NASA/TM–2020-220570

Wind Tunnel Balance Design: A NASA Langley Perspective

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March 2020
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**Nomenclature**

\( \delta \) displacement
\( \epsilon \) engineering strain
\( \nu \) Poisson’s ratio
\( \sigma \) uniaxial stress
\( \tau \) shear stress
\( \theta \) rotational displacement
\( A \) area
\( AF \) axial force
\( b \) relating to the base of a beam
\( bs \) bearing stress
\( d \) diameter
\( DB \) relating to a double bending mode of deflection
\( dwl \) dowel
\( E \) Young’s modulus
\( f \) relating to the flex beams in the axial section
\( G \) shear modulus
\( gage \) related to the stress under the gage location
\( GF \) gage factor
\( GL \) distance of the centerline of the active grid of a strain gage from a given axis
\( h \) relating to the height of a beam or horizontal direction
\( I \) bending moment of inertia
\( i \) beam group i
\( k \) spring stiffness
\( l \) length of beam
\( m \) relating to the measurement beams in the axial section
$m, s$ relating to the measurement beam and strap in the axial section

$max$ maximum

$N$ load proportion

$n$ number of beams, etc.

$NF$ normal force

$PM$ pitching moment

$RM$ rolling moment

$s$ relating to the strap above the measuring beams in the axial section

$SB$ related to a single bending mode of deflection

$SF$ side force

$T$ total

$Ts$ relating to the T-section near the axial flex beams

$V$ volts

$v$ relating to vertical direction

$YM$ yawing moment
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1 Introduction

This document chronicles and captures force balance design principles accumulated over the last 50 years at NASA Langley Research Center (LaRC). Many design and guidance internal memorandums have been drafted over the years, some as informal as a design engineer’s notes for a particular balance design. To date, there has been no comprehensive balance design or best practices documentation available on LaRC’s extensive design expertise. Therefore, the primary goal of this knowledge capture document is to assemble and organize the presentation of this historical corpus of knowledge for future balance engineers.

1.1 Strain Gage Wind Tunnel Balance Terminology

Forces and moments represent the predominant measurements made in wind tunnel testing campaigns today. The majority of wind tunnel tests require some form of load measurements where the word load implies a force or moment. In the early years of aeronautical research, the instruments were purely mechanical in nature borrowing from the weight measurement systems of the era and they were located outside the model. Their design led to the use of the term “balance” as initially forces were measured by balancing known weights against steady aerodynamic loads, often via linkages that led outside the test section. An internal balance is installed inside a model versus externally to the test article. Figure 1 gives examples of internal and semi-span strain gage balance designs used at NASA.

![Figure 1: Internal and Semi-Span Strain Gage Balance Examples: (a) Small 20 lb. Normal Force Internal Balance used in 31-Inch Mach 10 Wind Tunnel (LaRC) (b) Large 5000 lb Normal Force Internal Balance used in 14 x 22 ft Subsonic Wind Tunnel (LaRC) (c) Large 30000 lb Normal Force Semi-span Balance used at 11 ft Transonic Tunnel (ARC).]

The term balance is now used to refer to both internal and semi-span designs Hufnagel et al. [1]. Figure 2 shows a half section view of a notional aircraft wind tunnel model with an internal balance and sting. The sting, which is the support arm for the model positioning system, is connected to the tapered, non-metric end of the balance. The aircraft is connected to the balance by a precision cylindrical fit to the metric end, meaning the measuring end.
1.2 A Brief History of Strain Gage Balance Design

Invention of the strain gage is credited to Arthur Ruge who in 1938 embedded a high-resistance wire, arranged in a zig-zag pattern, in a carrier material Hoffman [2]. Interest in strain gage development at NACA followed directly de Forest and Leaderman [3]. By 1952, Paul Eisler had successfully created an “electric resistance device” by etching a zig-zag pattern from a metal foil with a technique similar to printed circuit board manufacture Hoffman [2]. The foil strain gages used in many modern internal balances still resemble this design in that a metal foil is photo-etched on a plastic carrier backing. Constantan is the oldest and most widely used metal for foil strain gages, due to its relatively high strain sensitivity and low temperature sensitivity Micro-Measurements [4]. Gages are affixed to measurement elements with a thin coating of adhesive, and are typically coated with a moisture proofing sealcoat after wiring.

The concept of using strain-gage-based internal balances for aerodynamic measurements began in the 1940s. As an example, consider Figure 3, the patent of N.L. Miller filed in 1949 which describes a six-component strain-gage-based internal balance design shown in a missile model.

A LaRC balance design from the mid 1950s is depicted in Figure 4 and shares many of the features found on current balance designs including a separate axial force measurement section and cage section for measurement of all other loads Hansen [5]. The non-metric end, or ground support non-measuring end, of the balance attaches to a sting. The tapered connection with a key and locking nut is a standard sting attachment method.
By the 1960s the use of internal balances had become widespread Pankhurst and Holder [6]. Figure 5 shows a sting supported model of the Apollo Escape System with an internal strain gage balance in the NASA LaRC 16-Foot Transonic Wind Tunnel.

1.3 LaRC Balance Design

While there are a variety of balance design types applications, throughout this document will focus on the NASA LaRC single piece internal balances which are utilized in wind tunnel testing. An example of one of these balances, the NTF-113B, is shown in Figure 6. Single piece balances are fabricated by machining one piece of stock material and consist of many intricate structural flexures. These flexures are then instrumented with strain gages to sense the structural strain induced by the forces acting on the balance. Precise machining techniques paired with advanced data acquisition systems and statistically designed calibration procedures allow NASA LaRC balances to achieve accuracies under 0.05% of the full scale range of the balance.
Six-component strain gage balances have the ability to measure three moments (pitch, roll, and yaw) as well as three forces (normal, axial, and side). The forces measured by the balance are then converted to aerodynamic loads acting on the wind tunnel model. A balance is depicted in a model in Figure 7. The forces and moments shown represent positive polarities based on historical practice and will follow this convention throughout the document.

![Figure 7: Internal Balance in Model.](image-url)

In balance design, the primary objective is to ensure the transducer meets measurement quality specifications. A simultaneous design objective is to ensure the safety of the design through detailed stress analysis. Safety consideration in balance design is critical as the balance is the only structural link between the model and the tunnel structure.

In the design of a balance there are competing criteria. As a measurement device, it is desired to increase strain measured by the gage which in turn increases voltage output and subsequently resolution. However, from a mechanical design perspective, it is desired to limit the stress in the balance for increased factors of safety and fatigue life. This is one of the many design trade-offs encountered in balance design. As will be seen, the design of a balance is a multi-dimensional, highly correlated problem. It is unlikely that changing one design feature will not impact other parameters.

This guide covers the development of analytical mechanics formulations that estimate the stress level and output of the transducer. These mechanics formulations have been refined over the years and have been shown to provide reliable estimates of the gage outputs of traditional LaRC balances. These closed-form equations help the designer to quickly iterate to a practical design during the initial design phase. Most modern designs are then supplemented with finite element analysis to better estimate maximum stress levels and to identify stress concentration areas. The analytical mechanics formulations provide valuable validation information for the finite element model. Thus, the analytical formulations and finite element analysis are complementary tools.

### 1.4 Nomenclature

The following section outlines the nomenclature format that will be used throughout this document. Many of the symbols and subscripts are consistent with those generally accepted and utilized in traditional solid mechanics and mechanical design texts. All other symbols and subscripts utilized in this TM were either carried forward from legacy...
NASA memorandums or created specifically for use in this document. The general form of the nomenclature used throughout this document is of the form

\[ \text{Symbol}_{\text{subscript1}}_{\text{subscript2}}_{\text{subscript3}} \]  

where symbol represents the dependent variable being solved for (e.g. \( \sigma \) is a common variable throughout the document representing uniaxial stress), subscript1 is a description or abbreviation of the quantity (e.g. max), subscript2 is a location or feature (e.g. m stands for the measurement beam in the axial section throughout this document), and subscript3 is the applied load (e.g. AF for an applied axial force).

Taken together, \( \sigma_{\text{max}_{\text{mAF}}} \) represents the maximum uniaxial stress on the axial measuring beam due to an applied axial force. In some cases, only two of the three subscripts are required to completely define a variable; for instance, \( N_{\text{flex}_{\text{AF}}} \), represents the load proportion carried by the flex beams due to an applied axial force.

## 2 Electrical Design Considerations

The design of a balance is a parallel effort of mechanical and electrical system design. This section will discuss the electrical design considerations for a new balance. Foil strain gages act as the transducer elements converting measured strain on the balance to electrical output. This is accomplished through the elongation (or contraction) of the wire traces that make up the gage which causes a change in the overall resistance of the gage. A diagram of a strain gage is shown in Figure 8.

![Figure 8: Image of a Strain Gage Depicting the Active Grid, Backing Material, and Solder Pads.](image)

The strain gages are arranged in a Wheatstone bridge configuration. The Wheatstone bridge is a differential wiring strategy used in a broad range of measurement instrumentation. It is used to amplify the change in strain gage resistances. In a Wheatstone bridge configuration designed to maximize output, half of the resistors in the bridge increase in resistance while the other half decrease in resistance under the predominant load of interest. This is accomplished by locating one pair of strain gages in compression and the other pair of strain gages in tension. Therefore, for most balance measurements, a minimum of four separate strain gages are used. Figure 9 demonstrates that gaging strategy for a cantilever beam subject to a force P. The left side of the beam experiences tension subject to P while the right side of the beam experiences compression.
The output of the bridge is a function of the gage factor which can be thought of as how efficiently the gage converts strain to an electrical signal. The higher the gage factor, the better the gage can “amplify” the signal. Typical gages used at NASA LaRC have a gage factor of approximately 2.2. The output of the Wheatstone bridge, in microvolts per volt of excitation voltage is given by

$$\text{Output} \left( \frac{\mu V}{V} \right) = \sigma \left( \frac{GF}{E} \right) \times 10^6$$

(2)

where $GF$ is the gage factor, $\sigma$ is the stress measured by the gage, and $E$ is the modulus of elasticity of the balance material. Gages used for LaRC balances have generally ranged in size from an active grid size of 0.040 inches by 0.050 inches to 0.062 inches by 0.125 inches.

Strain gages are susceptible to changes in resistance with factors other than applied loadings, such as temperature and humidity. For a detailed discussion of the use and application of strain gaging, the reader is referred to NASA TM-110327 Moore [7].

At NASA LaRC, individual strain gages are wired into a Wheatstone bridge and balanced such that the natural voltage offset from zero is maintained within $\pm$ 400 microV/V. This is done primarily because the strain gage data acquisition meters generally have limited ranges over which they can measure. When the balance is exposed to varying temperatures, there is a shift in the bridge output which should be minimized. A change in output of no more than 3-5 microV/V is desired for temperatures varying from 80°F to 180°F. To obtain this tolerance, the balance outputs are recorded while it is heated in an oven. The output is then corrected using a sensitive resistor or thin wire in series with one of the bridge gages as shown in Figure 9. Iteration of the process may be required until the correct resistance value is found. This on-board, analog compensation process may be further enhanced through mathematical modeling of the temperature effect on the balance signal.

3 Materials

Maraging and precipitation hardened steels are commonly used materials to manufacture balances. They have excellent spring characteristics and are some of the strongest and
stiffest materials in tension and compression. Moreover, these materials have a good combination of machinability, ductility, and fracture toughness. Aluminum alloy 7075-T6 is also used at LaRC to manufacture balances. Table 1 lists materials commonly used to manufacture balances along with their respective Young’s modulus, yield strength, and ultimate tensile strength.

<table>
<thead>
<tr>
<th>Property</th>
<th>200 CVM</th>
<th>300 CVM</th>
<th>17-4PH, 15-5PH</th>
<th>7075-T6 Al</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s Modulus (psi)</td>
<td>26.2E6</td>
<td>27.5E6</td>
<td>28.5E6</td>
<td>1.04E7</td>
</tr>
<tr>
<td>Yield Strength (ksi)</td>
<td>207</td>
<td>290</td>
<td>175</td>
<td>73</td>
</tr>
<tr>
<td>UTS (ksi)</td>
<td>212</td>
<td>294</td>
<td>190</td>
<td>83</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.258</td>
<td>0.300</td>
<td>0.295</td>
<td>0.33</td>
</tr>
<tr>
<td>Shear Modulus (psi)</td>
<td>10.4E6</td>
<td>10.57E6</td>
<td>11E6</td>
<td>3.91E6</td>
</tr>
<tr>
<td>Stress for 1000 counts (ksi)</td>
<td>11.963</td>
<td>12.557</td>
<td>13.014</td>
<td>4.727</td>
</tr>
<tr>
<td>Heat Treatment</td>
<td>900/925°F for 6 hours; air cool (RC 41-45)</td>
<td>900/925°F for 6 hours; air cool (RC 52-55)</td>
<td>915/935°F for 4 hours; air cool (RC 40-42)</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Common Materials used to Manufacture Balances at NASA LaRC.

4 Axial Section Design

This section will introduce the methods of calculating stresses in the axial section as well as general guidelines and best practices for the design of the axial section of a NASA LaRC single piece balance.

4.1 Introduction

The axial section of the balance is often the most challenging to design and analyze. From an aerodynamicist’s perspective, the most stringent requirements on accuracy and precision of wind tunnel test data are placed on drag measurements, predominantly derived from axial force when the model is near zero attitude. Therefore, measurement of drag is primarily resolved from deformation in the axial section of the balance. When the conventional internal single piece balances were first designed, see Chandler and Guarino [8], axial sections resembled those shown in Figure 10(a) which was referred to as the “double cantilever” beam design. The nomenclature referred to both the way the beam was supported and the manner in which it deformed under an applied axial force. The double cantilever beam geometry tended to deform in a double or “S” bending manner. When double or “S” bending occurs, the stress at the centerline of the measurement beam (inflection point) is effectively zero. This presented a problem in small balances due to steep stress gradient making it challenging to obtain sufficient sensitivity due the gage length relative to the beam length. To mitigate the problems associated with the double cantilever beam, alternative designs of the axial section like those shown in Figures 10(b) and 10(c) were explored. However, each of the designs presents challenges with respect to gaging, Figure 10(c), and temperature effects, Figure 10(b). For each figure below, the flex beams are labeled with an “f” and the measurement beam with an “m” and positive axial force is to the right.
Figure 10: Evolution of the Axial Section. Positive Axial Force is to the Right. f = Flex Beam, m = Measurement Beam, and s = Strap.

The slotted T configuration (Figure 10(d)) has been used in NASA LaRC balances since the early 1960s (soon after electrical discharge machining techniques became available) as an axial force measuring element. This configuration has multiple advantages over the earlier designs of the “double cantilever” beam. The first advantage is in isolating the applied axial force from other force components, namely the applied normal force. The relatively low spring constant of the strap allows the strap to flex when a normal force is applied, limiting the amount of stress in the measurement beam due to the normal force. Second, the slotted T axial section design allows the measurement beam to deflect in more of a “single cantilever” manner. Furthermore, when the measurement beam deforms in this manner, the stress gradient is reduced, improving the sensitivity challenges compared to a double-cantilevered beam. A finite element model of the strap design showing the (exaggerated) deflection of the flex beams and measurement beam under an applied axial force is shown in Figure 11.
In the measurement beam of a slotted T configuration, the highest stress (measurable by a strain gage) occurs at the root of the measurement beam, opposite from the T-strap. Over short distances, and neglecting stress concentration effects, it is assumed the stress decreases linearly away from the bulkhead. Thus, the closer the gage can be bonded to the root of the measurement beam, the higher the stress that can be measured for a given applied force. The gages used for most LaRC balances (Micro-Measurements SK-01-050AH) have a distance from the edge of the gage to the center of the active grid of 0.050 in (50 thousandths) and a typical radius at the bulkhead-measurement beam junction is 0.020 in (20 thousandths). With this configuration, the shortest length from the center of the active grid of the strain gage to the end of the beam junction is 0.070 in (see Figure 12).

An annotated drawing of an axial section is shown in Figure 12 which shows the nomenclature used by LaRC balance engineers.
4.2 Axial Section Stress equations

This section outlines the derivation of the stress equations for the axial section of the balance. The analysis is divided into subsections pertaining to the stress analysis due to each separate component of applied force. Closed form analytical equations for the slotted-T axial section stresses were first developed by Guarino and Chandler [8] and later modified and updated by Scott [9], Scott [10], Rhew [11] and others. The general procedure for solving for the stresses is to first find the spring constants of the individual components from their displacements under an applied force, then the load proportion on each component can be found which in-turn leads to the calculation of stresses. The stress analysis equations allow for the estimation of strain measured by the gage which determines the balance precision. The analysis also gives an estimate of the factor of safety of the design by approximating the maximum stress. The calculation of the factor of safety estimation assumes that the maximum stresses due to all six components of load occur at the same location on the balance and are thus superimposed. This is a conservative estimate. However, stress concentration effects are not considered in this analysis. The designer should be aware of potential larger magnitudes of stress due to stress concentration factors.

4.2.1 Axial Section Subjected to Pure Axial Force

Conceptually, the analysis approach for all balance measuring sections is to calculate the following properties, sequentially.

- moment of inertia of the flexure(s)
- spring constants of the flexure(s)
- load proportion or load division among structural elements
- maximum stress in each flexure
• stress at the centerline of the active strain gage grid

For the axial section subjected to pure axial force, the free body diagram of the measurement beam and the strap under an applied axial force and exaggerated deformation is shown in Figure 13.

The inertia of the straps is given by equation 3.

\[ I_s = \frac{n_s b_s h_s^3}{12} \]  

where \( n_s \) is the number of straps, \( b_s \) is the base of the strap, and \( h_s \) is the height of the strap. For a typical configuration, there will be two straps and two measurement beams. The inertia of the flex beams is given by equation 4. The base dimension, \( b \), goes into the page in Figures 12 and 13 for the strap, measurement beam, and flex beams.

\[ I_f = \frac{n_f b_f h_f^3}{12} \]  

where \( n_f \) is the number of flex beams, \( b_f \) is the base of the flex beam, and \( h_f \) is the height of the flex beam. The cross sectional area of the flex beam is given by equation 5.

\[ A_f = b_f h_f \]  

The number of flex beams is a multiple of four. From a design perspective, multiple thin, wide flex beams allow for compliance in the axial force direction and maintains stiffness in the vertical direction to carry normal force and pitching moment. The relatively wide flex beams provides stiffness for carrying roll, yaw, and side. However, the cost to manufacture the balance increases with the number of flex beams. The flex beams are formed in the electro discharge machining process by removing the material between the
beams (flex beam gap in Figure 12). Therefore, the thinner the flex beam, the higher the risk of electrical arcing through the flex beam thickness resulting in a fabrication failure.

The inertia of the measurement beams, $I_m$, is given by equation 6

$$I_m = \frac{n_m b_m h_m^3}{12}$$

where $n_m$ is the number of measurement beams, $b_m$ is the base of the measurement beam, and $h_m$ is the height of the measurement beam. As given by reference Young et al. [12], the slope at the cantilevered end of the measurement beam, $\theta_1$ (Figure 13), is given by equation 7

$$\theta_1 = \frac{F l_m^2}{2 E I_m} - \frac{M_0 l_m}{E I_m}.$$  

(7)

To determine the slope at the midpoint of the strap, $\theta_2$, the moment-area method is employed, Hibbler [13] and Scott [9]. Generally, the moment-area formulations are as follows

$$\theta = \frac{1}{EI} [\text{Area of Moment Diagram}]_{A-B}$$

$$\delta_A = \frac{1}{EI} [\text{Area of Moment Diagram}]_{A-B} \bar{x}_A$$

where $\theta$ is the slope at A relative to B, $\delta_A$ is the deflection at point A relative to point B, and $\bar{x}_A$ is the composite centroid of the different moment diagram sections. The internal reaction force, $R_A$, and internal moment are shown in the section cut at point A in Figure 14.

![Figure 14: Internal Forces at Point A.](image)

The moment diagram is divided into the following five sections as shown in Figure 15. The areas were separated into simple shapes to easily calculate their areas.
Figure 15: Moment Diagram.

The moment-area method is used to determine the slope and deflection. The shaded region in Figure 15 has an elasticity of $EI_m$ and the unshaded region has an elasticity of $EI_s$. The area and centroid of each section are tabulated in Table 2. Note that the centroids and deflection are determined from the left end of the diagram (point A).

<table>
<thead>
<tr>
<th>Section</th>
<th>Area, $A_i$</th>
<th>Centroid, $\bar{x}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{T_0 h_m}{4}$</td>
<td>$\frac{h_m}{4}$</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{RA h_m^2}{8}$</td>
<td>$\frac{h_m}{3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{T_0 l_e}{4}$</td>
<td>$\frac{h_m}{2} + \frac{l_e}{4}$</td>
</tr>
<tr>
<td>4</td>
<td>$-\frac{RA h_m l_e}{4}$</td>
<td>$\frac{h_m}{2} + \frac{l_e}{4}$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{RA l_e^2}{8}$</td>
<td>$\frac{h_m}{2} + \frac{l_e}{3}$</td>
</tr>
</tbody>
</table>

Table 2: Area and Centroid Calculations based on Figure 15.

The slope, $\theta_2$, at point A relative to point B ($\theta_2=0$ at point B because of the fixed boundary condition) is given by

$$\theta_2 = \frac{1}{EI_m} [A_1 + A_2] + \frac{1}{EI_s} [A_3 + A_4 + A_5].$$

Under the assumption that $I_m >> I_s$, we can neglect the contribution of the first set of
\[ \theta_2 = \frac{1}{EI_s} \left[ \frac{T_0 l_e}{4} - \frac{R_A h_m l_e}{4} - \frac{R_A l^2_e}{8} \right]. \]  

(8)

In practice, it has been found that this formulation is relatively robust to the assumption that \( I_m >> I_s \). The deflection, \( \delta \), at point A relative to point B (\( \delta = 0 \) at point B because of the fixed boundary condition) is then given by

\[ \delta = \frac{1}{EI_m} \left[ A_1 \bar{x}_1 + A_2 \bar{x}_2 \right] + \frac{1}{EI_s} \left[ A_3 \bar{x}_3 + A_4 \bar{x}_4 + A_5 \bar{x}_5 \right]. \]

Again using the assumption that \( I_m >> I_s \), we can neglect the contribution of the first set of terms leaving

\[ \delta = \frac{1}{EI_m} \left[ T_0 h_m l_e + \frac{T_0 l^2_e}{16} - \frac{R_A h_m^2 l_e}{8} - \frac{R_A h_m l^2_e}{16} - \frac{R_A l^3_e}{24} \right] \]

which simplifies to

\[ \delta = \frac{1}{EI_s} \left[ T_0 h_m l_e + \frac{T_0 l^2_e}{16} - \frac{R_A h_m^2 l_e}{8} - \frac{R_A h_m l^2_e}{16} - \frac{R_A l^3_e}{24} \right]. \]

(9)

Setting equation 9 equal to zero because it is assumed that point A does not displace in the vertical direction relative to point B (because the measurement beam is very stiff in this orientation) and solving for \( R_A \) yields

\[ R_A = \frac{3T_0}{2l_e} \left( \frac{2 \frac{h_m}{l_e} + 1}{3 \left( \frac{h_m}{l_e} \right)^2 + 3 \frac{h_m}{l_e} + 1} \right). \]

(10)

Substituting equation 10 into equation 8 and simplifying yields

\[ \theta_2 = \frac{T_0 l_e}{EI_s \lambda} \left( \frac{1}{48 \left( \left( \frac{h_m}{l_e} \right)^2 + \frac{h_m}{l_e} + 1/3 \right)} \right) \]

or

\[ \theta_2 = \frac{T_0 l_e}{EI_s \lambda} \]  

(11)

where

\[ \lambda = 48 \left( \left( \frac{h_m}{l_e} \right)^2 + \frac{h_m}{l_e} + 1/3 \right). \]

From the free body diagram in Figure 13, it is seen that

\[ T_0 = \frac{F h_s}{2} + M_0. \]

(12)

Noting that for continuity, \( \theta_1 = \theta_2 \) (setting equations 11 and 7 equal) and substituting in equation 12 gives

\[ \frac{F l^2_m}{2EI_m} - \frac{M_0 l_m}{EI_m} = \frac{(F h_s + M_0)}{2EI_s \lambda} l_e. \]
Simplifying and solving for \( M_0 \) yields

\[
M_0 = \frac{F l_m^2 I_s \lambda - F h_s I_m l_e}{2 l_m I_s \lambda + 2 l_e I_m}
\]

\[
M_0 = F l_m \left( \frac{\lambda - h_s I_m}{2 \lambda + 2 l_e I_m} \right) = F l_m \beta
\]

(13)

where

\[
\beta = \left( \frac{48 \left( \frac{h_m}{l_e} \right)^2 + \frac{h_m}{l_e} + \frac{1}{3} - \frac{h_s l_e I_m}{I_s}}{96 \left( \frac{h_m}{l_e} \right)^2 + \frac{h_m}{l_e} + \frac{1}{3} + 2 \left( \frac{l_e I_m}{l_m I_e} \right)} \right).
\]

(14)

To determine the spring constants of the measurement beam the deflection at point A must be found. The deflection at point A is determined from the four individual contributors:

- Bending of the measurement beam
- Shear deflection of the measurement beam
- Axial loading of the strap
- Rotation of the strap resulting from the thickness of the strap

The four contributors are summed to give the total deflection. The bending of the measurement beam is given from a cantilevered beam by superposition. In this case, the assumption is that the relative inertia of the strap is sufficiently small such that the strap offers no moment resistance at the strap-measurement beam junction (point A).

![Cantilever Beam Loading](image)

Figure 16: Cantilever Beam Loading.
As noted in reference Young et al. [12], the displacement at the end of the cantilever beam is
\[ \delta = \frac{P l^3}{3 EI} - \frac{M l^2}{2 EI}. \]

Therefore the deflection at point A is
\[ \delta_{\text{bending}} = \frac{F l_m^3}{3 E I_m} - \frac{M_0 l_m^2}{2 E I_m}. \] (15)

Timoshenko’s beam theory is used to derive the shear deflection in the measurement beam Timoshenko [14]. For rectangular beams, \( \kappa = 5/6 \). Therefore, the shear contribution to the overall deflection is
\[ \delta_{\text{shear}} = \frac{F l_m}{\kappa A_m G} = \frac{6 F l_m}{5 A_m G}. \] (16)
where \( A_m \) is the cross-sectional area of the measurement beam.

To find the axial loading, we know that stress and strain are related through Young’s Modulus. Substituting definitions of stress and strain, we get
\[ \sigma = E \epsilon \]
\[ \frac{F}{A_s} = E \frac{\delta}{l_e} \]
\[ \delta = \frac{F l_e}{A_s E}. \]

For the strap, the free-body diagram is shown in Figure 17. From the free-body diagram, equilibrium is given by
\[ F = R_C + R_B. \] (18)

\[ \delta_{C/B} = 0 = \frac{R_C l_e}{2 A_s E} - \frac{R_B l_e}{2 A_s E}. \]
Solving this equation for one of the reaction forces yields $R_C = R_B$. Therefore, $F = 2R_C = 2R_B$. Substituting this result into the deflection equation due to an axial load, we get (since $L = L/2$)

$$\delta_{Axial} = \frac{F l_e}{4 A_s E}. \quad (19)$$

The deflection due to the rotation of the strap resulting from the thickness of the strap is given by

$$\delta_{thickness} = \frac{h_s \theta_1}{2} = \frac{h_s}{2} \left( \frac{F l_m^2}{2 E I_m} - \frac{M_0 l_m}{E I_m} \right) = \frac{h_s}{4} \frac{F l_m^2}{E I_m} \left( 1 - 2 \beta \right). \quad (20)$$

The total deflection is the summation of all four contributions

$$\delta_T = \delta_{bending} + \delta_{shear} + \delta_{axial} + \delta_{thickness}$$

$$\delta_T = \frac{F l_m^3}{3 E I_m} - \frac{M_0 l_m^2}{2 E I_m} + \frac{6 F l_m}{5 A_m G} + \frac{F l_e}{4 A_s E} + \frac{h_s}{4} \frac{F l_m^2}{E I_m} \left( 1 - 2 \beta \right).$$

Simplifying and rearranging gives

$$\delta_T = \frac{F l_m^3}{3 E I_m} \left( 2 \frac{2 - 3 \beta}{2} + \frac{3 E}{10 G} \left( \frac{h_m}{l_m} \right)^2 + \frac{l_e h_s^2 I_m}{16 l_m^3 I_s} + \frac{3 h_s}{4 l_m} \left( 1 - 2 \beta \right) \right)$$

$$\delta_T = \frac{F l_m^3}{3 E I_m} \left( 1 + \frac{3 h_s}{4 l_m} \frac{3 \beta}{2} \left( 1 + \frac{h_s}{l_m} \right) + \frac{3 E}{10 G} \left( \frac{h_m}{l_m} \right)^2 + \frac{l_e h_s^2 I_m}{16 l_m^3 I_s} \right).$$

The spring constant for the measurement beam therefore becomes

$$k_m = \frac{F}{\delta} = \frac{3 E I_m}{l_m^3} \alpha \quad (21)$$

where

$$\alpha = \left( 1 + \frac{3 h_s}{4 l_m} \frac{3 \beta}{2} \left( 1 + \frac{h_s}{l_m} \right) + \frac{3 E}{10 G} \left( \frac{h_m}{l_m} \right)^2 + \frac{l_e h_s^2 I_m}{16 l_m^3 I_s} \right)^{-1}.$$

The spring constant for the flex beams is given as

$$k_f = \frac{n_f b_f h_f^3 E \gamma_{AF}}{l_f^3} = \frac{12 E I_f \gamma_{AF}}{l_f^3} \quad (22)$$

where $\gamma_{AF}$ is a non-dimensional spring constant correction factor for shearing deflection. The spring constant correction factor is a function of the height-to-length ratio of the flex beams. It is given as the ratio of the corrected spring constant, $k'_f$, to the spring constant uncorrected for shearing deflection $k_f$. According to reference Hubbard [15] for double bending,

$$\frac{1}{k_f} = \frac{1}{k'_f} = \frac{l_f^3}{12 E I} + \frac{6 l_f}{5 A G}.$$
where
\[ G = \frac{E}{2(1 + \nu)} \]
\[ \gamma_{AF} = \frac{k'_f}{k_f} = \frac{l_f^3}{12 E I} + \frac{6 l_f}{5 A G} \]

(23)
simplifying gives
\[ \gamma_{AF} = \left( \frac{h_f}{l_f} \right)^2 \frac{2.4(1 + \nu) + 1}{(1 + \nu)} . \]

(24)
The total spring constant and the load proportions of the flex beams and measurement beams are given as
\[ k_T = k_f + k_{m,s} \]

(25)
\[ N_{m,s} = k_{m,s}/k_T \]

(26)
\[ N_f = k_f/k_T. \]

(27)
In the design of the axial section, it is desired to have the load proportion ratio of 60% carried by the measurement beam \((N_{m,s} \times 100)\) and 40% carried by the flex beams \((N_f \times 100)\). The rationale for this is that a beam carrying more of the load proportion will be more robust to variability in the actual dimensions of the beams (considering fabrication tolerances).

The maximum stress in the measurement beam occurs at the end of the measurement beam. This stress is found from calculating the maximum moment in the measurement beam, along with the substitution of equation 13, which is equal to (refer back to the free body diagram in Figure 13)
\[ M_{max} = Fl_m - M_0 = Fl_m (1 - \beta) . \]

The maximum stress is derived from bending stress due to the moment, \(M_{max}\)
\[ \sigma_{max} = \frac{M_{max} c}{I} \]
where \(\sigma_{max}\) is the maximum stress due to bending, \(c\) is the distance from the neutral axis to the most distant fiber Hibbler [13], in this case, \(h_m/2\). Therefore the maximum stress in the measurement beam due to axial force, \(\sigma_{max_{m,AF}}\), accounting for the percentage of load carried by the flex beams is given by
\[ \sigma_{max_{m,AF}} = \frac{AF N_{m,s} l_m (1 - \beta)(h_m/2)}{(n_m b_m h_m^3)/12} = \frac{6 AF N_{m,s} l_m (1 - \beta)}{n_m b_m h_m^2} . \]

(28)
Again assuming the linear decrease in the moment away from the measurement beam-bulkhead junction, the stress under the gage on the measurement beam due to axial force, \(\sigma_{gage_{m,AF}}\), is given by
\[ \sigma_{gage_{m,AF}} = \frac{6 AF N_{m,s} (l_m (1 - \beta) - GL_{AF})}{n_m b_m h_m^2} . \]

(29)
The predicted full scale (FS) output in $\mu V$ (micro volts per volt of excitation voltage) is given in equation 30. The desired output in microvolts per volt is 1000 counts which requires a gage strain of approximately 500 microstrain. However, based on comparing predicted output to actual output from many balance calibrations, the predicted axial force is typically about 10% low. The difference can be attributed to the flex and measuring beams fabrication deviations toward the upper side of the tolerance interval, and it may also be due to simplifying assumptions in the analytical derivation. In practice, the axial force output is designed to be about 10% higher than required, and therefore the design goal of 1100 counts is specified.

$$Counts_{AF} = \frac{\sigma_{gage} \cdot AF}{E} \cdot GF \cdot 10^6$$

where $GF$ is the gage factor of the strain gage and $E$ is the material modulus of elasticity or Young’s modulus.

The maximum stress in the strap due to axial force, $\sigma_{maxs_{AF}}$, is due to both bending and pure compression and is located at point A. The stress is of the form

$$\sigma_{max} = \frac{M_{max} c}{I} + \frac{F}{A}.$$ 

Referring back to the free body diagram of the strap in Figure 17, the maximum stress in the strap becomes

$$\sigma_{maxs_{AF}} = \frac{M_A (h_s/2)}{(b_s h_s^3)/12} + \frac{F}{2 A_s} = \frac{M_A (h_s/2)}{(b_s h_s^3)/12} + \frac{F}{2 A_s}.$$ 

Noting that $A_s = b_s h_s$ and, from Figures 13 and 17, $M_A = T_0/2 - R_A h_m/2$,

$$\sigma_{maxs_{AF}} = \frac{6(T_0 - R_A h_m) + F}{2 b_s h_s^2}.$$ 

Substituting in for $R_A$ from equation 10 yields

$$\sigma_{maxs_{AF}} = \frac{1}{2 b_s h_s^2} \left( 6 T_0 - \frac{9 T_0 h_m}{l_e} \left( \frac{2(h_m/l_e) + 1}{3(h_m/l_e)^2 + 3(h_m/l_e) + 1} + \frac{2 T_0 h_s}{h_s + 2 l_m \beta} \right) \right).$$ 

Further substitution and simplification can occur noting that from equations 12 and 13, $T_0 = (F h_s)/2 + F l_m \beta$ and replacing $F$ with $AF$ gives

$$\sigma_{maxs_{AF}} = \frac{F h_s + 2 F l_m \beta}{2 b_s h_s^2} \left( 3 - \frac{9 h_m}{2 l_e} \left( \frac{2(h_m/l_e) + 1}{3(h_m/l_e)^2 + 3(h_m/l_e) + 1} + \frac{h_s}{h_s + 2 l_m \beta} \right) \right).$$

$$\sigma_{maxs_{AF}} = \frac{F}{2 b_s h_s} \left( 1 + \left( 1 + \frac{2 l_m \beta}{h_s} \right) \left( 3 - \frac{9 h_m}{2 l_e} \left( \frac{2(h_m/l_e) + 1}{3(h_m/l_e)^2 + 3(h_m/l_e) + 1} \right) \right) \right).$$

$$\sigma_{maxs_{AF}} = \frac{AF \cdot N_{m,s}}{2 n_s b_s h_s} \left( 1 + \left( 1 + \frac{2 l_m \beta}{h_s} \right) \left( 3 - \frac{9 h_m}{2 l_e} \left( \frac{2(h_m/l_e) + 1}{3(h_m/l_e)^2 + 3(h_m/l_e) + 1} \right) \right) \right).$$

The maximum stress in the flex beams occurs at the base of the flex beam-bulkhead junction similar to the measurement beam. However the calculation is slightly different due to the end constraints. The constraints on the flex beam are represented by one fixed
and one roller condition because the top and bottom of the flex beam are able to move relative to each other along the $y$ – direction in Figure 18.

\[ \sigma_{\text{max}} = \frac{M_{\text{max}} c}{I} = \frac{P(l/2)(h/2)}{I}. \]  

Therefore, the stress in the flex beams due to an applied axial force, $\sigma_{\text{max}_{AF}}$, taking into account the percentage of load carried by the flex beams and replacing $P$ with $AF$ is given by

\[ \sigma_{\text{max}_{AF}} = \frac{AFN_f(l_f/2)(h_f/2)}{n_f(b_f h_f^3)/12} = \frac{3 AF N_f l_f}{n_f b_f h_f^2}. \]

### 4.2.2 Axial Section Subjected to Pure Normal Force

This section will discuss the stress analysis of the axial section under an applied pure normal force. The stress in the flex beams due to the applied normal force, $\sigma_{\text{max}_{NF}}$, is derived from a pure compression case. Further, it is assumed that the flex beams carry the entire load under the pure normal force condition, because the spring constant for the measurement beam and strap is very low.

\[ \sigma_{\text{max}_{NF}} = \frac{NF}{n_f b_f h_f}. \]

### 4.2.3 Axial Section Subjected to Pure Pitching Moment

This section will discuss the stress analysis of the axial section under an applied pure pitching moment. As is the case for the pure normal force, the pure pitching moment load is carried by the flex beams. The stress calculated in the flex beams is for the set of flex beams furthest away from the balance moment center as these will have the highest stress magnitudes. The reference length $DP$ (see Figure 19) which is the effective length of the axial section is given by equation 35.

\[ DP = Len - \text{endgap}_1 - \text{endgap}_2 \]
where \( Len \) is the distance from the front edge of the forward endgap to the back edge of the aft endgap in the axial section. The reference length, \( DE \), is used in the calculation of the stress in the flex beams due to the applied pitching moment is given by equation 36.

\[
DE = DP - \frac{n_f}{4} h_f - \left( \frac{n_f}{4} - 1 \right) flxgap
\]  

(36)

The stress realized in the flex beams is estimated by decoupling the pitching moment as seen in Figure 19 above. A pure compressive load on the flex beams is generated by the decoupled force. The decoupled force is assumed to act at a distance \( DE/2 \) from the balance moment center, thus

\[
F = \frac{PM}{DE/2}.
\]  

(37)

The direct compressive stress due to the applied pitching moment is given by

\[
\sigma_{comp_{PM}} = \frac{PM}{DE/2} \left( \frac{1}{b_f h_f n_f} \right).
\]  

(38)

The maximum stress due to pitch in the flex beams, \( \sigma_{max_{PM}} \), is found by multiplying the stress in equation 38 by the ratio of \( DP/DE \) which adjusts the moment arm based on the area that the flex beams occupy.

\[
\sigma_{max_{PM}} = \frac{PM}{DE/2} \left( \frac{1}{b_f h_f n_f} \right) \left( \frac{DP}{DE} \right) = \frac{2PM DP}{DE^2 n_f b_f h_f}
\]  

(39)

4.2.4 Axial Section Subjected to Pure Side Force

This section will discuss the stress analysis of the axial section due to an applied pure side force. Historically, it was assumed that the flex beams in the axial section carried the entire side load; however, formulations developed later by Rhew [11] considered load
sharing by the measurement beam and strap. Rhew followed the same general approach as Scott in his development of load sharing due to axial force.

The slope at the end of the measurement beam due to the applied side force is given by (refer back to section 4.2.1)

$$\theta_1 = \frac{F l_m^2}{2EI_m} - \frac{M_0 l_m}{EI_m}.$$  \hspace{1cm} (40)

However in this case, the inertia of the measurement beam, $I_m$, differs from the inertia value used in the axial force case. The direction of the applied force causes the height to become the base.

$$I_m = \frac{b_m^3 h_m}{12}$$  \hspace{1cm} (41)

The slope at the midpoint of the strap is divided into two components, the twist on the axis of the strap, $\theta_{2a}$, and the twist around the axis of the strap, $\theta_{2b}$. Based on reference Young et al. [12], $\theta_{2a}$ (20(c)) is given by

$$\theta_{2a} = \frac{T L}{2 I_m} = \frac{T_0 l_e}{\beta b_s h_s^3 G} = \frac{T_0 l_e}{4 \beta b_s h_s^3 G}$$  \hspace{1cm} (42)

where $\beta$ is the torsional constant. For a rectangular cross section, $\beta$ is approximated by (valid for $b_s > h_s$)

$$\beta = \left( \frac{1}{3} - 0.21 \frac{h_s}{b_s} \left( 1 - \frac{h_s^4}{12 b_s^4} \right) \right).$$  \hspace{1cm} (43)
For a typical strap configuration, the ratio of the base of the strap to the height of the strap is 10. This results in a \( \beta \) value of 0.312. The twist around the axis of the strap, \( \theta_{2s} \), is given by Scott [16]

\[
\theta_{2s} = \frac{12 T L^3}{E b^3 h^3}.
\]  

(44)

Correcting for shear deflection in both \( x \) - and \( y \) - direction (see section 4.2.1) and substituting in the parameters from Figure 20(c) \( (T = T_0/2, L = l_e/2) \) gives

\[
\theta_{2s} = \frac{3 T_0 l_e^3}{4 E b^2 h^3} (\gamma_x + \gamma_y).
\]  

(45)

where the shear deflection constants are given by

\[
\gamma_x = \left( \left( \frac{h_s}{l_e} \right)^2 2.4(1 + \nu) + 1 \right)^{-1}, \quad \gamma_y = \left( \left( \frac{b_s}{l_e} \right)^2 2.4(1 + \nu) + 1 \right)^{-1}.
\]  

(46)

The total slope at the midpoint of the strap is the sum of the two components

\[
\theta_2 = \theta_{2a} + \theta_{2b} = \frac{T_0 l_e}{4 \beta b_s h^2 G} + \frac{3 T_0 l_e^3}{4 E b^2 h^3} (\gamma_x + \gamma_y).
\]  

(47)

Noting that \( T_0 = \frac{F h_s}{2} + M_0 \),

\[
\theta_2 = \frac{(F h_s + 2 M_0) l_e}{8 \beta b_s h^2 G} + \frac{3 (F h_s + 2 M_0) l_e^3}{8 E b^2 h^3} (\gamma_x + \gamma_y).
\]  

(48)

For continuity, the slopes at the midpoint of the strap and the end of the measurement beam must be equal \( (\theta_1 = \theta_2) \). The moment at the end of the measurement beam can be solved for as follows

\[
\frac{F l_m^2}{2 E I_m} - \frac{M_0 l_m}{E I_m} = \frac{(F h_s + 2 M_0) l_e}{8 \beta b_s h^2 G} + \frac{3 (F h_s + 2 M_0) l_e^3}{8 E b^2 h^3} (\gamma_x + \gamma_y)
\]  

(49)

rearranging and solving for \( M_0 \) gives

\[
M_0 = F l_e \left[ \frac{48}{96} \left( \frac{l_m l_e}{l_m l_e} \right) + \frac{6}{\gamma_x + \gamma_y} \left( \frac{l_m}{h_s l_m} \right)^2 + \frac{E}{G} \left( \frac{1}{b} \right) \left( \frac{b_s^2}{h_s} \right)^2 \right].
\]  

(50)

For simplicity, \( \eta \) is defined such that

\[
M_0 = F l_e \eta.
\]  

(51)

With the moment at the end of the measurement beam defined, the displacement at the same location can be derived. The total displacement at the end of the measurement beam is divided into the five components:

- Bending of the measurement beam
- Shear deflection of the measurement beam
- Double bending of the strap due to the center applied load
Rotation of the measurement beam translated to the strap
Shear deflection of the strap

The expression for the total deflection follows the order of itemized components and is given by

\[
\delta = \left( \frac{F l_m^3}{3 E I_m} - \frac{M_0 l_m^2}{2 E I_m} \right) + 6 \frac{F l_m}{5 A_m G} + \frac{F l_e^3}{192 E I_s} + \frac{h_s \theta_1}{2} + \frac{3 F l_e}{10 A_s G} \tag{52}
\]

simplifying gives

\[
\delta = \left( \frac{F l_m^3}{3 E I_m} \right) \left[ \frac{2 - 3 \eta}{2} + \frac{3}{10} \left( \frac{b_m}{l_m} \right)^2 \frac{E}{G} + \frac{1}{64} \left( \frac{l_e}{l_m} \right)^3 \frac{I_m}{I_s} + \ldots \right] + \frac{3 h_s}{4 l_m} (1 - 2 \eta) + \frac{3}{40} \left( \frac{E}{G} \right) \left( \frac{l_m}{I_s} \right) \left( \frac{l_e}{l_e} \right)^2 \tag{53}
\]

for simplicity, assign \( \alpha \) such that

\[
\delta = \frac{F l_m^3}{3 E I_m \alpha}. \tag{54}
\]

The spring constant for the measurement beam and strap, \( k_{m,s} \), can now be defined as

\[
k_{m,s} = \frac{F}{y} = \frac{3 E I_m \alpha}{l_m^3}. \tag{55}
\]

The spring constant for the flex beams is given by

\[
k_f = n_f \frac{b_f^3}{l_f} h_f E \gamma_{SF} \frac{l_f}{l_f} \tag{56}
\]

which is similar to equation 22, but with respect to the applied side force where \( \gamma_{SF} \) is the shear correction factor given by

\[
\gamma_{SF} = \left( \left( \frac{b_f}{l_f} \right)^2 \frac{2.4(1 + \nu) + 1}{1} \right)^{-1}. \tag{57}
\]

The total spring constant, \( k_T \), and the proportions of the load carried by the flex beam, \( N_f \), and measurement beam-strap combination, \( N_{m,s} \), are given, respectively, as

\[
k_T = k_m + k_f \quad N_f = \frac{k_f}{k_T} \quad N_{m,s} = \frac{k_{m,s}}{k_T}. \tag{58}
\]

With the displacements and subsequently the spring constants defined, the stresses in the components can be estimated. The maximum stress in the flex beams, \( \sigma_{max f_{SF}} \), due to the applied side force, \( SF \), is given by

\[
\sigma_{max f_{SF}} = \frac{3 SF N_f l_f}{n_f b_f^2 h_f}. \tag{59}
\]

The maximum stress in the measurement beam, \( \sigma_{max m_{SF}} \), due to the applied side force, \( SF \) is determined from the maximum moment \( (SFl_m - M_0 = SFl_m (1 - \eta)) \). This maximum moment occurs at the base of the measurement beam (see Figure 20(b)) generating
a bending stress. Again, this derivation parallels the previous details in section 4.2.1. Accounting for the percentage of the side force load carried by the measurement beam, this stress is then given by

$$\sigma_{m,s,F}^{\max} = \frac{M c}{I} = \frac{SF N_{m,s} l_m (1 - \eta) b_m/2}{(n_m h_m b_m^2)/12} = \frac{6 SF N_{m,s} l_m (1 - \eta)}{n_m h_m b_m^2}. \quad \text{(60)}$$

The stress in the strap due to the applied side force is more complex than the simple bending stress in the measurement beam. The strap stress is a summation of three stresses (where $F$ has been replaced with $SF$):

- Stress due to torque twist (shear stress)
- Stress due to torque bending around axis
- Stress due to double bending

As noted in references [11], Young et al. [12], the stress component due to torque twist in the strap is given by

$$\tau_{TT}^{s,F} = \frac{T}{\alpha b t^2} = \frac{T_0/2}{\alpha b_s h_s^2} = \frac{1/2(SF h_s/2 + M_0)}{4 \alpha b_s h_s^2} = \frac{SF(h_s + 2 l_m \eta)}{4 \alpha b_s h_s^2}. \quad \text{(61)}$$

The stress due to torque bending around axis, correcting for shearing deflection, is given by reference (Scott [16])

$$\sigma_{TB}^{s,F} = \frac{9SF l_c (h_s + 2 l_m \eta)}{4b_s^2 h_s^2} \left| \frac{\gamma_x - \gamma_y}{\gamma_x + \gamma_y} \right|. \quad \text{(62)}$$

As noted in reference Rhew [11], the stress due to double bending is due to the fixed endpoint constraints of the strap and the applied side force at the center of the strap and is given by

$$\sigma_{DB}^{s,F} = \frac{M c}{I} = \frac{SF L_c}{h_s b_s^2} = \left(\frac{SF l_c/2}{h_s b_s^2/12}\right)\frac{1}{2} = \frac{3 SF l_c}{4 h_s b_s^2}. \quad \text{(63)}$$

The summation of the two bending stresses gives the total stress in the strap, $\sigma_{T}^{s,F}$, due to the applied side force (accounting for the percentage of the load carried by the measurement beam and strap),

$$\sigma_{T}^{s,F} = N_{m,s} \left[ \frac{9(SF/2) l_c (h_s + 2 l_m \eta)}{4b_s^2 h_s^2} \left| \frac{\gamma_x - \gamma_y}{\gamma_x + \gamma_y} \right| + \frac{3(SF/2) l_c}{4 h_s b_s^2} \right]. \quad \text{(64)}$$

### 4.2.5 Axial Section Subjected to Pure Yawing Moment

This section will discuss the derivation of the stresses in the axial section under an applied yawing moment. Like the axial force and side forces, the yawing moment is carried both by the flex beams and the measurement beam and strap. The highest stresses in the axial section are most commonly due to yawing moment. To determine the percentage of the load carried by each of the components, the yawing moment is decoupled into vertical and horizontal components (in the balance $x-$ and $y-$directions respectively). A top view of an axial section showing, the flex beam groups, the measurement beams, the decoupling of the forces and the reaction forces is shown in Figure 21. The vertical component of
the decoupled yawing moment is carried by the flex beams and the T-section portion of the axial section. (For balances where the moment center lies at the center of the axial section, the measurement beams do not carry any decoupled vertical load because they lie on the balance $y$-axis) The front and rear flex beam groups and the T-sections act like springs in series to carry the decoupled vertical load. To determine their combined spring constant, each of the flex beam individual spring constants must be determined.

Figure 21: Decoupled Yaw Moment on Axial Section.
The flex beam spring constants, $k_f$, have been previously defined from the application of the side force (see section 4.2.4). The spring constant for the T-section, $k_{ts}$, is based on the total effective area of the T-section located on the balance moment center.

To calculate the total effective area for a cylindrical balance, the total T-section area is subdivided into three elements: a circular segment (Element 1), a rectangle (Element 2) and the projection of the circular wire access hole (Element 3) (see Figure 22). The area of the circular segment is given by the formula

$$A_{E1} = \frac{R^2}{2} \left( \theta_{E1} - \sin(\theta_{E1}) \right)$$  \hspace{1cm} (65)

where $R$ is the balance outer diameter divided by 2 and

$$\theta_{E1} = 2 \left( \frac{\pi}{2} - \cos^{-1} \left( \frac{clm}{R} \right) \right).$$  \hspace{1cm} (66)

The area of Element 2 is found by multiplying a base times a height where the base of the rectangular element, $b_{E2}$, is given by

$$b_{E2} = 2(clm - (bm + \text{gap behind measurement beam}))$$  \hspace{1cm} (67)

and the height of the rectangular element, $h_{E2}$, is given by

$$h_{E2} = l_m/2 + h_s + \text{gap under strap} - \left( \frac{\text{diag. slot width}}{2 \cos(\text{diag. slot angle})} \right).$$  \hspace{1cm} (68)

Therefore the area of Element 2, $A_{E2}$, is given by

$$A_{E2} = b_{E2} \times h_{E2}.$$  \hspace{1cm} (69)
To determine the area of Element 3, $A_{E_3}$, the angle of projection of the diagonal slot, $\theta_{E_3}$, on the wire access must be calculated as follows

$$\theta_{E_3} = \cos^{-1}\left(\frac{\text{diag. slot width}}{2 \cos(\text{diag. slot angle})}\right). \quad (70)$$

Since Element 3 is a hole, its area is effectively negative, and as a result its area is subtracted from the total area of Element 2. The area of Element 3 is given by

$$A_{E_3} = r^2 - (0.5 \left(2 \theta_{E_3} - 2 \sin(2 \theta_{E_3})\right)) \quad (71)$$

where $r$ is the radius of the wire access hole. Note that the complexity in the formulation of the area of Element 3 is due to the fact that the diagonal slot is non-orthogonal to the section shown in Figure 22. The total area for the effective T-section, $A_T$, is given by

$$A_T = A_{E_1} + A_{E_2} + A_{E_3}. \quad (72)$$

The spring constant for the effective T-section is therefore

$$k_{Ts_v} = \left(\left(\frac{L_{Ts}^3}{3 I_{Ts} E}\right) + \left(\frac{6 L_{Ts}}{5 A_{Ts} G}\right)\right)^{-1} \quad (73)$$

where $L_{Ts}$ is the length of the effective T-section which is the average of the overall length of the axial section and the reference dimension, $DE$ (equation 36), $L_{Ts} = 1/2 (\text{Len} + DE)$. The effective T-section is treated as a single cantilever beam for the calculation of the displacement and subsequently the spring constant. The area of the effective T-section, $A_{Ts}$, is the summation of the individual areas, $A_{E_1}, A_{E_2}$, and $A_{E_3}$. The total inertia of the T-section, $I_{Ts}$, is the summation of the inertia of each of the areas. As derived by reference Hibler [13], the moment of inertia for the circular first element, $I_{E_1}$ is given by

$$I_{E_1} = \frac{R^4}{12} \left(3 \frac{\theta}{2} - 2 \sin^3\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) - 3 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)\right). \quad (74)$$

The moment of inertia for the rectangular segment (Element 2) is simply

$$I_{E_2} = \frac{b_{E_2} h_{E_2}^3}{12}. \quad (75)$$

The third element’s moment of inertia, $I_{E_3}$ can be evaluated using equation 74 again; however, the radius term, $R$, and angle, $\theta_{E_1}$, will be replaced with the wire hole radius, $r$, and the angle, $\theta_{E_3}$, respectively. Recall that based on the definition of element three, the inertia is negative. The spring constant for the flex beams with respect to the yawing moment decoupled vertical force is given by

$$k_{f_v} = \left(\left(\frac{l_f^3}{12 \frac{m_f}{2} I_f E}\right) + \left(\frac{6 l_f}{5 \frac{m_f}{2} A_f G}\right)\right)^{-1} \quad (76)$$

The combined spring constant for the yawing moment decoupled vertical force from the effective T-section and flex beams in series is given by

$$k_v = \frac{k_{f_v} k_{Ts_v}}{k_{f_v} + k_{Ts_v}}. \quad (77)$$
The yawing moment horizontal decoupled force is carried by the flex beams and the strap-measurement beam. The spring constant for the flex beams is given by

\[ k_{hf} = \frac{b_f h_f^2 \gamma_{AF} E}{l_f^3} \]  \hspace{1cm} (78)

where the correction for shearing deflection, \( \gamma_{AF} \), is given in equation 23. The spring constant for the measurement beam - strap due to the decoupled horizontal force is given by

\[ k_{hm,s} = \frac{3 EI_m \alpha}{l_m^3} \]  \hspace{1cm} (79)

which is identical to the formula for the spring constant given in equation 21. After decoupled yawing moment’s spring constants have been defined, they can be converted to equivalent torsional spring constants. The torsional spring constant due to the decoupled vertical force, \( k_{\theta_vf} \), is given by

\[ k_{\theta_vf} = \frac{T}{\theta} = \frac{F_v r_{hf}}{(s/r_{hf})} \]  \hspace{1cm} (80)

where \( T \) is the applied torque, \( s \) is an arc length and \( r_{hf} \) is the horizontal distance (moment arm) from the balance moment center to the center of the flex beam grouping which is equal to \( DE/2 \). Taking into account the number of measurement beams and that \( k_v \) can alternately be defined as \( k_v = F_v/s \),

\[ k_{\theta_vf} = n_f k_v r_{hf}^2 \]  \hspace{1cm} (81)

The torsional spring constants for the horizontal decoupled yawing moment for the measurement beam - strap, \( k_{\theta_{hm,s}} \), and the flex beams, \( k_{\theta_{hf}} \), are given, respectively, as

\[ k_{\theta_{hm,s}} = n_m k_{hm,s} r_{vm}^2 \]  \hspace{1cm} (82)

\[ k_{\theta_{hf}} = n_f k_{hf} r_{vf}^2 \]  \hspace{1cm} (83)

where \( r_{vf} \) and \( r_{vm} \) are the vertical distances (moment arms) from the balance moment center to the center of the flex beams and measurement beams, respectively, given by

\[ r_{vf} = c l_f - b_f/2 \quad r_{vm} = c l_m - b_m/2. \]  \hspace{1cm} (84)

The load proportions can now be determined based on the torsional spring constants allowing for the calculation of stresses as a function of the applied yawing moment. The proportion of the yawing moment carried by the flex beams in the vertical direction is given as

\[ N_{vf} = \frac{k_{\theta_vf}}{k_{\theta_vf} + k_{\theta_{hm,s}} + k_{\theta_{hf}}}. \]  \hspace{1cm} (85)

Similarly, the proportion of the decoupled yawing moment carried by the flex beams in the horizontal direction is given as

\[ N_{hf} = \frac{k_{\theta_{hf}}}{k_{\theta_vf} + k_{\theta_{hm,s}} + k_{\theta_{hf}}}. \]  \hspace{1cm} (86)
Finally, the proportion of the decoupled yawing moment carried by the strap-measurement beam in the horizontal direction is

$$N_{hm,s} = \frac{k_{\theta_{hm,s}}}{k_{\theta_{ef}} + k_{\theta_{hm,s}} + k_{\theta_{hf}}}.$$  \hfill (87)

It follows that the distribution of the applied yawing moment to the flex beams in the vertical direction is given by

$$M_{vf} = N_{vf}YM$$  \hfill (88)

where $YM$ is the applied yawing moment. Similarly, the distribution of the yawing moment to the flex beams in the horizontal direction, $M_{hf}$, and the distribution of the yawing moment to the measurement beam - strap, $M_{hm,s}$, are given respectively by

$$M_{hf} = N_{hf}YM$$  \hfill (89)

$$M_{hm,s} = N_{hm,s}YM$$  \hfill (90)

Now the vertical and horizontal forces applied to the flex beams and measurement beam - strap can be back calculated. The vertical force applied to the flex beams from the applied yawing moment is given by

$$F_{vf} = \frac{M_{vf}}{2r_{hf}}.$$  \hfill (91)

Similarly, the horizontal force applied to the flex beams, $F_{hf}$, and the horizontal force applied to the measurement beam - strap, $F_{hm}$, are given, respectively, as

$$F_{hf} = \frac{M_{hf}}{2r_{vf}}$$  \hfill (92)

$$F_{hm,s} = \frac{M_{hm,s}}{2r_{vm}}.$$  \hfill (93)

Note that the factor of 1/2 in equations 91 through 93 is due to the formulation of the equations based on beam sets, that is the front and rear sets or the right and left beam sets (refer back to Figure 21).

With the yawing moment decoupled into the vertical and horizontal forces on the flex beams and measurement beams, the stresses in each of the components can be defined. Because the flex beams carry both the decoupled horizontal and vertical forces the stress in the flex beams from the applied yawing moment, $\sigma_{(F_v + F_h)_{YM}}$, is given as

$$\sigma_{(F_v + F_h)_{YM}} = \sigma_{F_v_{YM}} + \sigma_{F_h_{YM}}$$  \hfill (94)

where $\sigma_{F_v_{YM}}$ is the stress due to the decoupled vertical force and $\sigma_{F_h_{YM}}$ is the stress due to the decoupled horizontal force. They are given from a direct bending stress as

$$\sigma_{F_v_{YM}} = \frac{Mc}{I} = \frac{F_{vf}(l_f/2)b_f}{(n_f/2)\left(h_f b_f^2\right)/12} = \frac{6F_{vf}(l_f/2)}{(n_f/2)h_f b_f^2}$$  \hfill (95)

$$\sigma_{F_h_{YM}} = \frac{F_{hf}(l_f/2)h_f}{(n_f/2)\left(b_f h_f^2\right)/12} = \frac{6F_{hf}(l_f/2)}{(n_f/2)b_f h_f^2}.$$  \hfill (96)
The maximum stress on the measurement beam due to the applied yawing moment, \( \sigma_{\text{max}_{\text{Y}}M} \), can be calculated via the decoupled horizontal force. This can be done by ratioing the decoupled horizontal force (essentially an axial force) by the axial force exerted on the measuring beam and multiplying this ratio by the stress on the measurement beam due to axial force (refer back to section 4.2.1),

\[
\sigma_{\text{max}_{\text{Y}}M} = \frac{2F_{h_{m,s}}\sigma_{\text{max}_{\text{AF}}}}{N_mAF}.
\]  

(97)

Based on a similar load division, the stress in the strap due to the applied yawing moment, \( \sigma_{\text{max}_{\text{Y}}M} \), via the decoupled horizontal force is derived from the maximum stress in the strap due to an applied axial force and is given as

\[
\sigma_{\text{max}_{\text{Y}}M} = \frac{2F_{h_{m,s}}\sigma_{\text{max}_{\text{AF}}}}{N_mAF}.
\]  

(98)

### 4.2.6 Axial Section Subjected to Pure Rolling Moment

This section will detail the stress analysis of the axial section subjected to a pure rolling moment. As was the case for the pure normal and pitch cases, it is assumed that the flex beams carry the entire load of the rolling moment. This assumption of the measuring beam not carrying a roll load has proven to be useful in practice for hundreds of balance designs over decades of test experience, however finite element analysis of certain axial section designs has indicated non-negligible stresses due to roll on the measuring beam. The rolling moment stress in the flex beams, \( \sigma_{\text{max}_{\text{RM}}} \), is given simply as bending stress

\[
\sigma_{\text{max}_{\text{RM}}} = \frac{RM c_{lf}}{I_{L\text{flex}}}
\]  

(99)

where \( RM \) is the applied rolling moment, \( c_{lf} \) is the distance shown in Figure 23 (typically equal to \( c_{lm} \)), and \( I_{L\text{flex}} \) is the composite moment of inertia of the flex beams based on the parallel axis theorem, Hibler [13], \( I_x = I_x + Ay^2 \)

\[
I_{L\text{flex}} = \frac{n_f b_f h_f^3}{12} + n_f b_f h_f (r_{lf}f)^2
\]  

(100)

where \( r_{lf}f \) is the horizontal distance (moment arm) from the balance moment center to the center of the flex beams as given by

\[
r_{lf} = c_{lf} - (b_f/2).
\]  

(101)
4.2.7 T-Section Stress Calculations

In addition to the stress calculations performed for the measurement beam-strap and flex beams, the T-sections in the axial section are also stressed. The stress values reported are valid for the most extreme fibers or points furthest away from the balance moment center with respect to the applied force (points A and B in Figures 22 and 23). Although no gages are located in these areas, it is possible that the highest stressed regions are present in the T-sections. Separate stress calculations following the same equations are completed for each portion of the T-section, on-moment center and off-moment center. A large portion of the formulas for these stresses have been previously introduced in section 4.2.5 and will be referenced accordingly. The stress on each of the points, A and B, in the cross sections of the T-sections are given as totals of combined loading.

\[ \sigma_A = \sigma_{APM+NF} + \sigma_{AYM+SF} \]  
\[ \sigma_B = \sigma_{BPM+NF} + \sigma_{BYM+SF} \]

where \( \sigma_A \) is the total stress at point A, \( \sigma_{APM+NF} \) is the stress at point A due to applied pitching moment and normal force and \( \sigma_{AYM+SF} \) is the stress at point A due to applied yawing moment and side force. Similarly, the total stress at point B is given by

\[ \sigma_B = \sigma_{BPM+NF} + \sigma_{BYM+SF} \]

4.2.7.1 T-Section On Centerline

This section will detail the stress calculations for the T-section on the balance moment center. All elements and geometries referenced refer to Figure 22. The stress due to pitching moment and normal force at point A in the T-section on balance moment center is given by

\[ \sigma_{APM+NF} = \left( NF(\frac{DE}{4}) + \frac{PM}{2}(\bar{Y} - h_{E2}) \right) / I_{yy} \]
where $h_{E2}$ is the height of Element 2 given in equation 68, $\bar{Y}$ is the composite centroid given by

$$
\bar{Y} = \frac{\sum_{i} A_i}{\sum_{i} A_i \delta_i}
$$

(105)

where $A_i$ is the area of the $i^{th}$ element of the T-section (see Figure 22) and $\delta_i$ is the distance from the centroid of the $i^{th}$ element to the origin. The composite moment of inertia, $I_{yy'}$, based on the parallel axis theorem, is given by

$$
I_{yy'} = \sum_{i} I_{yyi} + \sum_{i} A_i \gamma_i^2
$$

(106)

where $I_{yyi}$ is the moment of inertia of the $i^{th}$ element of the T-section and $\gamma$ is equal to $\bar{Y} - d_i$. Similarly, the stress at point A due to yawing moment and side force is given by

$$
\sigma_{A_{YM+SF}} = \left( SF(DE/4) + (YM/2)(\bar{Y} - h_{E2}) \right) / I_{zz}.
$$

(107)

Note the change of axis with respect to which the moment of inertia was obtained in equation 107. The stress at point B due to pitching moment and normal force is given by

$$
\sigma_{B_{PM+NF}} = \left( NF(DE/4) + (PM/2) \bar{Y} \right) / I_{yy'}.
$$

(108)

The stress at point A due to side force and yawing moment is given by

$$
\sigma_{A_{YM+SF}} = \left( SF(DE/4) + (YM/2)(b_{E2}/2) \right) / I_{zz}.
$$

(109)

Based on the locations of points A and B in the T-section, it is expected that point A will have higher stresses due to yawing moment and side force, while point B will have higher stresses due to pitching moment and normal force.

### 4.2.7.2 T-Section Off Moment Center

This section will give the stress calculations for the T-section off the balance moment center, located at the end of the axial section. All elements and geometries referenced refer to Figure 23. The stress at point A in the T-section off the balance moment center is given by the maximum of either of the two following quantities

$$
\sigma_{APM+NF} = \left( (NF(Len/2) + PM) \left| h_{E2} + h_{E4} - \bar{Y} \right| \right) / I_{yy'}
$$

(110)

or

$$
\sigma_{APM+NF} = \left( (NF(Len/2) + PM) \left| h_{E2} - \bar{Y} \right| \right) / I_{yy'}
$$

(111)

where $Len$ is the overall length of the axial section. The two equations for the stress at point A allow for the proper estimation of stress regardless of the shape of the fourth element. The stress at point A due to side force and yawing moment is given by

$$
\sigma_{A_{YM+SF}} = \left( (SF(Len/2) + YM) \left| \bar{Y} \right| \right) / I_{zz}.
$$

(112)

The stress at point B in the T-section off the balance moment center due to pitching moment and normal force is given by

$$
\sigma_{B_{PM+NF}} = \left( (NF(Len/2) + PM) \bar{Y} \right) / I_{yy'}.
$$

(113)

The stress at point B in the T-section off the balance moment center due to side force and yawing moment is given by

$$
\sigma_{B_{YM+SF}} = \left( (SF(Len/2) + YM) b_{E2}/2 \right) / I_{zz}.
$$

(114)
5 Cage Section Design

Cage section(s) are typically located forward and/or aft of the balance axial section as shown in Figure 6. The geometry of the cage section varies considerably, in contrast to the axial sections which are relatively similar. The number, placement, area, and shape of beams as well as the number of cage sections (one or two) are all variables available to the balance designer to meet specific design requirements. Typically the five forces and moments normal, roll, pitch, yaw and side are read by Wheatstone bridges located in the cage section.

It is desirable to limit the interaction of responses for all components. For example, the output on the normal force should not be greatly influenced by the application of a pitching moment. In the design, the strain gages are placed to cancel the interactions, so that each component responds to its own load applied only. In practice, interactions (<5%) will likely be present primarily due to slight fabrication variations in beam dimensions and location, as well as slight variations in gage placement. These residual interactions are mathematically compensated for in the calibration process. There can also be large interactions (>5%) found in calibration that are due to simplifying assumptions in the analytical derivation, particularly in high load-to-diameter ratio balances. While these large interactions are not desirable, they are compensated for by the calibration mathematical model.

The procedure to solve for the predicted stresses and the bridge outputs in the cage section is similar to that of the axial section. The spring constants are used to calculate the amount of force carried by each beam (or beam set). Once the force on a particular beam has been found, the stress and strain fields can be determined. Cross-sections of some commonly-used cage designs are shown below in Figure 24.
The most common cage section design is based on sets of rectangular beams (Figure 24(a)). For a flow-through balance, an octagon beam is often employed as in (Figure 24(b)), which also features notched beams on the y-axis that are discussed in a subsequent section. In the case of large differences in the vertical and lateral forces and moments, employing two wafer beam cage sections as shown in (Figure 24(c)) is a typical strategy where multiple cage sections will have a 90-degree orientations of the wafer-style sections. For high load-to-diameter ratios, a cage section that fills the available volume may be employed show in (Figure 24(d)) and incorporates clipped-corner beams.

The optimization of the cage section can often be more difficult than the axial section optimization because five components (normal, pitch, side, yaw, and roll) are generally measured in the cage section require trade-offs in multiple axes. This means the outputs of all five have to be considered along with the maximum stress for the entire section and the total stress under each gage due to combined loads.
5.1 Cage Section Design

5.1.1 Cage Section Subjected to Pure Normal Force

This section will detail the stress analysis of the cage section under a pure normal force. The analysis for each of the subsections in this section will focus on the two-cage design. An example of a deformed cage section subject to applied normal force is shown in Figure 25. The beams in the cage section deflect in a double bending (DB) manner.

![Figure 25: Cage Section Deformation Under Applied Normal Force (Side View).](image)

The double bending deformation of the balance is shown for both normal force (DB an SB) and pitching moment (SB only) in figure 26 by the deflected balance $x - axis, x'$. 
Figure 26: Blue Lines Represent the Deformed Shape of the Balance under Applied Normal Force (Top) and Pitching Moment (Bottom). Note that the Forces and Moments are Applied through the Balance Front End but are Resolved about Moment Center. Thus, a Force Transferred from Moment Center to the Front End Produces a Restoring Moment.

To determine the spring constant for the application of the pure normal force, the moments of inertia for each of the beam sets in the cage section must be determined. The inertia for the $i^{th}$ beam group with respect to the applied normal force is given by

$$I_{i, NF} = \frac{n_i b_i h_i^3}{12} \quad (115)$$

where $n_i$ is the number of beams in the $i^{th}$ cage section group.
For example, in Figure 27, there are two beams in beam set one and four beams each in beam sets two and three \((n_1 = 2, n_2 = 4, n_3 = 4)\). Similar to the axial section, a correction for shearing deflection is included in the analysis. The shearing deflection for the applied normal force, \(\gamma_{zi}\) and the applied side force, \(\gamma_{yi}\) are given, respectively, as

\[
\gamma_{zi} = \left( \left( \frac{h_i}{l_i} \right)^2 (2.4)(1 + \nu) + 1 \right)^{-1} \quad \gamma_{yi} = \left( \left( \frac{b_i}{l_i} \right)^2 (2.4)(1 + \nu) + 1 \right)^{-1}
\]

where \(l_i\) is the length of the \(i^{th}\) cage section beam group. In a typical cage design, all cage beams groups are the same length. The spring constant for the \(i^{th}\) cage section beam group due to an applied normal force (generating a double bending mode) is given as

\[
k_{DB_{iNF}} = n_i \frac{12 E I_N \gamma_{zi}}{l_i^3}.
\]

It follows that the total spring constant for the applied normal force (generating a double bending mode) in the cage section is

\[
k_{T, DB_{iNF}} = \sum_{i}^{n} k_{DB_{iNF}}.
\]

Therefore the load proportions for each of the cage section beams due to the applied normal force is given by

\[
N_{DB_{iNF}} = \frac{k_{DB_{iNF}}}{k_{T, DB_{iNF}}}.
\]

The stress caused by the applied normal force is divided into single bending, \(\sigma_{SB_{iNF}}\), and double bending, \(\sigma_{DB_{iNF}}\). The double bending stress for the \(i^{th}\) cage section beam group is given by

\[
\sigma_{DB_{iNF}} = \frac{3NF l_i N_{DB_{iNF}}}{n_i b_i h_i^2}.
\]
(Refer back to Section 4.2.1 and the development of the forces acting on the flex beams and equation 33). The formulation for the stress due to single bending (refer back to Section 4.2.1) is based on the load proportions and inertias of an applied pitching moment, because the application of normal force at the moment center generates a pitching moment at the center of the cage beams. From the parallel axis theorem, the inertia of the \( i^{th} \) beam group due to an applied pitching moment is given by

\[
I_{m_i} = I_{N_i} + n_i b_i h_i r_{zi}^2
\]  

where \( r_{zi} \) is the distance from the origin to the center of the \( i^{th} \) beam group along the \( z - axis \) \( (r_{zi} = C_{zi} - h_i/2) \). The spring constant for the single bending case is then given by

\[
k_{SB_{i, NF}} = \frac{E I_{m_i}}{l_i^3}.
\]  

Similar to the total normal force spring constant, the total single bending spring constant is given as

\[
k_{T,SB_{i, NF}} = \sum_{i} k_{SB_{i, NF}}.
\]

Thus, the load proportions for each of the cage section beam groups due to single bending is given by

\[
N_{SB_{i, NF}} = \frac{k_{SB_{i, NF}}}{k_{T,SB_{i, NF}}}.
\]

The single bending stress generated by NF application is given by

\[
\sigma_{SB_{i, NF}} = \frac{NF L_C C_{zi} N_{SB_{i, NF}}}{I_{m_i}}
\]

where \( C_{zi} \) is the distance from the origin along the \( z - axis \) to the edge of the beam (see Figure 27), \( N_{SB_{i, NF}} \) is the proportion of the pitching moment load carried by the \( i^{th} \) beam group, \( I_{m_i} \) is the moment of inertia with respect to an applied pitching moment of the \( i^{th} \) beam group and \( L_C \) is the length from the balance moment center to the center of the cage section. The total stress on the \( i^{th} \) beam group due to the applied normal force is given by

\[
\sigma_{T_{i, NF}} = \sigma_{SB_{i, NF}} + \sigma_{DB_{i, NF}}.
\]

The stress under the normal force gage and output for the normal force gage, in microvolts per volt excitation, are given, respectively, by

\[
\sigma_{gage_{NF}} = \sigma_{SB_{i, NF}} + \sigma_{DB_{i, NF}} \left(1 - \frac{2GL_{NF}}{l_i}\right)
\]

\[
Counts_{NF} = \frac{\sigma_{gage_{NF}}}{E} * GF * 10^6
\]

where the gage length, \( GL \), is measured along the \( x - axis \) away from the bulkhead.
5.1.2 Cage Section Subjected to Pure Pitching Moment

This section details the stress analysis for the case of the cage section subjected to a pure pitching moment. The spring constants and load proportions for the applied pitching moment were previously defined in the section 5.1.1, where they were referred to as the single bending spring constants because of the single bending mode shape generated by a pitching moment. Therefore, the stress in the $i^{th}$ cage beam due the applied pitching moment is given from the bending stress equation ($Mc/I$) as

$$
\sigma_{\text{max}PM} = \frac{PM N_{SB_{PM}} C_{z_i}}{I_{M_i}}.
$$

(129)

Under a pure pitching moment, the cage section deflects only in a single bending manner. Therefore, there is no stress gradient along the length of the cage section beams. The output for the pitching moment gage, in microvolts per volt excitation, is given by

$$
\text{Counts}_{PM} = \frac{\sigma_{\text{gage}PM} E \times GF \times 10^6}{E}.
$$

(130)

Note that the output is based on the beam set which is actually gaged and not dependent on the location along the beam because the stress distribution is constant along the length of the beam from bulkhead to bulkhead.

5.1.3 Cage Section Subjected to Pure Side Force

This section presents the stress analysis for the cage section under a pure side force loading case. The spring constant for the side force case is similar to the normal force case but with the base and height dimensions switched. The spring constant for the $i^{th}$ cage section beam group is given by

$$
k_{DB_{SF}} = \frac{12 E I_{Y_i}}{l_i^3} \gamma_{y_i}.
$$

(131)

where the moment of inertia for the $i^{th}$ cage section beam group is given by

$$
I_{Y_i} = \frac{n_i h_i b_i^3}{12}
$$

(132)

and $\gamma_{y_i}$ is defined in section 5.1.1. It follows that the total spring constant for the side force case is given by

$$
k_{T,DB_{SF}} = \sum_{i} k_{DB_{SF}}.
$$

(133)

The load proportions on the $i^{th}$ cage section beam group under an applied side force is subsequently given by

$$
N_{DB_{SF}} = \frac{k_{DB_{SF}}}{k_{T,DB_{SF}}}.
$$

(134)

The double bending stress for the $i^{th}$ cage section beam group is given by

$$
\sigma_{DB_{SF}} = \frac{3 SF l_i N_{DB_{SF}}}{n_i h_i b_i^2}.
$$

(135)
(Refer back to Section 4.2.1 and the development of the forces acting on the flex beams and equation 33). Similar to the applied normal force case, the cage section beam groups are subjected to both double bending and single bending, due to the yawing moment generated by the side force. The single bending stress due to the applied side force on the \( i \)th cage section beam group is given by

\[
\sigma_{SB_{iSF}} = \frac{SF L_C C_{yi} N_{SB_{iSF}}}{I_{n_i}} \tag{136}
\]

where \( L_C \) is the distance from the moment center to the center of the cage section, \( C_{yi} \) is the distance from the origin along the \( y \)-axis to the edge of the \( i \)th beam group (see Figure 27), \( N_{SB_{iYM}} \) is the load proportion of the yawing moment carried by the \( i \)th beam group and \( I_{n_i} \) is the moment of inertia with respect to an applied yawing moment. The yawing moment of inertia is given by

\[
I_{n_i} = I_{Y_i} + n_i b_i h_i r_y^2 \tag{137}
\]

where \( r_y \) is the distance from the origin to the center of the \( i \)th beam group along the \( y \)-axis \((r_y = C_{yi} - b_i/2)\). The spring constant for the applied yaw moment on the \( i \)th beam group is given as

\[
k_{SB_{iSF}} = \frac{EI_{n_i}}{l_i} \tag{138}
\]

It follows the total spring constant for the applied yaw moment and the associated load proportions are

\[
k_{T,SB_{iSF}} = \sum_i^n k_{SB_{iSF}} \tag{139}
\]

\[
N_{SB_{iSF}} = \frac{k_{SB_{iSF}}}{k_{T,SB_{iSF}}} \tag{140}
\]

The total stress on the \( i \)th beam group is given by

\[
\sigma_{T_{iSF}} = \sigma_{SB_{iSF}} + \sigma_{DB_{iSF}} \tag{141}
\]

The stress and output for the side force gage are given, respectively, as

\[
\sigma_{gage_{SF}} = \sigma_{SB_{iSF}} + \sigma_{DB_{iSF}} \left( 1 - \frac{2GL_{SF}}{l_i} \right) \tag{142}
\]

\[
Counts_{SF} = \frac{\sigma_{gage_{SF}}}{E} \times GF \times 10^6 \tag{143}
\]

### 5.1.4 Cage Section Subjected to Pure Yawing Moment

This section outlines the stress analysis for the cage section subjected to a pure yawing moment. Similar to the pitching moment analysis of the cage section, the stress due to yawing moment is a direct bending stress \((Mc/I)\) calculation accounting for the respective cage beam load proportion. Also, because there is no stress gradient, the stress under the gage is also the total stress for that beam group where the gage is located.

\[
\sigma_{max_{iYM}} = \frac{YM N_{SB_{iYM}} C_{yi}}{I_{n_i}} \tag{144}
\]

The bridge output is then given by

\[
Counts_{YM} = \frac{\sigma_{gage_{YM}}}{E} \times GF \times 10^6 \tag{145}
\]
5.1.5 Cage Section Subjected to Pure Rolling Moment

This section describes the analysis of the stress encountered in the cage section when a pure rolling moment is applied, see Moore [18] and Scott [16]. Rolling moment is resolved as follows.

- Stress due to translation in they \( y \) – direction
- Stress due to translation in they \( z \) – direction
- Stress due to twisting of beam about its own axis in bending

Each one of the displacements or rotations will have an associated spring constant for each beam group. As noted in reference Scott [16], the spring constant for the \( i \)th beam group due to the translation in the \( y \) – direction is given by

\[
k_{tyiRM} = \frac{n_i h_i b_i^2 \gamma_y r_i^2}{l_i^3}
\]  

(146)

where the shear correction factor is calculated using equation 116, and \( r_i \) is the vertical distance from the center of the balance to the midpoint of each of the beam groups given by \( C_{zi} - (h_i/2) \). The spring constant for the \( i \)th beam group due to the translation in the \( z \) – direction is similarly given by

\[
k_{tziRM} = \frac{n_i b_i h_i^2 \gamma_z r_i^2}{l_i^3}
\]  

(147)

where \( r_y \) is the horizontal distance from the center of the balance to the midpoint of each of the beam groups given by \( C_{yi} - (b_i/2) \). The spring constant for the \( i \)th beam group due to the twisting of the beam about its own axis is given as

\[
k_{rxziRM} = \left( \frac{n_i E}{12} \right) \left( \frac{b_i h_i}{k_i} \right)^3 \left( \gamma_z + \gamma_y \right).
\]  

(148)

It follows that the total spring constant for the applied rolling moment, \( k_{T,\text{RM}} \), is given as

\[
k_{T,\text{RM}} = \sum_i \left( k_{tyiRM} + k_{tziRM} + k_{rxziRM} \right).
\]  

(149)

The load proportions are based on each individual beam spring constants relative to the total spring constant. This gives

\[
N_{tyiRM} = \frac{k_{tyiRM}}{k_{T,\text{RM}}}
\]  

(150)

\[
N_{tziRM} = \frac{k_{tziRM}}{k_{T,\text{RM}}}
\]  

(151)

\[
N_{rxziRM} = \frac{k_{rxziRM}}{k_{T,\text{RM}}}
\]  

(152)

As noted in reference Scott [16], the stress on the \( i \)th beam group due to the translation in the \( y \) – direction is given by

\[
\sigma_{tyiRM} = \frac{3 \ RM l_i N_{tyiRM}}{n_i h_i b_i^2 r_i^2}.
\]  

(153)
Similarly, the stress on the \(i^{th}\) beam group due to the translation in the \(z - direction\) is given by

\[
\sigma_{t_z, RM} = \frac{3 \, RM \, l_i \, N_{t_z, RM}}{n_i \, b_i \, h_i^2 \, r_{y_i}}. \tag{154}
\]

The stress on the \(i^{th}\) beam group due to the twisting of the beam about its own axis is given as

\[
\sigma_{t_x, RM} = \frac{18 \, RM \, l_i \, N_{t_x, RM} \, |\gamma_z - \gamma_y|}{n_i (b_i \, h_i)^2 (\gamma_z + \gamma_y)}. \tag{155}
\]

The total stress on the \(i^{th}\) beam group due to the applied rolling moment is then

\[
\sigma_{T, RM} = \sigma_{t_y, RM} + \sigma_{t_z, RM} + \sigma_{t_x, RM}. \tag{156}
\]

The stress under the gage due to the applied rolling moment is a function of the gage placement both down the length of the beam (\(x - direction\)) and across the face of the beam in either \(y\) or \(z\) direction. As an example, if the gage were placed on the upper or lower surfaces of the beam, there would be a stress gradient in the \(y\)-direction. The equation for stress under the gage for the applied rolling moment in this case, \(\sigma_{gage, RM}\), is given by

\[
\sigma_{gage, RM} = \left(\sigma_{t_y, RM} + \sigma_{t_x, RM}\right) \left(1 - \frac{2GL_{y, RM}}{l_i}\right) \left(1 - \frac{2GL_{x, RM}}{l_i}\right) + \sigma_{t_z, RM} \, GL_{RM, z}. \tag{157}
\]

where \(GL_y\) is the distance from the center of the active grid of the gage to the edge of the beam along the \(y - direction\) (width) and \(GL_x\) is the distance from the center of the active grid of the gage to the junction of the beam and the bulkhead along the \(x - direction\) (length). The gage placement location will be discussed in detail in section 5.2. The bridge output is then given by

\[
Counts_{RM} = \frac{\sigma_{gage, RM} \, E}{F} \ast GF \ast 10^6. \tag{158}
\]

### 5.1.6 Cage Section Subjected to Pure Axial Force

This section details the stress analysis for the case of a pure axial force applied to the cage section. The calculation of the axial stress in the cage section is a straightforward \(P/A\) stress. Instead of calculating spring constants based on deflection, spring constants with respect to the applied axial force in the cage section are derived from the beam group’s area ratios. This is due to the fact that the displacement along the balance \(x - axis\) is assumed to be negligible. The cross sectional area of concern for the \(i^{th}\) beam group is given by

\[
A_i = n_i \, b_i \, h_i. \tag{159}
\]

The total area is simply

\[
A_T = \sum_i^n \, A_i \tag{160}
\]

which gives load proportions for \(i^{th}\) beam group of

\[
N_{i, AF} = \frac{A_i}{A_T}. \tag{161}
\]

The stress on the \(i^{th}\) beam group in the cage section due to an applied axial force is

\[
\sigma_{max, i, AF} = \frac{AF \, N_{i, AF}}{A_i}. \tag{162}
\]
5.2 Cage Section Gaging Strategies

As previously discussed, there is a considerable design freedom when it comes to positioning the strain gages in the cage section relative to the axial section. This can be an advantage but also increases complexity. A target minimum output of 1000 uV/V is desired for all components. Up to 1500 uV/V is an approximate upper limit to constrain the maximum strain under the gage and prevent the potential of creep in the gage bond.

Another consideration in gaging the cage section is the strain gradients present under certain loading conditions. Moving the gage along a particular axis can either decrease or increase the amount of strain under the gage in some cases. For the pure force loading conditions, side force and normal force, double bending stress occurs in the cage section beam groups. The double bending stress causes a stress gradient along the length of the beam groups whereas single bending does not.

In addition to the stress gradients and the desired output ranges, the balancing of the bridge is also a concern. As introduced in section 2, half of the Wheatstone bridge must be in tension and the other half must be in compression. Following the LaRC convention, even gage numbers indicate a gage that is under compression and odd gage numbers indicate the gage is in tension. An example of a generic cage section with gage locations is given in Figure 28 and Figure 29. The two circles are cross sections of the front and rear cage sections taken at section A-A and section B-B in Figure 29. It is important to understand the location of the section cut due to the fact that gradients along the $x$-axis (into the page) are present. The gage call-outs with circles indicate that they are behind (into the page along the $x$-axis in Figure 28 the gages with no circles. Side, normal, pitch and yaw gages are split with half the bridge in the front cage section and the other half in the rear cage section (for two cage balances only). Conversely, the roll bridge is generally located in either the front or the rear cage section, with the preference toward the rear cage section to minimize the wiring crossing through the axial section. In addition, the roll gages are generally all located at the same end of a beam group.
Figure 28: Example Cage Section Gage Locations where the Circled Gage is Located Behind the Gage that is not Circled.

Figure 29: Example Cage Section Gage Locations for Pitching Moment and Normal Force. The Beam Bending Shapes show the Overall Deformation of the Cage Beams Subject to the Different Components of Load.

It is common practice to locate the force gages (side and normal) as close as possible to the bulkheads to measure both the double bending and single bending (moment induced by the NF acting over distance between the MC and centerline of the cage section).
The stress output from each of the bridges follows the equation

$$Output = [(1 + 3) - (2 + 4)] / 4$$  \hspace{1cm} (163)$$

where the numbers inside the brackets represent the gage numbers. The goal of the gaging strategy is to have a bridge be responsive to only one load component and be unresponsive to all others. As an example, Figure 29 shows the case of an applied normal force where general gage locations have been indicated to be in compression(-) or tension(+). Using 1s to represent the full scale gage output, the evaluation of equation 163 for the pitching moment bridge output under an applied normal force is

$$Output_{PM} = [((-1) + (+1)) - ((+1) + (-1))] / 4 = 0.$$  

For the same condition, an applied normal force, the normal force bridge output is

$$Output_{NF} = [((+1) + (+1)) - ((-1) + (-1))] / 4 = 1.$$  

Therefore, in theory, the pitching moment bridge will not output any change in voltage under the applied normal force case. As another example, the same exercise is completed for a case of an applied pitching moment. The pitching moment output under an applied pitching moment is given by

$$Output_{PM} = [((+1) + (+1)) - ((-1) + (-1))] / 4 = 1.$$  

With pitching moment applied, the normal force bridge output is

$$Output_{NF} = [((-1) + (+1)) - ((+1) + (-1))] / 4 = 0.$$  

These semi-quantitative tests help assure that the gaging strategy is properly employed. Figure 30 shows the gage placement for side force and yawing moment.

![Image](Figure 30: Example Cage Section Gage Locations for Yawing Moment and Side Force. The Beam Bending Shapes show the Overall Deformation of the Cage Beams Subject to the Different Components of Load.)
The same philosophy of bridge isolation is applied again in the side force and yawing moment case. Carrying out the same arithmetic as was done above, first for an applied side force gives

\[ Output_{SF} = \frac{[((+1) + (+1)) - ((-1) + (-1))]}{4} = 1. \]

For the same condition, an applied side force, the yawing moment bridge output is

\[ Output_{YM} = \frac{[((-1) + (+1)) - ((+1) + (-1))]}{4} = 0. \]

For consistency, the case of applied yawing moment is demonstrated. The output of the side force bridge under the applied yawing moment is given by

\[ Output_{SF} = \frac{[((-1) + (+1)) - ((+1) + (-1))]}{4} = 0. \]

For the same condition, an applied yawing moment, the yawing moment bridge output is

\[ Output_{YM} = \frac{[((+1) + (+1)) - ((-1) + (-1))]}{4} = 1. \]

The stress fields in the rolling moment case are more complex due to the three stresses contributing to the overall stress. In Figure 31 a four beam cage section is shown with each one of the beams in a separate quadrant. This four-quadrant design makes it easier to illustrate the general case of the stress gradients for the rolling moment. Note that this cage section differs from the cage section design referenced above; however, the location of the section cut, section A-A, still references Figure 29. The + / - symbols in Figure 29 indicate tension and compression at that location due to positive rolling moment shown at the forward end of the beam, away from the sectional cut.
The three stresses (equations 153 through 155) developed from the application of rolling stress are shown along with the respective signs for tension(+) and compression(-). The gages are located so that the three stresses are all the same sign to maximize bridge output. Note the gradient in the $x$ – direction (into the page) is not visible. See the memorandum written by Robert Moore for more information Moore [18].

6 Variable Inertia Beams

For manufacturing simplicity, the most common beams used for balances are typically constant inertia type beams. The purpose of this section is to introduce the generalized equations for the variable inertia type beam, which has advantages over the constant inertia beam in certain applications where optimum design is required Scott [19]. This material represents an abridged version of Scott [19]. Primarily, the basic beam used in balance design at LaRC is the constant inertia beam with fixed ends, which, with the inflection point being at the half-length, is referred to as a “double cantilever” beam. The variable inertia beams discussed in this section provide the designer with additional tools for design optimization when certain stress or deflection characteristics are required.
Given the inverse relationship between bending stress and the area moment of inertia, the designer can increase or decrease gage output accordingly with the use of variable inertia beams. Figure 32 shows examples of notched or beefed beam geometries utilized in transducer design including (a) end notch, (b) center beefed, (c) full notch, and (d) partial notch.

Figure 32: Examples of Variable Inertia Beam Types that have been Used Historically.

6.1 End Notch

The greatest advantage of the end notched beam, for design optimization, versus the “double cantilever” constant inertia beam is the shallow stress gradient generated along the beam length from the $I_1$ end of the beam in Figure 33. This has been particularly useful in the axial section of small-diameter transducers where the beam length is necessarily short. The shallower stress gradient allows a much greater percentage of the maximum strain at the $I_1$ end of the beam to be realized and sensed under the strain gage. The stress at the $I_1$ end of the beam is of greatest interest to the designer, but the stress at the $I_2$ end should also be determined. The moment at that end, $M_B$, may be small and the stress gradient steep. If the added inertia of a radius at the $I_2$ end is neglected, then the stress will generally be higher here than at the $I_1$ end.

Figure 33: Example of End Notch Beam Showing Boundary Conditions.
The end notched beam has a primary application in the transducer axial force section
where it is the measuring element flanked on either side by “double cantilever” flexures
that resist all forces and moments in this section other than axial force. For example, a
cylindrical balance with an end notched measurement beam is shown in Figure 34.

Figure 34: Example of End Notch Beam used in an Axial Section.

The end notched measurement beam is in parallel with the flexures; therefore, its spring
constant must be determined to obtain the percentage of load it carries versus the flex-
ures. Since the point of inflection for this type of beam is variable, it is first necessary to
determine an expression for the end moments before the spring constants can be deter-
mined.

From Figure 33, it can be seen that the beam is indeterminate in nature because the two
available static equations are not enough to determine the three unknowns \( M_A \), \( M_B \), and
\( F_B \). \( F_A \) is the percentage load of the applied force causing translation of the notched
beam. The two available static equations are below

\[
\sum F = 0 = -F_A + F_B = 0 \tag{164}
\]
\[
\sum M = 0 = M_A + M_B - FL = 0. \tag{165}
\]

One method for determining another relation for \( M_A \) is provided by the area-moment
theorem for determining slopes

\[
\theta_{AB} = \frac{1}{EI_1} (AREA)_{AB} \tag{166}
\]

where the (AREA) is the area of the moment diagram in Figure 35.

Figure 35: Moment-area Diagram for a End Notched Beam.
The beam slope is found from the moment-area diagram, by parts, for the end notched beam shown in Figure 35 and the area-moment theorem, as follows

\[
\theta = \frac{1}{EI_1} \left( MAW - \frac{FW^2}{2} \right) + \frac{1}{EI_2} \left[ MA(L - W) - FW(L - W) - F(L - W) \left( \frac{L - W}{2} \right) \right] = 0.
\]

This can be solved for \( MA \) yielding

\[
MA = \frac{I_1}{I_2} - \left( \frac{W}{L} \right)^2 \left( \frac{I_1}{I_2} \right) + \left( \frac{W}{L} \right)^2,
\]

Generally, the beam base \( b \) is held constant and the inertias \( I_1 \) and \( I_2 \) differ only by beam height \( h \). Since,

\[
I_1 = \frac{bh_1^3}{12}, \quad I_2 = \frac{bh_2^3}{12},
\]

the inertia ratio in equation 168 can be changed to \( (h_1/h_2)^3 \) yielding

\[
MA = FL \left[ \left( \frac{h_1}{h_2} \right)^3 - \left( \frac{W}{L} \right)^2 \left( \frac{h_1}{h_2} \right)^3 + \left( \frac{W}{L} \right)^2 \right] \]

The terms in brackets of equation 170 can be graphed, as shown in Figure 8 from reference Scott [19], such that

\[
MA = FL(\lambda).
\]

Since \( MA \) has now been determined, \( MB \) can be solved from equation 165

\[
MB = FL - MA
\]

\[
MB = FL - FL(\lambda)
\]

\[
MB = FL(1 - \lambda).
\]

The point of inflection, as measured from the \( I_1 \) end, can be obtained by setting

\[
MA - FX = 0
\]

\[
x = \frac{MA}{F} = \frac{FL\lambda}{F} = L\lambda.
\]

The designer is also interested in the moment that the strain gage senses, which is
\[ M_G = M_A - Fz \]
\[ M_G = FL(\lambda) - Fz \]
\[ M_G = F(L\lambda - z) \]  
(175)

where \((z)\) is the distance from the \(I_1\) end of the beam to the centerline of the grid on the strain gage.

The spring constant for the beam can be determined by, again, using the moment diagram and the area-moment theorem for determining deflections

\[ \delta_{A/B} = \frac{1}{EI_1} (\text{AREA})_{AB} x_A. \]  
(176)

For the end notched beam, this results in

\[ \delta = \frac{1}{EI_1} \left( \frac{MAW^2}{2} - \frac{FW^3}{3} \right) + \frac{1}{EI_2} \left[ \frac{MA}{2} (L^2 - W^2) - FW(L^2 - W^2) - \frac{F}{6} (W^3 - 3WL^2 + 2L^3) \right]. \]  
(177)

Substituting in \(FL(\lambda)\) for \(MA\) and simplifying, the deflection equation can be put in the form

\[ \delta = \frac{FL^3}{3EI_1} \left( \frac{3\lambda}{2} \left( \frac{W}{L} \right)^2 + \frac{I_1}{I_2} - \left( \frac{W}{L} \right)^2 \frac{I_1}{I_2} \right) - \frac{I_1}{I_2} + \left( \frac{W}{L} \right)^3 \left( \frac{I_1}{I_2} - 1 \right) \]  
(178)

The spring constant is then

\[ K = \frac{F}{\delta} = \frac{3EI_1}{L^3} \left( \frac{1}{3\lambda} \left[ \left( \frac{W}{L} \right)^2 + \frac{I_1}{I_2} - \left( \frac{W}{L} \right)^2 \frac{I_1}{I_2} \right] - \frac{I_1}{I_2} + \left( \frac{W}{L} \right)^3 \left( \frac{I_1}{I_2} - 1 \right) \right). \]  
(179)

As in equation 170, substitute \((h_1/h_2)^3\) in for \((I_1/I_2)\) such that equation 179 becomes

\[ K = \frac{F}{\delta} = \frac{3EI_1}{L^3} \left( \frac{1}{\frac{3\lambda}{2} \left[ \left( \frac{W}{L} \right)^2 + \frac{h_1}{h_2} \right]^3 - \left( \frac{W}{L} \right)^2 \left( \frac{h_1}{h_2} \right)^3 - \left( \frac{h_1}{h_2} \right)^3 + \left( \frac{W}{L} \right)^3 \left( \frac{h_1}{h_2} \right)^3 - 1 \right) \right). \]  
(180)

The terms within the brackets of equation 180 represented by \((\mu)\), may be graphed, as shown in Figure 9 of reference Scott [19], such that

\[ K = \frac{F}{\delta} = \frac{3EI_1}{L^3}(\mu). \]  
(181)

Equation 181 is arranged in this order so that the spring constant of the end notch beam is equal to the spring constant of a single cantilever beam of the \(I_1\) size multiplied by a factor \(\mu\).
6.2 Center Beefed Beam

The end moment characteristics of the center beefed or center notched beams do not differ from the constant inertia beam; however, the spring constant characteristics vary. Center beefed beams are used at LaRC to increase the beam buckling strength acting along the longitudinal axis of the “double cantilever” flexures in the axial section. Another use for the center beefed beams is to minimize transducer deflections where relatively long beams are used, e.g., where the W/L ratio is greater than 0.5.

Contrary to the center beefed beam, the center notched beam is used in transducers as an aerodynamic moment measuring element. Typically, center notched beams are used in pairs with the notch being the moment gaged area and the ends of the beam used to measure an aerodynamic force perpendicular to the beam length. For example, the pitching moment gage can be located in the notch while the normal force gage is located at the beam end nearest the bulkhead.

The center beefed and center notched beams, as shown in Figure 36, respectively, are cited from reference Scott [19]. Both beam configurations are not considered indeterminate because they are symmetrical about the midpoint of the beam.

It is obvious from the symmetry of Figure 37 that $M_A$ is equal to $M_B$ therefore, the static equations give

\[
\sum F = -F_A + F_B = 0
\]

\[
\sum M = M_A + M_B - FL = 0
\]

\[
M_A = M_B = \frac{FL}{2}.
\]

(182)

In addition, because of symmetry, the point of inflection is at the beam half-length, the same as the constant inertia “double cantilever” beam. The only unknowns for these two
types of variable inertia beams, therefore, are their respective spring constants. Since the inflection point is always located at the beam half-length, it is only necessary to work with half of the beam when determining the spring constant.

An inflection point is in effect a structural “hinge,” therefore, the force, $F$, as modeled in Figure 38, acts at the left of the half beam. The moment diagram, by parts, for this beam is also shown in Figure 38.

![Figure 38: Static Diagram of a Center Beeped Beam and the Moment Diagram Corresponding to the Static Conditions.](image)

Using the moment diagram in Figure 38 and the area-moment theorem for beam deflections

$$\delta_{O/B} = \frac{1}{EI_1} [\text{AREA}_{OB}] \bar{x}_O$$

$$\delta_{O/B} = \frac{FL^3}{EI_1} \left( -\frac{FW^3}{24} \right) + \frac{1}{EI_2} \left( -\frac{FL^3}{24} + \frac{FW^3}{24} \right).$$ \hspace{1cm} (183)

Again, based on symmetry, equation 183 can be multiplied by two to solve for the total beam deflection, simplified and arranged to give

$$\delta = \frac{FL^3}{12EI_2} \left[ -\frac{I_2}{I_1} \left( \frac{W}{L} \right)^3 + \left( \frac{W}{L} \right)^3 - 1 \right].$$ \hspace{1cm} (184)

Knowing the total beam deflection, the spring constant can now be determined using

$$K = \frac{F}{\delta} = \frac{12EI_2}{L^3} \left[ -\frac{I_2}{I_1} \left( \frac{W}{L} \right)^3 + \left( \frac{W}{L} \right)^3 - 1 \right].$$ \hspace{1cm} (185)

If the beam base is constant, substitute $\left( \frac{h_2}{h_1} \right)^3$ for $\left( \frac{I_2}{I_1} \right)$ and equation 185 becomes

$$K = \frac{F}{\delta} = \frac{12EI_2}{L^3} \left[ -\left( \frac{h_2}{h_1} \right)^3 \left( \frac{W}{L} \right)^3 + \left( \frac{W}{L} \right)^3 - 1 \right].$$ \hspace{1cm} (186)

Here, too, the variable within the brackets of equation 186, represented by $\alpha$, can be graphed, as shown in Figure 10 from reference Scott [19], such that
\[ K = \frac{F}{\delta} = \frac{12EI_2}{L^3} (\alpha). \] (187)

Therefore, equation 187 provides the spring constant for a conventional “double cantilever” beam having a constant inertia value of \( I_2 \) times some factor \( \alpha \), which will make the beam either stiffer or softer than the constant inertia beam, depending on whether the beam center is beefed or notched.

### 6.3 Full Notched Beam

The term “full notch” refers to a large radius machined into the outward facing surface of the beam, along its entire length in the longitudinal direction, typically the \( x - direction \) as in Figure 39. Doing so varies the cross-sectional area of the beam and the beam inertia, which are both at a minimum at the beam half-length, or midpoint. This is important to the designer where there is a need to increase the sensitivity at the gage for a specific load. Due to the inverse relationship between inertia and bending stress, the pitch and yaw gages are located at the midpoints of the fully notched beam in a single cage balance.

![Figure 39: Depiction of a Fully Notched Beam and an Associated Free Body Diagram.](image)

A free body diagram of a fully notched beam is also shown in Figure 39. Here, the pitch or yaw moment \( (M) \) is converted to a couple, \( Fc \), where \( c \) is the distance from the neutral axis at the point of inflection of the top beam to the neutral axis at the point of inflection of the bottom beam Scott [20]. As shown in Figure 39, additional moments \( M_A \) and \( M_B \) are induced in the top and bottom beams as a result of eccentricity created by the notch. Eccentricity \( (e) \) refers to the offset distance between the line of action of the force, \( F \), and the neutral axis, which produces an additional bending moment, \( Fe \). The bending stress in the curvature from \( M_A \) is additive to the normal stress \( (F/A) \) and thus provides a method for increasing the sensitivity to pitch or yaw in strain-gage balances.

A problem arises in determining what proportion of the moment, \( Fe \), is represented by the induced moment, \( M_A \). The ratio of \( M_A \) to \( M_B \) is dependent on the beam stiffness from the beam mid-length to the beam end. Assuming fixed end beams, the change in slope, \( \Delta \theta \), resulting from the induced moment is zero from the center to the beam end; therefore, the differential equation for the beam slope provides a method for determining \( M_A \).
Using the expression $z = h + Kx^2$, as shown in Figure 40, to approximate the beam curvature between heights $h$ and $H$ or between $x = 0$ to $x = L/2$ requires that $K = 4(H - h)/L^2$. Therefore, the beam height ($z$) is

$$z = h + \frac{4(H - h)x^2}{L^2} \quad (188)$$

and the beam slope ($\theta$) is

$$\theta = \frac{dy}{dx} = \frac{1}{E} \int_0^{L/2} \frac{MA}{I} dx - \frac{1}{E} \int_0^{L/2} \frac{Fe}{I} dx. \quad (189)$$

The general expression for $e$ and $I$ must be used over the entire beam length. The general expressions for eccentricity and inertia are $e = z/2 - h/2$ and $I = (bh^3)/12$, respectively. Substituting the beam height ($z$) into the eccentricity equations results in

$$e = \frac{1}{2} \left[ h + \frac{4(H - h)x^2}{L^2} \right] - \frac{h}{2} \quad (190)$$

Simplifying further, the equation becomes

$$e = 2(H - h)\frac{x^2}{L^2}. \quad (191)$$

And substituting $z$ in for $h$ the general expression for $I$ results in,

$$I = \frac{b}{12} \left[ h + \frac{4(H - h)x^2}{L^2} \right]^3. \quad (192)$$

Both $I$ and $e$ can be substituted back into the beam slope equation above, as follows

$$\frac{dy}{dx} = \frac{12MA}{Eb} \int_0^{L/2} \frac{1}{\left[ h + \frac{4(H - h)x^2}{L^2} \right]^3} dx - \frac{24F(H - h)}{EbL^2} \int_0^{L/2} \frac{x^2}{\left[ h + \frac{4(H - h)x^2}{L^2} \right]^3} dx. \quad (193)$$
Since the change in slope is zero ($\Delta \theta = 0$) between $x = 0$ and $x = L/2$, and $K = [4(H - h)]/L^2$ the above equation can be simplified and rearranged to solve for $M_A$ accordingly

$$\frac{12M_A}{Eb} \int_0^{L/2} \frac{1}{[h + Kx^2]^3} dx = \frac{6FK}{Eb} \int_0^{L/2} \frac{x^2}{[h + Kx^2]^3} dx$$

$$M_A = \frac{FK}{2} \int_0^{L/2} \frac{x^2}{[h + Kx^2]^3} dx$$

(194)

Integrating equation 194 gives

$$M_A = \frac{FK}{2} \left[ \frac{-L}{8K(h - K\frac{L^2}{4})^2} + \frac{1}{4K} \left( \frac{L}{4h(h + K\frac{L^2}{4})} + \frac{1}{2h} \left( \frac{1}{\sqrt{hK}} \tan^{-1} \frac{L\sqrt{hK}}{2h} \right) \right) \right].$$

(195)

Substituting $[4(H - h)/L^2]$ in for $K$ into the above equation and simplifying gives

$$M_A = \frac{F(H - h)}{2} \left[ \frac{H}{\sqrt{H^2 - 1}} - \frac{2h}{H} + 1 \right] \left[ 3 \left( \frac{H}{h} - 1 \right) \left( \frac{H}{\sqrt{H^2 - 1}} + \frac{2h}{3H} \right) + 1 \right].$$

(196)

Equation 196 can be further simplified to $M_A = Fe(\alpha)$ where

$$\alpha = \frac{H}{\sqrt{H^2 - 1}} - \frac{2h}{H} + 1 \left[ 3 \left( \frac{H}{h} - 1 \right) \left( \frac{H}{\sqrt{H^2 - 1}} + \frac{2h}{3H} + 1 \right) \right].$$

(197)

Values of $\alpha$ range from 0.333 for $(H/h) = 1$ to 0.0719 for $H/h = 5$. The values of alpha are plotted in Figure 3a of reference Scott [20]. Since,

$$M_A + M_B = Fe$$

$$M_B = Fe - M_A$$

$$M_B = Fe(1 - \alpha)$$

(198)

the moment at any point $(x)$ from the beam midlength is also valuable to the designer.

$$M_x = M_A - Fe$$

65
\[ M_x = F \frac{(H - h)}{2} \alpha - 2F(H - h) \frac{x^2}{L^2} \]

\[ M_x = F \frac{(H - h)}{2} \left[ \alpha - 4 \left( \frac{x}{L} \right)^2 \right] \]

\[ M_x = Fe \left[ \alpha - 4 \left( \frac{x}{L} \right)^2 \right] \quad (199) \]

Setting equation 199 equal to zero, the point of inflection \( x \) can be determined for the beam

\[ 4Fe \left( \frac{x}{L} \right)^2 = Fe(\alpha) \]

\[ x = \sqrt{\alpha} \left( \frac{L}{2} \right). \quad (200) \]

Until now, a single beam has been analyzed with an applied force, \( F \); however, to measure pitch or yaw moment it is necessary to resolve the moment, \( M \), into a force couple, \( F_c \), as shown below in Figure 41. The value of \( c \) must be determined at the point of inflection (where \( \theta = 0 \)) for both top and bottom beams to correctly obtain the force, \( F \).

Figure 41: Free Body Diagram Showing Representation of a Decoupled Moment on a Full Notched Beam.

At the point of inflection, where \( x = \sqrt{\alpha}(L/2) \), the values for \( c, z, \) and \( F \) can be determined using the equations below

\[ c = a + z \]

\[ z = h + 4(H - h) \left( \frac{x}{L} \right)^2 \]

\[ F = \frac{M}{a + h + \alpha(H - h)}. \quad (201) \]

When used in a balance cage section, the bending moment spring constant is required to determine the load proportions between the notched beam group and the other beam group(s) in the cage section. The general expression for the bending moment spring constant for a cantilever beam is
\[
\frac{M}{\theta} = \frac{EI}{L}.
\]  
(202)

Figure 42: Free Body Diagram showing the Nomenclature used for Representation of a Full Notched Beam.

For Figure 42, the spring constant would take the following form

\[
\frac{M}{\theta} = \frac{E}{2 \int_0^{L/2} \frac{1}{12} dx} - \frac{Eba^3}{12L} \tag{203}
\]

\[
\frac{M}{\theta} = \frac{E}{2 \int_0^{L/2} \left( \frac{1}{12} \left( \frac{1}{d+4(D-d)(\frac{x}{L})^2} \right) \right) dx} - \frac{Eba^3}{12L} \tag{204}
\]

letting \(K = 4(D - d)/L^2\), it becomes

\[
\frac{M}{\theta} = \frac{24}{6} \int_0^{L/2} \frac{1}{(d+Kx^2)^3} dx - \frac{Eba^3}{12L}. \tag{205}
\]

Integrating the above gives

\[
\frac{M}{\theta} = \left[ \frac{24}{6} \frac{L}{8d(d+Kx^2)^2} + \frac{3}{4d} \frac{L}{4d(d+Kx^2)} + \frac{1}{2d} \left( \frac{1}{\sqrt{d^2K}} \tan^{-1} \frac{L\sqrt{dK}}{2d} \right) \right] - \frac{Eba^3}{12L} \tag{206}
\]

and then substituting \((K)\) back into the equation results in

\[
\frac{M}{\theta} = \left[ \frac{24}{6} \frac{L}{8dD^2} + \frac{3}{4d} \frac{L}{4dD} + \frac{1}{2d} \left( \frac{L}{2d} \tan^{-1} \frac{\sqrt{2}-1}{\sqrt{2}-1} \right) \right] - \frac{Eba^3}{12L}. \tag{207}
\]

Then, the bending moment spring constant for the fully notched beam can be further simplified as
\[
\frac{M}{\theta} = \frac{2Eb}{9L} \left[ \frac{dD^2}{(\frac{L}{D})^2 \tan^{-1} \sqrt{D/d-1} + \frac{D}{d} + \frac{3}{3}} - \frac{3a^3}{8} \right].
\]

(208)

The maximum stress at the beam mid-length can be determined by combining the bending stress from the induced moment \(M_A\) with normal stress using the combined stress equation below:

\[
\sigma = \frac{F}{bh} + \frac{6M_A}{bh^2}
\]

(209)

\[
\sigma_{\text{maxmid}} = \frac{F}{bh} \left[ 1 + 3\alpha \left( \frac{H}{h} - 1 \right) \right].
\]

(210)

The usual method is to average the maximum stress \(\sigma_{\text{maxmid}}\) and the stress at the end of the strain-gage active grid \(\sigma_x\); therefore, the stress along the beam at any point \(x\) from the beam center is:

\[
\sigma_x = \frac{F}{bh} \left[ 1 + 4\left( \frac{H}{h} - 1 \right) \left( \frac{x}{L} \right)^2 \right] \left[ 1 + 3\alpha \left[ \frac{1 - \frac{4}{\alpha} \left( \frac{x}{L} \right)^2}{1 + 4 \left( \frac{H}{h} - 1 \right) \left( \frac{x}{L} \right)^2} \right] \right].
\]

(211)

For gage stress, the average stress under the strain-gage active grid will have to be determined using the appropriate \(M_x\) and \(z\) values from their respective equations above.

\[
\sigma_{\text{gage}} = \sigma_{\text{maxmid}} + \sigma_x
\]

(212)

### 6.4 Partial Notch

The partial notch figure is cited from Scott [21]. The pitch or yaw moment \((M)\) of Figure 43 can be resolved into a force couple, \(F\), where \(c\) is the distance from the neutral axis at the point of inflection of the top beam to the neutral axis at the point of inflection for the bottom beam. The term “partial” refers to the fact that the notch does not extend the entire length of the beam. As shown in Figure 44, additional moments \(M_A\) and \(M_B\)
are induced in the top and bottom beams as a result of eccentricity created by the notch. Eccentricity \( \varepsilon \) refers to the offset distance between the line of action of the force and the neutral axis, which produces an additional bending moment, \( Fe \). The bending stress in the curvature from \( M_A \) is additive to the normal stress \( (F/A) \) and thus provides a method for increasing the sensitivity to pitch or yaw in strain-gage balances.

Figure 44: Free Body Diagram Showing a Beam with a Partial Notch and Further Decoupling the Applied Forces.

The process for determining the spring constant of a partial notch beam is similar in theory to that of the full notch beam, therefore, the first step is the determination of \( M_A \). However, a challenge arises in determining what proportion of the moment \( Fe \) is represented by \( M_A \). The ratio of \( M_A \) to \( M_B \) is dependent on the beam stiffness from the beam mid-length to the beam end. Assuming fixed end beams, the change in slope, \( \Delta \theta \), resulting from the induced moment is zero from the mid-length to the beam end; therefore, the differential equation for the beam slope provides a method for determining \( M_A \).

Figure 45: Free Body Diagram Showing a Beam with a Partial Notch Annotated to Define Notch Features.

Using the expression \( z = h + Kx^2 \) in Figure 45, the constant \( K \) required to approximate the beam curvature between heights \( h \) and \( H \) or between \( x = 0 \) and \( x = w/2 \) requires that

\[
K = \frac{4(H - h)}{w^2}. \tag{213}
\]

Therefore, the beam height \( (z) \) at any point is

\[
z = h + \frac{4(H - h)x^2}{w^2}. \tag{214}
\]

The beam slope \( (\theta) \) at any point is determined with the following equation

\[
\frac{dy}{dx} = \frac{1}{E} \int_0^{L/2} \frac{M_A - Fe}{I} \, dx \tag{215}
\]
\[
\frac{dy}{dx} = \frac{1}{E} \int_0^{w/2} \frac{M_A - F e_1}{I_1} + \frac{1}{E} \int_{w/2}^{L/2} \frac{M_A - F e_2}{I_2}. \quad (216)
\]

The following general expressions for eccentricity and moment of inertia must be used to evaluate the beam from \( x = 0 \) to \( x = w/2 \). Substituting equation 214 into both equations for \( h \) and simplifying results in the following

\[
e_1 = \frac{2(H - h)x^2}{w^2},
\]

\[
I_1 = \frac{b}{12} \left[ h + \frac{4(H - h)x^2}{w^2} \right]^3. \quad (217)
\]

For \( x = w/2 \) to \( x = L/2 \), the corresponding equations for eccentricity and moment of inertia are

\[
e_2 = \frac{H - h}{2},
\]

\[
I_2 = \frac{bH^3}{12}. \quad (218)
\]

Substituting the above eccentricity and moment of inertia equations into equation 216 gives

\[
\frac{dy}{dx} = \frac{12M_A}{Eb} \int_0^{w/2} \frac{dx}{(h + Kx^2)^3} - \frac{24F(H - h)}{Eb w^2} \int_0^{w/2} \frac{x^2 dx}{(h + \left(\frac{4(H - h)x^2}{w^2}\right)^3} + \frac{12M_A - 6F(H - h)}{Eb H^3} \int_{w/2}^{L/2} dx. \quad (219)
\]

Since \( \Delta \theta = 0 \) between \( x = 0 \) and \( x = L/2 \) and \( K = (4(H - h))/w^2 \), equation 219 can be solved in terms of \( M_A \) and \( F \) as follows,

\[
\frac{12M_A}{Eb} \int_0^{w/2} \frac{dx}{(h + Kx^2)^3} + \frac{12M_A}{Eb H^3} \int_0^{L/2} dx = \frac{24F(H - h)}{Eb w^2} \int_0^{w/2} \frac{x^2 dx}{(h + Kx^2)^3} + \frac{6F(H - h)}{Eb H^3} \int_{w/2}^{L/2} dx. \quad (220)
\]

Integrating and simplifying further,

\[
M_A = F \left( \frac{H - h}{2} \right), \quad e = \frac{(H - h)}{2} \quad (221)
\]

\[
\alpha = \frac{\frac{H}{\pi} \tan^{-1} \sqrt{H/h-1} - 2h + 1 + 8 \left( \frac{H}{\pi} \right)^2 \left( \frac{L}{w} - 1 \right) \left( \frac{H}{h} - 1 \right)}{\left(3 \left( \frac{H}{\pi} - 1 \right) \left( \frac{H}{\pi} \tan^{-1} \sqrt{H/h-1} + \frac{2h}{\pi H + 1} + 1 \right) + 8 \left( \frac{H}{\pi} \right)^2 \left( \frac{L}{w} - 1 \right) \left( \frac{H}{h} - 1 \right) \right)} \quad (222)
\]

or

\[
M_A = Fe(\alpha). \quad (222)
\]
From the moment diagram and the static equations, $M_B$ can be determined

\[
M_A + M_B = Fe
\]
\[
M_B = Fe - M_A
\]
\[
M_B = Fe(1 - \alpha).
\] (223)

The moment at any point $x$ from the beam mid-length is also valuable to the designer. To start, from $x = 0$ to $= w/2$:

\[
M_x = M_A - Fe
\]
\[
M_x = \frac{F(H - h)}{2} (\alpha) - \frac{2F(H - h)x^2}{w^2}
\]
\[
M_x = \frac{F(H - h)}{2} \left( \alpha - \frac{4x^2}{w^2} \right) = Fe \left( \alpha - \frac{4x^2}{w^2} \right).
\] (224)

If $M_x$ is negative, it simply means that $x$ is past the point of inflection and $M_x$ is in the opposite direction from $M_A$. From $x = w/2$ to $x = L/2$

\[
M_x = \frac{F(H - h)}{2} (\alpha) - \frac{F(H - h)}{2}
\]
\[
M_x = \frac{F(H - h)}{2} (\alpha - 1) = Fe(\alpha - 1).
\] (225)

Again, the negative $M_x$ means that the moment is opposite of $M_A$. By setting equation 224 equal to zero, the point of inflection ($x$) can be determined for the beam

\[
M_x = 0 = Fe \left( \alpha - \frac{4x^2}{w^2} \right)
\]
\[
4Fe \frac{x^2}{w^2} = Fe(\alpha)
\]
\[
x = \sqrt{\frac{\alpha w}{2}}.
\] (226)

Until now a single beam has been analyzed with an applied force, $F$. To measure pitch or yaw moment, $M$, as shown in Figure 46 below from Scott [21], it is necessary to resolve the moment into a couple, $Fc$. The value of $c$ must be obtained at the point of inflection for the two beams to correctly obtain the force.

Figure 46: Free Body Diagram Showing a Beam with a Partial Notch Subjected to a Moment.
From Figure 46, the resolved moment can be deconstructed to find the force, $F$, at the point of inflection, $x$

$$c = a + z$$

$$z = h + \frac{4(H - h)x^2}{w^2}.$$  

(227)

Substituting in equation 227 for $x$ at the point of inflection, the beam height, $z$, becomes

$$z = h + \alpha(H - h)$$

(228)

and subsequently, the force, $F$, is

$$F = \frac{Mc}{c} = \frac{M}{a + h + \alpha(H - h)}.$$  

(229)

When used in a balance cage section, the bending moment spring constant is required to determine the load proportions between the notch beam group and the other beam group(s) in the cage section. The general expression for the bending moment spring constant for a cantilever beam is

$$\frac{M}{\theta} = \frac{EI}{L}$$

(230)

and for Figure 46, the spring constant equation would take the following form

$$\frac{M}{\theta} = \frac{E}{2\int_0^{L/2} \frac{dx}{x^2}} - \frac{Eba^3}{12L}$$

(231)

$$\frac{M}{\theta} = \frac{E}{2\left(\int_0^{w/2} \frac{dx}{b(d+Kx^2)} \right) + \int_0^{(L-w)/b} \frac{dx}{bD^{3/2}}} - \frac{Eba^3}{12L}$$

(232)

where $K = (4(D - d))/w^2$. After integration and simplification, it can be put into the following form

$$\frac{M}{\theta} = \frac{Eb}{12L} \left( \frac{D^3}{sLd} \left( \frac{D}{d} \right)^2 \tan^{-1} \left( \frac{D}{d-1} \right) + \frac{D}{d} \right) + 1 - \frac{w}{L}$$

(233)

or grouping terms

$$\frac{M}{\theta} = \frac{Eb}{12L}(D^3\gamma - a^3)$$

(234)

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where
\[
\gamma = \left( \frac{1}{\frac{w D}{L d} + \frac{(D/d)^2 \tan^{-1} \sqrt{D/d-1}}{\sqrt{D/d-1}} + \frac{D}{d} + \frac{3}{3}} + 1 - \frac{w}{L} \right).
\] (235)

This form is convenient since \( \gamma=1 \) represents the spring constant of an un-notched beam group. The values of \( \gamma \) are graphed in Figure 7 from reference Scott [21]. The maximum stress at the beam mid-length is
\[
\sigma = \frac{F}{bh} + \frac{6M_A}{bh^2}.
\] (236)

For the gage stress, the average stress under the strain-gage will have to be determined using the appropriate \( M_x \) and \( z \) values. The usual method is to average the maximum stress and the stress at the end of the gage active grid. Given that \( M_x = Fe(\alpha-(4x^2)/w^2) \) and \( z = h+(4(H-h)x^2)/w^2 \), the stress at any point from the beam center, or mid-length, at \( x = 0 \) to \( x = w/2 \) is
\[
\sigma_x = \frac{F}{bh} \left[ 1 + 3\alpha \left( \frac{H}{h} - 1 \right) \right].
\] (237)

and from \( x = w/2 \) to \( x = L/2 \), since the cross-section is constant, the stress is constant as follows
\[
\sigma_x = \frac{F}{bH} \left[ 1 + 3 \left( 1 - \frac{H}{h} \right) (\alpha - 1) \right].
\] (238)

A negative stress means that the top edge of the beam has changed from a compressive to a tensile stress, using the sign convention of Figure 2 from reference Scott [21].

### 6.5 Stress Riser

A stress riser can be added to a rectangular beam to increase the stress intensity at the strain gage location on top of the riser by increasing the distance of the extreme fiber on the top of the riser from the neutral axis without significantly increasing the moment of inertia of the beam as shown in Figure 47.
Since the moment of inertia is constant along the beam length, structural analysis of a
stress riser is simplified compared to the notched beams with variable cross sections. The
analysis requires consideration of the riser portion of the beam in the calculation of the
centroid of the beam cross section as well as the calculation of the moment of inertia.

7 Dowel, Model, Sting and Key Analysis

This section will discuss the stress analysis for the model-balance connections and the
balance-sting connections. These stresses can be limiting factors in the design if the
overall length and diameter to load ratios are high.

The bearing stress on the dowel that connects the model to the balance, is given by the
combination of the rolling moment and axial force loads.

$$\sigma_{bs,dwl,AF+RM} = \sqrt{\left( \frac{2RM}{n_d t_d d_m} \right)^2 + \left( \frac{AF}{n_d t_d d_m} \right)^2}$$

(239)

where $RM$ is the applied rolling moment, $n_d$ is the number of dowels (usually only one),
$t_d$ is the depth of the dowel and $d_m$ is the diameter of the balance at the location of the
dowel, and $d_d$ is the diameter of the dowel. The shear stress on the dowel is also given
by the combination of rolling moment and axial force loads.

$$\tau_{dwl,AF+RM} = \sqrt{\left( \frac{8RM}{n_d \pi d_d^2 d_m} \right)^2 + \left( \frac{4AF}{n_d \pi d_d^2} \right)^2}$$

(240)

The stress on the sting end of the balance is a combined calculation of the stress in both
the $x-$ and $y-$directions.

$$\sigma_{xSEPM+NF} = \frac{(PM + NF L_{SE}) C}{I}$$

(241)

$$\sigma_{ySEYM+SF} = \frac{(YM + SF L_{SE}) C}{I}$$

(242)
\[ \sigma_{\text{maxSE}} = \sqrt{\sigma_{\text{SEPM+NF}}^2 + \sigma_{\text{SEYM+SF}}^2} \quad (243) \]

where \( L_{SE} \) is the length from the balance moment center to the sting end relief (the landing point of the sting), and \( C \) is half the sting end diameter, \( D_{SE}/2 \). The moment of inertia, \( I \), is given by

\[ I = I_x = I_y = \frac{\pi}{64} (D_{SE}^4 - d_{SE}^4) \quad (244) \]

where \( d_{se} \) is the wire hole diameter. The outline drawing of the NTF-118 highlighting the relevant dimensions for the calculations is shown in Figure 48.

\[ \text{Figure 48: NTF-118 Outline.} \]

The key, which locks the balance into the sting, is subjected to both a bearing stress and a torsional shear stress. The bearing stress on the key is given by

\[ \sigma_{\text{bs keyRM}} = \frac{2 \cdot RM}{T \cdot L_{key} \cdot DTA} \quad (245) \]

where \( T \) is the depth of the key, \( DTA \) is the average diameter at the key centerline, and \( L_{key} \) is the length of the key. The torsional shear stress on the key is given by

\[ \tau_{\text{keyRM}} = \frac{2 \cdot RM}{B \cdot L_{key} \cdot DTA} \quad (246) \]

where \( B \) is the width of the key.

The stress on the model end is calculated similarly to the sting end stress. The model end stress is a combined calculation of the stress in both \( x \) and \( y \).

\[ \sigma_{x_{\text{MEPM+NF}}} = \frac{(PM + NF \cdot L_{ME}) \cdot D_{ME}/2}{I_{ME}} \quad (247) \]

\[ \sigma_{y_{\text{MEYM+SF}}} = \frac{(YM + SF \cdot L_{ME}) \cdot D_{ME}/2}{I_{ME}} \quad (248) \]

\[ \sigma_{\text{maxME}} = \sqrt{\sigma_{x_{\text{MEPM+NF}}}^2 + \sigma_{y_{\text{MEYM+SF}}}^2} \quad (249) \]

The inertia for the model end stress calculations is given by

\[ I_{ME} = \frac{\pi}{64} D_{ME}^4. \quad (250) \]

It is desirable for all stress values in the dowel, model, sting and key analysis to be under 60 ksi based on historical precedent.
8 Format of the Typical Stress Report Summary Tables

For each balance design, a stress report is generated which includes the final stress predictions as well as all intermediate calculations. This section illustrates typical summary tables from the stress generation report. The following table maps each cell, individual stress values, with the corresponding equation number from this document where that specific quantity can be found for the Axial section stress summary.

<table>
<thead>
<tr>
<th>LOADS</th>
<th>FLEX</th>
<th>MEAS</th>
<th>STRAP</th>
<th>OUTPUT</th>
</tr>
</thead>
<tbody>
<tr>
<td>NORMAL</td>
<td>$N_F$</td>
<td>$\sigma_{\text{max}f_{NF}}$ (EQ 34)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AXIAL</td>
<td>$A_F$</td>
<td>$\sigma_{\text{max}f_{AF}}$ (EQ 33)</td>
<td>$\sigma_{\text{max}m_{AF}}$ (EQ 28)</td>
<td>$\sigma_{\text{max}s_{AF}}$ (EQ 29)</td>
</tr>
<tr>
<td>PITCH</td>
<td>$P_M$</td>
<td>$\sigma_{\text{max}f_{PM}}$ (EQ 39)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ROLL</td>
<td>$R_M$</td>
<td>$\sigma_{\text{max}f_{RM}}$ (EQ 99)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>YAW</td>
<td>$Y_M$</td>
<td>$\sigma_{\text{max}f_{YM}}$ (EQ 94)</td>
<td>$\sigma_{\text{max}m_{YM}}$ (EQ 97)</td>
<td>$\sigma_{\text{max}s_{YM}}$ (EQ 98)</td>
</tr>
<tr>
<td>SIDE</td>
<td>$S_F$</td>
<td>$\sigma_{\text{max}f_{SF}}$ (EQ 60)</td>
<td>$\sigma_{\text{max}m_{SF}}$ (EQ 60)</td>
<td>$\sigma_{\text{max}s_{SF}}$ (EQ 64)</td>
</tr>
<tr>
<td>Totals</td>
<td></td>
<td>Sum FLEX</td>
<td>Sum MEAS</td>
<td>Sum STRAP</td>
</tr>
</tbody>
</table>

Table 3: Axial Section Summary Stress Report.

A typical design rule when the balance is made from 17-4 or maraging steel is that each of the individual stress values in Table 9 should be kept under 35 ksi. The cells labeled “Sum FLEX”, “Sum MEAS”, and “Sum STRAP” are direct summations of the all the individual stress values in the respective columns that should all be under 100 ksi. It is understood that the summation of the individual stress is a conservative approach as all of these stresses are not likely to occur at the same location within the flexure. The desired output or counts on the axial gage is between 1100 and 1500 counts. Note that higher output is usually better, but strain under the gage must not be excessive (usually under 60 ksi for steel balances). Also designers should take into consideration that these stress predictions do not include any stress concentration factors. For detailed information on stress concentration factors, the reader is referred to reference Pikey [22].

The table for the cage section stress summary is similar to the axial section table. However, as the number of beams in the cage section is dependent on the design, the number of columns in the table will vary. Only a generic representation of the $i^{th}$ beam group set is given in Table 10.
Table 4: Cage Section Summary Stress Report.

For the cases of applied pitching moment and yawing moment, the maximum stress is equal to the gage stress because the stress is assumed to be constant across the length of the beam group. The same general rules apply to the cage section stress summary as the axial stress summary. That is for a balance made of 17-4 PH or maraging steel each of the individual stress values should be under 35 ksi and the summation of the stresses on each beam should be under 100 ksi.

9 Quick Design Reference Guide

For a new balance design, it is recommended to begin by researching balances with a similar load range, diameter, and length, thereby leveraging the many decades of experience of previous balance design engineers. The NASA LaRC single piece balances have been utilized in a wide range of ground test applications. It is very likely that the design requirements for a new balance are similar to an existing balance in the library. Once a similar balance has been found, identify the differences between the existing design and the new balance requirements and use its design as a starting point for meeting the new requirements. Historically, balance designs have been a solution to the design criteria and iterating to an optimal solution was not the primary objective. If building a replicate of an existing balance is not an option, identify the design parameters that are the drivers of the differences between the two designs. Bear in mind, that changing one parameter will most likely cause adverse outcomes elsewhere, so numerous iterations may be needed before all requirements are satisfied.

10 Conclusions

The analytical formulations presented throughout this document have proven to be useful in the design of LaRC balances for decades and have supported aerospace research and breakthroughs influencing commercial and defense aircraft. These formulations are the starting point of the balance design process. Increasingly, finite element analysis is being used as a complement to the analytical formulations, offering cross validation of expected outputs and more comprehensive, higher fidelity estimations of the maximum stress. While the formulations presented are specific to the LaRC family of balances, the general approach is applicable force transducers design in general.
11 A Numerical Example

The following variables define an example balance. Inputting these variables into the relevant equations throughout this document will produce the stresses and gage outputs reported for the axial section (Table 9) and cage section (Table 10).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_s )</td>
<td>2</td>
</tr>
<tr>
<td>( b_s )</td>
<td>0.320 in</td>
</tr>
<tr>
<td>( h_s )</td>
<td>0.030 in</td>
</tr>
<tr>
<td>( l_s )</td>
<td>1.000 in</td>
</tr>
<tr>
<td>( \text{gap under strap} )</td>
<td>0.035 in</td>
</tr>
<tr>
<td>( n_f )</td>
<td>20</td>
</tr>
<tr>
<td>( b_f )</td>
<td>0.320 in</td>
</tr>
<tr>
<td>( h_f )</td>
<td>0.030 in</td>
</tr>
<tr>
<td>( l_f )</td>
<td>0.600 in</td>
</tr>
<tr>
<td>( \text{flx gap} )</td>
<td>0.035 in</td>
</tr>
<tr>
<td>( n_m )</td>
<td>2</td>
</tr>
<tr>
<td>( b_m )</td>
<td>0.320 in</td>
</tr>
<tr>
<td>( h_m )</td>
<td>0.130 in</td>
</tr>
<tr>
<td>( l_m )</td>
<td>0.600 in</td>
</tr>
<tr>
<td>( \text{gap behind measurement beam} )</td>
<td>0.035 in</td>
</tr>
<tr>
<td>( Len )</td>
<td>2.500 in</td>
</tr>
<tr>
<td>( \text{endgap}_1 )</td>
<td>0.045 in</td>
</tr>
<tr>
<td>( \text{endgap}_2 )</td>
<td>0.045 in</td>
</tr>
<tr>
<td>( \text{diag. slot width} )</td>
<td>0.035 in</td>
</tr>
<tr>
<td>( \text{diag. slot angle} )</td>
<td>16 degrees</td>
</tr>
<tr>
<td>( GL_{AF} )</td>
<td>0.070 in</td>
</tr>
<tr>
<td>( R )</td>
<td>0.6875 in</td>
</tr>
<tr>
<td>( r )</td>
<td>0.078 in</td>
</tr>
<tr>
<td>( clm )</td>
<td>0.5826 in.</td>
</tr>
<tr>
<td>( clf )</td>
<td>0.5826 in.</td>
</tr>
</tbody>
</table>

Table 5: Axial section parameter values.

<table>
<thead>
<tr>
<th>Material Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>27.5E6 psi</td>
</tr>
<tr>
<td>( G )</td>
<td>10.57E6 psi</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.3</td>
</tr>
<tr>
<td>( GF )</td>
<td>2.2</td>
</tr>
</tbody>
</table>

Table 6: Material and strain gage properties.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1$</td>
<td>2</td>
</tr>
<tr>
<td>$b_1$</td>
<td>0.210 in</td>
</tr>
<tr>
<td>$h_1$</td>
<td>0.465 in</td>
</tr>
<tr>
<td>$C_{z1}$</td>
<td>0.625 in</td>
</tr>
<tr>
<td>$C_{y1}$</td>
<td>0.105 in</td>
</tr>
<tr>
<td>$n_2$</td>
<td>4</td>
</tr>
<tr>
<td>$b_2$</td>
<td>0.180 in</td>
</tr>
<tr>
<td>$h_2$</td>
<td>0.335 in</td>
</tr>
<tr>
<td>$C_{z2}$</td>
<td>0.495 in</td>
</tr>
<tr>
<td>$C_{y2}$</td>
<td>0.370 in</td>
</tr>
<tr>
<td>$n_3$</td>
<td>4</td>
</tr>
<tr>
<td>$b_3$</td>
<td>0.090 in</td>
</tr>
<tr>
<td>$h_3$</td>
<td>0.115 in</td>
</tr>
<tr>
<td>$C_{z3}$</td>
<td>0.275 in</td>
</tr>
<tr>
<td>$C_{y3}$</td>
<td>0.540 in</td>
</tr>
<tr>
<td>$l_i$</td>
<td>0.600 in</td>
</tr>
<tr>
<td>$L_C$</td>
<td>2.050 in</td>
</tr>
</tbody>
</table>

Table 7: Cage section parameter values assuming a two cage balance with identical cage sections.

<table>
<thead>
<tr>
<th>Gage</th>
<th>Beam Number</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GL_{NF}$</td>
<td>1</td>
<td>0.300 in</td>
</tr>
<tr>
<td>$GL_{PM}$</td>
<td>1</td>
<td>0.300 in</td>
</tr>
<tr>
<td>$GL_{RM}$</td>
<td>2</td>
<td>length = 0.070 in</td>
</tr>
<tr>
<td></td>
<td></td>
<td>width = 0.032 in</td>
</tr>
<tr>
<td>$GL_{YM}$</td>
<td>3</td>
<td>0.300 in</td>
</tr>
<tr>
<td>$GL_{SF}$</td>
<td>3</td>
<td>0.070 in</td>
</tr>
</tbody>
</table>

Table 8: Cage Section Gage Locations.
### Table 9: Axial Section Summary Stress Report.

<table>
<thead>
<tr>
<th>DESIGN LOAD</th>
<th>FLEX (PSI)</th>
<th>MEASURING (PSI)</th>
<th>STRAP (PSI)</th>
<th>GAGE STRESS (PSI)</th>
<th>OUTPUT ($\mu$V/V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NORMAL</td>
<td>4,167</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>800 lb</td>
<td>8,656</td>
<td>17,842</td>
<td>14,438</td>
<td>15,617</td>
<td>1249</td>
</tr>
<tr>
<td>AXIAL</td>
<td>11,171</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>85 lb</td>
<td>6,486</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>PITCH</td>
<td>29,849</td>
<td>18,016</td>
<td>14,579</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>2000 in-lb</td>
<td>7,915</td>
<td>4,129</td>
<td>3,277</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>ROLL</td>
<td>24,142</td>
<td>21,197</td>
<td>12,216</td>
<td>16,082</td>
<td>1287</td>
</tr>
<tr>
<td>400 in-lb</td>
<td>178</td>
<td>178</td>
<td>178</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>YAW</td>
<td>19,612</td>
<td>15,533</td>
<td>8,629</td>
<td>19,612</td>
<td>1569</td>
</tr>
<tr>
<td>800 in-lb</td>
<td>15,516</td>
<td>20,265</td>
<td>13,842</td>
<td>12,592</td>
<td>1007</td>
</tr>
<tr>
<td>SIDE</td>
<td>2,759</td>
<td>9,721</td>
<td>14,187</td>
<td>14,187</td>
<td>1135</td>
</tr>
<tr>
<td>300 lb</td>
<td>8,609</td>
<td>13,474</td>
<td>14,497</td>
<td>13,659</td>
<td>1093</td>
</tr>
<tr>
<td>TOTAL</td>
<td>68,244</td>
<td>39,987</td>
<td>32,293</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

### Table 10: Cage Section Summary Stress Report.

<table>
<thead>
<tr>
<th>DESIGN LOAD</th>
<th>FLEX (PSI)</th>
<th>MEASURING (PSI)</th>
<th>STRAP (PSI)</th>
<th>GAGE STRESS (PSI)</th>
<th>OUTPUT ($\mu$V/V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NORMAL</td>
<td>24,142</td>
<td>21,197</td>
<td>12,216</td>
<td>16,082</td>
<td>1287</td>
</tr>
<tr>
<td>800 lb</td>
<td>178</td>
<td>178</td>
<td>178</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>AXIAL</td>
<td>19,612</td>
<td>15,533</td>
<td>8,629</td>
<td>19,612</td>
<td>1569</td>
</tr>
<tr>
<td>85 lb</td>
<td>15,516</td>
<td>20,265</td>
<td>13,842</td>
<td>12,592</td>
<td>1007</td>
</tr>
<tr>
<td>PITCH</td>
<td>2,759</td>
<td>9,721</td>
<td>14,187</td>
<td>14,187</td>
<td>1135</td>
</tr>
<tr>
<td>2000 in-lb</td>
<td>8,609</td>
<td>13,474</td>
<td>14,497</td>
<td>13,659</td>
<td>1093</td>
</tr>
<tr>
<td>ROLL</td>
<td>70,815</td>
<td>80,367</td>
<td>63,548</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>400 in-lb</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>YAW</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>800 in-lb</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIDE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300 lb</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOTAL</td>
<td>70,815</td>
<td>80,367</td>
<td>63,548</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
References


This document chronicles and captures force balance design principles accumulated over the last 50 years at NASA Langley Research Center (LaRC). Many design and guidance internal memorandums have been drafted over the years, some as informal as a design engineer's notes for a particular balance design. To date, there has been no comprehensive balance design or best practices documentation available on LaRC's extensive design expertise. Therefore, the primary goal of this knowledge capture document is to assemble