Gamma_{sv}} - n_x \left[\tilde{M} \frac{dv}{dx} \right]_{\Gamma_{sM}} + n_x \left[EI \frac{d^3 w}{dx^3} v \right]_{\Gamma_{sw}} - n_x \left[EI \frac{d^2 w}{dx^2} \frac{dv}{dx} \right]_{\Gamma_{s\theta}}.
\end{aligned} \tag{10}$$

As mentioned previously, n_x is the direction cosine of the unit outward drawn normal to Ω_s ; $n_x = 1$ if the boundary is on the right side of Ω_s , and $n_x = -1$ if the boundary is on the left side of Ω_s .

The trial functions are assumed as

$$w(x) = \sum_{j=1}^N (w_j \psi_j^{(w)}(x) + \theta_j \psi_j^{(\theta)}(x)), \tag{11a}$$

and the test functions are assumed as

$$v(x) = \mu_i^{(w)} \chi_i^{(w)}(x) + \mu_i^{(\theta)} \chi_i^{(\theta)}(x), \tag{11b}$$

where w_j and θ_j are the nodal values of deflection and slope at node j , ψ_j are the shape functions, N is the number of nodes in the model, $\mu_i^{(w)}$ and $\mu_i^{(\theta)}$ are the arbitrary constants for deflections and slopes of the test function, and $\chi_i^{(w)}(x)$ and $\chi_i^{(\theta)}(x)$ are components of the test functions. The trial and test functions will be discussed in detail later.

MLPG Equations

Substituting the trial and test functions of Eq. (11) into Eq. (10), and requiring that the weak form be valid for arbitrary values of $\mu_i^{(w)}$ and $\mu_i^{(\theta)}$ leads to the MLPG equations as

$$\mathbf{K}^{(\text{node})} \mathbf{d} + \mathbf{K}^{(\text{bdry})} \mathbf{d} - \mathbf{f}^{(\text{node})} - \mathbf{f}^{(\text{bdry})} = \mathbf{0} \tag{12}$$

where “bdry” denotes boundary and

$$\{\mathbf{d}\} = \{ w_1 \quad \theta_1 \quad w_2 \quad \theta_2 \quad w_3 \quad \theta_3 \cdots \cdots \cdots w_N \quad \theta_N \}^T \quad (13a)$$

are the nodal values of deflections, w , and slopes, θ , at all the N nodes of the model used to analyze the problem, and

$$\mathbf{K}^{(\text{node})} = \left[\mathbf{k}_{ij}^{(\text{node})} \right] \quad (13b)$$

$$\mathbf{K}^{(\text{bdry})} = \left[\mathbf{k}_{ij}^{(\text{bdry})} \right] \quad (13c)$$

with

$$\begin{aligned} \mathbf{k}_{ij}^{(\text{node})} = & EI \left[\begin{array}{cc} \int_{\Omega_i^{(j)}} \frac{d^2 \chi_i^{(w)}}{dx^2} \frac{d^2 \psi_j^{(w)}}{dx^2} dx & \int_{\Omega_i^{(j)}} \frac{d^2 \chi_i^{(w)}}{dx^2} \frac{d^2 \psi_j^{(\theta)}}{dx^2} dx \\ \int_{\Omega_i^{(j)}} \frac{d^2 \chi_i^{(\theta)}}{dx^2} \frac{d^2 \psi_j^{(w)}}{dx^2} dx & \int_{\Omega_i^{(j)}} \frac{d^2 \chi_i^{(\theta)}}{dx^2} \frac{d^2 \psi_j^{(\theta)}}{dx^2} dx \end{array} \right] \\ & + n_x EI \left[\begin{array}{cc} \chi_i^{(w)} \frac{d^3 \psi_j^{(w)}}{dx^3} & \chi_i^{(w)} \frac{d^3 \psi_j^{(\theta)}}{dx^3} \\ \chi_i^{(\theta)} \frac{d^3 \psi_j^{(w)}}{dx^3} & \chi_i^{(\theta)} \frac{d^3 \psi_j^{(\theta)}}{dx^3} \end{array} \right]_{\mathbf{r}_{ij}^{(j)}} \\ & - n_x EI \left[\begin{array}{cc} \frac{d \chi_i^{(w)}}{dx} \frac{d^2 \psi_j^{(w)}}{dx^2} & \frac{d \chi_i^{(w)}}{dx} \frac{d^2 \psi_j^{(\theta)}}{dx^2} \\ \frac{d \chi_i^{(\theta)}}{dx} \frac{d^2 \psi_j^{(w)}}{dx^2} & \frac{d \chi_i^{(\theta)}}{dx} \frac{d^2 \psi_j^{(\theta)}}{dx^2} \end{array} \right]_{\mathbf{r}_{ij}^{(j)}} \end{aligned} \quad (13d)$$

$$\begin{aligned}
\mathbf{k}_{\bar{y}}^{(\text{bdry})} = & \alpha_w \begin{bmatrix} \chi_i^{(w)} \psi_j^{(w)} & \chi_i^{(w)} \psi_j^{(\theta)} \\ \chi_i^{(\theta)} \psi_j^{(w)} & \chi_i^{(\theta)} \psi_j^{(\theta)} \end{bmatrix}_{\mathbf{r}^{(i)}_{sw}} \\
& + n_x EI \begin{bmatrix} \chi_i^{(w)} \frac{d^3 \psi_j^{(w)}}{dx^3} & \chi_i^{(w)} \frac{d^3 \psi_j^{(\theta)}}{dx^3} \\ \chi_i^{(\theta)} \frac{d^3 \psi_j^{(w)}}{dx^3} & \chi_i^{(\theta)} \frac{d^3 \psi_j^{(\theta)}}{dx^3} \end{bmatrix}_{\mathbf{r}^{(i)}_{sw}} \\
& + \alpha_\theta \begin{bmatrix} \frac{d \chi_i^{(w)}}{dx} \frac{d \psi_j^{(w)}}{dx} & \frac{d \chi_i^{(w)}}{dx} \frac{d \psi_j^{(\theta)}}{dx} \\ \frac{d \chi_i^{(\theta)}}{dx} \frac{d \psi_j^{(w)}}{dx} & \frac{d \chi_i^{(\theta)}}{dx} \frac{d \psi_j^{(\theta)}}{dx} \end{bmatrix}_{\mathbf{r}^{(i)}_{s\theta}} \\
& - n_x EI \begin{bmatrix} \frac{d \chi_i^{(w)}}{dx} \frac{d^2 \psi_j^{(w)}}{dx^2} & \frac{d \chi_i^{(w)}}{dx} \frac{d^2 \psi_j^{(\theta)}}{dx^2} \\ \frac{d \chi_i^{(\theta)}}{dx} \frac{d^2 \psi_j^{(w)}}{dx^2} & \frac{d \chi_i^{(\theta)}}{dx} \frac{d^2 \psi_j^{(\theta)}}{dx^2} \end{bmatrix}_{\mathbf{r}^{(i)}_{s\theta}}
\end{aligned} \tag{13e}$$

$$\mathbf{f}^{(\text{node})} = \begin{Bmatrix} \int_{\Omega_s^{(i)}} \chi_i^{(w)} f \, dx \\ \int_{\Omega_s^{(i)}} \chi_i^{(\theta)} f \, dx \end{Bmatrix} \tag{13f}$$

and

$$\begin{aligned}
\mathbf{f}^{(\text{bdry})} = & n_x \tilde{\mathbf{M}} \begin{Bmatrix} \frac{d \chi_i^{(w)}}{dx} \\ \frac{d \chi_i^{(\theta)}}{dx} \end{Bmatrix}_{\Gamma_{SM}^i} + n_x \tilde{\mathbf{V}} \begin{Bmatrix} \chi_i^{(w)} \\ \chi_i^{(\theta)} \end{Bmatrix}_{\Gamma_{SV}^i} \\
& + \alpha_w \begin{Bmatrix} \tilde{w} \chi_i^{(w)} \\ \tilde{w} \chi_i^{(\theta)} \end{Bmatrix}_{\Gamma_{sw}^i} + \alpha_\theta \begin{Bmatrix} \tilde{\theta} \frac{d \chi_i^{(w)}}{dx} \\ \tilde{\theta} \frac{d \chi_i^{(\theta)}}{dx} \end{Bmatrix}_{\Gamma_{s\theta}^i},
\end{aligned} \tag{13g}$$

where $i, j = 1, 2, \dots, N$.

The system of equations presented in Eqs. (12 – 13g) are the general equations valid for any set of trial and test functions. In this paper, a Petrov-Galerkin method is used; the test functions are chosen to be different from the trial functions. The choices for the trial and test functions are discussed next.

Trial Functions

As mentioned previously, orthogonal polynomials are chosen as trial functions. Chebyshev and Legendre polynomials are considered. For MLPG-Chebyshev formulation, the deflection interpolation is chosen as

$$w(x) = a_1 T_0(x) + a_2 T_1(x) + a_3 T_2(x) + \dots a_{2N} T_{(2N-1)}(x) \quad (14)$$

$$\theta(x) = \frac{dw}{dx} = a_1 \frac{dT_0(x)}{dx} + a_2 \frac{dT_1(x)}{dx} + a_3 \frac{dT_2(x)}{dx} + \dots a_{2N} \frac{dT_{(2N-1)}(x)}{dx}$$

or concisely as

$$\begin{bmatrix} w \\ \theta \end{bmatrix} = \begin{bmatrix} T_0(x) & T_1(x) & \dots & T_{(2N-1)}(x) \\ \{dT_0(x)/dx\} & \{dT_1(x)/dx\} & \dots & \{dT_{(2N-1)}(x)/dx\} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2N} \end{bmatrix} \quad (15)$$

or as

$$\{w\} = [T]\{a\} \quad (16)$$

where $T_i(x)$ are the Chebyshev polynomials and $\{a\}$ are undetermined coefficients. The Chebyshev polynomials are orthogonal polynomials in the interval $(-1 \leq \xi \leq 1)$ and are given by (see Ref. 10)

$$T_0 = 1 ; T_1 = \xi ; T_2 = 2\xi^2 - 1 ; T_3 = 4\xi^3 - 3\xi ; T_4 = 8\xi^4 - 8\xi^2 + 1 ; \dots \quad (17a)$$

with the recurrence relation

$$T_{n+1}(\xi) = 2\xi T_n(\xi) - T_{n-1}(\xi). \quad (17b)$$

The beam coordinate x can be converted to the coordinate ξ by

$$\xi = (2x / L) - 1. \quad (18)$$

The Legendre polynomials are

$$P_0 = 1 ; P_1 = 2\xi ; P_2 = \frac{1}{2}(3\xi^2 - 1) ; P_3 = \frac{1}{2}(5\xi^3 - 3\xi) ; P_4 = \frac{1}{8}(35\xi^4 - 30\xi^2 + 3) ; \dots (19)$$

with the recurrence relation

$$(n+1)P_{n+1}(\xi) = (2n+1)\xi P_n(\xi) - nP_{n-1}(\xi). \quad (19a)$$

The rest of the analysis is presented for the Chebyshev polynomials and the Legendre polynomial formulation follows in a similar manner.

Using Eq. (15) the nodal values of deflections and slopes at each of the nodes in the model can be expressed in terms of the undetermined coefficients $\{a\}$ as

$$\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_N \\ \theta_N \end{bmatrix} = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \dots & T_{(2N-1)}(x_1) \\ \{dT_0(x)/dx\}_1 & \{dT_1(x)/dx\}_1 & \dots & \{dT_{(2N-1)}(x)/dx\}_1 \\ T_0(x_2) & T_1(x_2) & \dots & T_{(2N-1)}(x_2) \\ \{dT_0(x)/dx\}_2 & \{dT_1(x)/dx\}_2 & \dots & \{dT_{(2N-1)}(x)/dx\}_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ T_0(x_N) & T_1(x_N) & \dots & T_{(2N-1)}(x_N) \\ \{dT_0(x)/dx\}_N & \{dT_1(x)/dx\}_N & \dots & \{dT_{(2N-1)}(x)/dx\}_N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{2N} \end{bmatrix} \quad (20)$$

Or concisely as

$$\{d\} = [R]\{a\} \quad (21)$$

Thus

$$\{a\} = [R]^{-1}\{d\} \quad (22)$$

Substituting Eq. (21) into Eq. (16) one obtains the trial functions as

$$\{w\} = [T][R]^{-1}\{d\} \quad (23)$$

Accurate shape functions in Eq. (23) are possible only if the inverse of $[R]$ is possible and can be computed accurately. Notice that, in general, the matrix in Eqs. (20 and 21) is not well conditioned. The first column is of the order 1 while the last column of the order $\xi^{(2N-1)}$. As the order of N increases, the matrix's conditioning gets worse and the accuracy of the shape functions in Eq (23) can become poor. To ensure the accuracy of the inverse of $[R]$, the product of $[R]$ and $[R]^{-1}$ is computed and the resulting matrix is examined for deviation from the unit matrix. If the off-diagonal terms are not machine zeros, and the diagonal terms deviate from unity, then the inverse may not be accurate. The conditioning of the $[R]$ matrix is examined for various N -node models and is discussed in the appendix.

Test Functions

As mentioned previously, the test function, v , is assumed as in Eq. (11b) as

$$v(x) = \mu_i^{(w)} \chi_i^{(w)}(x) + \mu_i^{(\theta)} \chi_i^{(\theta)}(x). \quad (11b)$$

The test function components, χ_i , are chosen to be distinctly different from the shape functions, ψ_i . A power weight function

$$\chi_i^{(w)}(x) = \begin{cases} [1 - (d_i / R_0)^2]^\beta & \text{if } 0 \leq d_i \leq R_0 \\ 0 & \text{if } d_i > R_0. \end{cases} \quad (23)$$

was chosen (see Ref. 6). In Eqs. (23), $d_i = \|(x - x_i)\|$, R_0 is a user-defined parameter that determines the extent of the test functions (and hence Ω_s - see Figure 1), and exponent β was chosen to be 3, 4 or 5. The components of the test functions chosen for θ are the first derivatives of the components of the test functions chosen for the primary variable w , i.e.

$$\chi_i^{(\theta)} = \frac{d\chi_i^{(w)}}{dx}, \quad (23a)$$

as $\theta = (dw/dx)$ is also a primary variable. Because test functions, χ_i , are zero for $d_i > R_0$, the integrations in Eq. (13) need to be carried out only over Ω_s (see Figure 1) and this makes the method local. Also, the test functions have zero values for the first, second, and third derivative at the ends of the domain Ω_s , i.e. when $d_i = R_0$.

Once Eq. (12) is solved for the unknown primary variables $\mathbf{d}$, the deflections, slopes, moments, and shear forces can be evaluated at any point on the beam using Eqs. (14) and (4).

NUMERICAL EVALUATIONS

A beam of constant flexural rigidity EI and a length of $L = 4l$ are considered. The integrations in the weak form are performed numerically using Gaussian quadrature. A 12-point Gaussian is found adequate to integrate the weak form accurately. First, the current formulations were applied to simple patch-tests and simple beam problems. Next, they were tested on problems with load discontinuities and interior supports. The numerical solutions were compared to the exact solutions to evaluate the effectiveness of these formulations.

Four models with 5, 9, 17, or 33 nodes were developed. Each of these models have equidistant nodes along the length of the beam. A 9-node model is shown in Figure 2. The distance between successive nodes is Δx , ($\Delta x = L/N$, in an N -node model) and the R_o is chosen to be $(2*\Delta x/l)$. Each of these models is used in the numerical studies on patch tests.

Patch Tests:

The current MLPG-Chebyshev formulation was evaluated by applying the formulations to simple patch-test problems. The problems considered were

(a) rigid-body translation:

$$w(x) = c_0, \quad \theta = \frac{dw}{dx} = 0, \quad (24a)$$

(b) rigid-body rotation:

$$w = c_1 x, \quad \theta = c_1 \quad (24b)$$

and (c) a constant-curvature condition:

$$w(x) = c_2 x^2/2, \quad \theta = c_2 x, \quad (24c)$$

where c_0 , c_1 , and c_2 are arbitrary constants. The third patch test may be looked upon as the problem of a cantilever beam with a moment, $M = EI (d^2 w/dx^2) = EI c_2$, applied at $x = 4l$. The deflection, w , and the slope, θ , corresponding to problems (a), (b), and (c) were prescribed as essential boundary conditions (EBCs) at $x = 0$ and $x = 4l$. For a displacement of $w(x) = c_0$ and $c_1 x$, the rigid body conditions (Eqs. 24a and 24b) were modeled with boundary conditions

$$\begin{aligned} (a) \quad & w|_{x=0} = c_0, \quad w|_{x=4l} = c_0, \\ & \theta|_{x=0} = 0, \quad \theta|_{x=4l} = 0 \\ & \text{for rigid body translation and} \\ (b) \quad & w|_{x=0} = 0, \quad w|_{x=4l} = 4c_1 l, \\ & \theta|_{x=0} = c_1, \quad \theta|_{x=4l} = c_1 \end{aligned}$$

for rigid body rotation.

Since the exact solutions using conditions (a) and (b) are constant and linear in x , respectively, the MLPG method developed with a Chebyshev and Legendre polynomial basis functions must reproduce the solutions exactly. As expected, the algorithm reproduced the exact solutions for w and θ to machine accuracy (for double precision) for both rigid body modes at all the nodes and at any arbitrary point in the beam for the 5-, 9-, and 17-node models.

For the constant-curvature condition, $w = c_2 x^2/2$, the problem was modeled with EBCs

$$(c) \quad \begin{aligned} w|_{x=0} &= 0, & w|_{x=4l} &= 8c_2 l^2, \\ \theta|_{x=0} &= c_1, & \theta|_{x=4l} &= 4c_2 l. \end{aligned}$$

Again since the exact solution is quadratic in x , the MLPG-Chebyshev and Legendre polynomials must reproduce the solution exactly. As expected, the algorithm reproduced the exact solution for the primary variables to machine accuracy at all nodes and at any arbitrary point in the beam for the 5-, 9-, and 17-node models.

All three patch tests failed when the 33-node model was used. Further examination showed that the matrix $[R]$ is highly ill conditioned for the 33-node model. As mentioned previously, the first column of the matrix $[R]$ is of the order unity, while the last column of this matrix is of the order ξ^{65} , and recall that $(-1 \leq \xi \leq 1)$. When ξ is less than unity the last column is an extremely small number that is near zero and this leads to the ill conditioning. Note that 5-, 9-, and 17-node models did not suffer from this ill conditioning. To accurately determine the inverse of $[R]$ and the shape functions in the 33-node model, the MLPG algorithm was reworked with quadruple precision. With quadruple precision, the inverse of $[R]$ is computed accurately and hence the shape functions are computed accurately. The resulting 33-node model passed all three patch tests with machine accuracy (for quadruple precision). The 33-node model is not used further in this paper as the 17- node model gave accurate results.

To ensure that all the results are comparable to each other, all the results presented in this paper were obtained with quadruple precision.

Simple Beam Problems:

Next the MLPG-Chebyshev and Legendre formulations were applied to simple beam problems such as cantilever and simply-supported beam problems with various loadings as shown in Figure 3. As in the patch test problems, three models with 5-, 9-, and 17- nodes equally placed along the length of the beam were used. For each of these models ($R_0 = 2 \Delta x/l$) was used.

For the cantilever with a concentrated load at the tip (Figure 3a), the EBCs at the clamped end are $w = \theta = 0$. At the tip, $x=4l$, the applied load is applied as natural boundary conditions (NBCs) $V=-P$ and $M=0$. The problem was also worked as applied load case by evaluating the

vector $\mathbf{f}^{(node)}$ from Eq. 13(f). As expected, all three models and both the NBC and loading cases produced exact results for Chebyshev and Legendre formulations.

For the cantilever with a moment at the tip (Figure 3b), the EBCs at the clamped end are $w = \theta = 0$. At the tip, $x=4l$, the applied moment is applied as natural boundary conditions (NBC) $V=0$ and $M=M_0$. As expected, all three models produced exact results for Chebyshev and Legendre formulations. Note that the concentrated moment at the end (Figure 3b) was already discussed as a patch test problem (Case b).

For the cantilever with a uniformly distributed load (UDL) of intensity q (Figure 3c), the EBCs at the clamped end are $w = \theta = 0$. In the MLPG models, the uniformly distributed load q is used in developing the right-hand vector $\mathbf{f}^{(node)}$ using Eq. 13f. As expected, all three models produced exact results for Chebyshev and Legendre formulations.

Next, a simply-supported beam with uniformly distributed loading (Figure 3d) was considered. The right-hand vector $\mathbf{f}^{(node)}$ using Eq. 13f is computed for the uniformly distributed load, q . The problem was also worked two ways – exploiting symmetry, and as a full beam. When only half the beam was considered, the boundary conditions at $x=0$ were $w=0$ and $M=0$; and at $x=2l$, $\theta=0$. When the full beam was considered, the boundary conditions at $x=0$ and $x=4l$ were $w=0$ and $M=0$. As expected, all three models and both the half beam and the full beam models produced exact results for Chebyshev and Legendre formulations.

Next, a simply-supported beam with central concentrated moment (Figure 3e) was considered. The beam is modeled exploiting anti-symmetry in the problem. Everything is anti-symmetric with respect to the center of the beam $x=2l$. As such, the EBCs are

$x=0$ and $2l$, $w=0$, and the NBCs are

$x=0$, $M=0$, and

$x=2l$, $M= M_0/2$.

As expected, all three models with these antisymmetric boundary conditions on the half beam produced exact results for deflections, slopes, moments, and shear forces for Chebyshev and Legendre formulations.

Problems with Load Discontinuities

To study the efficiency of the MLPG method for problems with concentrated loads, load discontinuities, and interior supports, three problems shown in Figure 4 were chosen. Two problems with load discontinuities are shown in Figure 4a and 4b. In Figure 4a, a simply supported beam with a central concentrated load is shown. In figure 4b, a cantilever beam with a UDL only on half of the beam is shown. The third problem is that of a continuous beam loaded with a uniformly distributed load (Figure 4c).

First, the problems were analyzed with models with equidistant nodes. Then, models with non-uniform nodal spacing were used to delineate the deflections and slopes near the load discontinuity. In all the problems considered in this section, the maximum distance, $(\Delta x / l)_{Max}$, between successive nodes in the models was computed and the R_o was chosen to be $(2 * \Delta x / l)_{Max}$.

Simply-supported beam with a Central Concentrated load:

Using symmetries in the problem, one-half of the beam was modeled. The EBCs are $w=0$ at $x=0$ and $\theta=0$ at $x=2l$ and the NBCs are $M=0$ at $x=0$ and $V=-P/2$ at $x=2l$. First, three models with 5, 9, and 17 equidistant nodes were developed. With these boundary conditions, the MLPG-Chebyshev and Legendre formulations reproduced the exact solution.

Next the full beam was modeled with the EBCs of $w=0$ at $x=0$ and $x=4l$, and NBCs of $M=0$ at $x=0$, and $4l$. External loads contribute to the $f^{(node)}$ of Eq. (13f). If a concentrated load P is applied at node j , the integrals in Eq. (13f) are evaluated with Dirac delta function as

$$f^{(node)} = \begin{pmatrix} \int_{\Omega_s^{(i)}} \chi_i^{(w)} f \cdot \delta(x = x_j) dx \\ \int_{\Omega_s^{(i)}} \chi_i^{(\theta)} f \cdot \delta(x = x_j) dx \end{pmatrix} = \begin{pmatrix} \chi_i^{(w)}(x_j) \cdot P \\ \chi_i^{(\theta)}(x_j) \cdot P \end{pmatrix} \quad (25)$$

To evaluate the $f^{(node)}$, all the nodes in the domain of influence $\Omega_s^{(i)}$ of node j need to be considered. The values of each test function, v_i , are evaluated using

$$v_i = \mu_i^{(w)} \chi_i^{(w)}(x_j) + \mu_i^{(\theta)} \chi_i^{(\theta)}(x_j).$$

The maximum deflection, w_{Max} , and $\theta_{x=0}$ are presented in Table 1 for the three models. The 17-node model gave inaccurate results and the accuracy progressively reduced because of load discontinuity. Therefore, two nonuniform mesh models with 11 and 21 nodes were developed with nodes that are closely spaced to handle the load discontinuity. These models are presented in Figure 5. The results from these models are presented in Table 1. Apparently, the refinement of the model near the load discontinuity improved the accuracy of both models compared to the uniformly spaced models. However, notice that the 21-node model results are not as accurate as the 11-node model results. This is probably due to worsening of the conditioning of the $[R]$ matrix as the number of nodes in the models increased beyond about 20 nodes. Thus, there appears to be a limitation on the order of the $[R]$ matrix and hence the number of nodes with these Chebyshev and Legendre polynomials as trial functions.

The analysis was repeated with Legendre polynomials as trial functions. The MLPG with Legendre polynomials produced nearly identical results.

Cantilever with partial loading:

Figure 4b shows a cantilever beam with $L=4l$ and with a UDL on a portion of the beam. The exact solution for this problem is:

For $0 \leq x \leq 2l$,

$$\begin{aligned} w &= \frac{q}{24EI} [-(2l-x)^4 - 32l^3x + 16l^4] \\ \theta &= \frac{q}{6EI} [(2l-x)^3 - 8l^3] \end{aligned} \quad (26)$$

For $2l \leq x \leq 4l$,

$$\begin{aligned} w &= \frac{q}{24EI} [-48l^4 - 32l^3(x-2l)] \\ \theta &= -\frac{q}{6EI} \end{aligned} \quad (27)$$

Four models, two with equidistant nodes (the 9-node and 17-node models) and two models with closely spaced nodes near the discontinuity at $x=2l$ (the 11-node and 13-node models) were developed. The 11- and 13-node models are presented in Figure 6. These four models were used to study the performance of the MLPG method. Table 2 presents the results for deflection and slope at the end of the loading, at $x=2l$, and the tip of the cantilever at $x=4l$. All four models yielded accurate deflections and slope. However, the 13-node model results are more accurate than the other models. Table 2 also shows that the 17-node uniformly spaced model's accuracy is worse than the other models. This confirms that judicious choice of nodal placement is needed to obtain accurate results with this formulation.

The analysis was repeated with Legendre polynomials as trial functions. The MLPG with Legendre polynomials produced nearly identical results.

Continuous beam:

A continuous beam loaded with a UDL of q/length is shown in Figure 4c. This problem represents a beam with an intermediate support. This is very similar to load discontinuity.

Two problems were considered. In the first case $L_1=2l$, $L_2=2l$. In the second case $L_1=l$, $L_2=2l$. The exact solution to the problem with $L_1 = L_2=2l$ is:

For $0 \leq x \leq 2l$,

$$\begin{aligned}
w(x) &= \frac{q}{48EI} [6lx^3 - 2x^4 - 8l^3x] \\
\theta(x) &= \frac{q}{48EI} [18lx^2 - 8x^3 - 8l^3]
\end{aligned} \tag{28}$$

For $2l \leq x \leq 4l$,

$$\begin{aligned}
w(x) &= \frac{q}{48EI} [26lx^3 - 2x^4 - 120l^2x^2 + 232l^3x - 160l^4] \\
\theta(x) &= \frac{q}{48EI} [78lx^2 - 8x^3 - 240l^2x + 232l^3]
\end{aligned} \tag{29}$$

This problem was analyzed exploiting the symmetry in the problem. The boundary conditions are: at $x=0$, $w=0$ and $M=0$; at $x=2l$, $w=0$ and $\theta=0$. The 5- node and 9-node uniformly spaced models produced the exact solution.

Next, the full beam was modeled with a 9-node uniformly spaced model and an 11-node non-uniform model, shown in Figure 5. To account for the additional beam support (Figure 4c) an additional penalty term is added to the weak form in Eq. 13 as,

$$\alpha_c [(w - \tilde{w}) \cdot v]_{\Gamma_c} \tag{30}$$

where α_c is the penalty parameter to enforce the continuous beam boundary condition, $w = \tilde{w}$ at Γ_c .

Table 3 compares the deflections and slopes along the length of the beam with exact values. The 11-node model with closely spaced nodes near the support produced accurate results compared to the 9-node uniformly spaced node model. The analysis was repeated with Legendre polynomials as trial functions. The MLPG with Legendre polynomials produced nearly identical results.

Next, the problem with $L_1=l$, $L_2=2l$ was analyzed.

The exact solution to this problem is as follows;

For $0 \leq x \leq l$,

$$\begin{aligned}
w(x) &= \frac{q}{48EI} [-2x^4 + lx^3 + l^3x] \\
\theta(x) &= \frac{q}{48EI} [-8x^3 + 3lx^2 + l^3]
\end{aligned} \tag{31}$$

and for $l \leq x \leq 3l$,

$$\begin{aligned}
w(x) &= \frac{q}{96EI} [-4x^4 + 35lx^3 - 99l^2x^2 + 101l^3x - 33l^4] \\
\theta(x) &= \frac{q}{96EI} [-16x^3 + 105lx^2 - 198l^2x + 101l^3]
\end{aligned} \tag{32}$$

Three models were developed: a model using 7 equally spaced nodes, and 15- node and 17-node models with closely spaced nodes near the support at $x=l$. Figure 7 shows the 7- and 15- node models. The 17-node model, not shown in Figure 7, is obtained by adding to additional nodes on either side of the support at $x=l$. Figure 8 compares the deflections and slopes obtained from these models. The results of the three models are shown as discrete symbols and the exact solutions as the solid line. Excellent agreement is observed from this figure.

Figure 9 compares the moments in the continuous beam. For most locations along the beam, reasonable agreement with the exact solution was obtained by the MLPG-Chebyshev method. Also the 15- node and 17-node moment results are more accurate than the 7-node results. However, at the far end of the beam, near $x > 2l$, the moment results are inaccurate. Models with further refinement are needed in the region $x > 2l$. Although the deflections are accurate, as shown in Figure 8, the inaccuracies of moments shown in Figure 9 are probably because moments are second derivatives of the deflections.

As in the previous problems, the analysis was repeated with Legendre polynomials as trial functions. The MLPG with Legendre polynomials produced nearly identical results.

DISCUSSION

The numerical evaluations presented clearly demonstrate that the MLPG-Chebyshev and MLPG-Legendre methods do not perform well for problems with discontinuities and sharp changes in loads. Near regions of such sharp load gradients, most of the numerical methods utilize large numbers of closely spaced nodes. However, MLPG-Chebyshev and MLPG-Legendre methods appear to have limits on the number of nodes in the models because of the ill-conditioning inherent in these methods. The results presented in references 6 and 7 suggest that the moving least square and radial basis trial functions appear to yield more accurate results, as the number of nodes in the beam problems could be increased easily in the models. (Reference 6 and 7 show models with 100 plus nodes.) No ill-conditioning was observed with these formulations even for models that have more than three or four times the number of nodes used in this paper. The limitation on the number of nodes appears to be a serious drawback for the MLPG-Chebyshev and MLPG-Legendre Methods.

CONCLUDING REMARKS

Meshless methods were first introduced with moving least square approximating functions as trial functions. The first forms of these methods were Galerkin methods, where trial and test functions came from the same space. A meshless local Petrov-Galerkin method (MLPG) was proposed that chooses the trial and test functions from different spaces. These methods were very attractive and performed well for C^0 problems. These methods were later extended for C^1 problems.

In this paper, the feasibility of orthogonal polynomials in the MLPG method is studied. The orthogonal polynomials, Chebyshev and Legendre polynomials, are used in this MLPG method as trial functions. The test functions used were power functions with smooth derivatives at their ends. The performance of these methods is studied by applying these methods to several Euler-Bernoulli beam problems.

The MLPG-Chebyshev and Legendre methods passed all the patch tests for simple beam problems. Problems with load discontinuities and additional supports require special attention. Judicious choice of nodal placement is needed to obtain accurate solutions. Models with nodes that are closely spaced near discontinuities are needed to account for sharp changes in slopes and deflections. Both Chebyshev and Legendre methods are capable of yielding accurate values of deflections, slopes, moments, and shear forces. However, as with all numerical methods, judicious choice of nodes and nodal placement are needed to obtain accurate results with these formulations.

Increasing the number of nodes in the model can result in a transformation matrix that is ill-conditioned, which can lead to problems with its inversion. The conditioning worsens as the number of nodes are increased beyond 20. Quadruple precision was needed for models to obtain accurate solutions. Even with quadruple precision, the accuracy of the method suffers as the number of nodes are increased beyond 20. This appears to be a drawback of the MLPG-Chebyshev and MLPG-Legendre methods.

REFERENCES

1. Nayroles, B., Touzot, G., and Villon, P., "Generalizing the finite element method: diffuse approximation and diffuse elements," *Computational Mechanics*, Vol. 10, pp. 307-318, 1992.
2. Belytschko, T., Lu, Y. Y., and Gu, L., "Element-free Galerkin methods," *International Journal of Numerical Methods in Engineering*, Vol. 37, pp. 229-256, 1994.
3. Atluri, S. N. and Shen, S., *The Meshless Local Petrov-Galerkin (MLPG) Method*, Tech Science Press, Encino, CA, 2002.

4. Liu G. R., *Mesh Free Methods: Moving Beyond the Finite Element Method*, CRC press, Boca Raton, 2002.
5. Atluri, S. N., Cho, J. Y., and Kim, H. -G., “Analysis of thin beams, using the meshless local Petrov-Galerkin method, with generalized moving least squares interpolations,” *Computational Mechanics*, Vol. 24, pp. 334-347, 1999.
6. Raju, I. S. and Phillips, D. R., “Further developments in the MLPG method for beam problems,” *CMES: Computer Modeling in Engineering & Sciences*, Vol. 4, No. 1, pp. 141-159, 2003.
7. Raju, I. S., Phillips, D. R., and Krishnamurthy, T., “A Radial basis function approach in the meshless local Petrov-Galerkin Method for Euler-Bernoulli beam problems,” *Computational Mechanics*, Vol. 34, pp. 464-474. 2004
8. Smith, S. T., Radford, M. A., and Oehlers D. J., “ Numerical convergence of simple and orthogonal polynomials for the unilateral plate buckling problem using Rayleigh-Ritz method,” *International Journal of Numerical Methods in Engineering*, Vol 44, pp. 1685-1707, 1999.
9. Singhvi, S. and Kapania, R. K., “Comparison of simple and Chebyshev polynomials in Rayleigh-Ritz analysis,” *J. Engng. Mech. ASCE*, Vol 120 (10) pp. 2126-2135, 1994.
10. Alanbay, B. and Kapania, R. K., “On the use of classical Jacobi orthogonal Polynomials in the Ritz method,” 2018AIA/ASCE/AHS/ASC, Structures, Structural Dynamics and Materials Conference, 8-12 January 2018, Kissimmee, Florida. Paper AIAA-2018-1225, 2018.
11. Presss, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., *Numerical Recipes – The Art of Scientific Computing (FORTRAN version)*, Cambridge University Press, Cambridge, New York, NY, 1989.

APPENDIX: Conditioning of the $[R]$ matrix

As mentioned, the matrix $[R]$ is expected to be ill conditioned as the number of nodes in the MLPG models are increased. As such, in this appendix the conditioning of the $[R]$ matrix is examined.

A $M \times N$ matrix can be decomposed using Singular Value Decomposition (SVD) as

$$[R] = [U][W][V]^T \quad (A1)$$

where $[U]$ is a $M \times N$ column-orthogonal matrix, $[W]$ is a diagonal $N \times N$ eigenvalue matrix with positive or zero elements, and $[V]$ is a $N \times M$ orthogonal matrix¹¹. Because $[U]$ and $[V]$ are

orthogonal their inverses are just their respective transposes and the inverse of $[W]$ is a diagonal matrix with elements that are the reciprocals of the eigenvalues.

Once the decomposition of Eq. (A1) is completed, the conditioning number of $[R]$ is simply the ratio of the largest to the smallest eigenvalues in the $[W]$ matrix. Reference 11 points out, “A matrix is singular if the condition number is infinite and it is ill-conditioned if its conditioning number is too large, that is, if its reciprocal approaches the machine’s floating point precision (for example, less than 10^{-6} for single precision or 10^{-12} for double)”.

A FORTRAN version of the SVD algorithm given in reference 11, *SVDCMP*, was used to decompose the $[R]$ matrices for various models. From each of the eigenvalue matrices $[W]$, the conditioning numbers were computed and are presented in Table A1. Clearly, the matrix becomes ill-conditioned as the number of nodes in the model is increased. Comparison of the conditioning number for 17-node and 33-node models, shows that the 33-node model has a $[R]$ matrix that is nearly singular. As such, quadruple precision was needed for this MLPG algorithm to work.

Table A1: Eigenvalues of the $[R]$ for various N-node beam models ($L=4l$)

Nodes in the Beam model	Largest Eigenvalue	Smallest Eigenvalue	Condition Number
5	43.97	0.0851	517
9	140.7	-0.2455	573
17	487.7	-0.508×10^{-4}	975×10^4
33	2166	0.519×10^{-12}	418×10^{13}

Table 1. Comparison of maximum deflection and slope for various models for a simply supported beam with a central concentrated load.

N-Node model	$(\frac{EIw_{Max}}{Pl^3})$	$(\frac{EI\theta_{x=0}}{Pl^2})$
Models with uniformly placed nodes		
5	-1.337	-0.995
9	-1.316	-0.984
17	-1.440	-1.086
Models with non-uniformly placed nodes		
11	-1.333	-1.000
21	-1.3303	-1.000
Exact Values	-1.333	-1.000

Table 2. Comparison of values of deflection and slopes for partially loaded cantilever beam subjected to UDL, q . (See Figure 3c and 6)

N-node Model	$\frac{EIw_A}{qL^4}$	$\frac{EI\theta_A}{qL^3}$	$\frac{EIw_B}{qL^4}$	$\frac{EI\theta_B}{qL^3}$
Models with uniformly placed nodes				
9	-2.005	-1.338	-4.683	-1.340
17	-1.984	-1.318	-4.611	-1.309
Models with non-uniformly placed nodes				
11	-1.997	-1.330	-4.655	-1.328
13	-2.002	-1.335	-4.674	-1.336
Exact values	-2.000	-1.333	-4.667	-1.333

Table 3. Comparison of values of deflection and slopes along the length of a continuous beam subjected to UDL, q . ($L_1=2l$, $L_2=2l$)

x/l	$\frac{EIw}{ql^4}$	$\frac{EI\theta}{ql^3}$	Exact Values	
			$\frac{EIw}{ql^4}$	$\frac{EI\theta}{ql^3}$
9-node model with non-uniformly placed nodes				
0	0	-0.1747	0	-0.1667
1	-0.0847	0.0453	-0.0833	0.04167
2	0	0	0	0
3	-0.0847	-0.0453	-0.0833	-0.04167
4	0	0.1747	0	0.1667
11-node model with non-uniformly placed nodes				
0	0	-0.1638	0	-0.1667
1	-0.0825	0.04133	-0.0833	0.04167
2	0	0	0	0
3	-0.0825	-0.04133	-0.0833	-0.04167
4	0	0.1638	0	0.1667

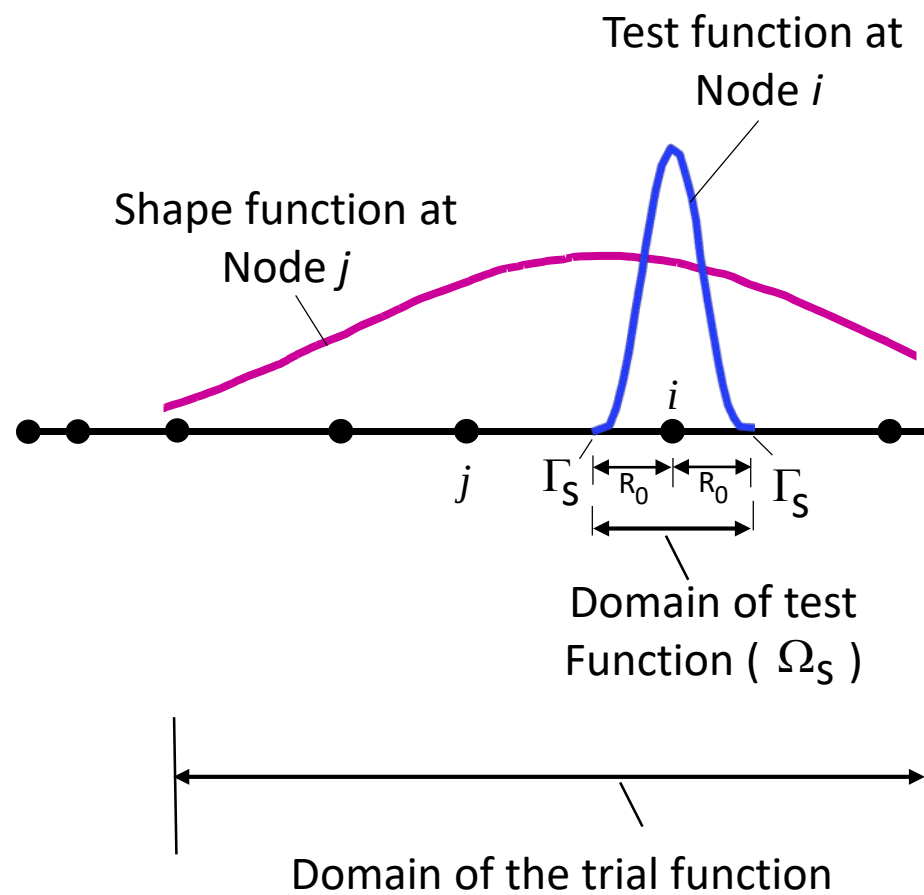
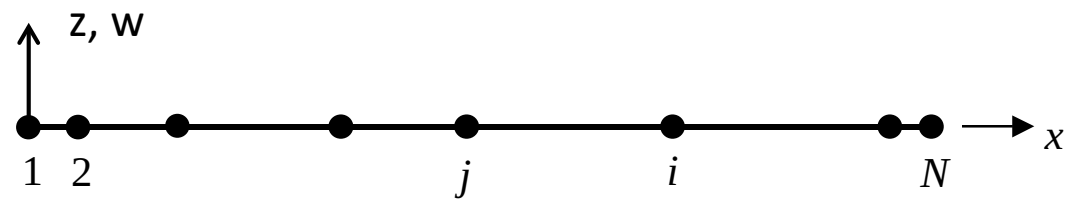


Figure 1. Comparison of the domains of trial and test functions

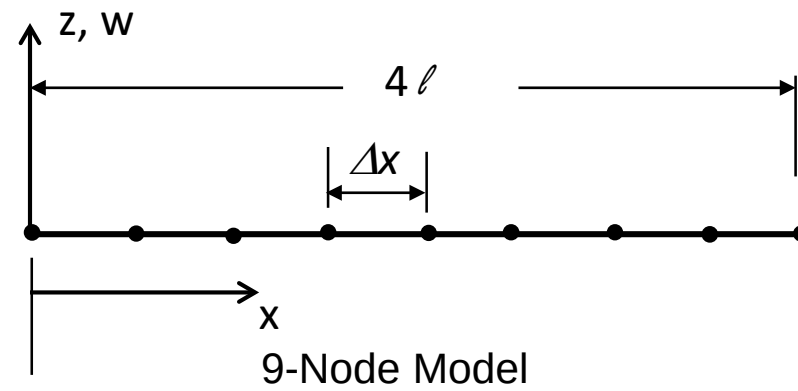
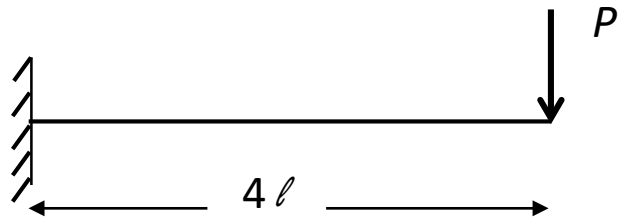
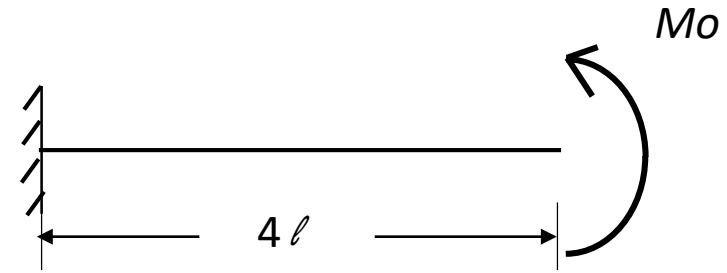


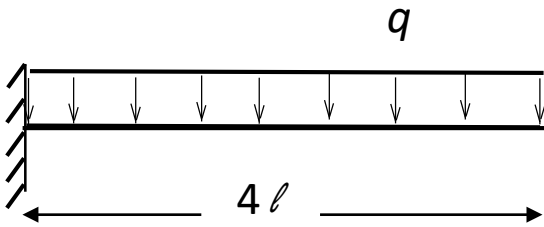
Figure 2. A 9-node model with equidistant nodes



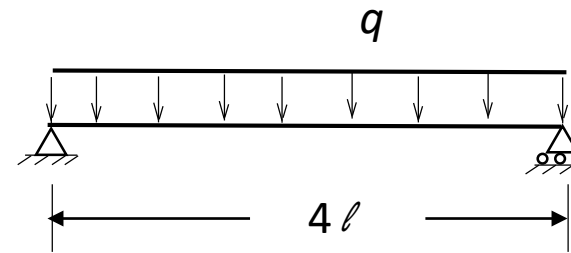
(a) Cantilever with a tip load



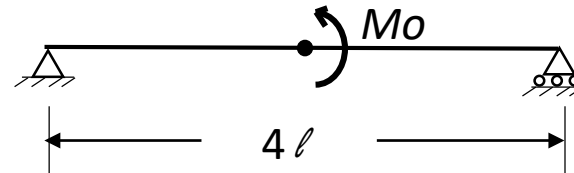
(b) Cantilever with an end moment



(c) Cantilever with a uniformly distributed load (UDL)

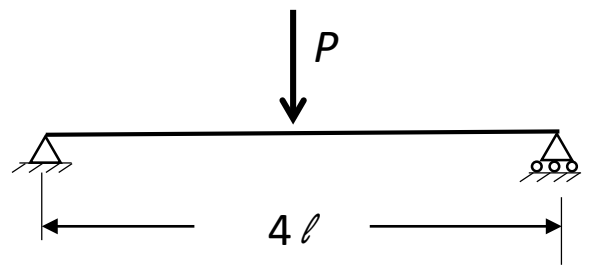


(d) SS beam with UDL

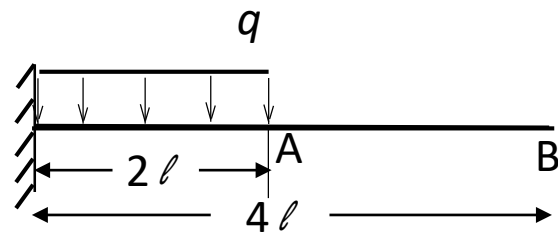


(e) SS beam with a concentrated moment

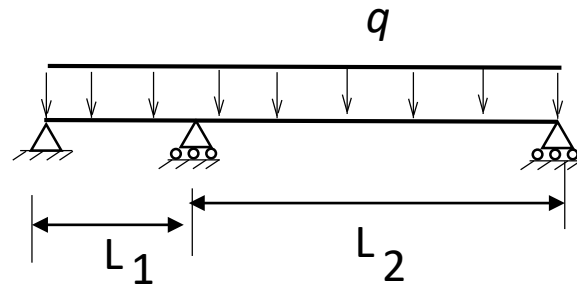
Figure 3: Various simple beam problems analyzed



(a) SS beam with a central concentrated load



(b) Partially loaded Cantilever



(c) Continuous beam with UDL

Figure 4: Various beam problems with discontinuities

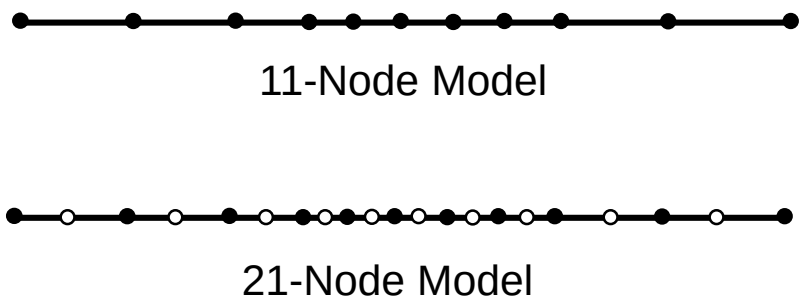


Figure 5. 11- and 21 -node models used for SS beam with a central concentrated load

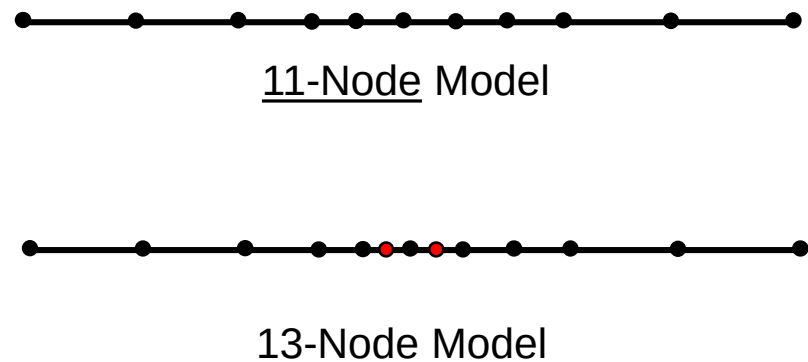


Figure 6. 11- and 13 -node models used for cantilever beam with partial UDL

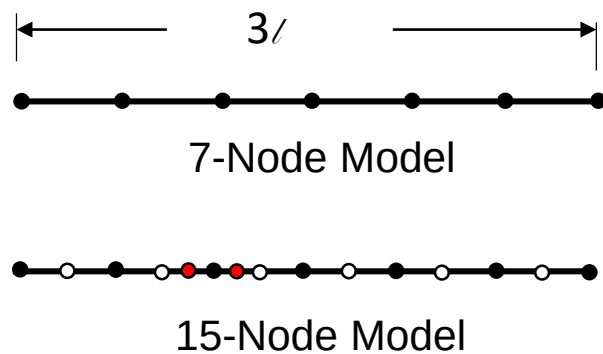


Figure 7. 7- and 15 -node models used for the continuous beam with UDL

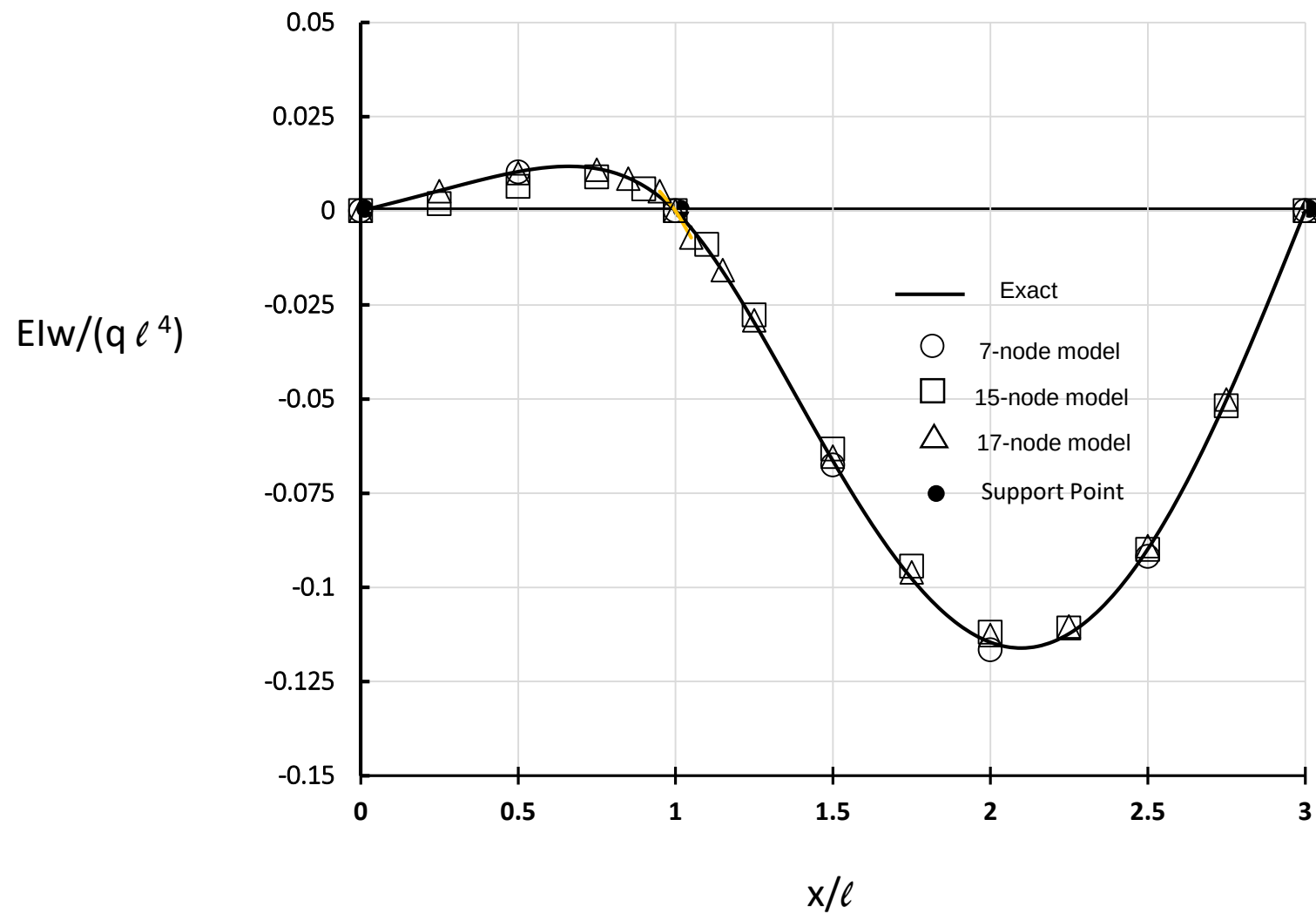


Figure 8. Comparison of deflections for continuous beam

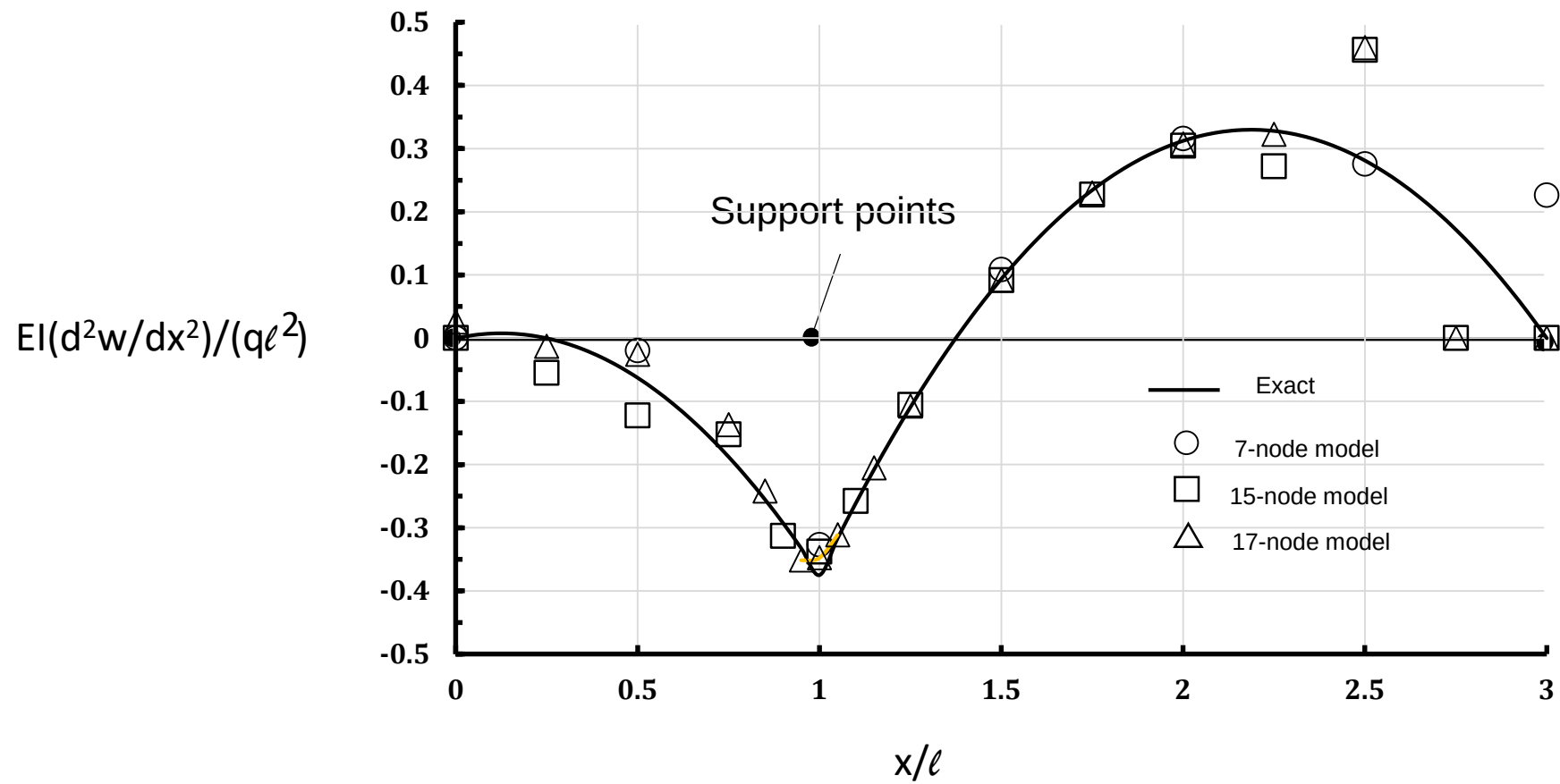


Figure 9. Comparison of moments for continuous beam