Study network-related optimization problems using quantum alternating optimization ansatz

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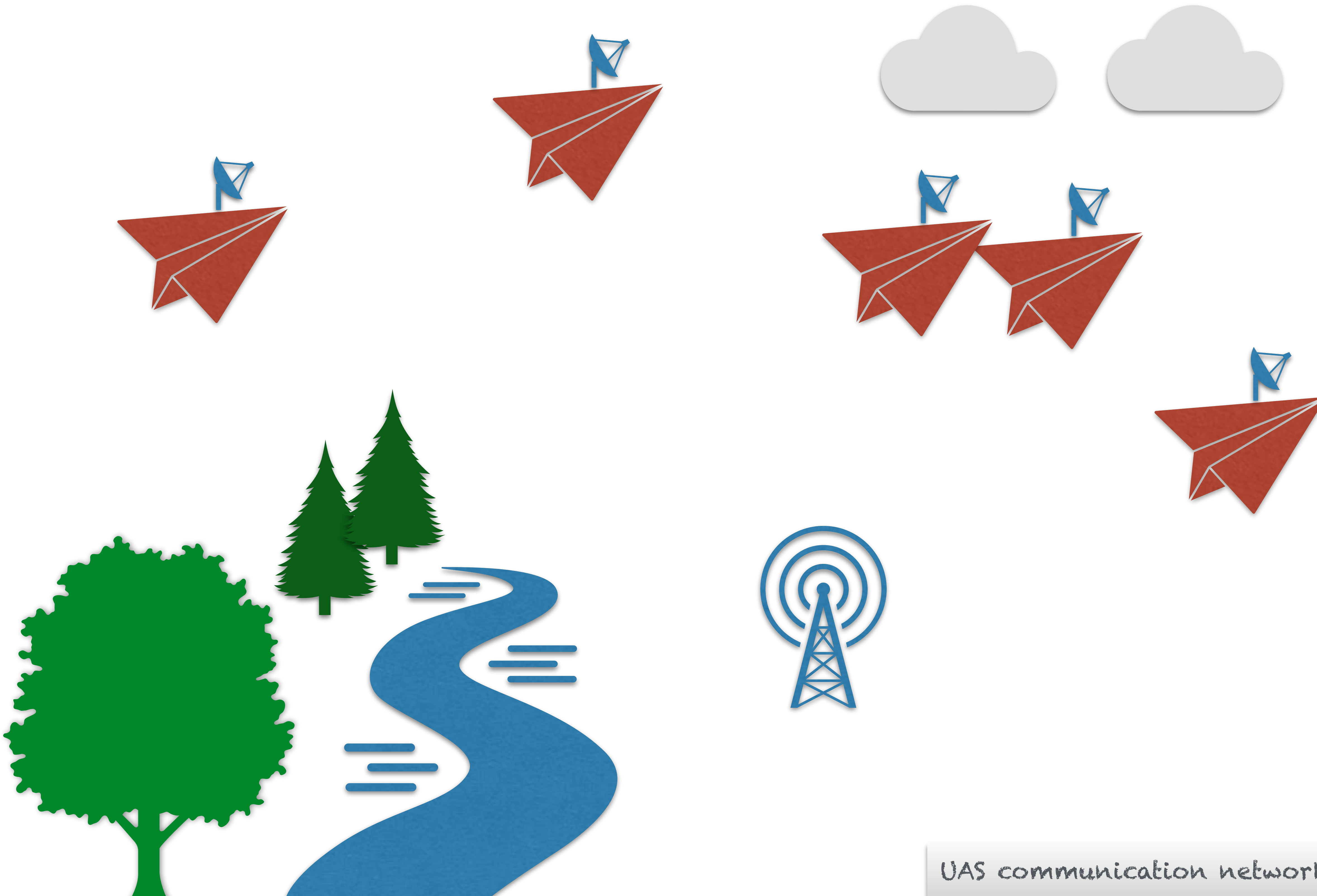


UAS communication network problem
NASA QuAIL

Relevance of spanning-tree/graph problem to applications

- Network connectivity problem is key to many applications.
  We study quantum heuristics (QAOA) on some network problems.

  Two basic models:
  - Spanning tree, a super basic model: minimum spanning tree. A spanning tree guarantees communication, and further cost can be minimized over the set of spanning trees.
  - Spanning graph, a model for robustness: while maintain an instantaneous spanning tree ensures connectivity at the moment, when temporal relation is considered, extra connectivity may be in favor for an overall optimal network.

  A variety of cost function based on constraints can piggyback on these two models.

Relevance of spanning-tree/graph problem to applications

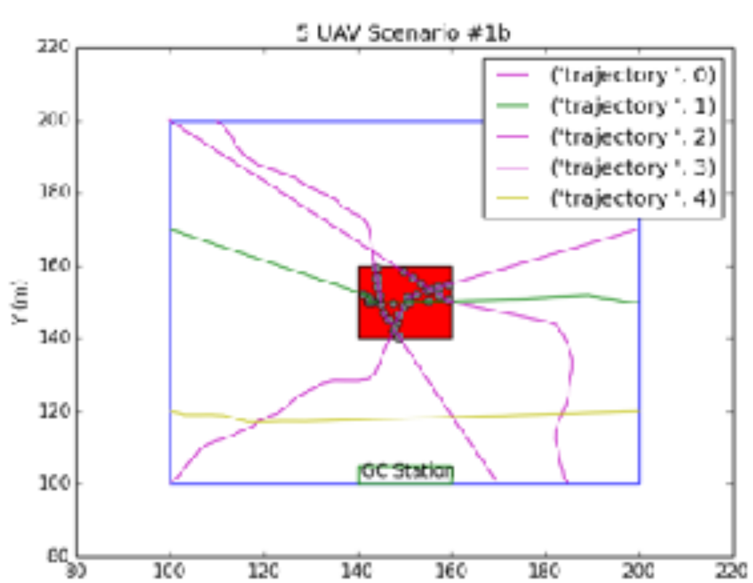
Example Testing model: Pre-determined network disruption area

- known trajectory and know disruption areas *S*
- Vehicle outside *S* can talk directly to ground control station
- Vehicle inside *S* will need to communicate with vehicles not in *S* to talk to the ground station

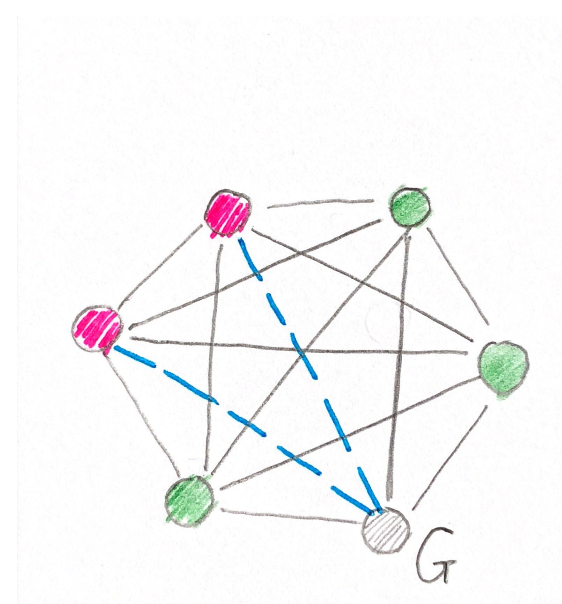
The requirement “stay connected” is equivalent to the connectivity-problem for graph *G*, in which each vehicle (as well as the ground station) corresponds to a node, and an edge exist between two vehicles can directly communicate.

Questions:

- What is the minimum communication distance? — minimizing edge weight on a **spanning tree** of *G*
- Over time, if each time the routing is changed, a cost is incurred, what is the least expensive routing plan? — **spanning subgraph** could be favorable.



Initial set of trajectories for a 5 UAV scenario for testing the quantum algorithms.



Red nodes: in disruption area
Green nodes: not in disruption area
Grey node: ground station
Blue lines: missing connections, not included in the graph

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Quantum Approximate Optimization Algorithms (QAOA)

- Quantum Heuristics suitable for near-term quantum hardware
- Hard Constraints: have to be satisfied in a solution
- Soft Constraints: cost to be optimized
- Underlying idea: search in a ***feasible*** subspace
  - Start from a ***feasible*** state: one that satisfies all hard constraints
  - Mixing (create coherent superposition) of all feasible states
- Challenges:
  - Come up with classical mixing rules that maintains the feasible subspace and corresponding quantum mixing operators
  - Acquire a good set of parameters (angles)

[Farhi, Goldstone, and Gutmann, '14]

[Hadfield, arXiv 1709.03489]

One formula summary of the algorithm

$$U = \underline{e^{-i\beta_p H_M}} e^{-i\gamma_p H_C} \ldots e^{-i\beta_2 H_M} \underline{e^{-i\gamma_1 H_C}}$$

Mixing: explore the solution space   Cost: associate with problem characteristics

Mixer for the Spanning Tree Problem

- Hard constraints:

  the solution being a spanning tree. — No loops

  Mixer:

  Basic move: For a node *v*, swap its parents between (*p*, *q*),

  conditioned on *p*, *q* are not descendants of *v.*

  Realized through manipulation of two binary matrices.

  Descendent Matrix ***D***: whether *u* is *v*'s descendant.

  Parent Matrix ***A***: whether *u* is *v*'s parent.

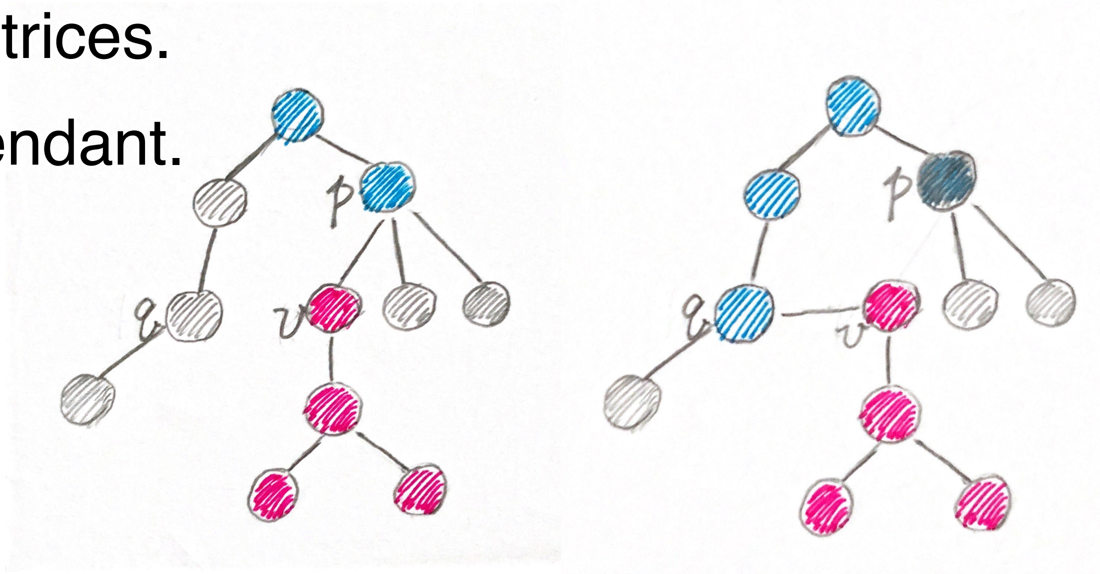
  Reversible move:

  Swap parents of *v*

$$\Lambda[\bar{D}_{vp} \wedge \bar{D}_{vq}]\ \mathrm{SWAP}(A_{pv}, A_{qv})$$

  The descendant relation between any descendants of v and any (ancestor of *p* XOR ancestor of *q*), is negated.

$$\Lambda\left[D_{v,l} \wedge (D_{kp} \neq D_{kq}) \wedge \bar{D}_{vp} \wedge \bar{D}_{vq} \wedge (A_{pv} \neq A_{qv})\right] \mathrm{NOT}(D_{kl})$$



Mixer for the Spanning Tree Problem

For each (v, p, q)

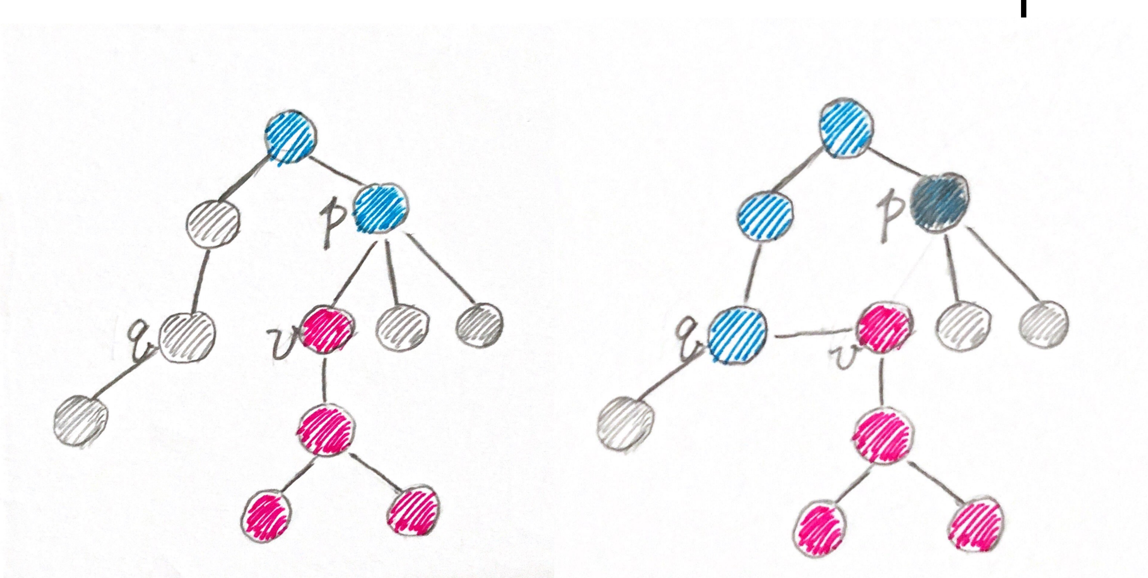
Classical operation	**Quantum operators**
$\Lambda[\bar{D}_{vp} \wedge \bar{D}_{vq}]$ SWAP(A_{pv}, A_{qv})	$M^{(A)}_{v;p,q} = \lvert 00\rangle\langle 00 \rvert_{D_{vp}D_{vq}} \otimes (\lvert 01\rangle\langle 10\rvert + \lvert 10\rangle\langle 01\rvert)_{A_{pv},A_{qv}}$

Further for each (k, l)

$\Lambda\Big[D_{v,l} \wedge (D_{kp} \neq D_{kq}) \wedge \bar{D}_{vp} \wedge \bar{D}_{vq} \wedge (A_{pv} \neq A_{qv})\Big]$ NOT(D_{kl})	$M^{(D)}_{v;p,q;k,l} = P_{A_{pv}A_{qv}} \otimes \lvert 00\rangle\langle 00\rvert_{D_{vp}D_{vq}} \otimes \Big(Q_{D_{kp}D_{kq}} \otimes \lvert 1\rangle\langle 1\rvert_{D_{vl}} \otimes X_{D_{kl}} + (I - Q_{D_{kp}D_{kq}} \otimes \lvert 1\rangle\langle 1\rvert_{D_{vl}}) \otimes I_{D_{kl}}\Big)$ where $P \equiv \lvert 10\rangle\langle 10\rvert + \lvert 01\rangle\langle 01\rvert$ and $Q \equiv I - \lvert 00\rangle\langle 00\rvert$

Mixing operator for (v, p, q): A and D updated together

$$
\begin{aligned}
H^M_{v;p,q} &= M^{(A)}_{v;p,q} \prod_{(k\neq v, l\notin\{p,q\})} M^{(D)}_{v;p,q;k,l} \\
&= |00\rangle\langle 00|_{D_{vp}D_{vq}} \\
&\quad \otimes(|01\rangle\langle 10| + |10\rangle\langle 01|)_{A_{pv},A_{qv}} \\
&\quad \otimes \prod_{(k\neq v, l\notin\{p,q\})} \Big(Q_{D_{kp}D_{kq}} \otimes |1\rangle\langle 1|_{D_{vl}} \otimes X_{D_{kl}} \\
&\quad + (I_{D_{kp}D_{kq}} I_{D_{vl}} - Q_{D_{kp}D_{kq}} \otimes |1\rangle\langle 1|_{D_{vl}}) \otimes I_{D_{kl}} \Big)
\end{aligned}
$$



Mixer for the **Spanning Graph** Problem

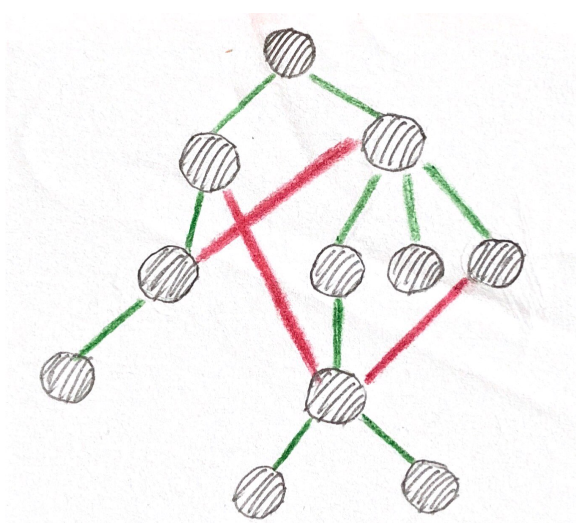
Spanning tree + more connectivity   Potential applications: more robust network

Underlying idea for a mixer:

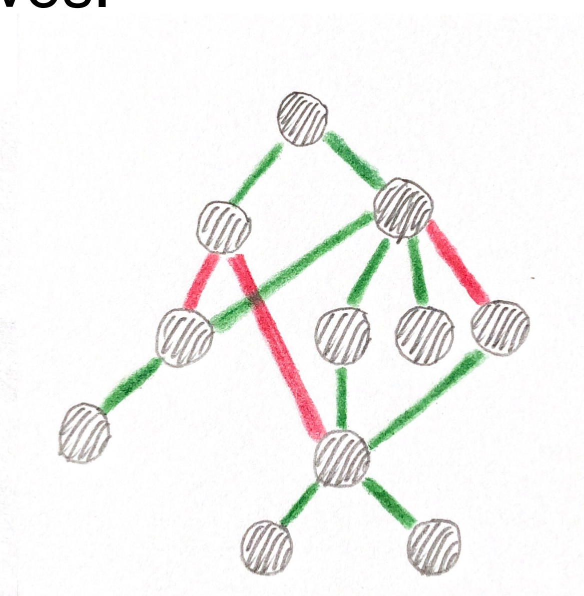
Each spanning graph *G'* can be partitioned into a spanning tree ***T***, and the set of extra edges ***Ø***. There are multiple partitions.

With a spanning tree *T* fixed, adding/removing different sets of extra edges (edges that are not in T) lead to different spanning graphs.

Therefore, mediated by the a mixer for spanning tree, we can explore all spanning graphs through these two moves.



$E(G') = (\mathbf{E(T)}, \boldsymbol{\emptyset})$



$E(G') = (\mathbf{E(T')}, \boldsymbol{\emptyset'})$

- Mixer between spanning graphs with the same backbone spanning tree

$$G' = (T, \Phi) \longleftrightarrow G'' = (T, \Phi')$$

For each edge $\{u, v\}$

$$H_{(u,v)}^{(\Phi)} = \underbrace{|00\rangle\langle 00|_{A_{uv}A_{vu}}}_{\text{edge } \{u,v\} \text{ not in the tree}} \otimes \underbrace{X_{\Phi_{u,v}}}_{\text{add/remove } \{u,v\} \text{ from } \Phi}$$

Corresponding unitary

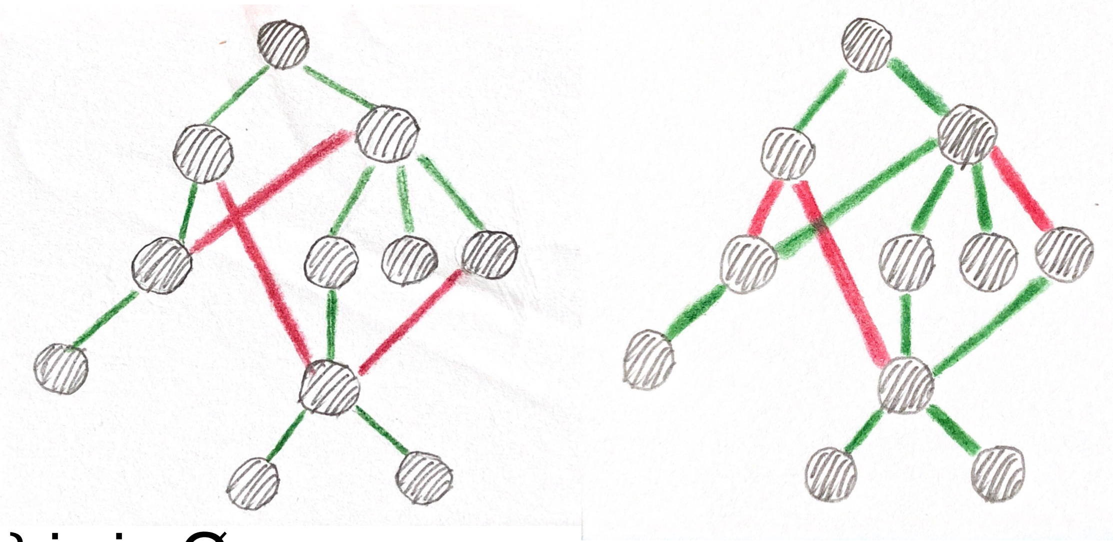
$$\begin{aligned} U_{(u,v)}^{(\Phi)}(\beta) &= \exp[-i\beta H_{(u,v)}^{(\Phi)}] \\ &= |00\rangle\langle 00|_{A_{uv}A_{vu}} \otimes e^{-i\beta X_{\Phi_{u,v}}} \\ &\quad +(I_{A_{uv},A_{v,u}} - |00\rangle\langle 00|_{A_{uv},A_{v,u}}) \otimes I_{\Phi_{u,v}} \end{aligned}$$

Total: $m = |E|$ doubly controlled single-qubit gates, each $\sim$ 12 C-NOT gates

- Mixer between partitions of a spanning $G' = (T, \Phi) = (T', \Phi')$

  Quantum register: $|D, A, \Phi\rangle$

  G' and *G* share spanning-trees, thus the spanning-tree mixer can be applied to any *G*'



Modification in parents swapping (v, p, q):

Consider *p to be parent* of *v* only when edge {*v*, *p* } is in Ø;
Disable a parental relation between q and v, and place the edge {*v*, *q* } in Ø;
i.e., *E*(*T*) and Ø swaps elements {*v*, *p* } and {*v*, *q* }

Mixing Hamiltonian:

$$H^M_{(v,p,q)} = \underbrace{|00\rangle\langle 00|_{D_{vp}D_{vq}}}_{} \otimes M^{(A\Phi)}_{pv,qv} \otimes M^{(D)}_{v,p,q}$$

neither *p* or *q* is descendant of *v*

$$M^{(A\Phi)}_{pv,qv} \equiv (|0110\rangle\langle 1001| + |1001\rangle\langle 0110|)_{A_{pv},A_{qv},\Phi_{pv},\Phi_{qv}}$$ ***E*(*T*) and Ø swaps edges (*v*, *p*) and (*v*, *q*)**

$$M^{(D)}_{v,p,q} \equiv \prod_{\underbrace{(k\neq v,\ l\notin\{p,q\})}_{}} M^{(D)}_{v;p,q;k,l}$$ **For each (*v,p,q*), all *(k,l*) need to be updated together.**

$$M^{(D)}_{v;p,q;k,l} = \Big(Q_{D_{kp}D_{kq}} \otimes |1\rangle\langle 1|_{D_{vl}} \otimes X_{D_{kl}}$$ **Update *D for* relevant *l's***

$$+ (I_{D_{kp}D_{kq}} \otimes I_{D_{vl}} - Q_{D_{kp}D_{kq}} \otimes |1\rangle\langle 1|_{D_{vl}}) \otimes I_{D_{kl}}\Big)$$ **Identity for irrelevant *l's***

Mixing Unitary:

$$
\begin{aligned}
U^{M}_{(v,p,q)}(\beta) &= \exp(-i\beta H^{M}_{(v,p,q)}) \\
&= \underline{|00\rangle\langle 00|_{D_{vp}D_{vq}}} \otimes \exp\left[-i\beta M^{(A\Phi)}_{pv,qv} \otimes M^{(D)}_{v,p,q}]\right)\Big] \\
&\quad +\big(I - |00\rangle\langle 00|_{D_{vp}D_{vq}}\big) \otimes I_{\text{others}}
\end{aligned}
$$

double-control

$$M^{(A\Phi)}_{pv,qv} \equiv (|0110\rangle\langle 1001| + |1001\rangle\langle 0110|)_{A_{pv},A_{qv},\Phi_{pv},\Phi_{qv}}$$

—> 4 qubits gates: XXYY + XYXY + XYYX +…

$$M^{(D)}_{v,p,q} \equiv \prod_{\underline{(k\neq v,\ l\notin\{p,q\})}} M^{(D)}_{v;p,q;k,l}$$

For each (*v,p,q*), all (*k,l*) need to be updated together.
Leads to high-locality of physical qubits and deep circuit depth

$$
\begin{aligned}
M^{(D)}_{v;p,q;k,l} = \Big(&Q_{D_{kp}D_{kq}} \otimes |1\rangle\langle 1|_{D_{vl}} \otimes X_{D_{kl}} \\
&+\big(I_{D_{kp}D_{kq}} \otimes I_{D_{vl}} - Q_{D_{kp}D_{kq}} \otimes |1\rangle\langle 1|_{D_{vl}}\big) \otimes I_{D_{kl}}\big)\Big)
\end{aligned}
$$

single-qubit operation controlled by a 3-q projector

$$H^{(\Phi)}_{(u,v)} = |00\rangle\langle 00|_{A_{uv}A_{vu}} \otimes X_{\Phi_{u,v}}$$

Resource requirement

Number of qubits $N_{qubit} = (n-1)^2 + 2m + m$

Qubits connectivity

Total number of connection between qubits: $\sim n^4$

Maximum qubit degree: $\sim n^2$

— High locality caused by simultaneous update of $\sim n^2$ elements of D

Break-up:

D: form a clique of size $(n-1)^2$

$D - A, \Phi$: Each D_{uv} is connected to all elements of A, and Φ

Therefore, each element of D is connected to all other qubits

$\rightarrow$ Num. links: $(n-1)^2(N_{\text{qubit}} - 1) \sim n^4$

$A_{pv}A_{qv} \rightarrow$ All edges injecting to vertex v (elements of column A_v) form a clique of size d_v. Num. links: $\sum_v \frac{d_v(d_v-1)}{2}$

$A_{uv}A_{vu} \rightarrow$ A link exists between A_{uv} and A_{vu} for any edge $\{u, v\}$. Num links: m

$A_{pv}\Phi_{qv}$ and $A_{pv}\Phi_{pv} \rightarrow$ A link exists between A_{pv} and Φ_{qv} for all q. Num. links: $\sum_v d_v^2$

$\Phi_{pv}\Phi_{qv} \rightarrow$ All edges that connects v form a clique of size d_v. Num. links: $\sum \frac{d_v(d_v-1)}{2}$

Resource requirement, example

Assume all-to-all connectivity between physical qubits on hardware, and a complete problem graph

n	Num. of physical qubits required
3	13
4	27
5	46
6	70
7	99
8	133
9	172
10	216

Application relevant (demo number of UAV's) problem size
Current / near-term gate-model quantum hardware size

Summary

QAOA for constrained optimization problem could greatly benefit from using a problem-dependent mixer.

We developed a QAOA mapping for the problem of degree-constrained spanning-tree and spanning-graph problems that could be feasible to test on NISQ devices.

Next:

simulation;

test runs on quantum hardware

performance analysis