Welcome!

Welcome to the Ph.D. Thesis Defense of Michelle E. Rodio

23 March 2020

Agenda

- Presentation of dissertation work
- Questions from general attendees
- Questions from dissertation committee
- Committee deliberation and decision

We will be starting promptly at 1:05 PM.

Investigating the Feasibility and Stability for Modeling Acoustic Wave Scattering Using a Time-Domain Boundary Integral Equation with Impedance Boundary Condition

Michelle E. Rodio

Dissertation Defense
23 March 2020

Dedication

Dr. Shahrdad G. Sajjadi

February 1, 1961 - January 6, 2020

Outline

- **Chapter 1**: Introduction and Related Work
- **Chapter 2**: Stable Time-Domain Boundary Integral Equation
- **Chapter 3**: Basis Functions and Spatial Resolution
- **Chapter 4**: Scalability and Performance Using CPUs
- **Chapter 5**: Time-Domain Liner Impedance Boundary Conditions
- **Chapter 6**: Stability Analysis and Numerical Results
- **Chapter 7**: Concluding Remarks

Outline

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Introduction

- Reducing aircraft noise is a major objective in field of computational aeroacoustics.
- Acoustic scattering problems can be modeled by reformulating convective wave equation as a boundary integral equation (BIE).
- BIEs reduce spatial dimension by one, from three-dimensional partial differential equation to a two-dimensional surface integral equation.
- Numerical solutions can be obtained by discretizing surface and solving using boundary element methods (BEMs).
- BEMs effectively handle singular and infinite fields, ultimately saving computing memory and maintaining high computational efficiency.

Time- vs. Frequency-Domain Solvers

- Methods for solving BIEs have been studied extensively in both frequency- and time-domain.
- Frequency-domain solvers are the most used and researched within literature.
- There are several distinct advantages to using a time-domain solver, including they:
  - Allow for simulation and study of broadband sources and time-domain transient signals [studying broadband sources in frequency-domain carries high computational cost].
  - Allow for scattering solutions at all frequencies to be obtained within a single computation using sparse matrices [frequency-domain requires inverting a large dense linear system].
  - Naturally couples with nonlinear computational fluids dynamics simulations.

Time-Domain Numerical Instability

- Time-domain BIEs (TD-BIEs) have been used since the 1960s to study wave propagation.
- As computers advanced and processing power improved, researchers discovered numerical instabilities when solving TD-BIEs over longer run times.
- Instability tends to occur at large time steps as a result of the existence of internal modes of resonance of the body corresponding to time harmonic solutions of the integral equation.
  - In frequency-domain, resonant modes near frequency of interest yield an ill-conditioned matrix due to the existence of non-unique solutions at resonant frequencies.
  - Resonant modes result in numerical instabilities in the time-domain solution.

Burton-Miller Reformulation

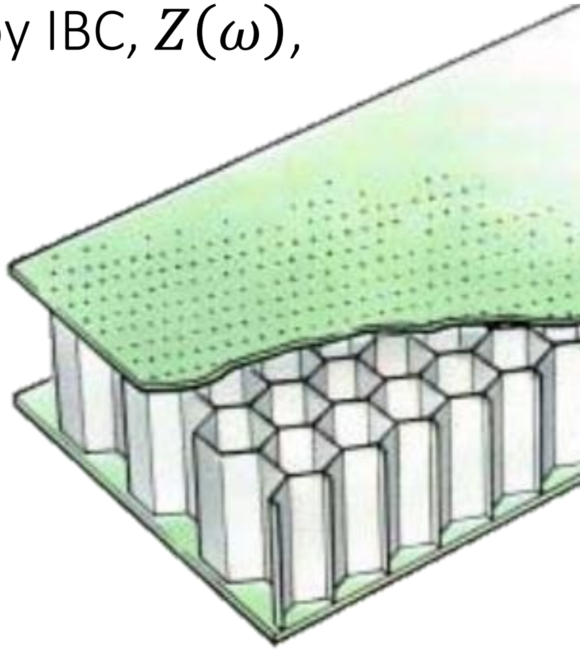
- Burton-Miller method is effective for eliminating resonant frequencies.
- Supplementary integral equation is derived, resulting in unique solution for exterior problem.
- Results in formation of hypersingular integrals, mitigated by regularization process.
- Application of this method increases already high computational cost associated with time-domain solvers, and is reduced using fast algorithms and high performance computing.

Research Objective

Investigate feasibility and stability for modeling acoustic wave scattering using a Burton-Miller-type TD-BIE with IBC.

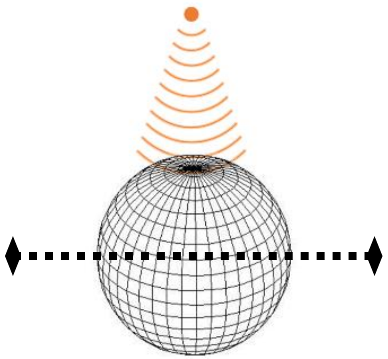
Impedance Boundary Conditions

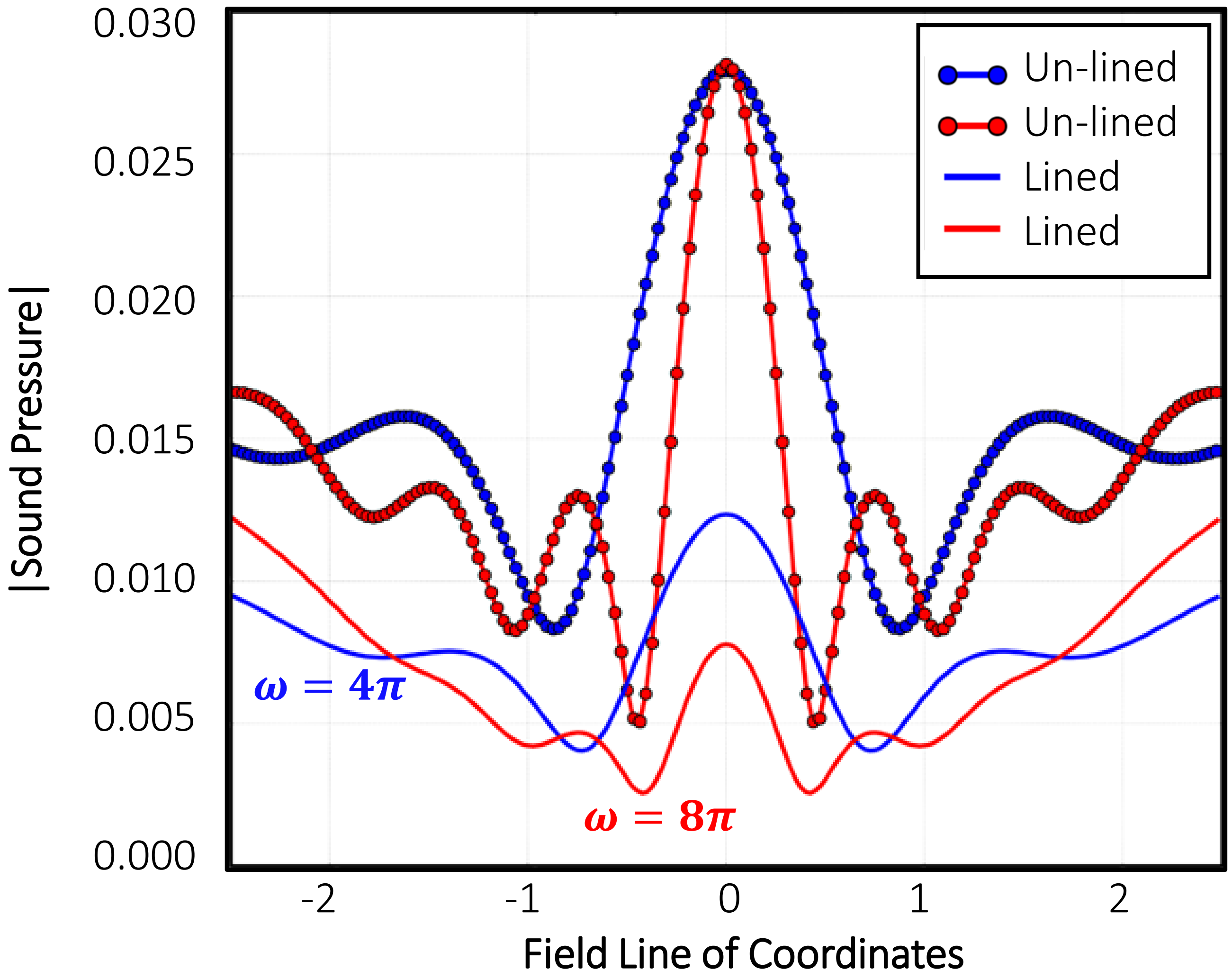
- BCs are defined through terms involving normal derivative of pressure.
  - On rigid bodies, normal derivative of pressure is equal to zero.
  - On soft bodies, normal derivative of pressure is non-zero and defined by IBC, $Z(\omega)$, or admittance BC (ABC), $Y(\omega) = 1/Z(\omega)$.
- Impedance is a complex-valued, measured quantity.
  - $\mathrm{Re}(Z)$ is acoustic resistance.
  - $\mathrm{Im}(Z)$ is acoustic reactance.
- Soft bodies imply that an acoustic liner is installed on the surface.
- Acoustic liners are typically composed of an array of Helmholtz resonators.
- Transformed into the time-domain using Fourier transforms, an IBC or ABC is coupled with TD-BIE to model acoustic wave scattering by soft bodies.



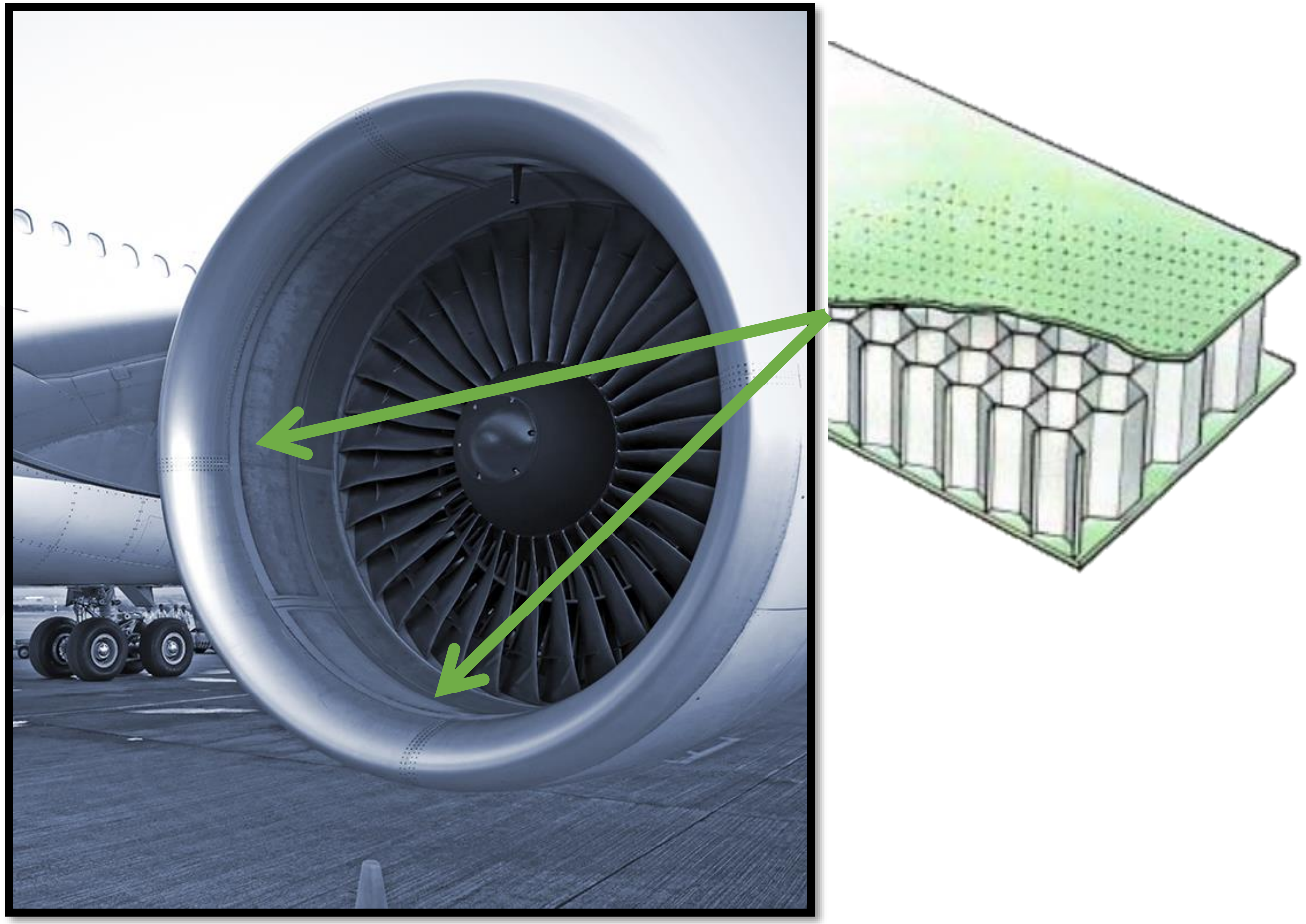
Lined, Un-Lined Comparison

- Consider scattering of an acoustic point source by a sphere with rigid (un-lined) and soft (lined) surfaces.
- Frequency-domain solution along field line of coordinates $-2.5 \leq x \leq 2.5$, $y = 0, z = 0$ converted from time-domain at frequencies $\omega = 4\pi, 8\pi$.





Application of Acoustic Liners on Aircraft



Acoustic Liner Models

- It is necessary to study TD-BIE to ensure stability of the system once coupled with IBC.
- *Extended Helmholtz Resonator Model* and *Three-Parameter Impedance Model* each represent acoustic liner impedance at a single frequency.
- *Broadband Impedance Model* simulates multiple frequencies simultaneously.
- For time-domain IBC to be physical, each model is required to be causal, real, and passive.

Research Proposal

Study numerical stability of coupling a IBC with a Burton-Miller-type TD-BIE using either the *Extended Helmholtz*, *Three-Parameter*, or *Broadband Impedance* model.

Stability Assessment

- In literature, convolution quadrature methods have been numerically proven stable up to second-order but no theoretical proof has yet been provided for other methods.
- Eigenvalue analysis is current standard for studying the stability of TD-BIEs.
- Though eigenvalue analysis alone is not sufficient for proving stability, it is necessary that numerical scheme has maximum eigenvalues no greater than unity.

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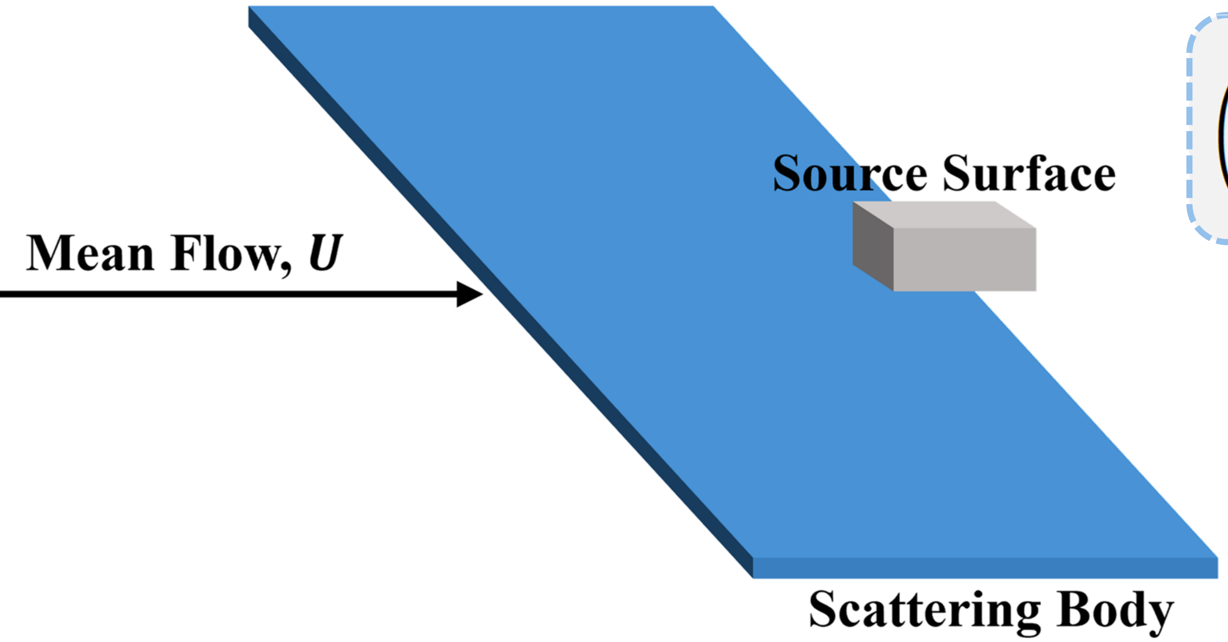
Governing Equation

- Acoustic waves are assumed to be disturbances of small amplitudes.
- With a uniform mean flow, acoustic disturbances are governed by linear convective wave equation, with homogeneous initial conditions:



Linear Convective Wave Equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla\right)^2 p\left(\boldsymbol{r}, t\right) - c^2 \nabla^2 p\left(\boldsymbol{r}, t\right) = q\left(\boldsymbol{r}, t\right)$$

Homogeneous Initial Conditions

$$p(\boldsymbol{r}, 0) = \frac{\partial p}{\partial t}(\boldsymbol{r}, 0) = 0, \;\; t = 0$$

Green's Function

- Introducing free-space adjoint Green's with homogeneous initial conditions:

Free-Space Adjoint Green's Function

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)^2 \tilde{G}(\boldsymbol{r},t;\boldsymbol{r'},t') - c^2\nabla^2\tilde{G}(\boldsymbol{r},t;\boldsymbol{r'},t') = \delta\left(\boldsymbol{r}-\boldsymbol{r'}\right)\delta\left(t-t'\right)$$

$$\tilde{G}(\boldsymbol{r},t;\boldsymbol{r'},t') = \frac{\partial \tilde{G}}{\partial t}(\boldsymbol{r},t;\boldsymbol{r'},t') = 0,\ t > t'$$

Initial Conditions

$$\tilde{G}(\boldsymbol{r},t;\boldsymbol{r'},t') = \frac{G_0}{4\pi c^2}\delta\left(t' - t + \boldsymbol{\beta}\cdot(\boldsymbol{r'}-\boldsymbol{r}) - \frac{\overline{R}}{c\alpha^2}\right)$$

$$\overline{R}(\boldsymbol{r},\boldsymbol{r'}) = \sqrt{\left|\boldsymbol{M}\cdot(\boldsymbol{r}-\boldsymbol{r'})\right|^2 + \alpha^2\left|\boldsymbol{r}-\boldsymbol{r'}\right|^2},\ G_0 = \frac{1}{\overline{R}(\boldsymbol{r},\boldsymbol{r'})}$$

$$\boldsymbol{M} = \frac{\boldsymbol{U}}{c},\ \alpha = \sqrt{1-M^2},\ \boldsymbol{\beta} = \frac{\boldsymbol{U}}{c^2-U^2} = \frac{\boldsymbol{M}}{c\alpha^2}$$

- Wave propagation problem is reformulated into TD-BIE by considering operation:

$$\tilde{G}\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)^2 p - p\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)^2 \tilde{G}$$

Resulting TD-BIE

TD-BIE is representative of acoustic field in presence of a uniform mean flow

$$
\begin{aligned}
4\pi C_s p(\boldsymbol{r}', t') = Q(\boldsymbol{r}, t') &+ \int_S G_0 \left(1 - M_n^2\right) \frac{\partial p}{\partial n} \left(\boldsymbol{r}_s, t'_R\right) d\boldsymbol{r}_s \\
&- \int_S \frac{\partial G_0}{\partial \overline{n}} \left[p\left(\boldsymbol{r}_s, t'_R\right) + \frac{\overline{R}}{c\alpha^2} \frac{\partial p}{\partial t} \left(\boldsymbol{r}_s, t'_R\right) \right] d\boldsymbol{r}_s \\
&- \int_S M_n G_0 \left[\boldsymbol{M_T} \cdot \nabla p\left(\boldsymbol{r}_s, t'_R\right) + \frac{1}{c} \frac{\partial p}{\partial t} \left(\boldsymbol{r}_s, t'_R\right) \right] d\boldsymbol{r}_s
\end{aligned}
$$

$Q(\boldsymbol{r}, t')$

Source Contribution

t'_R

Retarded Time Values

Eliminating Resonant Frequencies

- TD-BIE has an intrinsic numerical instability due to resonant frequencies resulting from non-trivial solutions in interior domain.
- Resonant frequencies eliminated and stability achieved using Burton-Miller reformulation.
- Reformulation is applied by taking derivative in form of: $a\frac{\partial}{\partial t} + bc\frac{\partial}{\partial \overline{n}'}$
- Constants a and b define stability condition: $a/b < 0$

Stable Burton-Miller Reformulation

$$4\pi a C_s \frac{\partial p}{\partial t}(\boldsymbol{r}'_s, t') + 4\pi bc C_s \frac{\partial p}{\partial n'}(\boldsymbol{r}'_s, t') + 4\pi bc \frac{\partial C_s}{\partial n'} p(\boldsymbol{r}'_s, t')$$
$$= a\frac{\partial Q}{\partial t'}(\boldsymbol{r}'_s, t') + bc\frac{\partial Q}{\partial n'}(\boldsymbol{r}'_s, t') + a\left(A_{\mathrm{INT}}\right) + bc\left(B_{\mathrm{INT}}\right) + \frac{b}{\alpha^2}\left(C_{\mathrm{INT}}\right)$$

The reformulation contains both hyper and weak singularities that are reduced prior to discretization.

$$A_{\mathrm{INT}} = \int_S G_0\left(1 - M_n^2\right)\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial n}(\boldsymbol{r}_s, t'_R)\right) d\boldsymbol{r}_s - \int_S \frac{\partial G_0}{\partial n}\left[\frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R) + \frac{\overline{R}}{c\alpha^2}\frac{\partial^2 p}{\partial t^2}(\boldsymbol{r}_s, t'_R)\right] d\boldsymbol{r}_s$$
$$- \int_S M_n G_0\left[\boldsymbol{M_T}\cdot\nabla\frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R) + \frac{1}{c}\frac{\partial^2 p}{\partial t^2}(\boldsymbol{r}_s, t'_R)\right] d\boldsymbol{r}_s,$$

$$B_{\mathrm{INT}} = \int_S \left(1 - M_n^2\right)\left[\frac{\partial G_0}{\partial n'}\frac{\partial p}{\partial n}(\boldsymbol{r}_s, t'_R) + G_0\frac{\partial}{\partial n'}\left(\frac{\partial p}{\partial n}\right)\right](\boldsymbol{r}_s, t'_R)\, d\boldsymbol{r}_s$$
$$- \int_S \frac{\partial^2 G_0}{\partial n \partial n'}\left[p(\boldsymbol{r}_s, t'_R) + \frac{\overline{R}}{c\alpha^2}\frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R)\right] d\boldsymbol{r}_s$$
$$- \int_S M_n \frac{\partial G_0}{\partial n'}\left[\boldsymbol{M_T}\cdot\nabla p(\boldsymbol{r}_s, t'_R) + \frac{1}{c}\frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R)\right] d\boldsymbol{r}_s$$

$$C_{\mathrm{INT}} = -\int_S \frac{\partial G_0}{\partial n}\left[(\boldsymbol{M}\cdot\overline{\boldsymbol{n}}')\frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R) + \frac{\overline{R}}{c\alpha^2}\left(\boldsymbol{M}\cdot\overline{\boldsymbol{n}}' - \frac{\partial\overline{R}}{\partial n'}\right)\frac{\partial^2 p}{\partial t^2}(\boldsymbol{r}_s, t'_R)\right] d\boldsymbol{r}_s$$
$$- \int_S M_n G_0\left(\boldsymbol{M}\cdot\overline{\boldsymbol{n}}' - \frac{\partial\overline{R}}{\partial n'}\right)\left[\boldsymbol{M_T}\cdot\nabla\frac{\partial p}{\partial t}(\boldsymbol{r}_s, t'_R) + \frac{1}{c}\frac{\partial^2 p}{\partial t^2}(\boldsymbol{r}_s, t'_R)\right] d\boldsymbol{r}_s$$

Impedance Boundary Condition

$$4\pi a C_s \frac{\partial p}{\partial t}(\boldsymbol{r}'_s, t') + 4\pi b c C_s \boxed{\frac{\partial p}{\partial \overline{n}'}(\boldsymbol{r}'_s, t')} + 4\pi b c \frac{\partial C_s}{\partial \overline{n}'} p(\boldsymbol{r}'_s, t')$$
$$= a\frac{\partial Q}{\partial t'}(\boldsymbol{r}'_s, t') + bc\frac{\partial Q}{\partial \overline{n}'}(\boldsymbol{r}'_s, t') + a\left(A_{\mathrm{INT}}\right) + bc\left(B_{\mathrm{INT}}\right) + \frac{b}{\alpha^2}\left(C_{\mathrm{INT}}\right)$$

The normal derivative of pressure is automatically zero on rigid surfaces.

- It is assumed that scattering surface S is decomposed into rigid S_0 and soft S_l surfaces.
  - On rigid surfaces, a Zero Energy Flux BC is imposed.
  - On soft surfaces, $\partial p/\partial \overline{n}'$ of pressure is non-zero, and assuming $M = 0$: $\partial p/\partial \overline{n}' = \partial p/\partial n$.

$$\frac{\partial p}{\partial n}(\boldsymbol{r}_s, \omega) = \left\{ \begin{array}{ll} P_n(\boldsymbol{r}_s, \omega), & \boldsymbol{r}_s \in S_l \\ 0, & \boldsymbol{r}_s \in S_0 \end{array} \right\}$$

- Acoustic pressure is related to impedance by: $\rho_0(i\omega)p(\boldsymbol{r}_s, \omega) = \frac{\partial p}{\partial n}(\boldsymbol{r}_s, \omega)Z(\omega)$

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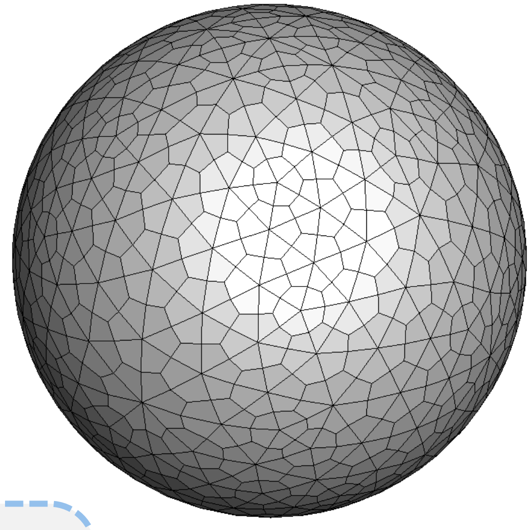
Collocation Method

- Stable Burton-Miller reformulation discretized using collocation methods.
- Surface S is divided into N_e boundary elements $\{E_j, j = 1, \ldots, N_e\}$ where collocation point $\boldsymbol{r}_j$ is located at centroid of each element E_j.
- Time-domain is divided into N_t time steps where $t_k = k\Delta t$.
- Solution is obtained by approximating p and $\partial p/\partial n$ terms using spatial basis functions ϕ_j and temporal basis functions ψ_k.
- $\boldsymbol{u}^k$ and $\boldsymbol{v}^k$ denote vector of all unknowns on S_0 and S_l.



$$p(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} u_j^k \phi_j(\boldsymbol{r}_s)\psi_k(t)$$

$$\frac{\partial p}{\partial n}(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} v_j^k \phi_j(\boldsymbol{r}_s)\psi_k(t)$$

Basis Functions

- The spatial and temporal basis functions are defined as follows:

$$\phi_j(\boldsymbol{r}_s) = \begin{cases} 1, & \boldsymbol{r}_s \text{ on element } E_j \text{ that contains node } \boldsymbol{r}_j \\ 0, & \text{otherwise} \end{cases}$$

Constant Spatial

$$\psi_k(t) = \Psi\left(\frac{t - t_k}{\Delta t}\right), \quad \Psi(\tau) = \begin{cases} 1 + \frac{11}{6}\tau + \tau^2 + \frac{1}{6}\tau^3, & -1 < \tau \le 0 \\ 1 + \frac{1}{2}\tau - \tau^2 - \frac{1}{2}\tau^3, & 0 < \tau \le 1 \\ 1 - \frac{1}{2}\tau - \tau^2 + \frac{1}{2}\tau^3, & 1 < \tau \le 2 \\ 1 - \frac{11}{6}\tau + \tau^2 - \frac{1}{6}\tau^3, & 2 < \tau \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Third-Order Temporal

- Integrations are computed by high-order Gauss quadrature on a 6×6 grid.
- The solution at t_n is interpolated using time steps $t_{n-3}, t_{n-2}, t_{n-1}, t_n$.

Burton-Miller System of Equations

- Evaluating Burton-Miller-type reformulation, it is cast into the following system:

$$p(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} u_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t)$$

$$\frac{\partial p}{\partial n}(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} v_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t)$$

$$\phi_j(\boldsymbol{r}_s) = \begin{cases} 1, & \boldsymbol{r}_s \text{ on element } E_j \text{ that contains node } \boldsymbol{r}_j \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_k(t) = \Psi\left(\frac{t - t_k}{\Delta t}\right), \quad \Psi(\tau) = \begin{cases} 1 + \frac{11}{6}\tau + \tau^2 + \frac{1}{6}\tau^3, & -1 < \tau \le 0 \\ 1 + \frac{1}{2}\tau - \tau^2 - \frac{1}{2}\tau^3, & 0 < \tau \le 1 \\ 1 - \frac{1}{2}\tau - \tau^2 + \frac{1}{2}\tau^3, & 1 < \tau \le 2 \\ 1 - \frac{11}{6}\tau + \tau^2 - \frac{1}{6}\tau^3, & 2 < \tau \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$4\pi a C_s \frac{\partial p}{\partial t}(\boldsymbol{r}'_s, t') + 4\pi bc C_s \frac{\partial p}{\partial n'}(\boldsymbol{r}'_s, t') + 4\pi bc \frac{\partial C_s}{\partial n'} p(\boldsymbol{r}'_s, t') = a \frac{\partial Q}{\partial t'}(\boldsymbol{r}'_s, t') + bc \frac{\partial Q}{\partial n'}(\boldsymbol{r}'_s, t') + a\left(A_{\mathrm{INT}}\right) + bc\left(B_{\mathrm{INT}}\right) + \frac{b}{\alpha^2}\left(C_{\mathrm{INT}}\right)$$

$$\boldsymbol{B}_0 \boldsymbol{u}^n + \boldsymbol{C}_0 \boldsymbol{v}^n = \boldsymbol{q}^n - \boldsymbol{B}_1 \boldsymbol{u}^{n-1} - \boldsymbol{C}_1 \boldsymbol{v}^{n-1} - \boldsymbol{B}_2 \boldsymbol{u}^{n-2} - \boldsymbol{C}_2 \boldsymbol{v}^{n-2} - \ldots - \boldsymbol{B}_J \boldsymbol{u}^{n-J} - \boldsymbol{C}_J \boldsymbol{v}^{n-J}$$

Burton-Miller System of Equations

- Evaluating Burton-Miller-type reformulation, it is cast into the following system:

$$p(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} u_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t)$$

$$\frac{\partial p}{\partial n}(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} v_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t)$$

$$\phi_j(\boldsymbol{r}_s) = \begin{cases} 1, & \boldsymbol{r}_s \text{ on element } E_j \text{ that contains node } \boldsymbol{r}_j \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_k(t) = \Psi\left(\frac{t - t_k}{\Delta t}\right), \quad \Psi(\tau) = \begin{cases} 1 + \frac{11}{6}\tau + \tau^2 + \frac{1}{6}\tau^3, & -1 < \tau \le 0 \\ 1 + \frac{1}{2}\tau - \tau^2 - \frac{1}{2}\tau^3, & 0 < \tau \le 1 \\ 1 - \frac{1}{2}\tau - \tau^2 + \frac{1}{2}\tau^3, & 1 < \tau \le 2 \\ 1 - \frac{11}{6}\tau + \tau^2 - \frac{1}{6}\tau^3, & 2 < \tau \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$4\pi a C_s \frac{\partial p}{\partial t}(\boldsymbol{r}'_s, t') + 4\pi bc C_s \frac{\partial p}{\partial n'}(\boldsymbol{r}'_s, t') + 4\pi bc \frac{\partial C_s}{\partial n'} p(\boldsymbol{r}'_s, t') = a\frac{\partial Q}{\partial t'}(\boldsymbol{r}'_s, t') + bc\frac{\partial Q}{\partial n'}(\boldsymbol{r}'_s, t') + a\left(A_{\text{INT}}\right) + bc\left(B_{\text{INT}}\right) + \frac{b}{\alpha^2}\left(C_{\text{INT}}\right)$$

For rigid body scattering, $\boldsymbol{v}^k = \boldsymbol{0}$.

$$\boldsymbol{B}_0\boldsymbol{u}^n + \boldsymbol{C}_0\boldsymbol{v}^n = \boldsymbol{q}^n - \boldsymbol{B}_1\boldsymbol{u}^{n-1} - \boldsymbol{C}_1\boldsymbol{v}^{n-1} - \boldsymbol{B}_2\boldsymbol{u}^{n-2} - \boldsymbol{C}_2\boldsymbol{v}^{n-2} - \ldots - \boldsymbol{B}_J\boldsymbol{u}^{n-J} - \boldsymbol{C}_J\boldsymbol{v}^{n-J}$$

Burton-Miller System of Equations

- Evaluating Burton-Miller-type reformulation, it is cast into the following system:

A second system is needed to solve for both $\boldsymbol{u}^k$ and $\boldsymbol{v}^k$.

$$p(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} u_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t)$$

$$\frac{\partial p}{\partial n}(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} v_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t)$$

$$\phi_j(\boldsymbol{r}_s) = \begin{cases} 1, & \boldsymbol{r}_s \text{ on element } E_j \text{ that contains node } \boldsymbol{r}_j \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_k(t) = \Psi\left(\frac{t - t_k}{\Delta t}\right), \quad \Psi(\tau) = \begin{cases} 1 + \frac{11}{6}\tau + \tau^2 + \frac{1}{6}\tau^3, & -1 < \tau \le 0 \\ 1 + \frac{1}{2}\tau - \tau^2 - \frac{1}{2}\tau^3, & 0 < \tau \le 1 \\ 1 - \frac{1}{2}\tau - \tau^2 + \frac{1}{2}\tau^3, & 1 < \tau \le 2 \\ 1 - \frac{11}{6}\tau + \tau^2 - \frac{1}{6}\tau^3, & 2 < \tau \le 3 \\ 0, & \text{otherwise} \end{cases}$$

$$4\pi a C_s \frac{\partial p}{\partial t}(\boldsymbol{r}'_s, t') + 4\pi bc C_s \frac{\partial p}{\partial n'}(\boldsymbol{r}'_s, t') + 4\pi bc \frac{\partial C_s}{\partial n'} p(\boldsymbol{r}'_s, t')$$
$$= a\frac{\partial Q}{\partial t'}(\boldsymbol{r}'_s, t') + bc\frac{\partial Q}{\partial n'}(\boldsymbol{r}'_s, t') + a\left(A_{\text{INT}}\right) + bc\left(B_{\text{INT}}\right) + \frac{b}{\alpha^2}\left(C_{\text{INT}}\right)$$

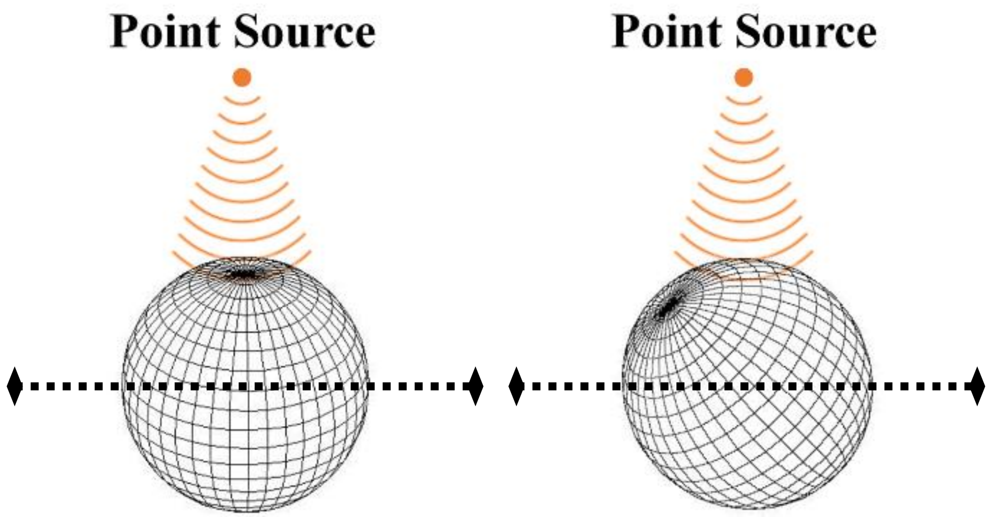
$$\boldsymbol{B}_0\boldsymbol{u}^n + \boldsymbol{C}_0\boldsymbol{v}^n = \boldsymbol{q}^n - \boldsymbol{B}_1\boldsymbol{u}^{n-1} - \boldsymbol{C}_1\boldsymbol{v}^{n-1} - \boldsymbol{B}_2\boldsymbol{u}^{n-2} - \boldsymbol{C}_2\boldsymbol{v}^{n-2} - \ldots - \boldsymbol{B}_J\boldsymbol{u}^{n-J} - \boldsymbol{C}_J\boldsymbol{v}^{n-J}$$

Spatial Resolution of BEM

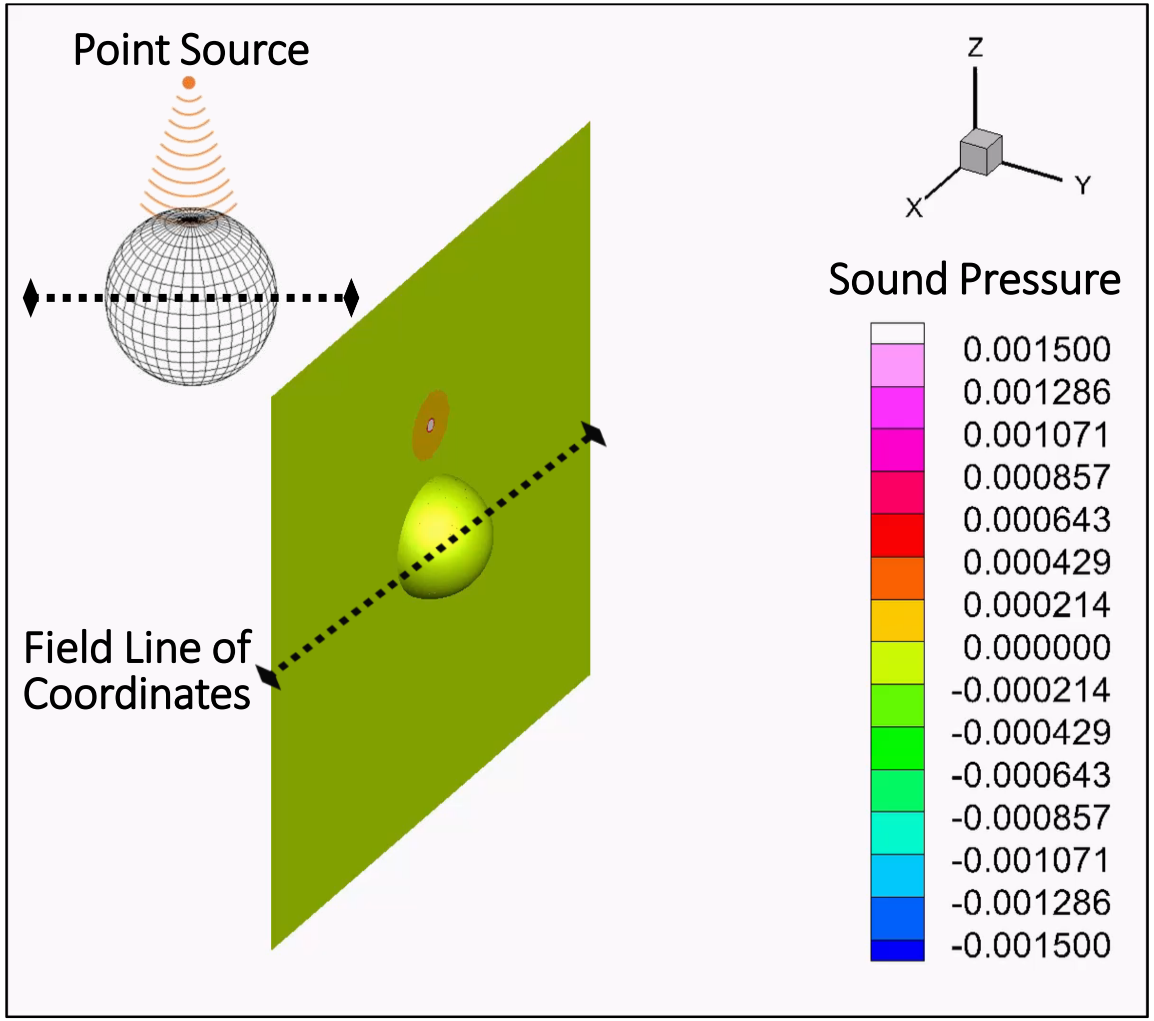
- The spatial resolution is studied by assessing metric points-per-wavelength-squared (PPW2).
- Consider scattering of acoustic point source by a rigid sphere centered at $(0,0,0)$ with radius of 0.5 and point source of 1 located at $(x, y, z) = (0, 0, 1)$.
- Computations carried out by increasing number of elements used from 729 to $72{,}901$.
- Solution along field line of coordinates $-2.5 \leq x \leq 2.5, y = 0, z = 0$ used for analysis.

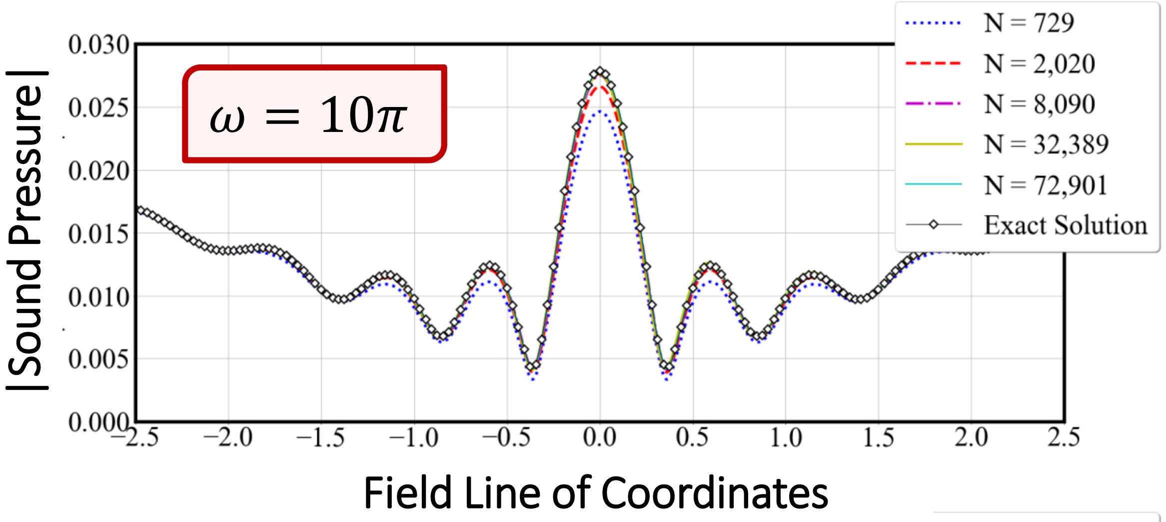
$$PPW2 = \frac{4\pi^2(p+1)^2 N}{k^2 S_A} \approx \frac{4\pi^2 \cdot \text{DOF}}{k^2 S_A}$$

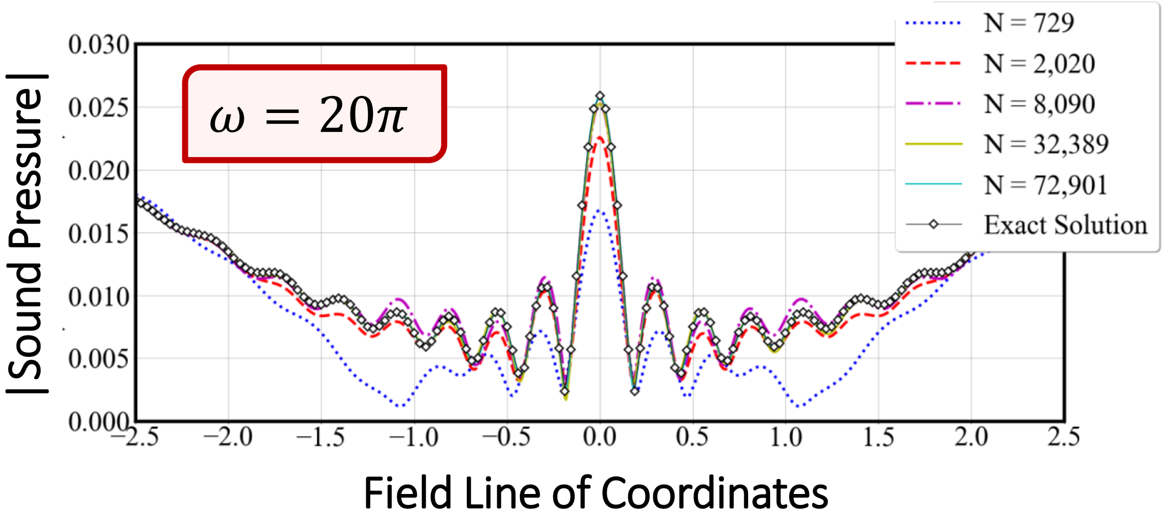
p : order of the basis function
k : wavenumber, $k = w/c$
N : surface elements
S_A : surface area



Sphere Simulation



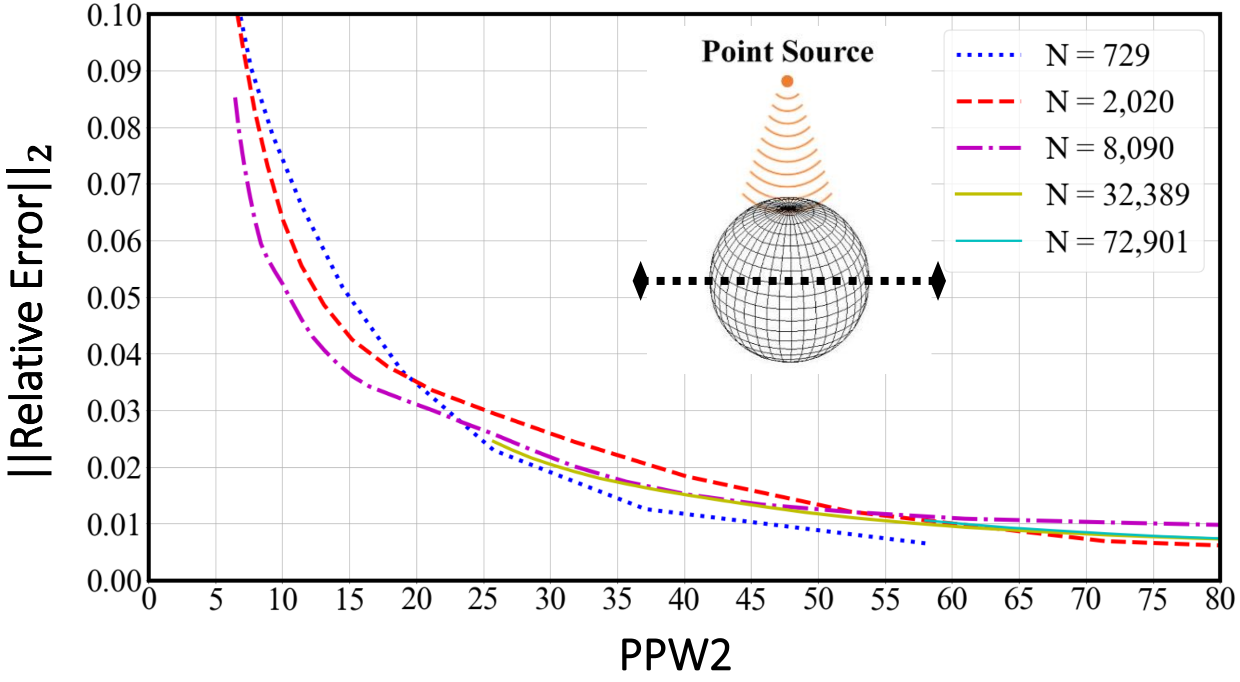


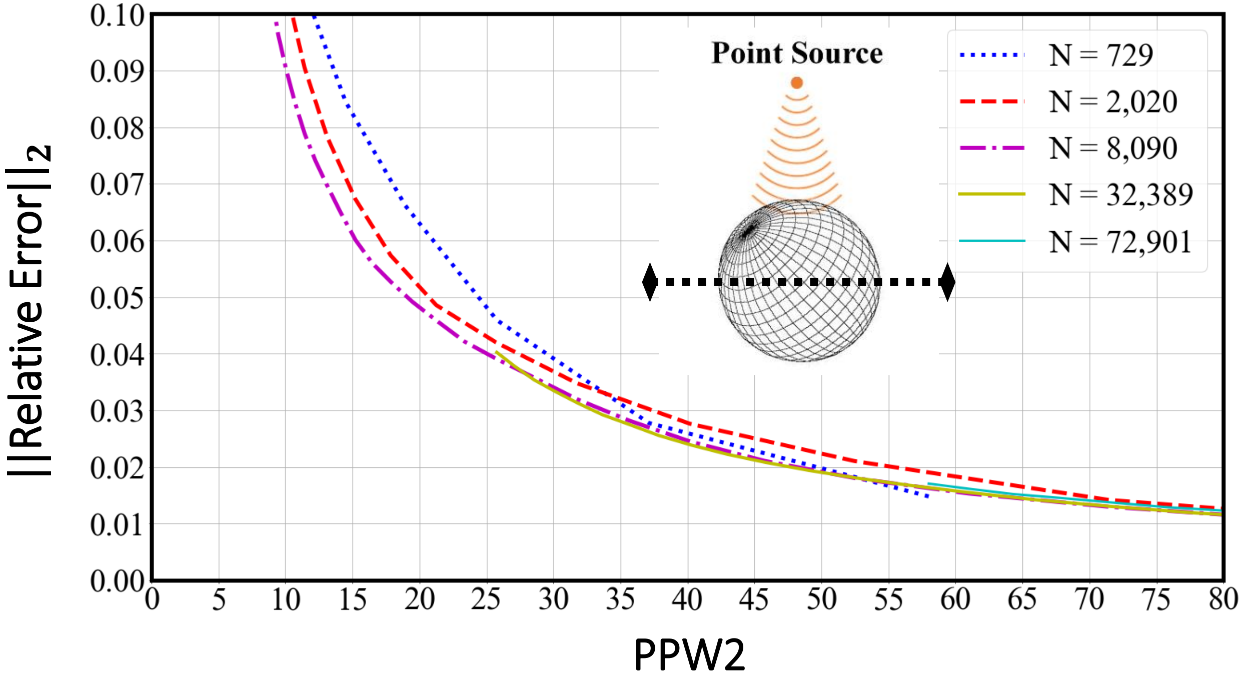


Sphere PPW2

$$PPW2 = \frac{4\pi^2(p+1)^2 N}{k^2 S_A} \approx \frac{4\pi^2 \cdot \mathrm{DOF}}{k^2 S_A}$$

- Using exact solution, the relative error in the L_2 norm is graphed as a function of $PPW2$.
- Excellent spatial resolution demonstrated with relative error in L_2 norm less than 3% and 5%, respectively, with 25 $PPW2$ (original and rotated, respectively) likely due to using high-order Gauss quadrature integration over a closed domain.





Outline

- **Chapter 1**: Introduction and Related Work
- **Chapter 2**: Stable Time-Domain Boundary Integral Equation
- **Chapter 3**: Basis Functions and Spatial Resolution
- **Chapter 4**: Scalability and Performance Using CPUs
- **Chapter 5**: Time-Domain Liner Impedance Boundary Conditions
- **Chapter 6**: Stability Analysis and Numerical Results
- **Chapter 7**: Concluding Remarks

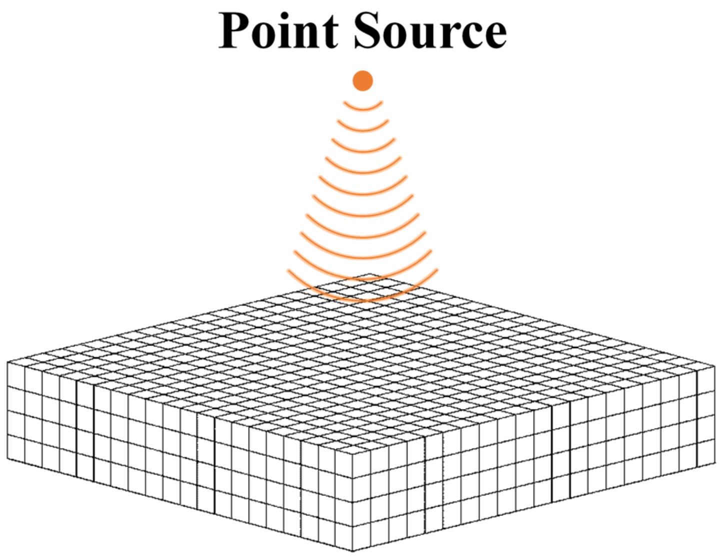
Scalability and Performance Using CPUs

- The Burton-Miller reformulation of the TD-BIE has been implemented in numerical algorithm called TD-FAST: Time-Domain Fast Acoustic Scattering Toolkit.
- TD-FAST has capability of performing large-scale parallel computations using either central processing units (CPUs) or graphics processing units (GPUs).
- TD-FAST has significant speed-up when utilizing GPU architecture, yet maintains ability to exploit parallelism with CPUs for instances when GPU hardware may be unavailable.
- It is important to study performance of TD-FAST when utilizing CPU architecture only.

Problem Size and Core Count

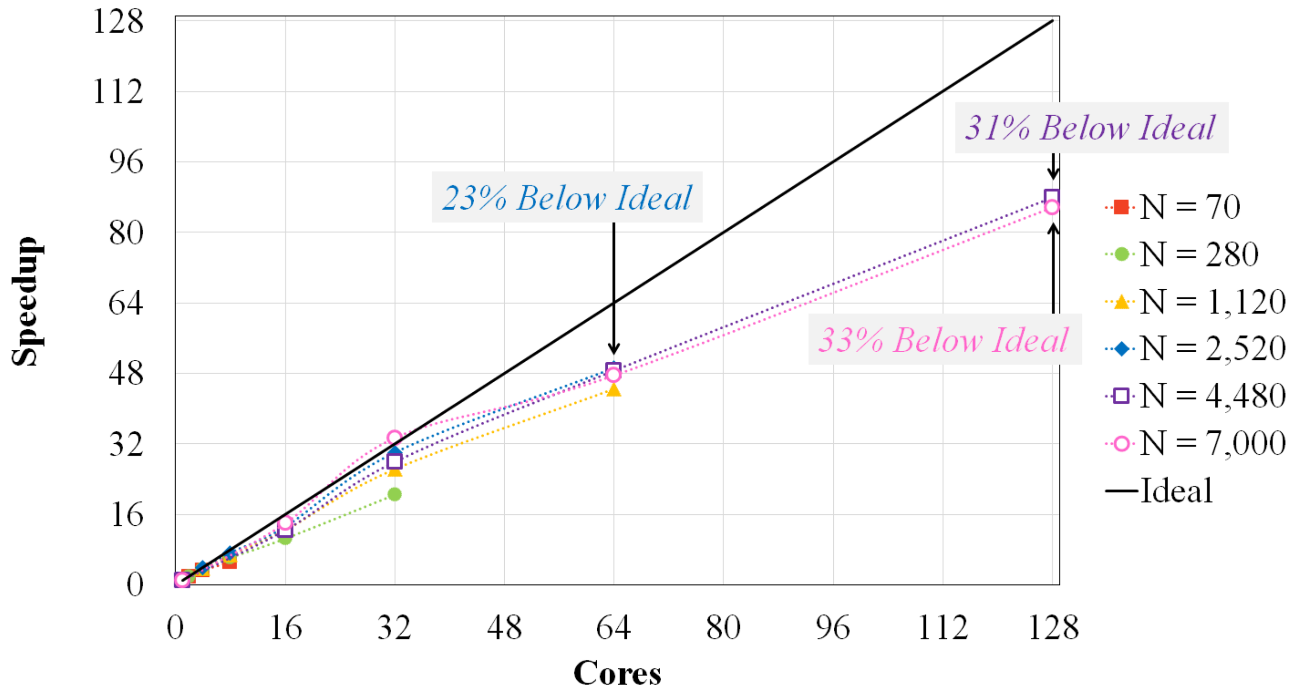
- Consider scattering of acoustic point source by a flat plate.
  - Dimension of flat plate : $[-0.5, 0.5] \times [-0.5, 0.5] \times [-0.1, 0.1]$
  - Point source location : $(x, y, z) = (0, 0, 1)$

- Assessed multiple problem sizes at differing processing powers.
  - Elements Range : 70 elements to $7{,}000$ elements
  - Processing Power : 1 to 4 nodes, exclusive
    Ranging 1 to 128 cores



Strong Scaling

- Indicates how solution time varies with an increasing core count for a fixed problem size.
- IDEAL: as core count increases, speedup is linearly proportional to problem size.
- Speedup measures how must faster an algorithm performs compared to serial processing.



As core count increases for a fixed problem size, there is an increase in time associated with performing parallel communications.

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Acoustic Pressure

- Consider geometric body with surface treated with acoustic liner.
- Assume model with no mean flow, i.e., Mach number $M = 0$.
- Acoustic pressure is related to impedance by: $\rho_0(i\omega)p(\boldsymbol{r_s},\omega) = \frac{\partial p}{\partial n}(\boldsymbol{r_s},\omega)Z(\omega)$
- By Fourier transforms, a relation for normal derivative of acoustic pressure is obtained:

$$\rho_0 \frac{\partial p}{\partial t}(\boldsymbol{r}_s, t) = \frac{1}{2\pi}\int_{-\infty}^{t} z(t-\tau)\frac{\partial p}{\partial n}(\boldsymbol{r}_s,\tau)d\tau$$

IBC Relating Impedance to the Time-Derivative of Pressure

Discretizing IBC

- IBC is discretized using collocation methods, same as with Burton-Miller.
  - Surface S is divided into N_e boundary elements $\{E_j, j = 1, \ldots, N_e\}$.
  - Time-domain is divided into N_t time steps where $t_k = k\Delta t$.
  - Solution is obtained by approximating terms using ϕ_j and ψ_k.
- Solution is cast into the following system of equations:

$$D_0 u^n + E_0 v^n = -D_1 u^{n-1} - E_1 v^{n-1} - D_2 u^{n-2} - E_2 v^{n-2} - \ldots - D_K u^{n-K} - E_K v^{n-K}$$

- And coupled with the Burton-Miller system:

$$B_0 u^n + C_0 v^n = q^n - B_1 u^{n-1} - C_1 v^{n-1} - B_2 u^{n-2} - C_2 v^{n-2} - \ldots - B_J u^{n-J} - C_J v^{n-J}$$

Coupled Matrix System

$$B_0 u^n + C_0 v^n = q^n - B_1 u^{n-1} - C_1 v^{n-1} - B_2 u^{n-2} - C_2 v^{n-2} - \ldots - B_J u^{n-J} - C_J v^{n-J}$$

$$D_0 u^n + E_0 v^n = -D_1 u^{n-1} - E_1 v^{n-1} - D_2 u^{n-2} - E_2 v^{n-2} - \ldots - D_K u^{n-K} - E_K v^{n-K}$$

$$\begin{bmatrix} B_0 & C_0 \\ D_0 & E_0 \end{bmatrix} \begin{bmatrix} u^n \\ v^n \end{bmatrix} = \begin{bmatrix} q^n \\ 0 \end{bmatrix} - \begin{bmatrix} B_1 & C_1 \\ D_1 & E_1 \end{bmatrix} \begin{bmatrix} u^{n-1} \\ v^{n-1} \end{bmatrix} - \cdots - \begin{bmatrix} B_K & C_K \\ D_K & E_K \end{bmatrix} \begin{bmatrix} u^{n-K} \\ v^{n-K} \end{bmatrix} - \cdots - \begin{bmatrix} B_J & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u^{n-J} \\ v^{n-J} \end{bmatrix}$$

- Due to limited temporal stencil width, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ are sparse.
- Matrices are diagonally dominant, each with size $N \times N$ for scattering body with N elements.
- $\boldsymbol{B}$, $\boldsymbol{C}$ are defined by Burton-Miller and $\boldsymbol{D}$, $\boldsymbol{E}$ are defined by the IBC.

Acoustic Liner Models

- Various liner models are investigated for simulating sound absorption of lined bodies.
  - Extended Helmholtz Resonator Model
  - Three-Parameter Impedance Model
  - Broadband Impedance Model

$$\rho_0 \frac{\partial p}{\partial t}(\boldsymbol{r}_s, t) = \frac{1}{2\pi} \int_{-\infty}^{t} z(t-\tau) \frac{\partial p}{\partial n}(\boldsymbol{r}_s, \tau) d\tau$$

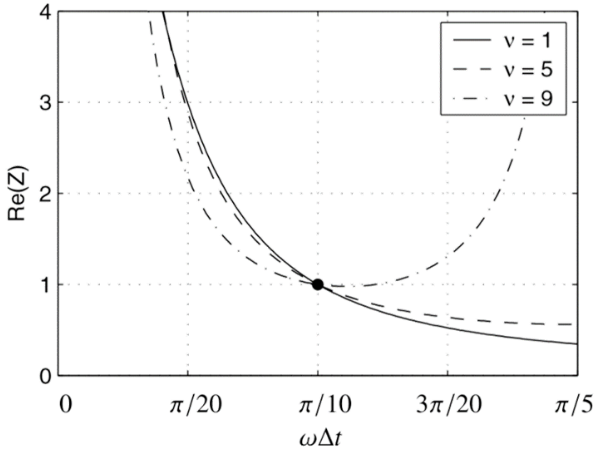
Measured Impedance Quantity Established by Each Liner Model

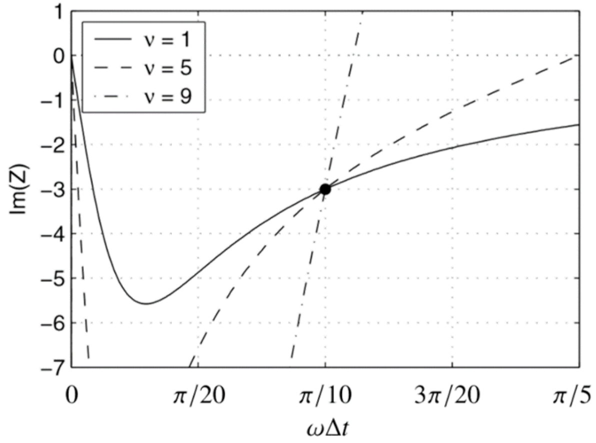
- *Extended Helmholtz* and *Three-Parameter* specify liner impedance at a single frequency ω.
- *Broadband Model* allows for the investigation of multiple frequencies simultaneously.

Extended Helmholtz Resonator Model

- Surface impedance is defined as:

$$Z(\omega) = F_R + i\omega m - iF_\beta \cot\left(\frac{1}{2}\omega\nu\Delta t - i\frac{1}{2}\epsilon\right)$$





Damping in cavity's fluid, ϵ

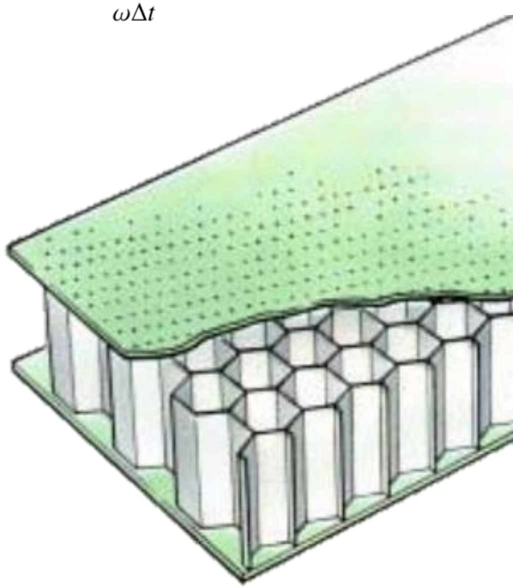
$\nu\Delta t = 2L/c$ for depth L, speed of sound c

Cavity Reactance, $-\cot\left(\frac{1}{2}\omega\nu\Delta t - i\frac{1}{2}\epsilon\right)$

Parameter used for varying cavity reactance, F_β

Face-Sheet Mass Reactance, ωm

Face-Sheet Resistance, F_R



Extended Helmholtz Resonator IBC

- For $\mathrm{Im}(\omega) < \epsilon/(\nu\Delta t)$:

$$Z(\omega) = F_R + i\omega m + F_\beta + 2F_\beta \sum_{N=1}^{\infty} e^{-i\omega N\nu\Delta t - \epsilon N}$$

Inverse Fourier Transform

$$z(t) = 2\pi \left[F_R\delta(t) + m\delta'(t) + F_\beta\delta(t) + 2F_\beta \sum_{N=1}^{\infty} e^{-\epsilon N}\delta(t - N\nu\Delta t) \right]$$

- Substituting $z(t)$ into normal derivative of acoustic pressure, discretizing, and letting the solutions for p and $\partial p/\partial n$ be expanded in the same manner as with Burton-Miller BEM:

$$\rho_0 \sum_{k=0}^{N_t}\sum_{j=1}^{N_e} u_j^k\, \delta_{ij}\psi_k'(t_n) = \sum_{k=0}^{N_t}\sum_{j=1}^{N_e} v_j^k\, \delta_{ij} \left[(F_R + F_\beta)\,\psi_k(t_n) + m\psi_k'(t_n) + 2F_\beta \sum_{N=1}^{\infty} e^{-\epsilon N}\psi_k(t_n - N\nu\Delta t) \right]$$

Time-Domain IBC Discretization

$$\boldsymbol{D}_0\boldsymbol{u}^n + \boldsymbol{E}_0\boldsymbol{v}^n = -\boldsymbol{D}_1\boldsymbol{u}^{n-1} - \boldsymbol{E}_1\boldsymbol{v}^{n-1} - \boldsymbol{D}_2\boldsymbol{u}^{n-2} - \boldsymbol{E}_2\boldsymbol{v}^{n-2} - \ldots - \boldsymbol{D}_K\boldsymbol{u}^{n-K} - \boldsymbol{E}_K\boldsymbol{v}^{n-K}$$

Three-Parameter Impedance Model and IBC

- Surface impedance is defined as:

$$Z(\omega) = R_0 + h_0(-i\omega) + \frac{A_1}{-i\omega}$$

Inverse Fourier Transform

$$z'(t) = 2\pi\left(R_0\,\delta'(t) - h\,\delta''(t) - A_1\delta(t)\right)$$

$$\rho_0 \sum_{k=0}^{N_t}\sum_{j=1}^{N_e} u_j^k\,\delta_{ij}\psi_k''(t_n) = \sum_{k=0}^{N_t}\sum_{j=1}^{N_e} v_j^k\,\delta_{ij}\left[R_0\,\psi_k'(t_n) + h_0\,\psi_k''(t_n) - A_1\,\psi_k(t_n)\right]$$

Time-Domain IBC Discretization

$$D_0u^n + E_0v^n = -D_1u^{n-1} - E_1v^{n-1} - D_2u^{n-2} - E_2v^{n-2} - \ldots - D_Ku^{n-K} - E_Kv^{n-K}$$

Broadband Impedance Model

- Surface impedance is defined as:

$$Z(\omega) = (-i\omega)h_0 + R_0 + \sum_{\ell=1}^{J_1} \frac{A_\ell}{\gamma_\ell - i\omega} + \frac{1}{2}\sum_{\ell=1}^{J_2}\left[\frac{B_\ell + iC_\ell}{\alpha_\ell + i\beta_\ell - i\omega} + \frac{B_\ell - iC_\ell}{\alpha_\ell - i\beta_\ell - i\omega}\right]$$

- To facilitate easy conversion to time-domain, the following terms are defined:

$$p_\ell^{(0)}(\boldsymbol{r}_s,\omega) = \frac{1}{\gamma_\ell - i\omega} P_n(\boldsymbol{r}_s,\omega) \text{ for all } \ell = 1,\dots,J_1,$$

$$p_\ell^{(1)}(\boldsymbol{r}_s,\omega) = \frac{\alpha_\ell - i\omega}{(\alpha_\ell - i\omega)^2 + \beta_\ell^2} P_n(\boldsymbol{r}_s,\omega) \text{ for all } \ell = 1,\dots,J_2, \text{ and}$$

$$p_\ell^{(2)}(\boldsymbol{r}_s,\omega) = \frac{\beta_\ell}{(\alpha_\ell - i\omega)^2 + \beta_\ell^2} P_n(\boldsymbol{r}_s,\omega) \text{ for all } \ell = 1,\dots,J_2.$$

Broadband IBC

- Simplifying and using Fourier transforms, time-domain IBC and subsequent partial differential equations are given to be:

$$0 = \rho_0 \frac{\partial p}{\partial t}(\boldsymbol{r_s}, t) + h_0 \frac{\partial P_n}{\partial t}(\boldsymbol{r_s}, t) + R_0 P_n(\boldsymbol{r_s}, t) + \sum_{\ell=1}^{J_1} A_\ell p_\ell^{(0)}(\boldsymbol{r}_s, t) + \sum_{\ell=1}^{J_2} B_\ell p_\ell^{(1)}(\boldsymbol{r}_s, t) + \sum_{\ell=1}^{J_2} C_\ell p_\ell^{(2)}(\boldsymbol{r}_s, t)$$

$$0 = P_n(\boldsymbol{r_s}, t) - \frac{\partial p_\ell^{(0)}}{\partial t}(\boldsymbol{r_s}, t) - \gamma_\ell p_\ell^{(0)}(\boldsymbol{r_s}, t)$$

$$0 = \beta_\ell p_\ell^{(1)}(\boldsymbol{r}_s, t) - \frac{\partial p_\ell^{(2)}}{\partial t}(\boldsymbol{r}_s, t) - \alpha_\ell p_\ell^{(2)}(\boldsymbol{r}_s, t)$$

$$p_\ell^{(m)}(\boldsymbol{r}_s, t) = \sum_{k=0}^{N_t} \sum_{j=1}^{N_e} \left(p_\ell^{(m)}\right)_j^k \phi_j(\boldsymbol{r}_s) \psi_k(t), m = 0, 1, 2$$

IBC Discretization

- Unlike the *Helmholtz* and *Three-Parameter* models where discretized IBC is cast into a single equation and coupled with Burton-Miller reformulation, **the *Broadband* model has three additional equations that must be included in coupled system**.
- The following vectors are additionally defined:

$$\boldsymbol{P}_{(0)} = [\boldsymbol{p}_1^{(0)} \boldsymbol{p}_2^{(0)} \cdots \boldsymbol{p}_{J_1}^{(0)}]^T, \boldsymbol{P}_{(1)} = [\boldsymbol{p}_1^{(1)} \boldsymbol{p}_2^{(1)} \cdots \boldsymbol{p}_{J_1}^{(1)}]^T, \text{ and } \boldsymbol{P}_{(2)} = [\boldsymbol{p}_1^{(2)} \boldsymbol{p}_2^{(2)} \cdots \boldsymbol{p}_{J_1}^{(2)}]^T$$

- $\boldsymbol{p}_j^{(0)}, \boldsymbol{p}_j^{(1,2)}$ denote vectors that contain auxiliary variables from all points where IBC is applied.
- Coupled system provides solutions for $\boldsymbol{u}^k, \boldsymbol{v}^k, \boldsymbol{P}_{(0,1,2)}^k$.

Broadband System of Equations

$$\begin{bmatrix} B_0 & C_0 & 0 & 0 & 0 \\ D_0 & E_0 & F_0 & G_0 & H_0 \\ 0 & \mathcal{J}_0 & \mathcal{K}_0 & 0 & 0 \\ 0 & \mathcal{L}_0 & 0 & \mathcal{M}_0 & \mathcal{N}_0 \\ 0 & 0 & 0 & \mathcal{P}_0 & \mathcal{Q}_0 \end{bmatrix} \begin{bmatrix} u^n \\ v^n \\ P^n_{(0)} \\ P^n_{(1)} \\ P^n_{(2)} \end{bmatrix} = \begin{bmatrix} q^n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 \\ D_1 & E_1 & F_1 & G_1 & H_1 \\ 0 & \mathcal{J}_1 & \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{L}_1 & 0 & \mathcal{M}_1 & \mathcal{N}_1 \\ 0 & 0 & 0 & \mathcal{P}_1 & \mathcal{Q}_1 \end{bmatrix} \begin{bmatrix} u^{n-1} \\ v^{n-1} \\ P^{n-1}_{(0)} \\ P^{n-1}_{(1)} \\ P^{n-1}_{(2)} \end{bmatrix} - \begin{bmatrix} B_2 & C_2 & 0 & 0 & 0 \\ D_2 & E_2 & F_2 & G_2 & H_2 \\ 0 & \mathcal{J}_2 & \mathcal{K}_2 & 0 & 0 \\ 0 & \mathcal{L}_2 & 0 & \mathcal{M}_2 & \mathcal{N}_2 \\ 0 & 0 & 0 & \mathcal{P}_2 & \mathcal{Q}_2 \end{bmatrix} \begin{bmatrix} u^{n-2} \\ v^{n-2} \\ P^{n-2}_{(0)} \\ P^{n-2}_{(1)} \\ P^{n-2}_{(2)} \end{bmatrix}$$

$$- \cdots - \begin{bmatrix} B_K & C_K & 0 & 0 & 0 \\ D_K & E_K & F_K & G_K & H_K \\ 0 & \mathcal{J}_K & \mathcal{K}_K & 0 & 0 \\ 0 & \mathcal{L}_K & 0 & \mathcal{M}_K & \mathcal{N}_K \\ 0 & 0 & 0 & \mathcal{P}_K & \mathcal{Q}_K \end{bmatrix} \begin{bmatrix} u^{n-K} \\ v^{n-K} \\ P^{n-K}_{(0)} \\ P^{n-K}_{(1)} \\ P^{n-K}_{(2)} \end{bmatrix} - \cdots - \begin{bmatrix} B_J & C_J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^{n-J} \\ v^{n-J} \\ P^{n-J}_{(0)} \\ P^{n-J}_{(1)} \\ P^{n-J}_{(2)} \end{bmatrix}$$

System has dimension $N(2 + J_1 + 2J_2) \times N(2 + J_1 + 2J_2)$, each submatrix is banded diagonal, and when solved iteratively, provides solutions for $\boldsymbol{u}^k, \boldsymbol{v}^k, \boldsymbol{P}^k_{(0,1,2)}$.

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Eigenvalue Analysis

- **Recall**: direct numerical solution of TD-BIE without Burton-Miller is prone to instabilities.
- To study stability of coupled Burton-Miller TD-BIE / IBC system, a numerical eigenvalue study is conducted.
- Denote the coupled systems:

$$\begin{bmatrix} B_0 & C_0 \\ D_0 & E_0 \end{bmatrix}\begin{bmatrix} u^n \\ v^n \end{bmatrix} = \begin{bmatrix} q^n \\ 0 \end{bmatrix} - \begin{bmatrix} B_1 & C_1 \\ D_1 & E_1 \end{bmatrix}\begin{bmatrix} u^{n-1} \\ v^{n-1} \end{bmatrix} - \cdots - \begin{bmatrix} B_K & C_K \\ D_K & E_K \end{bmatrix}\begin{bmatrix} u^{n-K} \\ v^{n-K} \end{bmatrix} - \cdots - \begin{bmatrix} B_J & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u^{n-J} \\ v^{n-J} \end{bmatrix}$$

$$\begin{bmatrix} B_0 & C_0 & 0 & 0 & 0 \\ D_0 & E_0 & F_0 & G_0 & H_0 \\ 0 & \mathcal{J}_0 & \mathcal{K}_0 & 0 & 0 \\ 0 & \mathcal{L}_0 & 0 & \mathcal{M}_0 & \mathcal{N}_0 \\ 0 & 0 & 0 & \mathcal{P}_0 & \mathcal{Q}_0 \end{bmatrix}\begin{bmatrix} u^n \\ v^n \\ P^n_{(0)} \\ P^n_{(1)} \\ P^n_{(2)} \end{bmatrix} = \begin{bmatrix} q^n \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} B_1 & C_1 & 0 & 0 & 0 \\ D_1 & E_1 & F_1 & G_1 & H_1 \\ 0 & \mathcal{J}_1 & \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{L}_1 & 0 & \mathcal{M}_1 & \mathcal{N}_1 \\ 0 & 0 & 0 & \mathcal{P}_1 & \mathcal{Q}_1 \end{bmatrix}\begin{bmatrix} u^{n-1} \\ v^{n-1} \\ P^{n-1}_{(0)} \\ P^{n-1}_{(1)} \\ P^{n-1}_{(2)} \end{bmatrix} - \begin{bmatrix} B_2 & C_2 & 0 & 0 & 0 \\ D_2 & E_2 & F_2 & G_2 & H_2 \\ 0 & \mathcal{J}_2 & \mathcal{K}_2 & 0 & 0 \\ 0 & \mathcal{L}_2 & 0 & \mathcal{M}_2 & \mathcal{N}_2 \\ 0 & 0 & 0 & \mathcal{P}_2 & \mathcal{Q}_2 \end{bmatrix}\begin{bmatrix} u^{n-2} \\ v^{n-2} \\ P^{n-2}_{(0)} \\ P^{n-2}_{(1)} \\ P^{n-2}_{(2)} \end{bmatrix}$$

$$- \cdots - \begin{bmatrix} B_K & C_K & 0 & 0 & 0 \\ D_K & E_K & F_K & G_K & H_K \\ 0 & \mathcal{J}_K & \mathcal{K}_K & 0 & 0 \\ 0 & \mathcal{L}_K & 0 & \mathcal{M}_K & \mathcal{N}_K \\ 0 & 0 & 0 & \mathcal{P}_K & \mathcal{Q}_K \end{bmatrix}\begin{bmatrix} u^{n-K} \\ v^{n-K} \\ P^{n-K}_{(0)} \\ P^{n-K}_{(1)} \\ P^{n-K}_{(2)} \end{bmatrix} - \cdots - \begin{bmatrix} B_J & C_J & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u^{n-J} \\ v^{n-J} \\ P^{n-J}_{(0)} \\ P^{n-J}_{(1)} \\ P^{n-J}_{(2)} \end{bmatrix}$$

- as:

$$A_0 w^n = q_0^n - A_1 w^{n-1} - A_2 w^{n-2} - \ldots - A_J w^{n-J}$$

Generalized Eigenvalue Problem

- Look for solutions of the form $\boldsymbol{w}^{\boldsymbol{n}} = \lambda^n \boldsymbol{e_0}$ to the corresponding homogenous system.
- Obtain a polynomial eigenvalue problem:

$$\left[A_0\lambda^J + A_1\lambda^{J-1} + A_2\lambda^{J-2} + \cdots + A_{J-1}\lambda + A_J\right] e_0 = 0$$

- Cast into a generalized eigenvalue problem, such that $\boldsymbol{e_j} = \lambda^j \boldsymbol{e_0}$:

$$\begin{bmatrix} -A_1 & -A_2 & \cdots & \cdots & -A_{J-1} & -A_J \\ I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} e_{J-1} \\ e_{J-2} \\ \cdot \\ \cdot \\ e_1 \\ e_0 \end{bmatrix} = \lambda \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \begin{bmatrix} e_{J-1} \\ e_{J-2} \\ \cdot \\ \cdot \\ e_1 \\ e_0 \end{bmatrix}$$

Stability Condition

- Look for solutions of the form $\boldsymbol{w^n} = \lambda^n \boldsymbol{e_0}$ to the corresponding homogenous system.
- Obtain a polynomial eigenvalue problem:

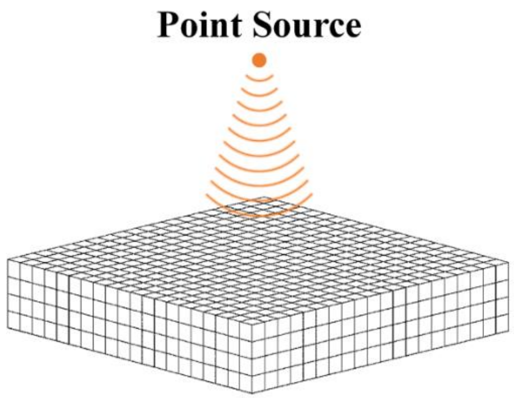
$$\left[A_0\lambda^J + A_1\lambda^{J-1} + A_2\lambda^{J-2} + \cdots + A_{J-1}\lambda + A_J \right] e_0 = 0$$

- Cast into a generalized eigenvalue problem, such that $\boldsymbol{e_j} = \lambda^j \boldsymbol{e_0}$:

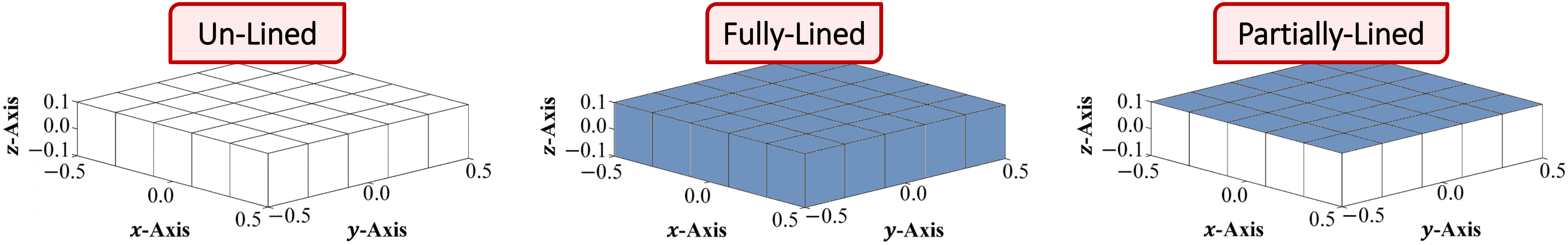
$$\begin{bmatrix} -A_1 & & & & & \\ I & & & & & \\ 0 & I & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \cdot \\ \cdot \\ e_1 \\ e_0 \end{bmatrix} = \lambda \begin{bmatrix} & & & & & \\ & & & & & \\ 0 & 0 & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \begin{bmatrix} e_{J-1} \\ e_{J-2} \\ \cdot \\ \cdot \\ e_1 \\ e_0 \end{bmatrix}$$

Stable if $|\lambda| \leq 1$ for all eigenvalues.

Stability Problem Statement



- Consider scattering of acoustic point source by a flat plate.
  - Dimension of flat plate : $[-0.5, 0.5] \times [-0.5, 0.5] \times [-0.1, 0.1]$
  - Point source location : $(x, y, z) = (0, 0, 1)$



- The values of maximum eigenvalue solved iteratively by a code written in MATLAB.
  - Liner Models : *Helmholtz*, *Three-Parameter*, *Broadband*
  - Surface Discretizations : $5 \times 5 \times 1$, $10 \times 10 \times 2$, $20 \times 20 \times 4$, $30 \times 30 \times 6$
  - Time steps : $\Delta t = 1/12, 1/24$

Fully-Lined Eigenvalue Results

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Matrix Exceeds MATLAB Memory Bounds

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5\times5\times1$	$10\times10\times2$	$20\times20\times4$	$30\times30\times6$	$5\times5\times1$	$10\times10\times2$	$20\times20\times4$	$30\times30\times6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5\times5\times1$	$10\times10\times2$	$20\times20\times4$	$30\times30\times6$	$5\times5\times1$	$10\times10\times2$	$20\times20\times4$	$30\times30\times6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5\times5\times1$	$10\times10\times2$	$20\times20\times4$	$30\times30\times6$	$5\times5\times1$	$10\times10\times2$	$20\times20\times4$	$30\times30\times6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Did Not Converge

λ did not converge to a tolerance of $\delta = 10^{-9}$ within 5,000 iterations.

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Largest Eigenvalue Exceeds Unity

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Extended Helmholtz Resonator Model

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Three-Parameter Impedance Model

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Broadband Impedance Model

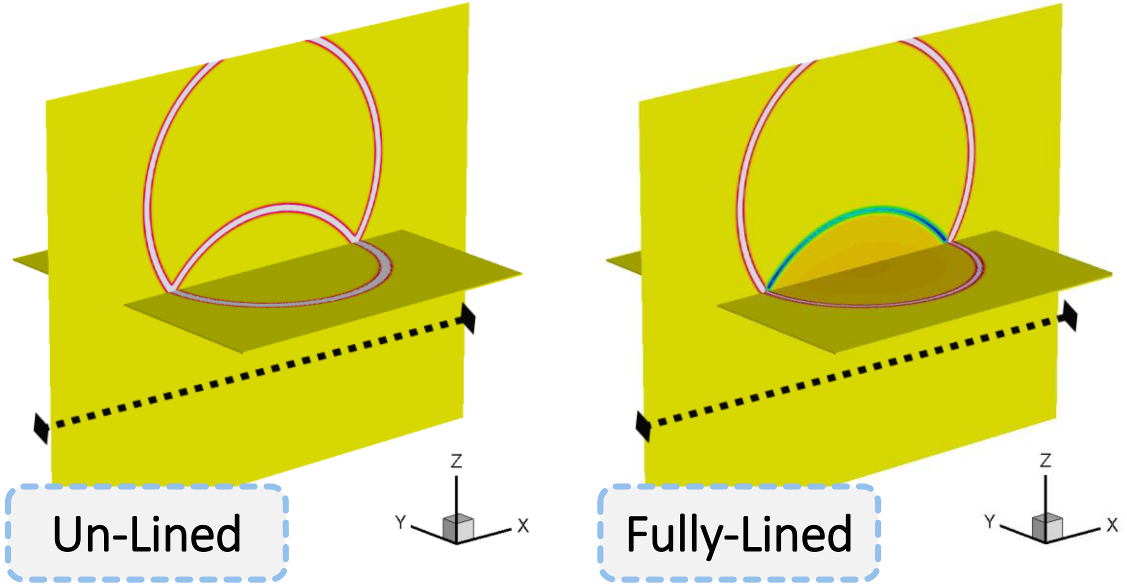
Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	5 × 5 × 1	10 × 10 × 2	20 × 20 × 4	30 × 30 × 6	5 × 5 × 1	10 × 10 × 2	20 × 20 × 4	30 × 30 × 6
1	1.000000	1.000000	1.000000	1.000000	dnc †	1.000000	1.000000	1.000000
2	1.017881	dnc †	dnc †	dnc	0.999772	1.000000	0.999999	0.999999
3	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
4	1.000000	1.000000	1.000000	1.000000	dnc †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	dnc †	1.000000	1.000000	1.000000
6	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
7	1.000000	1.000000	1.000000	1.000000	dnc †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	dnc †	1.000000	1.000000	0.999999
9	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
10	1.002829	dnc †	dnc †	dnc	dnc †	1.0006875	dnc †	dnc
11	1.000000	1.000000	1.000000	1.000000	dnc †	1.000000	1.000001	0.999999
12	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
13	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
14	dnc †	dnc †	dnc †	dnc	dnc †	0.999999	dnc †	dnc
15	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
16	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
17	dnc †	dnc †	dnc †	dnc	dnc †	dnc †	dnc †	dnc
18	dnc †	dnc †	dnc †	dnc	dnc †	0.997547	dnc †	dnc

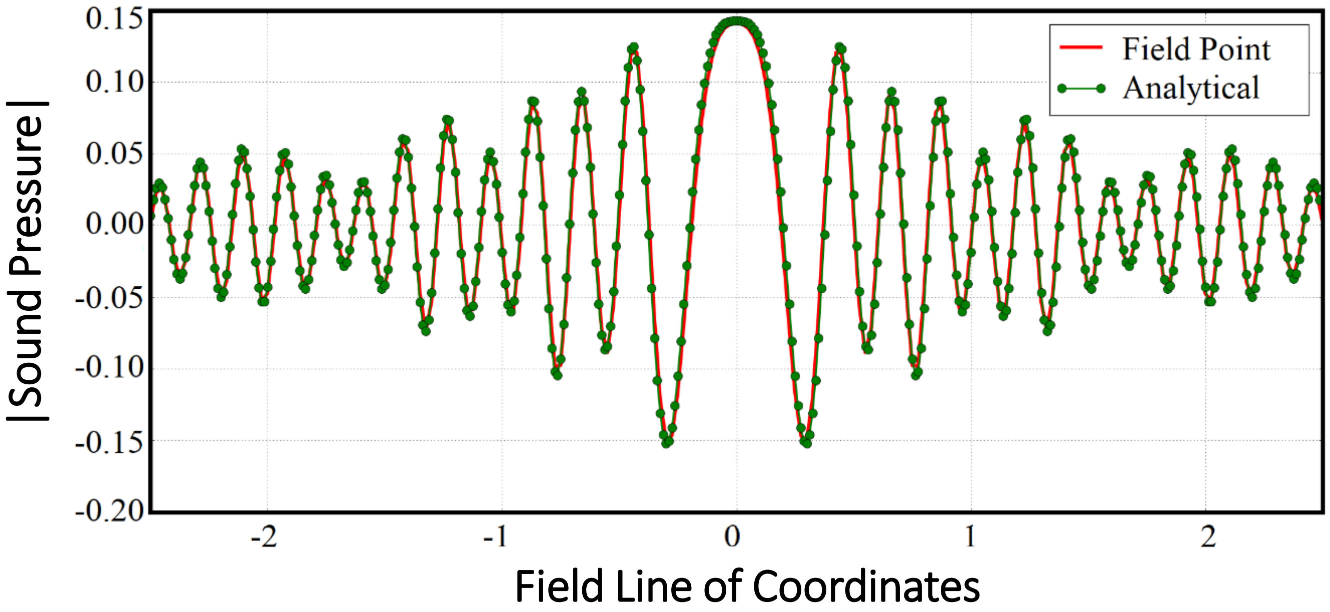
Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	5 × 5 × 1	10 × 10 × 2	20 × 20 × 4	30 × 30 × 6	5 × 5 × 1	10 × 10 × 2	20 × 20 × 4	30 × 30 × 6
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	5 × 5 × 1	10 × 10 × 2	20 × 20 × 4	30 × 30 × 6	5 × 5 × 1	10 × 10 × 2	20 × 20 × 4	30 × 30 × 6
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	dnc †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	emb	1.000000	1.000000	1.000000	emb
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	emb	dnc †	1.001096	1.000000	emb
16	1.000000	1.000000	1.000000	emb	0.999999	1.000000	1.000000	emb
17	1.000000	1.000000	1.000000	emb	1.000000	1.000000	1.000000	emb
18	1.000000	1.000000	1.000000	emb	1.000000	1.000000	1.000000	emb
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

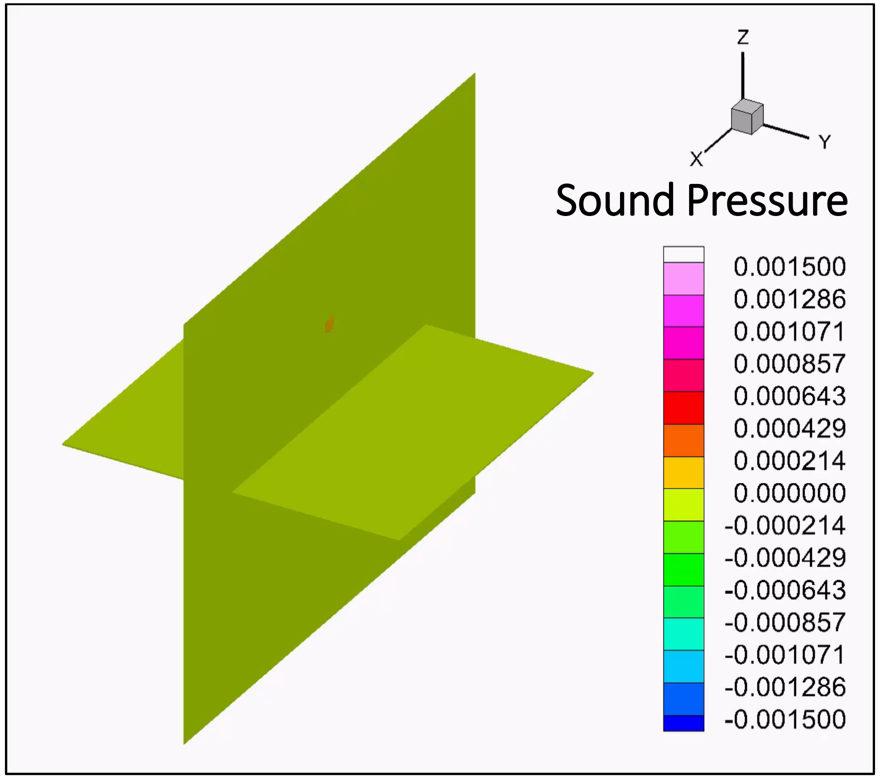
Numerical Example of Lined Body

- Consider scattering of acoustic point source by a flat plate.
  - Dimension of flat plate : $[-2,2] \times [-2,2] \times [-0.02,0]$
  - Point source location : $(x, y, z) = (0, 0, 1)$
- *Three-Parameter* model applied on all elements with $Z(\omega_0) = 0.5 + 0.2i$.
- Graph illustrates frequency-domain solution converted from time-domain at $\omega = 12\pi$.

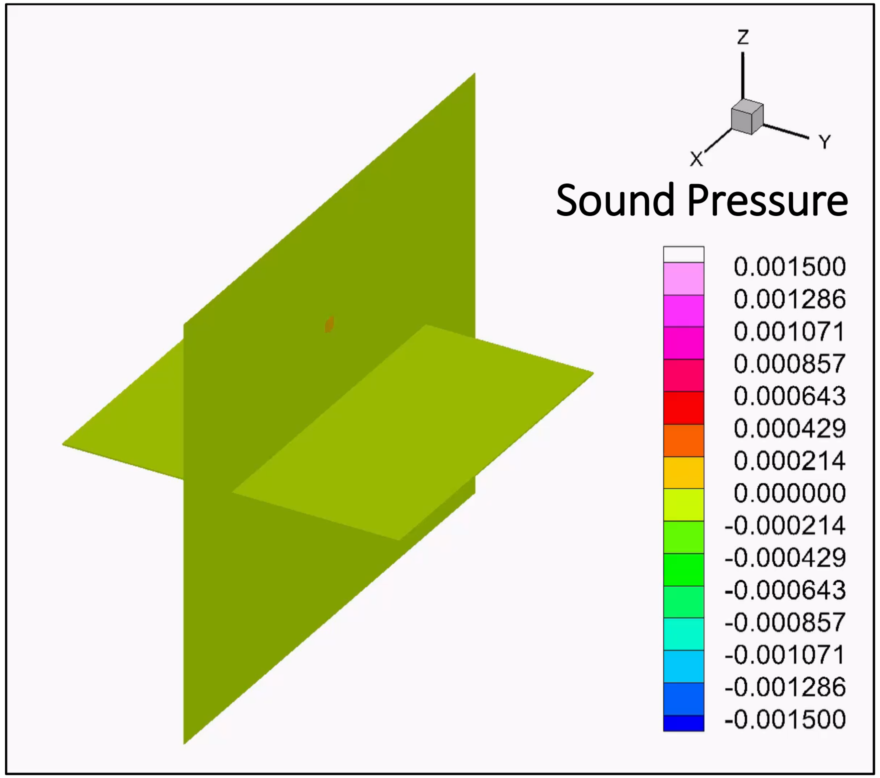




Lined Body Simulation



Un-Lined



Fully-Lined

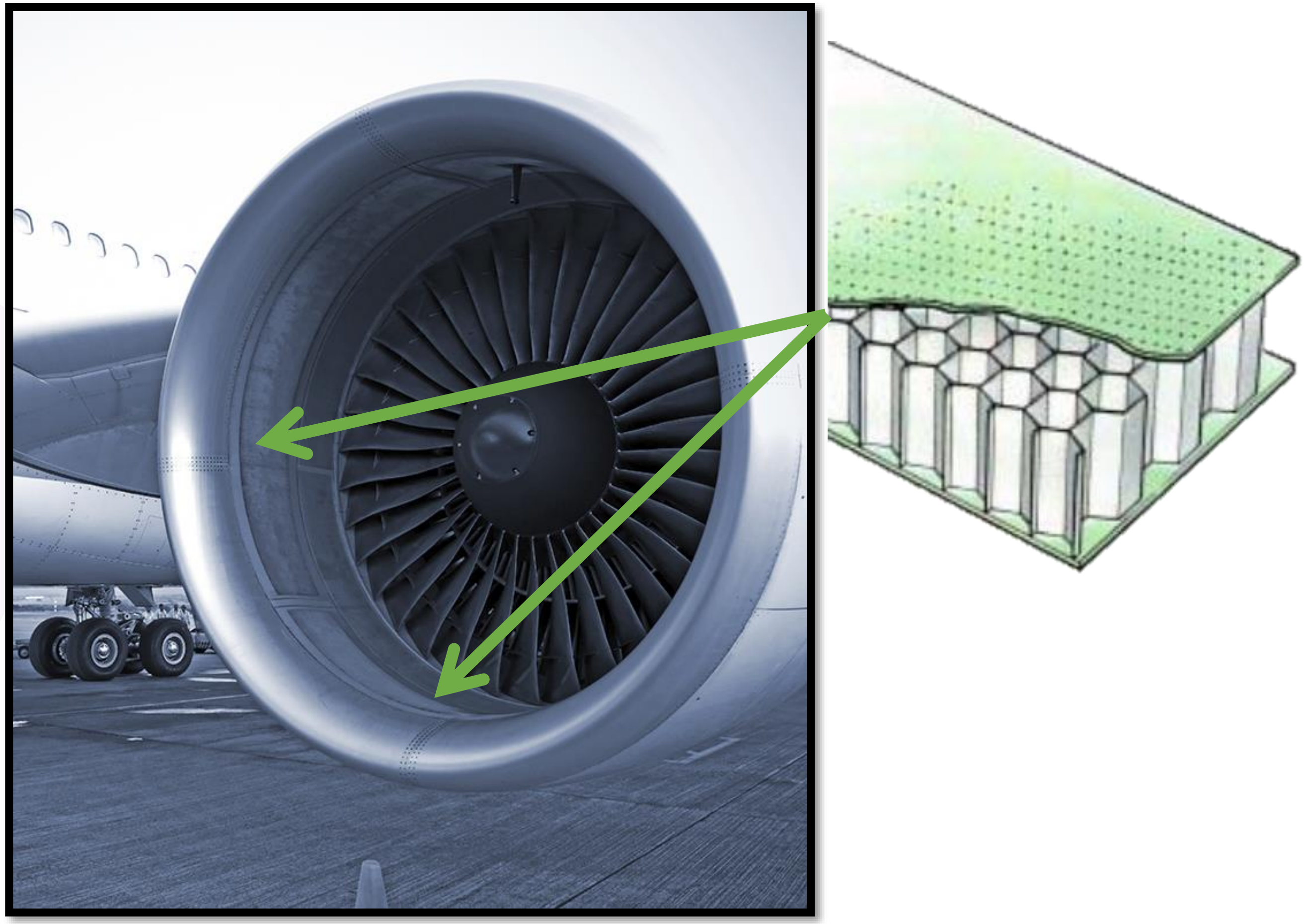
Concluding Remarks

- Proposed formulation of acoustic wave scattering of bodies treated with acoustic liners.
- Coupled time-domain IBC with a TD-BIE, stabilized by Burton-Miller type reformulation.
- Presented March-On-in-Time scheme for the solution of coupled system using spatial and temporal basis functions.
- Excellent spatial resolution demonstrated with relative error less than 3% and 5%, respectively, with 25 PPW2 (original and rotated, respectively).
- Algorithm scaled well with fewer CPUs for small problem sizes and performance suffered as processing power increased due to the costs associated with parallel overhead.
- Three acoustic liner models and their IBCs discussed: *Extended Helmholtz Resonator Model*, *Three-Parameter Impedance Model*, *Broadband Impedance Model.*
- Stability assessment reinforced that eigenvalue analysis is necessary to show stability.

Major Contributions

- Proposed formulation of acoustic wave scattering of bodies treated with acoustic liners.
- Coupled time-domain IBC with a TD-BIE, stabilized by Burton-Miller type reformulation.
- Presented March-On-in-Time scheme for the solution of coupled system using spatial and temporal basis functions.
- Excellent spatial resolution demonstrated with relative error less than 3% and 5%, respectively, with 25 PPW2 (original and rotated, respectively).
- Algorithm scaled well with fewer CPUs for small problem sizes and performance suffered as processing power increased due to the costs associated with parallel overhead.
- Three acoustic liner models and their IBCs discussed: *Extended Helmholtz Resonator Model, Three-Parameter Impedance Model, Broadband Impedance Model.*
- Stability assessment reinforced that eigenvalue analysis is necessary to show stability.

Application of Acoustic Liners on Aircraft



ACKNOWLEDGMENTS

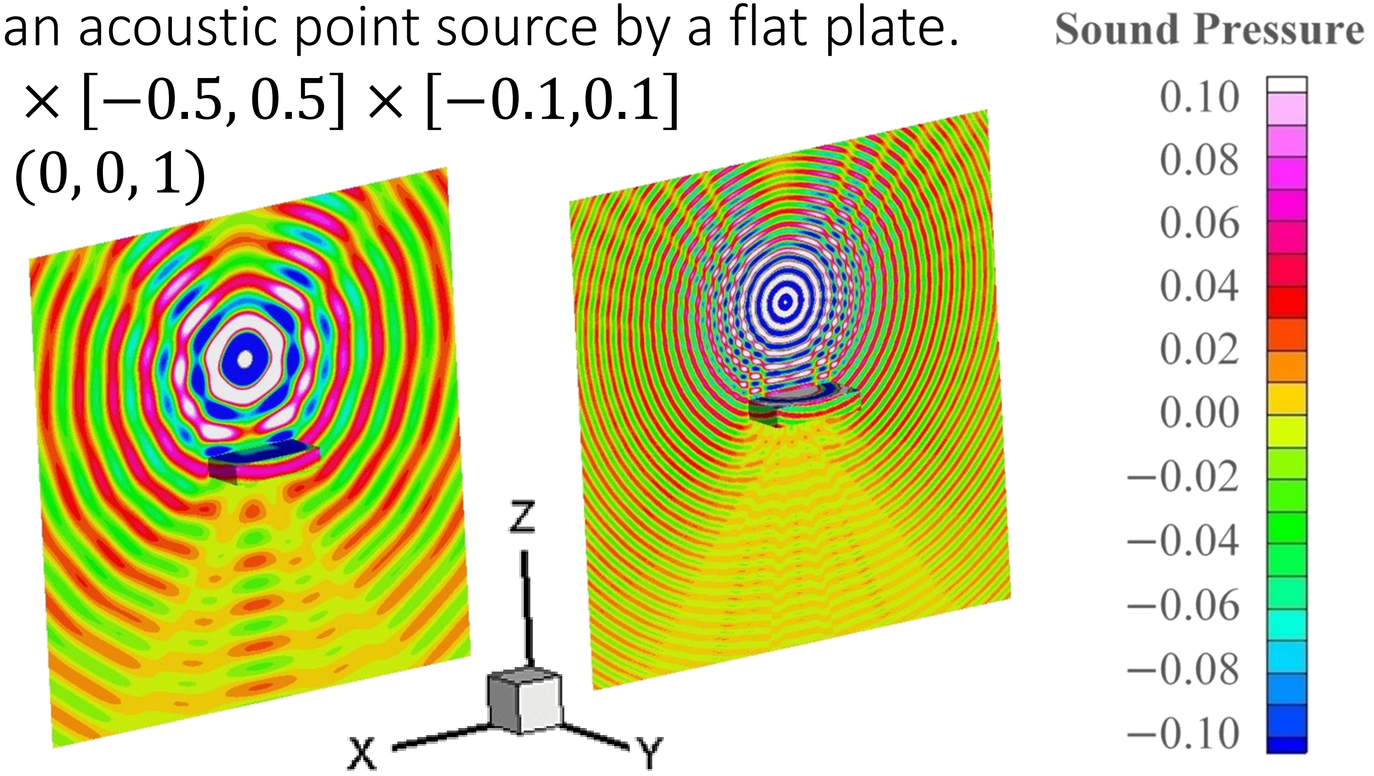
- *Principal Advisor*
  - Dr. Fang Q. Hu, Old Dominion University
- *Dissertation Committee*
  - Dr. Douglas M. Nark, NASA Langley Research Center
  - Dr. Yan Peng, Old Dominion University
  - Dr. John Tweed, Old Dominion University
  - Dr. Ruhai Zhou, Old Dominion University
- *Funding Sources*
  - NASA Cooperative Agreement, NNX11AI63A
  - Old Dominion University Modeling and Simulation Fellowship
  - Old Dominion University ITS Turing Cluster and Extreme Science and Engineering Discovery Environment, supported by National Science Foundation Grant Number OCI-1053575

Thank you!
Questions?

Spatial Resolution of BEM

- The spatial resolution of the TD-BEM with respect to the spatial basis functions is studied by considering PPW and PPW2.

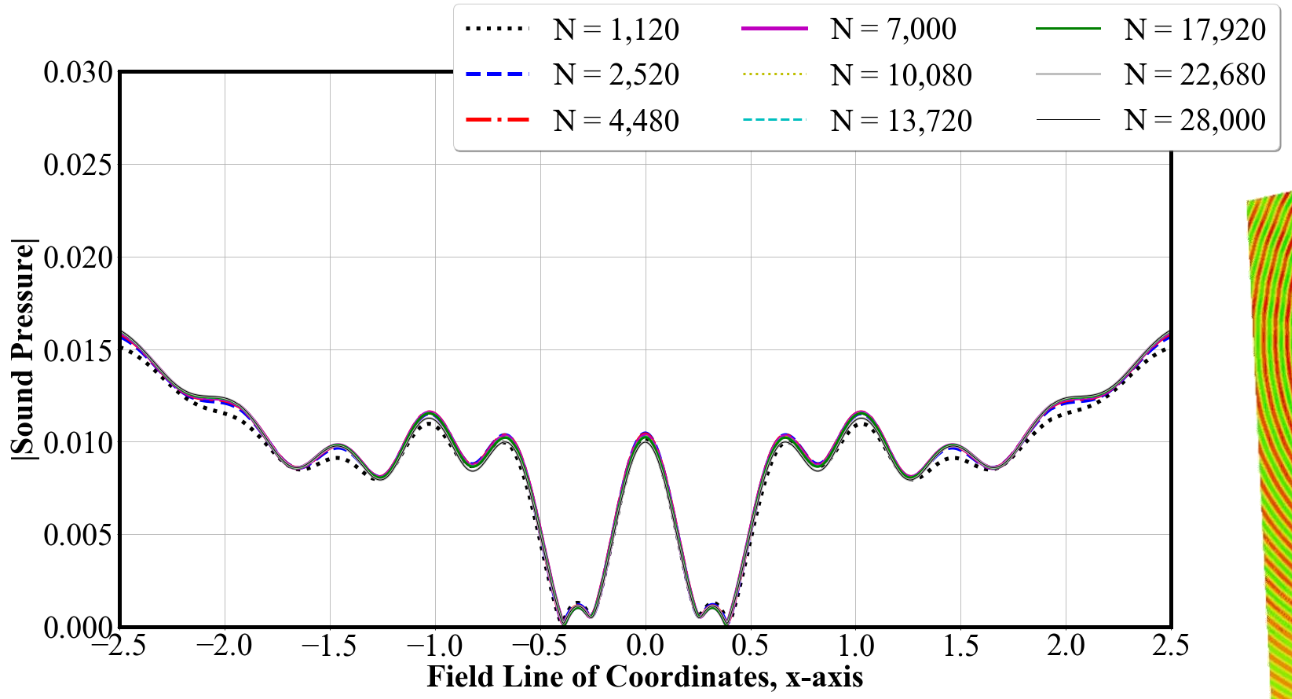
- Consider the scattering and shielding of an acoustic point source by a flat plate.
- Dimension of flat plate : $[-0.5, 0.5] \times [-0.5, 0.5] \times [-0.1, 0.1]$
- Point source location : $(x, y, z) = (0, 0, 1)$

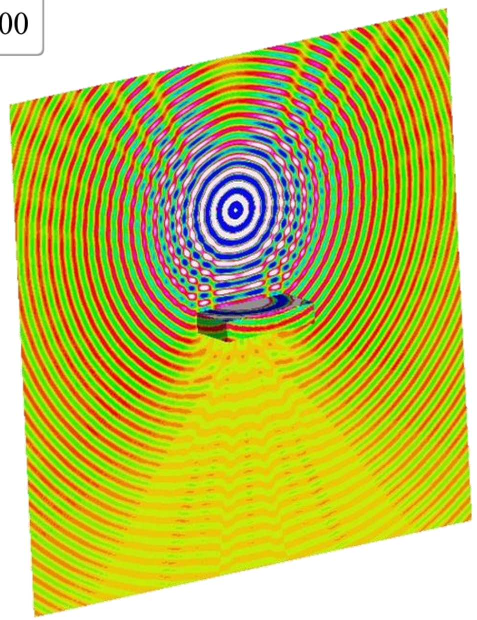


Contour plots of the frequency-domain solutions converted from time-domain solution at $\omega = 5\pi$ [left] and $\omega = 15\pi$ [right].

Flat Plate Solution

- The top and bottom of plate (at $z = 0.1$ and $z = -0.1$) are discretized by $N_x N_y$ elements
- A series of computations are carried out by increasing the number of elements used from $N_x \times N_y \times N_z = 20 \times 20 \times 4$ (1,120 elements) to $100 \times 100 \times 20$ (28,000 elements).





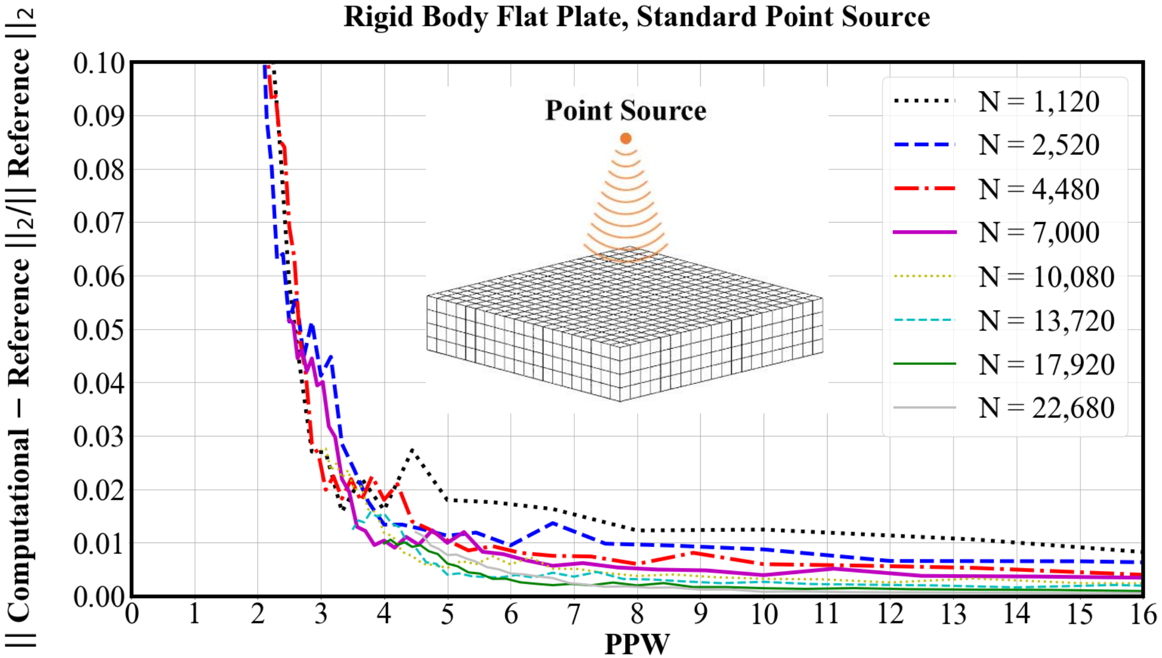
Frequency-domain solution along $-2.5 \leq x \leq 2.5$, $y = 0$, $z = -2.5$ converted from the time-domain solution at $\omega = 15\pi$.

Flat Plate PPW

- PPW is a metric used to measure spatial resolution along one direction on the surface.
- Using solution computed by $100 \times 100 \times 20$ as the reference, the relative error in the L_2 norm is plotted as a function of PPW.

$$PPW = \frac{2\pi(p+1)N_x}{kL_x}$$

p : order of the basis function
k : wavenumber, $k = w/c$
L_x : plate length in x-direction
N_x : elements in x-direction

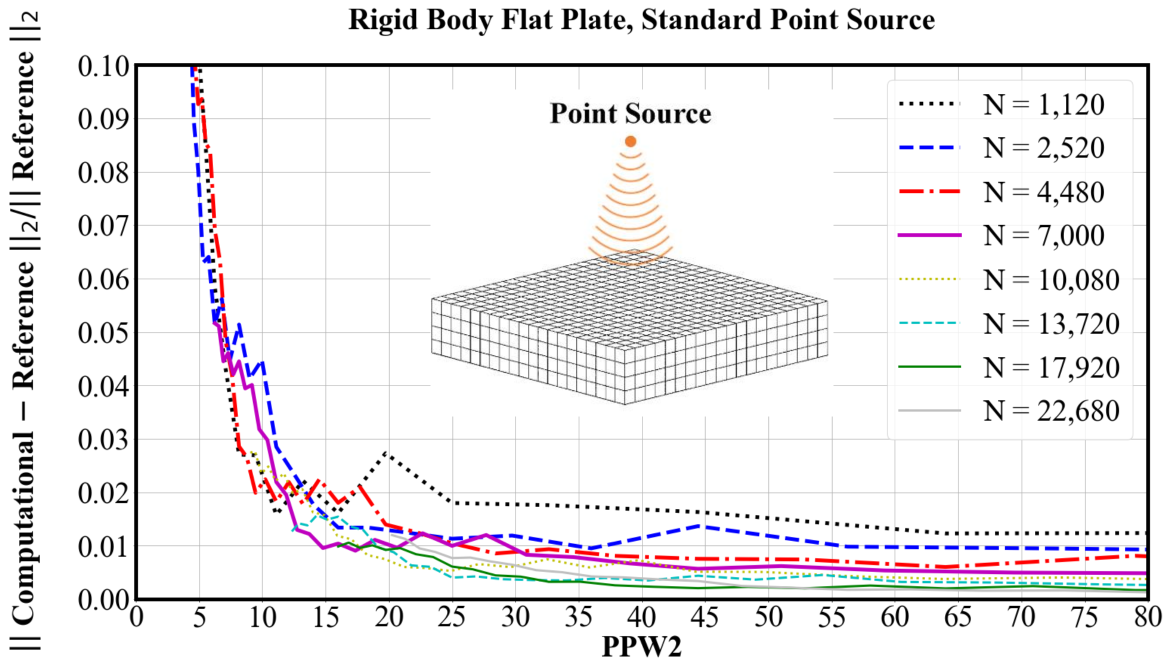


Flat Plate PPW2

- PPW2 is a metric used to measure spatial resolution over the entire surface.
- Using solution computed by $100 \times 100 \times 20$ as the reference, the relative error in the L_2 norm is plotted as a function of PPW2.

$$PPW2 = \frac{4\pi^2 (p+1)^2 [2N_x N_y + 2(N_x + N_y) N_z]}{k^2 [2L_x L_y + 2(L_x + L_y) L_z]}$$

p : order of the basis function
k : wavenumber, $k = w/c$
L_x : plate length in x-direction
N_x : elements in x-direction

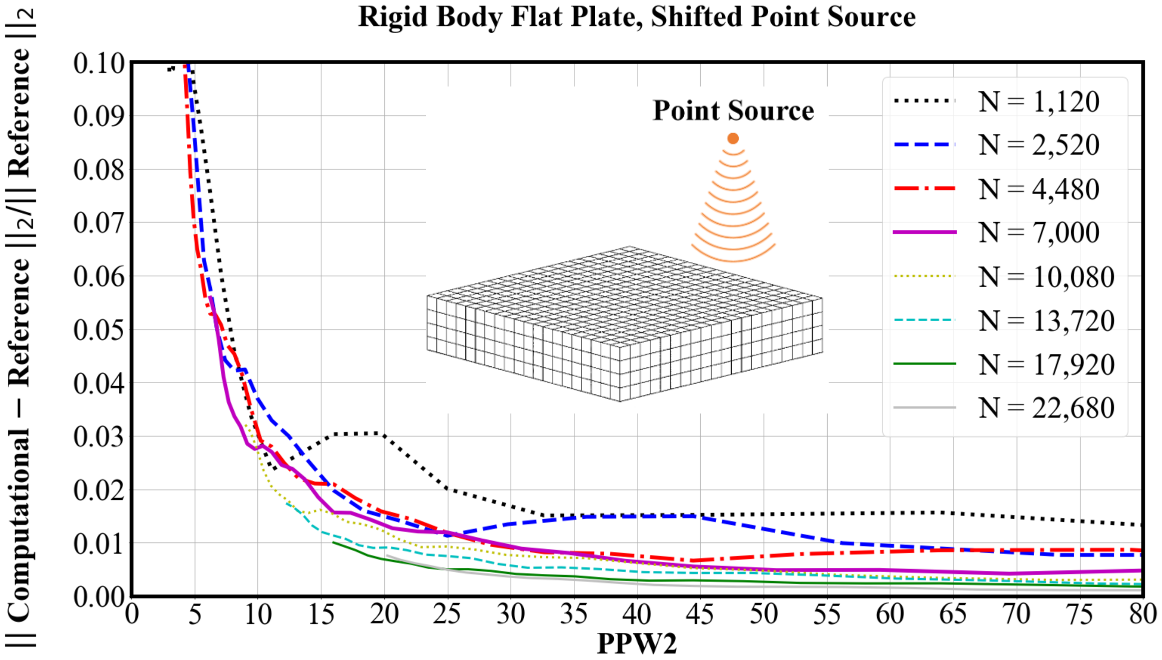


Flat Plate PPW2

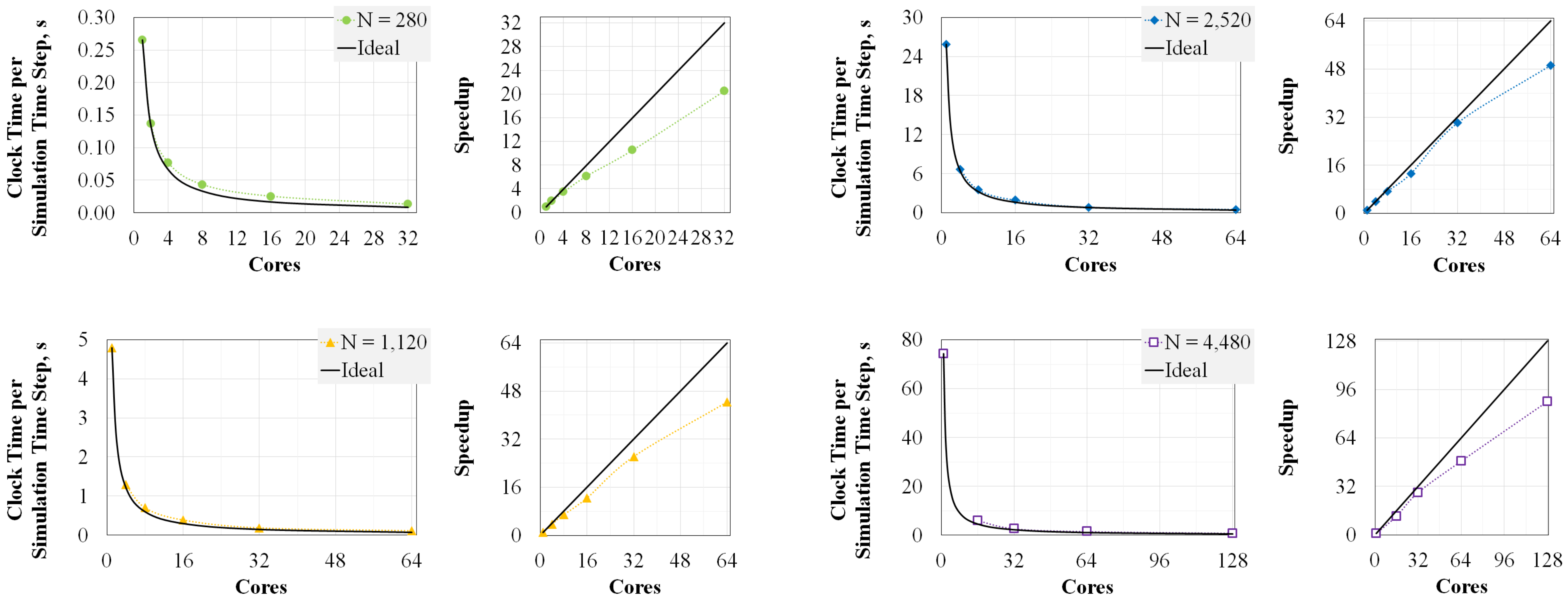
- To investigate whether the location of the point source affects the accuracy of the solution, a shifted point source located at $(x, y, z) = (0.5, 0, 1)$ is considered.
- Results further demonstrate that the relative error in the L_2 norm becomes less than 2% for all discretizations with 25 PPW2.

$$PPW2 = \frac{4\pi^2(p+1)^2[2N_xN_y + 2(N_x+N_y)N_z]}{k^2[2L_xL_y + 2(L_x+L_y)L_z]}$$

p : order of the basis function
k : wavenumber, $k = w/c$
L_x : plate length in x-direction
N_x : elements in x-direction



Speedup for 280 through 4,480 Elements



Matrix Power Iteration

- The matrix power iteration method is used for finding the largest eigenvalue of:

$$\left[\boldsymbol{A}_0\lambda^J + \boldsymbol{A}_1\lambda^{J-1} + \boldsymbol{A}_2\lambda^{J-2} + \cdots + \boldsymbol{A}_{J-1}\lambda + \boldsymbol{A}_J\right] e_0 = 0$$

$$\boldsymbol{A} = \begin{bmatrix} -\boldsymbol{A}_0{}^{-1}\boldsymbol{A}_1 & -\boldsymbol{A}_0{}^{-1}\boldsymbol{A}_2 & \cdots & -\boldsymbol{A}_0{}^{-1}\boldsymbol{A}_{J-2} & -\boldsymbol{A}_0{}^{-1}\boldsymbol{A}_{J-1} & -\boldsymbol{A}_0{}^{-1}\boldsymbol{A}_J \\ \boldsymbol{I} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \boldsymbol{I} & \cdots & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \boldsymbol{I} & 0 \end{bmatrix}$$

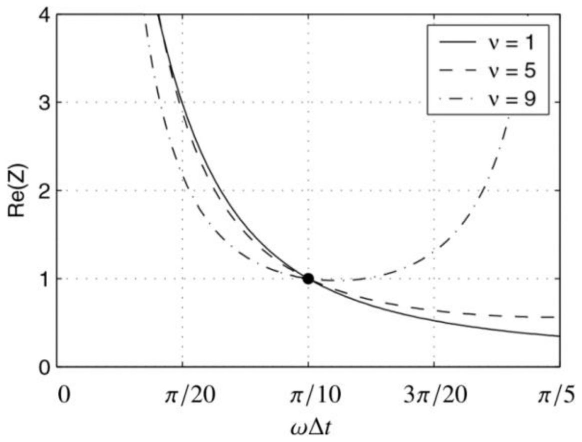
- Given an arbitrary unit vector $\boldsymbol{e}^{(0)}$, and for $k = 1,2,\ldots$ compute:

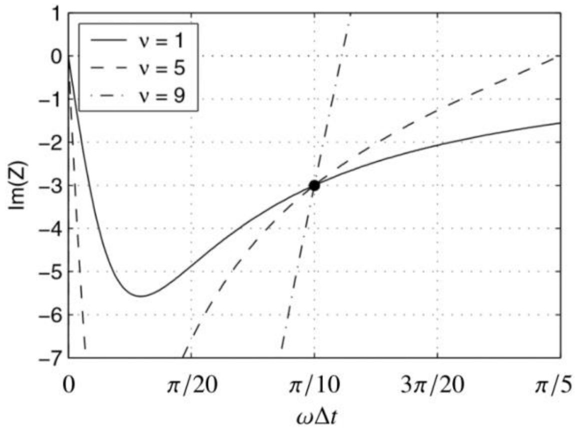
$$\boldsymbol{w}^{(k)} = \boldsymbol{A}\boldsymbol{e}^{(k-1)},\ \boldsymbol{e}^{(k)} = \frac{\boldsymbol{w}^{(k)}}{||\boldsymbol{w}^{(k)}||_2},\ \text{and eigenvalue } \lambda^{(k)} = \left[\boldsymbol{e}^{(k)}\right]^T \boldsymbol{A}\boldsymbol{e}^{(k)} = \left[\boldsymbol{e}^{(k)}\right]^T \boldsymbol{w}^{(k+1)}$$

Iteration stops when $\left|\lambda^k - \lambda^{k-1}\right| / \left|\lambda^k\right| < \delta$, where $\delta = 10^{-9}$.
Iteration converges to largest eigenvalue of $\boldsymbol{A}$, i.e., $|\lambda|_{\max}$.

Helmholtz, Three-Parameter IBC Data

- IBC constants generated using numerical data from Rienstra 2006.
- Rienstra proposed eighteen different impedance curves.
- Model data obtained from curves at single frequency $\omega\Delta t = \pi/10$.

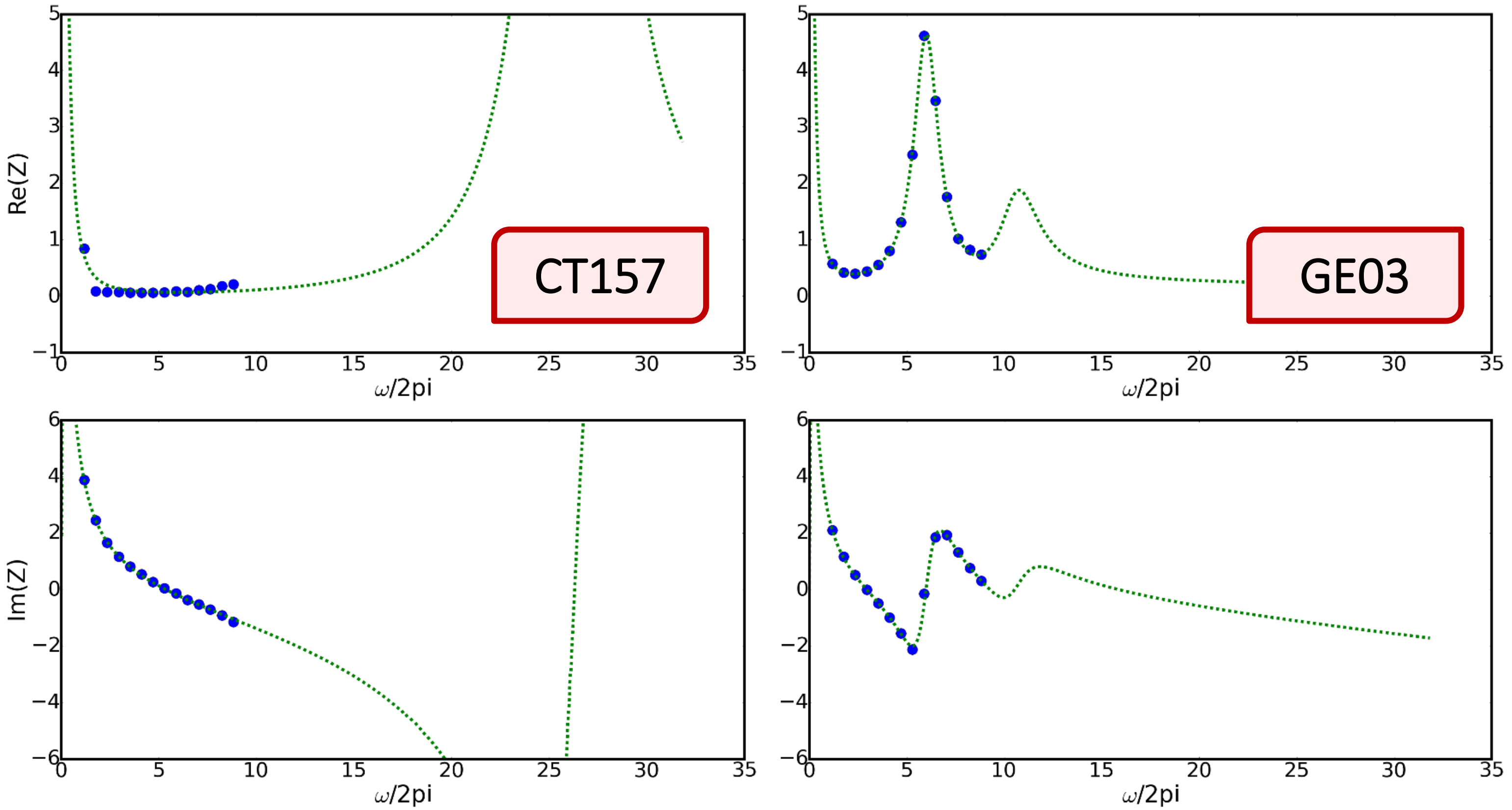




Case	$Z(\omega_0\Delta t) = Z_R + iZ_I$	ν	Case	$Z(\omega_0\Delta t) = Z_R + iZ_I$	ν	Case	$Z(\omega_0\Delta t) = Z_R + iZ_I$	ν
1		1	**7**		1	**13**		19
2	$1-3i$	5	**8**	$1-i$	5	**14**	$1+2i$	15
3		9	**9**		9	**15**		11
4		1	**10**		19	**16**		19
5	$1-2i$	5	**11**	$1+i$	15	**17**	$1+3i$	15
6		9	**12**		11	**18**		11

Broadband Frequency IBC Data

- Two acoustic liners, CT157 and GE03, tested in the Grazing Flow Impedance Tube at NASA Langley Research Center.
- Impedance values measured along a broad range of frequencies.
- Using measured data, twenty-five numerical models were generated using least squares.



Time Step $\Delta t = 1/12$

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*
Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Time Step $\Delta t = 1/24$

Case	*Extended Helmholtz Resonator Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
2	1.017881	*dnc* †	*dnc* †	*dnc*	0.999772	1.000000	0.999999	0.999999
3	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
4	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
5	1.000000	1.000000	0.999990	1.000000	*dnc* †	1.000000	1.000000	1.000000
6	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	1.000000
8	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000000	0.999999
9	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
10	1.002829	*dnc* †	*dnc* †	*dnc*	*dnc* †	1.0006875	*dnc* †	*dnc*
11	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000001	0.999999
12	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
13	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
14	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.999999	*dnc* †	*dnc*
15	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
16	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
17	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	*dnc* †	*dnc* †	*dnc*
18	*dnc* †	*dnc* †	*dnc* †	*dnc*	*dnc* †	0.997547	*dnc* †	*dnc*

Case	*Three-Parameter Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000001	1.000000	1.000000	1.000000	1.000004	1.000000	1.000000	0.995652
4	1.000003	1.000000	1.000000	1.000000	0.995680	0.995693	0.995715	0.995715
7	1.000008	1.000000	1.000000	1.000000	1.000000	0.995758	0.995777	0.995780
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
16	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Case	*Broadband Impedance Model*, Impedance Boundary Condition, Soft Body							
	$\Delta t = 1/12$				$\Delta t = 1/24$			
	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$	$5 \times 5 \times 1$	$10 \times 10 \times 2$	$20 \times 20 \times 4$	$30 \times 30 \times 6$
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
2	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
3	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
4	0.999976	1.000231	1.000079	1.000029	1.000017	0.999965	0.999958	0.999994
5	1.000006	1.000000	1.000000	1.000000	1.000026	1.000000	1.000000	1.000000
6	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
7	1.000000	1.000000	1.000000	1.000000	*dnc* †	1.000000	1.000041	1.000000
8	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
9	1.000000	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
10	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
11	1.000000	1.000056	1.000001	1.000000	1.000056	1.000001	0.997911	1.000000
12	0.997825	0.999678	0.998873	0.999674	0.999630	0.999641	0.999758	0.999650
13	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
14	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
15	0.999965	1.000000	1.000000	*emb*	*dnc* †	1.001096	1.000000	*emb*
16	1.000000	1.000000	1.000000	*emb*	0.999999	1.000000	1.000000	*emb*
17	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
18	1.000000	1.000000	1.000000	*emb*	1.000000	1.000000	1.000000	*emb*
19	1.000065	1.000000	1.000000	1.000000	1.000006	1.000000	1.000000	1.000000
20	1.000001	1.000165	1.000023	1.000003	1.000165	1.000076	1.000009	1.000002
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
22	1.000060	1.000003	1.000000	1.000000	1.000045	1.000000	1.000000	1.000000
23	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
24	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
25	1.000134	1.000102	1.000091	1.000091	1.000077	1.000048	1.000043	1.000043

Partially-Lined

$\Delta t = 1/12$ (left), $\Delta t = 1/24$ (right)

Case	*Broadband, Impedance*		Case	*Helmholtz, Impedance*		*Helmholtz, Admittance*	
	$5 \times 5 \times 1$	$10 \times 10 \times 2$		$5 \times 5 \times 1$	$10 \times 10 \times 2$	$5 \times 5 \times 1$	$10 \times 10 \times 2$
1	1.000000	1.000000	1	1.000000	1.000000	1.000292	1.0000641
2	1.000000	1.000000	2	*dnc* †	*dnc* †	*dnc* †	*dnc* †
3	1.000000	1.000000	3	*dnc* †	*dnc* †	⋆⋆⋆	⋆⋆⋆
4	1.000148	1.000173	4	1.000000	1.000000	1.000169	1.000367
5	1.000000	0.993455	5	1.000000	1.000000	*dnc* †	*dnc* †
6	1.000000	1.000000	6	*dnc* †	*dnc* †	1.739504	⋆⋆⋆
7	1.000000	1.000000	7	1.000000	1.000000	1.000021	1.000045
8	0.989167	1.000000	8	1.000000	1.000000	*dnc* †	*dnc* †
9	1.000000	1.000000	9	*dnc* †	*dnc* †	1.672663	⋆⋆⋆
10	1.000000	1.000000	10	0.971815	*dnc* †	1.029051	1.037838
11	1.000111	1.000056	11	1.000000	1.000000	0.999950	*dnc* †
12	0.999540	0.999678	12	*dnc* †	*dnc* †	1.688297	⋆⋆⋆
13	1.000000	1.000000	13	*dnc* †	*dnc* †	1.014432	1.024535
14	1.000000	1.000000	14	*dnc* †	*dnc* †	*dnc* †	*dnc* †
15	1.000000	1.000000	15	*dnc* †	*dnc* †	1.751776	⋆⋆⋆
16	1.000000	1.000000	16	*dnc* †	*dnc* †	1.007172	1.015423
17	1.000000	1.000000	17	*dnc* †	*dnc* †	*dnc* †	*dnc* †
18	1.000000	1.000000	18	*dnc* †	*dnc* †	⋆⋆⋆	⋆⋆⋆
19	1.000004	1.000000	Case	*Three-Parameter, Impedance*		*Three-Parameter, Admittance*	
20	0.998459	1.000165	1	1.000000	1.000000	0.998224	0.996788
21	1.000000	1.000000	4	1.000000	1.000000	0.998092	0.996584
22	1.000000	1.000003	7	1.000000	1.000000	0.997977	0.996411
23	1.000000	1.000000	10	1.000000	1.000000	0.999874	0.999739
24	1.000000	1.000000	13	1.000000	1.000000	0.999873	0.999736
25	1.000099	1.000102	16	1.000000	1.000000	0.999873	0.999736

Case	*Broadband, Impedance*		Case	*Helmholtz, Impedance*		*Helmholtz, Admittance*	
	$5 \times 5 \times 1$	$10 \times 10 \times 2$		$5 \times 5 \times 1$	$10 \times 10 \times 2$	$5 \times 5 \times 1$	$10 \times 10 \times 2$
1	1.000000	1.000000	1	*dnc* †	1.000000	1.000064	1.000128
2	1.000000	1.000000	2	0.999999	1.000000	*dnc* †	*dnc* †
3	1.000000	1.000000	3	*dnc* †	*dnc* †	1.394840	⋆⋆⋆
4	0.999999	0.999981	4	*dnc* †	1.000000	1.000017	1.000028
5	1.000000	1.000000	5	0.999999	1.000000	*dnc* †	*dnc* †
6	1.000000	1.000000	6	*dnc* †	*dnc* †	*dnc* †	1.762173
7	0.999994	1.000000	7	*dnc* †	1.000000	1.000001	1.000009
8	1.000000	1.000000	8	1.000000	1.000000	*dnc* †	*dnc* †
9	1.000000	1.000000	9	1.014022	*dnc* †	*dnc* †	1.693648
10	1.000000	1.000000	10	*dnc* †	1.006875	1.000000	1.000000
11	1.000006	1.000001	11	1.000000	1.000000	*dnc* †	*dnc* †
12	0.999200	0.999641	12	*dnc* †	*dnc* †	1.213335	1.708369
13	1.000000	1.000000	13	*dnc* †	*dnc* †	1.000000	1.000000
14	1.000000	1.000000	14	0.999976	0.999999	*dnc* †	*dnc* †
15	*dnc* †	1.001096	15	*dnc* †	*dnc* †	1.253654	1.7773645
16	1.000000	1.000000	16	*dnc* †	*dnc* †	1.000000	0.999999
17	1.000000	1.000000	17	*dnc* †	*dnc* †	*dnc* †	*dnc* †
18	1.000000	1.000000	18	*dnc* †	0.997547	1.439175	⋆⋆⋆
19	1.000000	1.000000	Case	*Three-Parameter, Impedance*		*Three-Parameter, Admittance*	
20	0.999999	1.000076	1	1.000001	1.000000	*dnc* †	0.997620
21	1.000000	1.000000	4	1.000002	0.995693	*dnc* †	0.997570
22	0.997424	1.000000	7	1.000002	0.995758	*dnc* †	0.997527
23	1.000000	1.000000	10	1.000000	1.000000	0.569258	0.999869
24	1.000000	1.000000	13	1.000000	1.000000	0.999935	0.999868
25	1.000057	1.000048	16	1.000000	1.000000	0.999937	0.999868