Affine Generalized Inverse for Optimal Control Allocation

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This research is a follow on to the "Optimal Control Prediction Method for Control Allocation" paper in which the Prediction Method iterative algorithm was introduced. Previously, the Prediction Method was shown to provide optimal control allocation solutions over the entire Attainable Moment Set for the Moore-Penrose and the generalized (weighted) inverse. As an extension to the Prediction Method, this paper introduces a family of Moore Penrose Affine Generalized Inverses, applicable for all moments, which compute control allocation solutions using a constant matrix and fixed null-space vector. The Moore-Penrose Affine Generalized Inverse is proven to yield equivalent solutions to those of the Prediction Method and therefore is guaranteed to yield Moore-Penrose optimal control allocation solutions. While the Prediction Method is applicable for any moment along an a priori specified moment direction, the Affine Generalized Inverse is shown to yield optimal control allocation solutions in a neighborhood of the given moment which is not restricted to a specified moment direction. Furthermore, the Affine Generalized Inverse is shown to provide the time derivative of optimal control allocation solutions and to facilitate maintaining solutions within control effector rate limitations. The Moore-Penrose Affine Generalized Inverse is broadened to encompass any arbitrary (weighted) Affine Generalized Inverse. Finally, a method of creating a moment lookup table is outlined to utilize the Affine Generalized Inverse as an offline control allocation solution for all moments in the Attainable Moment Set.

I. Nomenclature

$\delta(\Omega)$	=	boundary of Allowable Control Set
$\delta(\Phi)$	=	boundary of Attainable Moment Set
Ω	=	Allowable Control Set (ACS), $\Omega \subset \Re^m$
Φ	=	Attainable Moment Set (AMS), $\Phi \subset \Re^n$
B	=	control effectiveness matrix, $B \in \Re^{n\times m}$
B_1	=	columns of B corresponding with unsaturated control effector indices (S_1), $B_1 \in \Re^{n\times(m-k)}$
B_2	=	columns of B corresponding with saturated control effector indices (S_2), $B_2 \in \Re^{n\times k}$
$\hat{\mathscr{B}}$	=	complete orthogonal basis vector matrix containing $\hat{u}_{des}$ and $\mathcal{N}(B)$ basis vectors
$\hat{B}$	=	orthogonal basis vector matrix containing $\mathcal{N}(B)$ basis vectors
$\hat{B}_1$	=	rows of $\hat{B}$ corresponding with unsaturated control effector indices (S_1), $\hat{B}_1 \in \Re^{(m-k)\times(m-n)}$
$\hat{B}_2$	=	rows of $\hat{B}$ corresponding with saturated control effector indices (S_2), $\hat{B}_2 \in \Re^{k\times(m-n)}$
$\vec{c}_0$	=	null-space offset vector used with P_{aff}, $\vec{c}_0 \in \Re^m$
$\vec{c}_1$	=	a null-space component of $\vec{u}_{pred}$ whose magnitude is moment dependent, $\vec{c}_1 \in \Re^m$
k	=	Number of saturated control effectors or number of elements in set S_2
$\vec{m}_{des}$	=	desired moment, $\vec{m}_{des} \in \Re^n$
$\hat{m}_{des}$	=	unit vector in the direction of desired moment, $\hat{m}_{des} \in \Re^n$
P_{aff}	=	Moore-Penrose affine generalized inverse, $P_{aff} \in \Re^{m\times n}$
P_{Iso}	=	Moore-Penrose Iso surface, orthogonal projection of ACS on $\mathcal{N}(B)$
P_{min}	=	Moore-Penrose generalized inverse of control effectiveness matrix B, $P_{min} \in \Re^{m\times n}$
$\vec{r}_{lwr}$	=	Lower bounds of control effector rate limitations, $\vec{r}_{lwr} \in \Re^m$
$\vec{r}_{upr}$	=	Upper bounds of control effector rate limitations, $\vec{r}_{upr} \in \Re^m$
S_1	=	Set of indices for all currently unsaturated control effectors, $S_1 = \{1, 2, \ldots, m\} \setminus S_2$
S_2	=	Set of indices for all currently saturated control effectors (e.g. $S_2 = \{1, 5, 10\}$)
$\vec{s}_2$	=	Vector of active control effectors position limits for all indices in S_2

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$\hat{u}_{des}$	=	Control unit vector in the direction of $P_{min}\vec{m}_{des}$, $\hat{u}_{des} \in \Re^m$
$\vec{u}_{lwr}$	=	Lower bounds of control effector position limitations, $\vec{u}_{lwr} \in \Re^m$
$\vec{u}_{opt}$	=	Optimal control allocation solution, $\vec{u}_{opt} \in \Re^m$
$\vec{u}_{pred}$	=	Moore-Penrose optimal control allocation solution found using Prediction Method, $\vec{u}_{pred} \in \Re^m$
$\vec{u}_{upr}$	=	Upper bounds of control effector position limitations, $\vec{u}_{upr} \in \Re^m$
W_{gi}	=	Positive definite weighting matrix used for generation of arbitrary generalized inverse, $W_{gi} \in \Re^{m\times m}$

II. Introduction

Most modern flight control systems make extensive use of linear and/or linearized analysis, due to the extensive volume of linear theory and analysis tools. A large majority of these analyzes are performed by linearizing a non-linear system around a point of interest and expressing the system as a time varying (or invariant) system which in state space form yields:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

$$y(t) = C(t)x(t) + D(t)u(t) \tag{2}$$

Legacy flight control methodologies typically used groups or "ganged" flight controls to generate the time history of the desired control signal $u(t)$ such that the resulting states are stable, provide the desired performance, stability margins etc. The last few decades have shown an increased interest in control allocation which is the process to solve for $u(t)$ while treating the control effectors individually. The control allocation problem is often cast using the system states to solve for $\vec{u}(t)$ given $\vec{m}(t)$ [1], or for each time step in discrete systems as:

$$\min_{\vec{u}} J(\vec{u}) = \vec{u}^T W_{gi} \vec{u} \text{ such that} \tag{3}$$

$$\vec{m} = B\vec{u}_{opt}, \quad \vec{u}_{lwr} \le \vec{u}_{opt} \le \vec{u}_{upr} \text{ elementwise} \tag{4}$$

where $W_{gi} > 0$ and thus the state derivative equation is of the form:

$$\dot{x}(t) = A(t)x(t) + m(t) \tag{5}$$

Alternatively, the controls allocation problem can be established for the output state [2] such as the Moore-Penrose allocation in discreet form:

$$\min_{\vec{u}} J(\vec{u}) = \vec{u}^T \vec{u}, \text{ such that} \tag{6}$$

$$\vec{a}_d = (CB)\,\vec{u}_{opt}, \quad \vec{u}_{lwr} \le \vec{u}_{opt} \le \vec{u}_{upr} \text{ elementwise} \tag{7}$$

Regardless of whether the control allocation problem formulation uses system states or system outputs, it is desirable to achieve control allocation solutions of linear form. Generalized inverses, such as the Moore-Penrose generalized inverse P_{min}, are a well known method to provide linearly weighted optimal solutions (e.g. $\vec{u}_{opt} = P_{min}\vec{m}$ where $P_{min} \in \Re^{n\times m}$). Some of the benefits of linear control allocation solutions include: simplicity and speed in computation of solutions (matrix vector multiplication), derivative of control allocation solution is readily available and they facilitate linear stability analysis. Unfortunately, it has been shown that generalized inverses (including the Moore-Penrose generalized inverse) are only applicable on a strict subset of the Attainable Moment Set (AMS)[1, 3]. The research in this paper, extends the previous work introducing the Prediction Method (PM) for optimal control allocation[4]. In particular, this research introduces a proposed method to compute a family of affine generalized inverses (constant matrix plus vector offset) which expands the coverage of generalized inverses to the entire Attainable Moment Set, thereby extending the linear allocation solution to the entire AMS.

III. Affine Moore-Penrose Generalized Inverse

The affine generalized inverse refers to a constant matrix $P_{aff} \in \Re^{m\times n}$ and constant vector $\vec{c}_0 \in \Re^m \cap \mathcal{N}(B)$ such that:

$$\vec{u}_{opt} = P_{aff}\vec{m}_{des} + \vec{c}_0, \quad \vec{m}_{des} \in \Phi \tag{8}$$

It should be noted that in the sequel, P_{aff} is shown to be a particular choice among the available generalized inverses. We are already aware of the existence of one Moore-Penrose affine generalized inverse that is applicable to a subset of Φ. Recalling from [4], that the AMS consists of two non-intersecting subsets $\Phi = \Phi_1 \cup \Phi_3$, then for $\vec{m}_{des} \in \Phi_1$ the choice of $P_{aff} = P_{min}$ and $\vec{c}_0 = \vec{0}$ satisfies Eq. (8). It has been shown [1, 3] that no choice of generalized inverse matrix is applicable to all of Φ. Therefore, we seek additional affine general inverses for the remaining portion of Φ or $\Phi_3 = \Phi \setminus \Phi_1$. In this section, the existence of a family of affine Moore-Penrose generalized inverses is rigorously derived $\forall \vec{m}_{des} \in \Phi_3$ and is proven equivalent to the solutions obtained by the PM which was previously shown to yield the Moore Penrose optimal control allocation solution [4]. The use of these affine MP generalized inverse matrices, in conjunction with the Moore-Penrose generalized inverse, enable the offline (non-iterative) computation of MP optimal controls allocation solution throughout the entire AMS as follows:

$$\vec{u}_{opt} = P_{min}\vec{m}_{des}, \quad \forall \vec{m}_{des} \in \Phi_1 \tag{9}$$

$$\vec{u}_{opt} = P_{aff_i}\vec{m}_{des} + \vec{c}_{0_i}, \quad \forall \vec{m}_{des} \in \phi_i \subset \Phi_3 \tag{10}$$

In the sequel, a numerical process using the prediction method and linearly independent moments vectors is shown to compute an affine generalized inverse. Next, some mathematical preliminaries are completed which enable a proof by construction method of analytically finding Moore-Penrose affine generalized inverse P_{aff_i} and the associated offset vector $\vec{c}_{0_i}$ which yields solutions identical to those of the Prediction Method.

A. Numerical Computation of Affine Generalized Inverse using the Prediction Method

This section documents a numerical process to compute P_{aff_i} for a given $\vec{m}_{des} \in \phi_i$ using the Prediction Method. The basics of the process are to use the Prediction Method on $\vec{m}_{des}$ and nearby permutations of $\vec{m}_{des} \in \phi_i$ to generate MP optimal control allocation solutions $\vec{u}_{pred}$. Expressing these linearly independent moments and the resulting control solutions offset by the vector $\vec{c}_0$ as $\mathcal{M}$ and $\mathcal{U}$ respectively, then P_{aff} is found using:

$$P_{aff}\mathcal{M} = (\mathcal{U} - I\vec{c}_0) \implies P_{aff} = (\mathcal{U} - I\vec{c}_0)\,\mathcal{M}^{-1} \tag{11}$$

Without a priori knowledge of the boundaries of $\phi_i \subset \Phi_3$, then the challenge is to permutate $\vec{m}_{des}$ while ensuring that the permutations remain within the selected ϕ_i. While the nature of P_{aff} has yet to be derived, it is subsequently shown that P_{aff_i} associated with ϕ_i requires a fixed set of saturated controls S_2, and Eq. (10) shows that we also require a fixed $\vec{c}_{0_i}$. Thus, for a given $\vec{m}_{des}$, any moment perturbations ($\vec{m}_j$) that result in a consistent set of saturated controls for each $\vec{u}_{pred_j}$ and yield a consistent $\vec{c}_0$ will enable the solution of P_{aff} as shown in Eq. (11). Recalling the general form of the predicted optimal control allocation solution from [4] as:

$$\vec{u}_{pred_i} = \hat{B}\begin{bmatrix} C_{02} \\ C_{03} \\ \vdots \\ C_{(m-n)+1} \end{bmatrix} + \|\vec{m}_i\|_2\,\hat{\mathcal{B}}\begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \\ \vdots \\ C_{(m-n)+1} \end{bmatrix}, \text{ where } \vec{c}_0 = \hat{B}\begin{bmatrix} C_{02} \\ C_{03} \\ \vdots \\ C_{(m-n)+1} \end{bmatrix} \tag{12}$$

Therefore for a particular $\phi_i \subset \Phi_3$, rewriting Eq. (11) in vector form using $\mathcal{M} := [\vec{m}_1, \vec{m}_2, \ldots, \vec{m}_n]$ and corresponding $\mathcal{U} := \left[\vec{u}_{pred_1}, \vec{u}_{pred_2}, \ldots, \vec{u}_{pred_n}\right]$ yields:

$$P_{aff}\begin{bmatrix}\vec{m}_1 & \vec{m}_2 & \ldots & \vec{m}_n\end{bmatrix} = \begin{bmatrix}\vec{u}_{pred_1} - \vec{c}_0 & \vec{u}_{pred_2} - \vec{c}_0 & \ldots & \vec{u}_{pred_n} - \vec{c}_0\end{bmatrix} \tag{13}$$

Since $\vec{m}_j$ are chosen such that they are linearly independent then $\begin{bmatrix}\vec{m}_1 & \vec{m}_2 & \ldots & \vec{m}_n\end{bmatrix}^{-1}$ exists which implies:

$$P_{aff} = \begin{bmatrix}\vec{u}_{pred_1} - \vec{c}_0 & \vec{u}_{pred_2} - \vec{c}_0 & \ldots & \vec{u}_{pred_n} - \vec{c}_0\end{bmatrix}\begin{bmatrix}\vec{m}_1 & \vec{m}_2 & \ldots & \vec{m}_n\end{bmatrix}^{-1} \tag{14}$$

So then for fixed $\phi_i \subset \Phi_3$, we have P_{aff} and can readily compute any optimal control solutions for $\vec{m}_{des} \in \phi_i$ as:

$$\vec{u}_{pred} = P_{aff}\vec{m}_{des} + \vec{c}_0 \tag{15}$$

Equation (14) has been shown in numerical examples to generate P_{aff}. Moreover, using Eq. (15), it has been shown to provide $\vec{u}_{pred}$ solutions for moments in the neighborhood of those used to generate P_{aff} with the same very high degree of precision ($\approx 10^{-12}$) as the solutions found using the Prediction Method.

B. Construction of Moore-Penrose Affine Generalized Inverse Equivalent to Prediction Method

Previously, a numerical method using linearly independent moment vectors and corresponding MP optimal control allocation solutions in the vicinity of a selected moment $\vec{m}_{des}$ was demonstrated. This methodology required n-linearly independent moment vectors and optimal control solutions in order to generate P_{aff}. While the prediction method has been proven to yield MP optimal control allocation solutions[4], the aforementioned numerically generated P_{aff} has not been shown to provide control allocation solutions which are MP optimal. In this section, an MP affine generalized inverse is constructed and it is proven to yield MP optimal solutions. More specifically, the MP affine generalized inverse is shown as equivalent to the PM solution which was shown to provide MP optimal control allocation solutions. As with the prediction method, the MP optimality using P_{aff} generated solutions is conditional upon having the correct lists of saturated control effectors (S_2,$\vec{s}_2$) for $\vec{m}_{des} \in \Phi_3$.

First, some mathematical preliminaries (including notation and two Propositions) are required which will enable the construction of an affine MP generalized inverse. For notation, there is utility in expressing associated vectors and matrices by dividing them into saturated and unsaturated portions (each in ascending order of indices). The subscript 1 will denote unsaturation while subscript 2 will denote saturation. To that end the following divisions are defined:

$$\vec{u} = \begin{bmatrix} \vec{u}_{unsat} \\ \vec{u}_{sat} \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \tag{16}$$

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \tag{17}$$

$$\hat{B}_1 = \hat{B}_{unsat} = \begin{bmatrix} \hat{b}_{1unsat} & \hat{b}_{2unsat} & \dots & \hat{b}_{(m-n)unsat} \end{bmatrix}$$

$$\hat{B}_2 = \hat{B}_{sat} = \begin{bmatrix} \hat{b}_{1sat} & \hat{b}_{2sat} & \dots & \hat{b}_{(m-n)sat} \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \tag{18}$$

$$P_{min} = \begin{bmatrix} P_{min_1} \\ P_{min_2} \end{bmatrix} \tag{19}$$

$$P_{aff} = \begin{bmatrix} P_{aff_1} \\ P_{aff_2} \end{bmatrix} \tag{20}$$

Proposition 1. *Given the generalized inverse $P_{aff} \in \Re^{m\times n}$ and the Moore-Penrose generalized inverse $P_{min} := B^T \left(BB^T\right)^{-1} \in \Re^{n\times m}$, then $\left(P_{aff} - P_{min}\right) = \hat{B}\hat{B}^T P_{aff}$.*

Proof. Recall that the columns of P_{min} form a basis (neither orthogonal nor unit vectors) for the range for $P_{min} \in \Re^n$. Additionally, $\hat{B} := \mathcal{N}(B)$ forms a basis with column vectors that are orthogonal unit vectors and which has previously been shown perpendicular to the columns of P_{min}. Then by linear algebra, together the columns of P_{min} and $\hat{B}$ form a basis that spans all of $\Re^m$. So then for the given affine generalized inverse P_{aff}, we seek to express the affine generalized matrix using the stated basis for $\Re^m$ or equivalently we seek to determine the matrix C such that :

$$P_{aff} = \begin{bmatrix} P_{min} & \hat{B} \end{bmatrix} C \tag{21}$$

Multiplying Eq. (21) by $\begin{bmatrix} P_{min} & \hat{B} \end{bmatrix}^T$ on the left yields:

$$\begin{bmatrix} P_{min} & \hat{B} \end{bmatrix}^T P_{aff} = \begin{bmatrix} P_{min} & \hat{B} \end{bmatrix}^T \begin{bmatrix} P_{min} & \hat{B} \end{bmatrix} C \tag{22}$$

Since $P_{min} \perp \hat{B}$ and the columns of $\hat{B}$ are orthogonal unit vectors ($\hat{B}^T\hat{B} = I$) then:

$$\begin{bmatrix} P_{min} & \hat{B} \end{bmatrix}^T P_{aff} = \begin{bmatrix} P_{min}^T P_{min} & 0^{n\times(m-n)} \\ 0^{(m-n)\times n} & I^{(m-n)\times(m-n)} \end{bmatrix} C \tag{23}$$

Now the matrix on the right is of block diagonal form and since $\left(P_{min}^T P_{min}\right)^{-1}$ exists, then its inverse is easily found as:

$$\begin{bmatrix} P_{min}^T P_{min} & 0^{n\times(m-n)} \\ 0^{(m-n)\times n} & I^{(m-n)\times(m-n)} \end{bmatrix}^{-1} = \begin{bmatrix} \left(P_{min}^T P_{min}\right)^{-1} & 0^{n\times(m-n)} \\ 0^{(m-n)\times n} & I^{(m-n)\times(m-n)} \end{bmatrix} \tag{24}$$

So therefore

$$C = \begin{bmatrix} \left(P_{min}^T P_{min}\right)^{-1} & 0^{n\times(m-n)} \\ 0^{(m-n)\times n} & I^{(m-n)\times(m-n)} \end{bmatrix} \begin{bmatrix} P_{min}^T \\ \hat{B}^T \end{bmatrix} P_{aff} = \begin{bmatrix} \left(P_{min}^T P_{min}\right)^{-1} P_{min}^T \\ \hat{B}^T \end{bmatrix} P_{aff} \tag{25}$$

Now using the Moore-Penrose generalized inverse $P_{min} = B^T \left(BB^T\right)^{-1}$ shows that:

$$\left(P_{min}^T P_{min}\right)^{-1} P_{min}^T = \left(\left(BB^T\right)^{-1} BB^T \left(BB^T\right)^{-1}\right)^{-1} \left(BB^T\right)^{-1} B = B \tag{26}$$

but since P_{aff} is a generalized inverse then $BP_{aff} = I$ which applied with Eq. (26) to Eq. (25) yields

$$C = \begin{bmatrix} I^{n\times n} \\ \hat{B}^T P_{aff} \end{bmatrix} \tag{27}$$

so then Eq. (27) applied to Eq.(21) yields

$$P_{aff} = \begin{bmatrix} P_{min} & \hat{B} \end{bmatrix} \begin{bmatrix} I^{n\times n} \\ \hat{B}^T P_{aff} \end{bmatrix} \implies \tag{28}$$

$$\left(P_{aff} - P_{min}\right) = \hat{B}\hat{B}^T P_{aff} \tag{29}$$

as claimed. □

Some examination of the nature of the vector $\left(P_{aff} - P_{min}\right)\vec{m} = \hat{B}\hat{B}^T P_{aff}\vec{m}$ is warranted. Since for any generalized inverse $BP_{gi} = I, \forall P_{gi}$, then $BP_{aff}\vec{m} = BP_{min}\vec{m} = \vec{m}, \forall \vec{m} \in \Phi$. then Eq. (29) implies:

$$B\left(P_{aff} - P_{min}\right)\vec{m} = (\vec{m} - \vec{m}) = 0 = B\left(\hat{B}\hat{B}^T P_{aff}\vec{m}\right) \tag{30}$$

which shows that $\hat{B}\hat{B}^T P_{aff}\vec{m} \in \mathcal{N}(B)$. For rationale that is explained subsequently, the special case of the generalized inverse P_{aff} is shown to be constructed by setting the columns of B_2 to zero vectors or equivalently:

$$\begin{aligned} P_{aff} &= \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} \left(\begin{bmatrix} B_1 & 0 \end{bmatrix} \begin{bmatrix} B_1^T \\ 0 \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} B_1^T \\ 0 \end{bmatrix} \left(B_1 B_1^T\right)^{-1} \implies \\ P_{aff_1} &= B_1^T \left(B_1 B_1^T\right)^{-1}, \quad P_{aff_2} = 0^{k\times n} \text{ where k = number saturated controls} \end{aligned} \tag{31}$$

So then revisiting Eq. (29) expressed in saturated and unsaturated components yields:

$$\begin{aligned} P_{min} &= \left(I - \hat{B}\hat{B}^T\right) P_{aff} \equiv \\ \begin{bmatrix} P_{min_1} \\ P_{min_2} \end{bmatrix} &= \left(I - \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}^T\right) \begin{bmatrix} P_{aff_1} \\ 0 \end{bmatrix} \\ \begin{bmatrix} P_{min_1} \\ P_{min_2} \end{bmatrix} &= \left(I - \begin{bmatrix} \hat{B}_1\hat{B}_1^T & \hat{B}_1\hat{B}_2^T \\ \hat{B}_2\hat{B}_1^T & \hat{B}_2\hat{B}_2^T \end{bmatrix}\right) \begin{bmatrix} P_{aff_1} \\ 0 \end{bmatrix} \implies \end{aligned}$$

$$P_{min_1} = \left(I - \hat{B}_1\hat{B}_1^T\right) P_{aff_1} \tag{32}$$

$$P_{min_2} = -\hat{B}_2\hat{B}_1^T P_{aff_1} \tag{33}$$

and therefore Eq. (28) can be written as:

$$\left(P_{aff} - P_{min}\right)\vec{m} = \hat{B}\hat{B}_1^T P_{aff_1}\vec{m} \tag{34}$$

Proposition 2. *Given* $\left(P_{aff} - P_{min}\right)\vec{m} = \hat{B}\hat{B}_1^T P_{aff_1}\vec{m}$, *then* $\hat{B}\hat{B}_1^T P_{aff_1}\vec{m} = -\hat{B}\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\vec{m}, \quad \forall\vec{m}\in\Phi$

Proof. The equivalence is shown by proving that

$$\hat{B}\hat{B}_1^T P_{aff_1}\vec{m} = -\hat{B}\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\vec{m} \iff \left(\hat{B}\hat{B}_1^T P_{aff_1} + \hat{B}\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\right)\vec{m} = 0, \quad \forall\vec{m}\in\Phi \tag{35}$$

Noting from Eq. (33) that $P_{min_2} = -\hat{B}_2\hat{B}_1^T P_{aff_1}$ then substituting into Eq. (35) and rearranging yields:

$$\hat{B}\left(\hat{B}_1^T - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2\hat{B}_1^T\right) P_{aff_1}\vec{m} = 0 \tag{36}$$

Now noting that $\mathcal{N}\left(\hat{B}\right) = \vec{0}$, then

$$\hat{B}\left(\hat{B}_1^T - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2\hat{B}_1^T\right) P_{aff_1}\vec{m} = 0 \iff \tag{37}$$

$$\left(\hat{B}_1^T - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2\hat{B}_1^T\right) P_{aff_1}\vec{m} = 0 \tag{38}$$

Now pulling out $\hat{B}_1^T$ from Eq. (38) in order to provide geometric insight yields:

$$\left(I - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2^T\right)\hat{B}_1^T P_{aff_1}\vec{m} = 0 \tag{39}$$

But $\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2^T$ is readily recognized as the projection operator $\mathcal{P}_{\hat{B}_2^T}$, which in turn shows that

$$\left(I - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2^T\right) = \mathcal{P}_{\hat{B}_2^{T\perp}} \tag{40}$$

As projection operators, $\mathcal{P}_{\hat{B}_2^T}$ and $\mathcal{P}_{\hat{B}_2^{T\perp}}$ have eigenvalues in $\{0,1\}$ and thus $\mathcal{N}\left(I - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2^T\right) \neq \vec{0}$. Similarly, $\mathcal{N}\left(\hat{B}_1^T\right) \neq \vec{0}$ and therefore we seek to show that:

$$P_{aff_1}\vec{m} \in \mathcal{N}\left(\hat{B}_1^T - \hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\hat{B}_2\hat{B}_1^T\right), \forall\vec{m} \tag{41}$$

Now for the choice of P_{aff} such that $P_{aff_2} = 0$, then from Eq. (31) we have that $P_{aff_1} = B_1^T\left(B_1B_1^T\right)^{-1}$, then we define $B_{1s} = B_1B_1^T$ which shows that:

$$P_{aff_1}B_{1s} = B_1^T\left(B_1B_1^T\right)^{-1}B_1B_1^T = B_1^T \tag{42}$$

Since we know that B times any column vector in the span of the null-space of B is identically $\vec{0}$, or equivalently that $B\hat{B} = 0$, then

$$\vec{0} = B\hat{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix}\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \implies -B_1\hat{B}_1 = B_2\hat{B}_2 \iff -\hat{B}_1^T B_1^T = \hat{B}_2^T B_2^T \tag{43}$$

Now thinking of B_{1s} as a matrix consisting of $\begin{bmatrix} \vec{m}_1 & \vec{m}_2 & \cdots & \vec{m}_n \end{bmatrix}$, then applying B_{1s} to Eq. (38) and utilizing Eq. (42) and Eq. (43) twice yields:

$$\left(\hat{B}_1^T - \hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} \hat{B}_2\hat{B}_1^T\right) P_{aff_1} B_{1s} \implies \tag{44}$$

$$\left(\hat{B}_1^T - \hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} \hat{B}_2\hat{B}_1^T\right) B_1^T \implies$$

$$\left(\hat{B}_1^T B_1^T - \hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} \hat{B}_2 \left(-\hat{B}_2^T B_2^T\right)\right) \implies$$

$$\left(\hat{B}_1^T B_1^T + \hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} \left(\hat{B}_2\hat{B}_2^T\right) B_2^T\right) \implies$$

$$\left(\hat{B}_1^T B_1^T + \hat{B}_2^T B_2^T\right) \implies$$

$$\left(\hat{B}_1^T B_1^T - \hat{B}_1^T B_1^T\right) = \begin{bmatrix} \vec{0}_1 & \vec{0}_2 & \cdots & \vec{0}_n \end{bmatrix} \tag{45}$$

Equation (45) shows that $\forall \vec{m}_i \in B_{1s}, i \in \{0, 1, \ldots, n\}$ we have $\left(\hat{B}_1^T - \hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} \hat{B}_2\hat{B}_1^T\right) P_{aff_1}\vec{m}_i = \vec{0}$, so then all linear combinations of the columns of B_{1s} (i.e. $B_{1s}\vec{x}, x \in \Re^n$) also yield $\vec{0}$. Now since $rank\left(B_1\right) = n$ then $rank(B_1) = rank\left(B_1 B_1^T\right) = rank\left(B_{1s}\right) = n$ where $B_{1s} \in \Re^{n\times n}$ and thus $\left(B_{1s}\right)^{-1}$ exists [5]. So finally,

$$\forall \vec{m} \in \Re^n, !\exists \vec{x} \in \Re^n, \text{ such that } \vec{m} = B_{1s}\vec{x}, \text{ where } \vec{x} := \left(B_{1s}\right)^{-1} \vec{m} \tag{46}$$

Thus $\forall \vec{m} \in \Re^n$, Eq. (41) holds. Since the zero matrix is the only matrix for which the null-space consists of the entire space ($\Re^n$), then $\hat{B}\hat{B}_1^T P_{aff_1}\vec{m} = -\hat{B}\hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\vec{m}$ as claimed. □

Finally, recalling from [4] the general form of the Prediction Method optimal control allocation solution for given S_2 and $\vec{s}_2$:

$$\vec{u}_{opt} = \vec{c}_0 + \vec{c}_1 + P_{min}\vec{m}_{des}, \text{ where} \tag{47}$$

$$\vec{c}_0 := \hat{B}\hat{B}_{sat}^T \left(\hat{B}_{sat}\hat{B}_{sat}^T\right)^{-1} \vec{s}_2 \tag{48}$$

$$\vec{c}_1 := -\hat{B}\hat{B}_{sat}^T \left(\hat{B}_{sat}\hat{B}_{sat}^T\right)^{-1} \left(P_{min}\vec{m}_{des}\right)_{sat} \tag{49}$$

Now separating Eqs. (50-52) into saturated and unsaturated components yields:

$$\vec{u}_{opt} = \vec{c}_0 + \vec{c}_1 + P_{min}\vec{m}_{des}, \text{ where} \tag{50}$$

$$\vec{c}_0 := \hat{B}\hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} \vec{s}_2 \tag{51}$$

$$\vec{c}_1 := -\hat{B}\hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\vec{m}_{des} \tag{52}$$

and therefore by Propositions (1,2), we have that

$$\vec{c}_1 = \left(P_{aff} - P_{min}\right)\vec{m} = -\hat{B}\hat{B}_2^T \left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\vec{m} \tag{53}$$

with $\vec{c}_1$ as computed by the Prediction Method.

Some physical insight is available with further examination. Since, the columns of $\hat{B}$ are orthogonal unit vectors then $\hat{B}^T\hat{B} = I$, so then pre-multiplying Eq. (53) by $\hat{B}_2\hat{B}^T$ yields:

$$\hat{B}_2\hat{B}^T\vec{c}_1 = -P_{min_2}\vec{m} \tag{54}$$

Recalling from [4], that multiplying a control vector in $P_{iso} \subset \mathcal{N}\left(B\right)$ by $\hat{B}^T$ converts the vector to the Gain Subspace, and then multiplying by $\hat{B}_2$ or only the saturated control portions of $\hat{B}$, converts the gains into only the saturated components in the Control Subspace. So separating $\vec{c}_1$ into unsaturated and saturated components, $\vec{c}_1 := \begin{bmatrix} \vec{c}_{1_1} \\ \vec{c}_{1_2} \end{bmatrix}$, then Eq. (54) implies that $\vec{c}_{1_2} = \hat{B}_2 \hat{B}^T \vec{c}_1 = -P_{min_2} \vec{m}$ or that the saturated components of $\vec{c}_1$ are equivalent to $-P_{min_2} \vec{m}$. Following a similar analysis for Eq. (51), we see that $\vec{s}_2 = \hat{B}_2 \hat{B}^T \vec{c}_0$, which shows that the saturated components of $\vec{c}_0$ are identically the saturated limits. Finally, in the Proposition 2 we have seen that:

$$\hat{B} \hat{B}_1^T P_{aff_1} \vec{m} = \hat{B} \hat{B}_2^T \left(\hat{B}_2 \hat{B}_2^T \right)^{-1} \hat{B}_2 \hat{B}_1^T P_{aff_1} \vec{m}, \quad \forall \vec{m} \in \Re^m \tag{55}$$

where $\hat{B}_2^T \left(\hat{B}_2 \hat{B}_2^T \right)^{-1} \hat{B}_2 = \mathcal{P}_{\hat{B}_2^T}$. Since Eq. (55) has been shown true, then $\hat{B}_1^T P_{aff_1} \vec{m} \in$ span$\{\hat{B}_2^T\}, \forall \vec{m}$.

Now construction of P_{aff} which yields solutions equivalent to those using the Prediction Method is according to the following theorem.

Theorem 1. *Given $\vec{m}_{des} \in \phi \subset \Phi$, the set of associated saturated controls S_2 and the corresponding vector of saturated limits $\vec{s}_2$, then $\exists P_{aff}$ (Moore-Penrose Affine Generalized Inverse) so that $!\exists \vec{u}_{opt}$ which is 2-norm minimal such that $\vec{u}_{opt} = \vec{c}_0 + P_{aff} \vec{m}_{des}$ where $\vec{c}_0 \in \mathcal{N}(B)$, $B\vec{u}_{opt} = \vec{m}_{des}$ and $\vec{u}_{opt} \in \Omega$.*

Proof. This proof is shown by construction. Specifically, for a given set of saturated controls, a MP affine generalized inverse is constructed and then shown to yield equivalent solutions to the Prediction Method which was shown in [4] to yield MP optimal control allocation solutions.

Case 1: $\vec{m}_{des} \in \phi = \Phi_1$

Choose $P_{aff} = P_{min} := B^T \left(BB^T \right)^{-1}$ and $\vec{c}_0 = \vec{0}$ then as shown in [4], $!\exists \vec{u}_{opt} = P_{min} \vec{m}_{des} + \vec{0}$ which is 2-norm minimal, where $\vec{c}_0 \in P_{Iso}$ and $B\vec{u}_{opt} = \vec{m}_{des}$ and $\vec{u}_{opt} \in \Omega$.

Case 2: $\vec{m}_{des} \in \phi \subset \Phi_3$

Recalling the general form of the generalized inverse (with positive semi-definite weighting matrix):

$$P_{gi} := W_{gi} B^T \left(B W_{gi} B^T \right)^{-1} \tag{56}$$

then the Moore-Penrose generalized inverse is the special case for which $W_{gi} = I$. Now reorganizing P_{gi} into saturated and unsaturated components which rewriting Eq. (56) yields:

$$\begin{bmatrix} P_{gi_1} \\ P_{gi_2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \left(\begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \right)^{-1} \tag{57}$$

Now we define P_{aff} by setting those diagonal elements of weight matrix which correspond to the saturated components to zero (or equivalently setting the columns of the B_2 matrix to all zeros) we have $W_{gi} \geq 0$ and then:

$$P_{aff} = \begin{bmatrix} P_{aff_1} \\ P_{aff_2} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \left(\begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \right)^{-1} \implies \tag{58}$$

$$P_{aff} = \begin{bmatrix} P_{aff_1} \\ P_{aff_2} \end{bmatrix} = \begin{bmatrix} B_1^T \left(B_1 B_1^T \right)^{-1} \\ 0 \end{bmatrix} \tag{59}$$

Now choosing a solution of the form $\vec{u}_{opt} = \vec{c}_0 + \vec{c}_1 + P_{min} \vec{m}_{des}$, then in particular we define $\vec{c}_0$ as:

$$\vec{c}_0 = \hat{B} \hat{B}_2^T \left(\hat{B}_2 \hat{B}_2^T \right)^{-1} \vec{s}_2 \tag{60}$$

and choose $\vec{c}_1$ as:

$$\vec{c}_1 = \left(P_{aff} - P_{min} \right) \vec{m} \tag{61}$$

then for these choices of $\vec{c}_0$ and $\vec{c}_1$ we have the desired MP Affine Generalized inverse solution:

$$\vec{u}_{opt} = \vec{c}_0 + \left(P_{aff} - P_{min} \right) \vec{m}_{des} + P_{min} \vec{m}_{des} = \vec{c}_0 + P_{aff} \vec{m}_{des} \tag{62}$$

but cannot yet make any claims on 2-norm optimality nor uniqueness. Now noting for the special case of $P_{aff_2} = 0$, we showed that $\vec{c}_1 = \left(P_{aff} - P_{min}\right)\vec{m} = -\hat{B}\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1} P_{min_2}\vec{m}_{des}$ (see Proposition 2). Therefore we have:

$$\vec{u}_{opt} = \vec{c}_0 + P_{aff}\vec{m}_{des} \iff \tag{63}$$

$$= \hat{B}\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\left(\vec{s}_2 - P_{min_2}\vec{m}_{des}\right) + P_{min}\vec{m}_{des} \tag{64}$$

which is equivalent to the Prediction Method solutions in Eqs. (50-52). Thus for a given $\vec{m}_{des}$, saturated control set S_2 and associated saturated limits vector $\vec{s}_2$, with the choice of W_{gi} with the saturated diagonal elements set to zero, then $\exists P_{aff}$, and $\vec{c}_0 = \hat{B}\hat{B}_2^T\left(\hat{B}_2\hat{B}_2^T\right)^{-1}\vec{s}_2$, such that $!\exists\vec{u}_{opt} = \vec{c}_0 + P_{aff}\vec{m}_{des}$ where $\vec{c}_0 \in \mathcal{N}(B)$, $B\vec{u}_{opt} = \vec{m}_{des}$ and $\vec{u}_{opt} \in \Omega$ with $\vec{u}_{opt}$ 2-norm minimal as claimed. □

IV. Derivative

Typical flight applications for the controls allocation problem require adherence to both control effector position ($\vec{u}_{lwr}$ and $\vec{u}_{upr}$) and rate limitations ($\vec{r}_{lwr}$ and $\vec{r}_{upr}$). While both the prediction and affine methods for computing MP optimal control allocation solutions ensure compliance with control effector position requirements ($\vec{u}_{lwr} \leq \vec{u}_{opt} \leq \vec{u}_{upr}$), the analytical tools to ensure rate limitation compliance are detailed in this section. First, various forms of the time-derivative of MP optimal control allocation solutions ($\frac{d}{dt}\vec{u}_{opt}$) are derived $\forall\vec{m}_{des} \in \Phi$ and limitations on the existence of the derivative are discussed. Subsequently, the derived forms of the time derivative are used to aid in control effector rate limitation compliance.

A. Computation of Total Derivative

Recalling the chain rule of differentiation for a quantity of several variables: $M = \mathcal{F}(t, p_1, p_2, \ldots p_n)$

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \sum_{i=1}^{n}\frac{\partial M}{\partial t}\frac{dp_i}{dt} \tag{65}$$

The quantity we seek to differentiate is the vector $\vec{u}_{opt}$ and it would generally be expressed as a function of:

$$\vec{u}_{opt} = \mathcal{F}(t, \vec{m}_{des}) \tag{66}$$

In this case, as $\vec{u}_{opt}$ is not expressly a function of time:

$$\frac{\partial\vec{u}_{opt}}{\partial t} = \vec{0} \tag{67}$$

so then:

$$\dot{u}_{opt} = \frac{d\vec{u}_{opt}}{dt} = \frac{\partial\vec{u}_{opt}}{\partial\vec{m}_{des}}\frac{\partial\vec{m}_{des}}{\partial t} \tag{68}$$

Note that $\vec{u}_{opt}$ differentiable requires that all the partial derivatives exist and that these are continuous. It is shown subsequently that while the partial derivatives exist throughout Φ, they are not continuous everywhere. In particular, the partial derivative discontinuities occur at the boundaries of the subsets within Φ (e.g. $\Phi_1 \cap \phi_i$ or $\phi_i \cap \phi_j$) where the set of saturated controls S_2 changes.

B. Total Derivative Using Moore-Penrose Affine Generalized Inverse

Previously, it was shown that an affine generalized inverse exists $\forall\vec{m}_{des} \in \Phi$, where in particular the MP optimal control allocation solution is found using:

$$\vec{u}_{opt} = P_{min}\vec{m}_{des}, \quad \forall\vec{m}_{des} \in \Phi_1 \tag{69}$$

$$\vec{u}_{opt} = P_{aff_i}\vec{m}_{des} + \vec{c}_{0_i}, \quad \forall\vec{m}_{des} \in \phi_i \subset \Phi_3 \tag{70}$$

Then applying Eq. (68) for $\vec{m}_{des} \in \Phi_1$ yields:

$$\frac{\partial \vec{u}_{opt}}{\partial \vec{m}_{des}} (P_{min} \vec{m}_{des}) = P_{min} \implies \tag{71}$$

$$\frac{d\vec{u}_{opt}}{dt} = P_{min} \frac{\partial \vec{m}_{des}}{\partial t}, \forall \vec{m}_{des} \in \Phi_1 \tag{72}$$

Similarly, $\vec{m}_{des} \in \phi_i$ Eqs. (68,70) yield:

$$\frac{\partial \vec{u}_{opt}}{\partial \vec{m}_{des}} \left(P_{aff_i} \vec{m}_{des} + \vec{c}_{0_i} \right) = P_{aff_i} \implies \tag{73}$$

$$\frac{d\vec{u}_{opt}}{dt} = P_{aff_i} \frac{\partial \vec{m}_{des}}{\partial t}, \forall \vec{m}_{des} \in \phi_i \tag{74}$$

which shows the existence of the vector partial derivative $\partial \vec{u}_{opt} / \partial \vec{m}_{des}, \forall \vec{m}_{des} \in \Phi$. Recalling that previously we have shown (see Eq. 59) that the rows of each P_{aff_i} can sorted into unsaturated and saturated components. Similarly, we can sort $\vec{u}_{opt}$ such that:

$$\frac{d}{dt} \vec{u}_{opt} = \dot{u}_{opt} = \begin{bmatrix} \dot{u}_{opt_1} \\ \dot{u}_{opt_2} \end{bmatrix} \tag{75}$$

(76)

then we obtain:

$$\frac{d\vec{u}_{opt}}{dt} = \begin{bmatrix} \dot{u}_{opt_1} \\ \dot{u}_{opt_2} \end{bmatrix} = \begin{bmatrix} P_{aff_1} \\ 0 \end{bmatrix} \frac{\partial \vec{m}_{des}}{\partial t} \tag{77}$$

where the time derivative of the MP optimal control allocation solution for the saturated components ($\dot{u}_{opt_2}$) is zero as expected.

However while the partial derivatives exist, the nature of the boundaries between subsets of Φ ensures that these partial derivatives are not continuous. In particular, the transition from Φ_1 to ϕ_i, requires transition from P_{min} to P_{aff} for a change in the set of saturated limits S_2, or similarly the change from one subset ϕ_i to an adjacent one ϕ_j requires a change in the set of saturated controls S_2. This change typically occurs as one of the components $\vec{u}_{opt_k}$ of the MP optimal control allocation vector saturates (or unsaturates).

For example, when transitioning from ϕ_i to ϕ_j along a constant $\hat{m}_{des}$, given $\vec{m}_{des} = m\hat{m}_{des}$, then as the scalar $m \in \Re$ increases at a constant rate ($\dot{m}$ = const), from Eq. (74) we see that for $\vec{m}_{des} \in \phi_i$:

$$\frac{d\vec{u}_{opt}}{dt} = \dot{m} P_{aff_i} \hat{m}, \implies \dot{u}_{opt_k} = a_k(\text{constant}), \forall k \in S_1 \text{ and } \dot{u}_{opt_k} = 0, \forall k \in S_2 \tag{78}$$

Now as m continues to increase, then $m\hat{m}_{des} \in \phi_i \cap \phi_j$, and changes occur to the sets of unsaturated controls S_1', saturated controls S_2' and the affine generalized inverse P_{aff_j}, so then we have:

$$\frac{d\vec{u}_{opt}}{dt} = \dot{m} P_{aff_j} \hat{m}, \implies \dot{u}_{opt_k} = b_k(\text{constant}), \forall k \in S_1' \text{ and } \dot{u}_{opt_k} = 0, \forall k \in S_2' \tag{79}$$

In this simple example, the changes across the Φ boundary cause a "step" function in the time derivatives for each component of $\dot{u}_{opt_k}$. Each component $\dot{u}_{opt_k}$ transitions from one constant to another (e.g. a_k changes to either 0 or b_k) and therefore the partial derivatives are not continuous.

However, in the interior of any subset of Φ, the partial derivatives are continuous since neither the P_{min},P_{aff_i} nor S_1,S_2 change. To demonstrate this, starting with the definition of continuity, the vector function f is continuous at the point $\vec{p}$ if given $\epsilon > 0, \exists \delta > 0$ such that:

$$\|\vec{p} - \vec{x}\|_2 < \delta \implies \|f(\vec{p}) - f(\vec{x})\|_2 < \epsilon \tag{80}$$

Note also for matrix norms we know that for $A\vec{x}$, that $\|A\vec{x}\|_2 \leq \|A\|_2 \, \|\vec{x}\|_2$. Now we seek to examine the derivative continuity at the point $\dot{u}_{opt}$. So $\forall \dot{m}'$, such that $\vec{m}_{des} + \dot{m}' dt \in \Phi_1$, then given $\epsilon > 0$ we choose $\delta = \epsilon / (\|P_{min}\|_2 \, \|B\|_2)$

so then by Eq. (72):

$$\begin{aligned}\dot{u}_{opt} - \dot{u}'_{opt} &= P_{min}\left(\dot{m}_{des} - \dot{m}'\right),\ \text{since } BP_{min} = I \implies \\ B\left(\dot{u}_{opt} - \dot{u}'_{opt}\right) &= \left(\dot{m}_{des} - \dot{m}'\right) \implies \\ \left\|\left(\dot{m}_{des} - \dot{m}'\right)\right\|_2 &= \left\|B\left(\dot{u}_{opt} - \dot{u}'_{opt}\right)\right\|_2 \leq \|B\|_2 \left\|\dot{u}_{opt} - \dot{u}'_{opt}\right\|_2 \end{aligned} \tag{81}$$

So then

$$\begin{aligned}&\left\|P_{min}\left(\dot{m}_{des} - \dot{m}'\right)\right\|_2 \leq \|P_{min}\|_2 \left\|\dot{m}_{des} - \dot{m}'\right\|_2 \leq \|P_{min}\|_2 \|B\|_2 \left\|\dot{u}_{opt} - \dot{u}'_{opt}\right\|_2 \\ &\quad \text{But } \left\|\dot{u}_{opt} - \dot{u}'_{opt}\right\|_2 < \delta \implies \\ &\left\|P_{min}\left(\dot{m}_{des} - \dot{m}'\right)\right\|_2 \leq \|P_{min}\|_2 \|B\|_2 \frac{\epsilon}{\|P_{min}\|_2 \|B\|_2} = \epsilon \text{ shows that}\end{aligned} \tag{82}$$

$$\left\|\dot{u}_{opt} - \dot{u}'_{opt}\right\|_2 < \delta \implies \left\|P_{min}\left(\dot{m}_{des} - \dot{m}'\right)\right\|_2 = \left\|P_{min}B\left(\dot{u}_{opt}\right) - P_{min}B\left(\dot{u}'_{opt}\right)\right\|_2 < \epsilon \tag{83}$$

which shows that the partial derivative $\frac{\partial \vec{u}_{opt}}{\partial \vec{m}_{des}}$ is continuous. Thus the existence and continuity of the partial derivative shows that $d\vec{u}_{opt}/dt$ is differentiable $\forall \vec{m} \in \Phi_1$. The same logic holds for $\phi_i \subset \Phi$ using Eq. (73), and therefore $d\vec{u}_{opt}/dt$ is differentiable within individual subsets of $\phi_i \subset \Phi_3$, however, differentiability fails at the points of intersection between adjacent subsets.

C. Incorporating Control Effector Rate Limitations

In this section, the incorporation of control effector rate limitations is discussed. Given minimum $\vec{r}_{lwr}$ and maximum $\vec{r}_{upr}$ control effector rate limitation vectors, with $\vec{r}_{lwr} < \vec{r}_{upr}$ where:

$$\vec{r}_{lwr} = \begin{bmatrix} r_{lwr_1} & r_{lwr_2} & \dots & r_{lwr_m} \end{bmatrix}^T \text{ with } r_{lwr_i} \in \Re \forall i = 1\dots m \tag{84}$$

$$\vec{r}_{upr} = \begin{bmatrix} r_{upr_1} & r_{upr_2} & \dots & r_{upr_m} \end{bmatrix}^T \text{ with } r_{upr_i} \in \Re \forall i = 1\dots m \tag{85}$$

then reordering the control effector elements into unsaturated and saturated groups respectively yields:

$$\vec{r}_{lwr} = \begin{bmatrix} \vec{r}_{lwr_1} \\ \vec{r}_{lwr_2} \end{bmatrix} \tag{86}$$

$$\vec{r}_{upr} = \begin{bmatrix} \vec{r}_{upr_1} \\ \vec{r}_{upr_2} \end{bmatrix} \tag{87}$$

$$\tag{88}$$

So similarly reordering the time-derivative of the MP optimal control vector:

$$\frac{d}{dt}\vec{u}_{opt} = \dot{u}_{opt} = \begin{bmatrix} \dot{u}_{opt_1} \\ \dot{u}_{opt_2} \end{bmatrix} \tag{89}$$

$$\tag{90}$$

since we desire

$$\begin{bmatrix} \vec{r}_{lwr_1} \\ \vec{r}_{lwr_2} \end{bmatrix} \leq \begin{bmatrix} \dot{u}_{opt_1} \\ \dot{u}_{opt_2} \end{bmatrix} \leq \begin{bmatrix} \vec{r}_{upr_1} \\ \vec{r}_{upr_2} \end{bmatrix} \tag{91}$$

Now by Eqs. (72,74), for the interior of the respective subsets (where the derivative exists), we have:

$$\begin{bmatrix} \vec{r}_{lwr_1} \\ \vec{r}_{lwr_2} \end{bmatrix} \leq \begin{bmatrix} P_{min_1} \\ P_{min_2} \end{bmatrix} \frac{\partial \vec{m}_{des}}{\partial t} \leq \begin{bmatrix} \vec{r}_{upr_1} \\ \vec{r}_{upr_2} \end{bmatrix}, \forall \vec{m}_{des} \in \Phi_1 \setminus \delta\left(\Phi_1\right) \tag{92}$$

$$\vec{r}_{lwr_1} \leq P_{aff_1} \frac{\partial \vec{m}_{des}}{\partial t} \leq \vec{r}_{upr_1},\ \text{with } \dot{u}_{opt_2} = \vec{0}, \forall \vec{m}_{des} \in \phi_i \setminus \delta\left(\phi_i\right) \tag{93}$$

Noting that $BP_{min} = I$ and similarly $B_1 P_{aff_1} = I$, then Eqs. (92,93) show

$$B\vec{r}_{lwr} \leq \frac{\partial \vec{m}_{des}}{\partial t} \leq B\vec{r}_{upr}, \forall \vec{m}_{des} \in \Phi_1 \setminus \delta(\Phi_1) \tag{94}$$

$$B_1\vec{r}_{lwr_1} \leq \frac{\partial \vec{m}_{des}}{\partial t} \leq B_1\vec{r}_{upr_1}, \forall \vec{m}_{des} \in \phi_i \setminus \delta(\phi_i) \tag{95}$$

Equations (92,93) show that management of the time rate of change of the desired moment vector (element wise) will enable compliance with control effector rate limitations.

V. Offline Methodology for Moore-Penrose Affine Generalized Inverse

The Prediction Method detailed in [4] is an iterative algorithm that for a given $\vec{m}_{des}$, starts at the origin and proceeds along $\hat{m}_{des} := \vec{m}_{des}/\|\vec{m}_{des}\|$ until the maximum moment is achieved in the direction of $\hat{m}_{des}$. As described, the Prediction Method is therefore relegated to be an online control allocation routine. However, with the proposed affine generalized inverse presented in this research, an opportunity is available to obtain optimal Moore-Penrose optimal control allocation throughout the entire AMS using an offline algorithm.

The following is an outline of a proposed offline affine generalized inverse algorithm. To implement an offline algorithm, then for a given $\vec{m}_{des}$, the algorithm would require the ability to determine which subset of the AMS ($\vec{m}_{des} \in \Phi_1$ or $\vec{m}_{des} \in \phi_i \subset \Phi_3$) contains the given moment vector or equivalently which portion of the $\delta(\Omega)$ is nearest. For any $\vec{m}_{des} \in \Phi_1$, the process could be accomplished by utilizing the P_{min} matrix to compute the optimal solution $\vec{u}_{opt}$ and then verifying that $\vec{u}_{opt} \in \Omega$. For $\vec{m}_{des} \in \Phi_3$, the determination of the correct subset ϕ_i could be accomplished with a moment lookup table, however this requires knowledge of the moment boundaries for each subset $\phi_i \in \Phi_3$.

Establishment of the moment boundaries for all subsets $\phi_i \in \Phi_3$ requires multiple phases. First, the determination of the particular subsets ϕ_i is required such that the union of the subsets is identically Φ_3. The determination of the total number of subsets required is a combinatorics problem. Specifically, the number of ϕ_i subsets required for a given number of saturated controls k is the k-combination defined as:

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} \tag{96}$$

where n is the total number of control effectors. So for example, the total number of subsets h required for $B \in \Re^{3\times 10}$ would be:

$$h = \begin{pmatrix} 10 \\ 2 \end{pmatrix} + \begin{pmatrix} 10 \\ 3 \end{pmatrix} + \cdots + \begin{pmatrix} 10 \\ 7 \end{pmatrix} \tag{97}$$

Now that the individual subsets are determined, then the moment boundaries of each subset are required. A process in [6] details a method to determine the moment area (n=2) or volume (n=3) associated with any generalized inverse. This process yields the knowledge of the moment boundaries that we seek. This process would be applied to the affine subset (including the offset vector $\vec{c}_0$) to determine the moment boundaries. Lastly, as noted in [4], there are a large number of instances of control effector unsaturation which occur along a specified $\hat{m}_{des}$. Since each P_{aff_i} is applicable over the subset ϕ_i, then the moment boundaries of each ϕ_i may be further constrained by control effector unsaturation which occurs based on the nearest portion of the boundary of Ω. Therefore, based on the large number of moment subsets and the work required to determine each subset's moment boundaries, the process of generating a moment lookup table for $\vec{m}_{des} \in \Phi_3$ is non-trivial.

VI. Conclusions

This research introduced a family of proposed affine Moore-Penrose generalized inverses which provide Moore-Penrose optimal control allocation solutions equivalent to those of the Prediction Method detailed in [4]. These affine MP generalized inverses are valid over a subset $\phi_i \subset \Phi_3$ which removes the requirement of validity along a given $\hat{m}_{des}$ inherent with the Prediction Method. Additionally, these affine MP generalized inverses are shown to provide the derivative of the optimal control solutions and further aid in ensuring control effector rate compliance. A potential offline control algorithm $\forall \vec{m}_{des} \in \Phi$ using the affine generalized inverse is outlined. Finally, the affine Moore-Penrose generalized inverse is expanded to any arbitrary weighted affine generalized inverse in the Appendix of this paper.

Follow on research by this author explores in detail the relationship between the Prediction Method and Cascading Generalized Inverse algorithms. Various methods to ensure the local optimization problem utilizes the nearest portion of the boundary of the AMS (or equivalently updates the list of saturated control effectors S_2) are discussed. Techniques to minimize the Prediction Method computation times for the Moore-Penrose optimal control allocation are described. Additionally, the problem of bounding the neighborhoods ϕ_i or equivalently determining the region for which P_{aff_i} is applicable are detailed.

Appendix

Affine Arbitrary Generalized Inverse

The previous work demonstrating the existence of an Affine Moore-Penrose Generalized Inverse is restated (for completeness) as the Affine Arbitrary Generalized Inverse. The relevant equations are:

$$\vec{c}_0 = \hat{B}\left(\hat{B}^T W_{gi}\hat{B}\right)^{-1}\hat{B}_2^T\left(\hat{B}_2\left(\hat{B}^T W_{gi}\hat{B}\right)^{-1}\hat{B}_2^T\right)^{-1}\vec{s}_2 \tag{98}$$

$$\vec{c}_1 = -\hat{B}\left(\hat{B}^T W_{gi}\hat{B}\right)^{-1}\hat{B}_2^T\left(\hat{B}_2\left(\hat{B}^T W_{gi}\hat{B}\right)^{-1}\hat{B}_2^T\right)^{-1} P_{gi_2}\vec{m}_{des} \tag{99}$$

$$\vec{u}_{opt} = \vec{c}_0 + \vec{c}_1 + P_{gi}\vec{m}_{des} \tag{100}$$

$$\vec{u}_{opt} = \vec{c}_0 + P_{gi_{aff}}\vec{m}_{des} \tag{101}$$

where

$$P_{gi_{aff}} := \begin{bmatrix} P_{gi_{aff_1}} \\ P_{gi_{aff_2}} \end{bmatrix} \tag{102}$$

$$P_{gi_{aff_1}} := \left(W_{gi_1}^T\right)^{-1} B_1\left(B_1\left(W_{gi_1}^T\right)^{-1} B_1^T\right)^{-1} \tag{103}$$

$$P_{gi_{aff_1}} := 0 \tag{104}$$

where W_{gi_1} consists of the unsaturated rows and columns of W_{gi}. For the case of $W_{gi} = I$, the MP affine generalized inverse form is recovered as expected.

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