

The summation-by-parts framework for the design and analysis of stable discretizations

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A little bit about myself

Background:

- PhD 2015 University of Toronto supervisor David W. Zingg
- Postdoc + teaching 2015-2016 University of Toronto
- Postdoc 2017-2018 NASA LaRC supervisor Mark H. Carpenter
- 2019–present research scientist NASA LaRC

I have two feet and one hand each in a different discipline

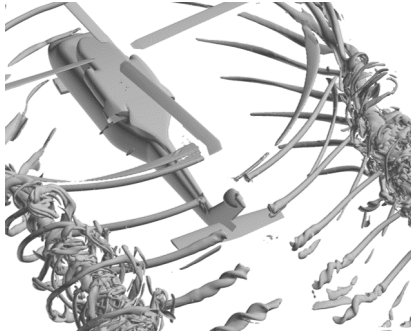
- Applied Mathematics: Algorithms with provable properties
- Computer Science: Algorithms suited for HPC
- Engineering: Application

Today I will concentrate on the mathematics

Motivation

Numerical approximation of problems that are:

- Nonlinear
- Time-dependent
- Multiscale
- Complex geometries



Example applications at NASA:

- Rotorcraft
- Fluid structure interactions
- High-lift configurations



Images courtesy of Mark Potsdam (Army Aeroflight Dynamics Division at NASA Ames)

Goal: Develop life-cycle efficient algorithms

- Robust $\implies$ provable stability
- Efficient:
 - ▶ Accurate $\implies$ high order/ high accuracy
 - ▶ Adaptive $\implies$ h/p -adaptation, r -adaptation, ...

Motivation

Goal: Develop life-cycle efficient algorithms

● Robust $\implies$ provable stability

3D Taylor-Green vortex: Successful (green) and failure (red)

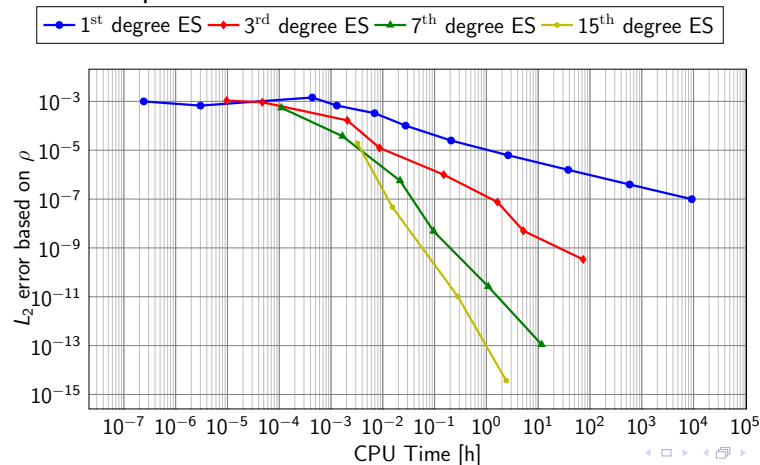
	Degree	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ES	3x3x3	Successful (green)														
	6x6x6															
	12x12x12															
	24x24x24															
	48x48x48															
Conv.	3x3x3	Successful (green)														
	6x6x6															
	12x12x12															
	24x24x24															
	48x48x48															
	96x96x96	Successful (green)														

Motivation

Goal: Develop life-cycle efficient algorithms

- Efficient: Accurate $\implies$ high order/ high accuracy

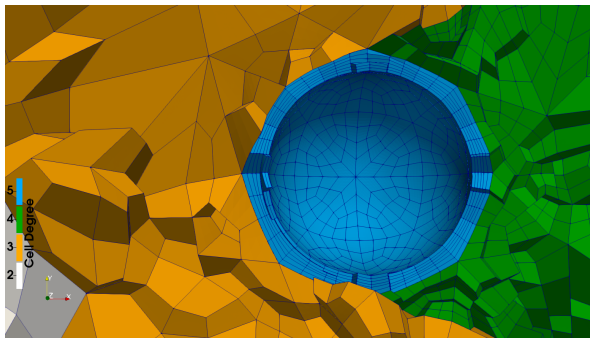
3D isentropic vortex



Motivation

Goal: Develop life-cycle efficient algorithms

- Efficient: Adaptive $\implies h/p$ -adaptation, r -adaptation, ...



Goal: Develop life-cycle efficient algorithms

- Robust $\implies$ provable stability
- Efficient:
 - ▶ Accurate $\implies$ high order/ high accuracy
 - ▶ Adaptive $\implies$ h/p -adaptation, r -adaptation, ...

At the continuous level, stability depend on integration by parts

Construct methods that come equipped with a discrete counterpart to integration by parts

Summation by parts (SBP)

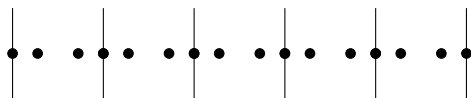
- Importance of provable properties (particularly stability)
 - ▶ Linear advection
 - ▶ Burgers' equation
- SBP framework
- Energy stability analysis linear advection: key tool integration by parts
- SBP operators: key property mimics integration by parts and is telescopic
- Semidiscret stability analysis: Linear advection
- Entropy stability analysis of the continuous and semidiscrete Burgers' equation
- Some recent work on entropy stable schemes

When discretizations blow up: Linear PDEs

Consider the linear advection equation

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= 0, \quad x \in [x_L, x_R], \quad t \geq 0 \\ u(x_L, t) &= \mathcal{G}(t), \quad u(x, 0) = \mathcal{I}(x)\end{aligned}\tag{1}$$

Divide the domain into N equally spaced elements ($x = \text{fcn}(\xi)$)



On the κ^{th} element (1) reads

$$\frac{\partial u}{\partial t} + \frac{\partial \xi}{\partial x} \frac{\partial u}{\partial \xi} = 0, \quad \xi \in [-1, 1], \quad \mathcal{J}^{-1} \equiv \frac{\partial \xi}{\partial x} = \frac{2N}{x_R - x_L}\tag{2}$$

When discretizations blow up: Linear PDEs

Generic strong-form discontinuous Galerkin semidiscretization of (2):

$$\frac{d\mathbf{u}_\kappa}{dt} + [\mathcal{G}]_\kappa^{-1} D_\xi \mathbf{u}_\kappa = \mathbf{CT} + \mathbf{BC}, \quad (3)$$

- D_ξ is a differentiation matrix, i.e., $D_\xi \mathbf{u} \approx \frac{\partial u}{\partial \xi}(\xi)$
- $[\mathcal{G}]_\kappa \text{diag}(\mathcal{G}(\xi(1)), \dots, \mathcal{G}(\xi(n))) = \frac{2N}{x_R - x_L} \mathbf{I}$
- $\mathbf{CT}$ inter-element coupling
- $\mathbf{BC}$ imposition of boundary condition

Consider two variations (identical in form):

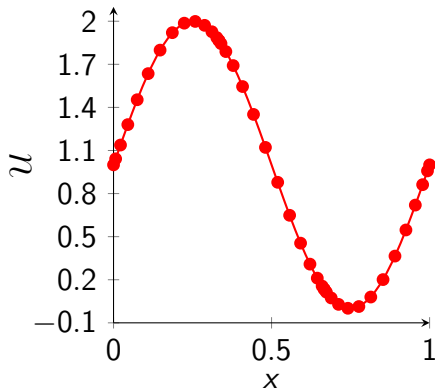
- Diagonal-norm (mass lumped)
- Dense-norm (not mass lumped)

When discretizations blow up: Linear PDEs

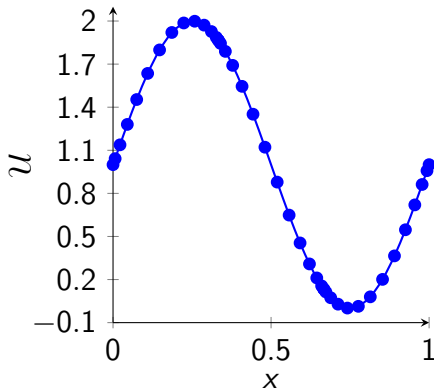
Numerical simulations: $x \in [0, 1]$, $t \in [0, 1]$, $\mathcal{I} = 1 + \sin(2\pi x)$

RK4: CLF = 0.99, $n = 14$, $N = 3$

Diagonal-norm



Dense-norm



When discretizations blow up: Linear PDEs

Now let us consider the curvilinear case (variable coefficient case):

$$x = \text{fcn}(\hat{\xi}), \quad \hat{\xi} = \text{fnc}(\xi)$$

Thus $\mathcal{G}^{-1} = \frac{\partial \hat{\xi}}{\partial x} \frac{\partial \xi}{\partial \hat{\xi}}$ is no longer constant

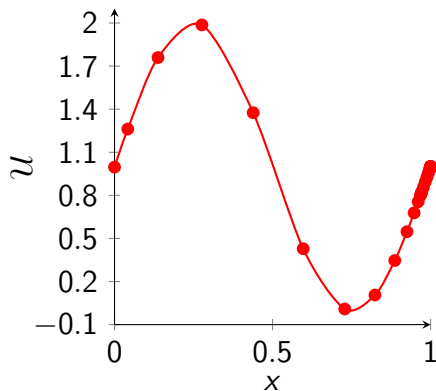
This is the critical difference

Mesh for $n = 14$, $N = 3$

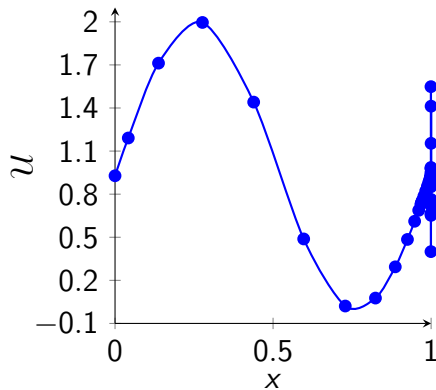


When discretizations blow up: Linear PDEs

Diagonal-norm



Dense-norm



$t = 0.01137195$

Observations

- The two schemes are very similar
 - ▶ Have the same form
 - ▶ Design order different
- The dense-norm scheme, intuitively, should be the same or better (no approximation in the norm matrix)
- Cartesian context: Both schemes behaved identically (they are probably stable)
- Curvilinear context: The diagonal-norm worked, the dense-norm scheme blew up

What did we learn?

- Intuitive improvements to a scheme can have unintended consequences
- Stability in one context may not translate to stability in another
- We need a rigorous way of determining when our schemes are stable
- Also want guidance to what design order changes we should make when they are not

When discretizations blow up: Nonlinear PDEs

Consider the Burger's equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = 0 \quad (4)$$

Generic strong-form (collocated diagonal-norm) semidiscretization of (4) on a Cartesian grid

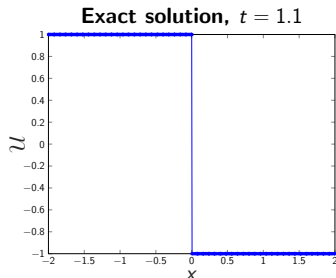
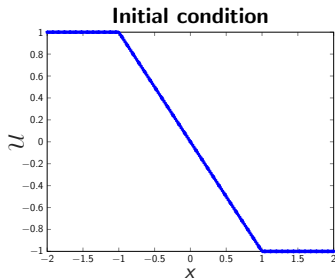
$$\frac{d\mathbf{u}_\kappa}{dt} + \frac{1}{2} [\mathcal{g}]_\kappa^{-1} D_\xi [\mathbf{u}_\kappa] \mathbf{u}_\kappa = \mathbf{CT} + \mathbf{BC} \quad (5)$$

Entropy conservative/stable strong-form (collocated diagonal-norm) semidiscretization of (4)

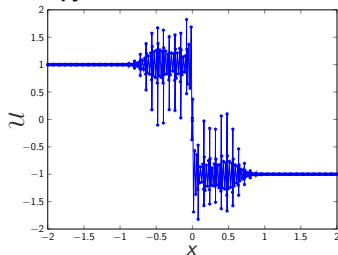
$$\frac{d\mathbf{u}_\kappa}{dt} + \frac{1}{3} [\mathcal{g}]_\kappa^{-1} D_\xi [\mathbf{u}_\kappa] \mathbf{u}_\kappa + \frac{1}{3} [\mathcal{g}]_\kappa^{-1} [\mathbf{u}_\kappa] D_\xi [\mathbf{u}_\kappa] \mathbf{u}_\kappa = \widetilde{\mathbf{CT}} + \widetilde{\mathbf{BC}} \quad (6)$$

When discretizations blow up: Nonlinear PDEs

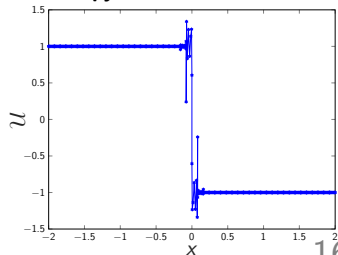
RK4, number of time steps 80,000



Entropy conservative solution, $t = 1.1$



Entropy stable solution, $t = 1.1$



When discretizations blow up: Nonlinear PDEs

Stability

Entropy conservative scheme

p	Over integration					
	0	1	2	3	4	5
1	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓	✓
4	✓	✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓	✓
6	✓	✓	✓	✓	✓	✓
7	✓	✓	✓	✓	✓	✓
8	✓	✓	✓	✓	✓	✓
9	✓	✓	✓	✓	✓	✓

Nodal DG scheme

p	Over integration					
	0	1	2	3	4	5
1	✗	✓	✓	✓	✓	✓
2	✗	✓	✓	✓	✓	✓
3	✗	✗	✓	✓	✓	✓
4	✗	✗	✓	✓	✓	✓
5	✗	✗	✗	✓	✓	✓
6	✗	✗	✗	✓	✓	✓
7	✗	✗	✗	✗	✓	✓
8	✗	✗	✗	✗	✓	✓
9	✗	✗	✗	✗	✗	✓

When discretizations blow up: Nonlinear PDEs

Observations

- The two schemes are fundamentally different
- Different volume and interface coupling terms
- The entropy conservative/stable scheme holds together while the collocated scheme does not

What did we learn?

- Linearly stable schemes do not necessarily translate into stable schemes for nonlinear problems
- We need a rigorous way of determining stability in the nonlinear context as well

The summation-by-parts (SBP) framework

Most numerical methods can be distilled into matrix operations

A high level of abstraction is the matrix form (perhaps highest?)

SBP Framework: A matrix analysis methodology

- Systematic approach to the design of new schemes that are
 - ▶ Stable, conservative, and accurate
- Simple: requires basic linear algebra/calculus to do proofs
- Discretization agnostic (FD, FV, DG, CG, FR/CPR, etc.)
 - ▶ Applicable a large set of numerical methods
- Design order correction of existing schemes
- Easily handles any variational crimes that are committed

The summation-by-parts (SBP) framework

Core concept: Prove stability at the continuous level

- Mimic in a one-to-one fashion at the semi/fully-discrete level

Stability: Norm of the solution is bounded by the data of the problem

- Data = initial condition, boundary conditions, etc.

Continuous stability analysis:

- For linear and variable coefficient problems: Energy method
- For nonlinear problems: Entropy stability analysis

Energy method: Linear advection

1) Multiply the PDE by the solution and integrate over the domain:

$$\int_{-1}^1 \left(u \frac{\partial u}{\partial t} g + u \frac{\partial u}{\partial \xi} \right) d\xi = 0$$

2) Volume integral: Integration by parts (IBP)

$$\int_{-1}^1 u \frac{\partial u}{\partial t} g d\xi + \frac{1}{2} \oint u^2 n_\xi d\Gamma = 0, \quad \int_{-1}^1 v \frac{\partial u}{\partial \xi} d\xi + \int_{-1}^1 u \frac{\partial v}{\partial \xi} d\xi = \oint v u n_\xi d\Gamma$$

3) Temporal term: product rule and Leibniz' rule

$$\frac{1}{2} \frac{d\|u\|^2}{dt} + \frac{1}{2} \oint u^2 n_\xi d\Gamma = 0, \quad \|u\|^2 \equiv \int_{-1}^1 u^2 g d\xi$$

4) Integrate in time and apply data

$$\|u(t)\|^2 = \|g\|^2 - \int_0^t u^2(x_R, t) d\tau + \int_0^t \mathcal{G}^2 d\tau \rightarrow \|u(t)\|^2 \leq \|g\|^2 + \int_0^t \mathcal{G}^2 d\tau$$

Summation-by-parts operators

Degree p matrix difference operator D_{ξ_l} acting on the ξ approximating $\frac{\partial}{\partial \xi_l}$:

- ① Accuracy: $D_{\xi_l} \mathbf{p}_j = \delta_{\xi_l} \mathbf{p}_j, \forall \mathbf{p} \in \mathbb{P}_j^p$
- ② Structure:
 - ▶ $D_{\xi_l} \equiv P^{-1} Q_{\xi_l}$, where the norm matrix $P > 0$
 - ▶ $Q_{\xi_l} = (S_{\xi_l} + \frac{1}{2} E_{\xi_l}), S_{\xi_l} = -S_{\xi_l}^T, E_{\xi_l} = E_{\xi_l}^T$
- ③ Accuracy on E_{ξ_l}

$$\mathbf{v}^T E_{\xi_l} \mathbf{u} = \oint_{\Gamma} v u n_{\xi_l} d\Gamma, \quad \forall v, u \in \mathbb{P}^r, r \geq p$$

- ④ $\mathbf{v}^T E_{\xi_l} \mathbf{u}$ telescopes point wise (separable into point values on surfaces)

Properties:

$$\mathbf{v}^T \mathbf{P} \mathbf{u} \approx \int_{\Omega} \mathcal{V} \mathcal{U} d\Omega, \quad \mathbf{v}^T \mathbf{Q}_{\xi_l} \mathbf{u} \approx \int_{\Omega} \mathcal{V} \frac{\partial \mathcal{U}}{\partial \xi_l} d\Omega,$$

$$\mathbf{v}^T \mathbf{Q}_{\xi_l}^T \mathbf{u} \approx \int_{\Omega} \mathcal{U} \frac{\partial \mathcal{V}}{\partial \xi_l} d\Omega, \quad \mathbf{v}^T \mathbf{E}_{\xi_l} \mathbf{u} \approx \int_{\Gamma} \mathcal{V} \mathcal{U} n_{\xi_l} d\Gamma,$$

Integration by parts:

$$\int_{\Omega} \mathcal{V} \frac{\partial \mathcal{U}}{\partial \xi_l} d\Omega + \int_{\Omega} \mathcal{U} \frac{\partial \mathcal{V}}{\partial \xi_l} d\Omega = \oint_{\Gamma} \mathcal{V} \mathcal{U} n_{\xi_l} d\Gamma$$

Summation by parts:

$$\underbrace{\approx \int_{\Omega} \mathcal{V} \frac{\partial \mathcal{U}}{\partial \xi_l} d\Omega}_{\mathbf{v}^T \mathbf{P} \mathbf{D}_{\xi_l} \mathbf{u}} + \underbrace{\approx \int_{\Omega} \mathcal{U} \frac{\partial \mathcal{V}}{\partial \xi_l} d\Omega}_{\mathbf{u}^T \mathbf{P} \mathbf{D}_{\xi_l} \mathbf{v}} = \mathbf{v} \left(\mathbf{Q}_{\xi_l} + \mathbf{Q}_{\xi_l}^T \right) \mathbf{u} = \underbrace{\approx \oint_{\Gamma} \mathcal{V} \mathcal{U} n_{\xi_l} d\Gamma}_{\mathbf{v}^T \mathbf{E}_{\xi_l} \mathbf{u}}$$

Pointwise telescoping: Simple example Gauss nodes



$$E_{\xi_I} = R_R^T R_R - R_L^T R_L, \quad R_R \mathbf{u} \approx \mathcal{U}(1), \quad R_L \mathbf{u} \approx \mathcal{U}(-1)$$

$$\underbrace{\approx \oint \nu u n_{\xi_I} d\Gamma}_{\mathbf{v}^T E_{\xi_I} \mathbf{u}} \approx \underbrace{\oint_{x_R} \nu u d\Gamma = \nu(1)u(1)}_{(R_R \mathbf{v})^T R_R \mathbf{u}} + \underbrace{\approx \oint_{x_L} \nu u d\Gamma = -\nu(-1)u(-1)}_{\left\{ - (R_L \mathbf{v})^T R_L \mathbf{u} \right\}}$$

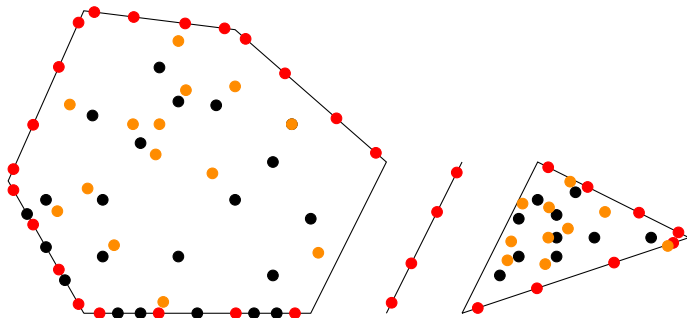
Why is this important?

- Telescope to the boundary for provable stability
- Apply appropriate boundary conditions
- Traverse interior surfaces and apply appropriate upwinding (dissipation)

Incomplete list of extensions to SBP concept

- **Original**: Kreiss and Scherer (finite difference (FD) single block)
- **Spectral element**: Carpenter and Gottlieb
- **Multiblock/element**: Carpenter, Nordström, and Gottlieb
- **General tensor-product**: Del Rey Fernández, Boom, and Zingg
- **Staggered tensor-product**: Carpenter, Parsani, Fisher, and Nielsen
- **General element**: Hicken, Del Rey Fernández, and Zingg
- **General element curvilinear**: Crean, Hicken, Del Rey Fernández, Carpenter, and Zingg
- **Staggered general element**: Del Rey Fernández, Crean, Hicken, and Carpenter
- **Modal + decoupled surface and volume cubature nodes**: Jesse Chan
- **Dense-norm, global SBP etc.**: Del Rey Fernández, Boom, Carpenter, and Zingg

State of the art in SBP operators



- solution nodes/modes
- flux nodes (volume quadrature/cubature nodes)
- surface quadrature/cubature nodes

The SBP concept can be applied to a rich set of numerical methods

More extensions are likely

The local SBP property is extended to the global SBP property via

- Appropriate coupling procedures (discontinuous approaches)
- Appropriate imposition of boundary conditions
- Additions such as dissipation are constructed so as not to destroy stability estimates

We consider one element (or an SBP operator constructed over the 1D mesh)

Strong form discretization:

$$\frac{d\mathbf{u}}{dt} + [\mathcal{J}]^{-1} D_{\xi} \mathbf{u} = -[\mathcal{J}]^{-1} P^{-1} R_L^T (R_L \mathbf{u} - \mathcal{G})$$

- LHS: discretization of the PDE
- RHS: Design order zero term (weakly imposes the boundary condition)

Semidiscrete analysis of the linear advection equation

Continuous:

1) Multiply by u and integrate over the domain

$$\int u \frac{\partial u}{\partial t} \mathcal{G} d\xi + \int u \frac{\partial u}{\partial \xi} d\xi = 0$$

Semidiscrete:

1) multiply by $\mathbf{u}^T \mathbf{P}[\mathcal{G}]$

$$\mathbf{u}^T \mathbf{P}[\mathcal{G}] \frac{d\mathbf{u}}{dt} + \mathbf{u}^T \mathbf{Q}_\xi \mathbf{u} = -\mathbf{u}^T \mathbf{R}_L^T (\mathbf{R}_L \mathbf{u} - \mathcal{G})$$

Note: for dense norm $\mathcal{G}$ $\mathbf{P}$ is not generally a norm

Continuous:

2) Use integration by parts

$$\int u \frac{\partial u}{\partial t} g d\xi = -\frac{1}{2} \oint u^2 n_\xi d\Gamma$$

Semidiscrete:

2) Use summation-by-parts

$$\mathbf{u}^T \mathbf{Q}_\xi \mathbf{u} + \mathbf{u}^T \mathbf{Q}_\xi \mathbf{u} = \mathbf{u}^T \mathbf{E}_\xi \mathbf{u} \rightarrow \mathbf{u}^T \mathbf{Q}_\xi \mathbf{u} = \frac{1}{2} \mathbf{u}^T \mathbf{E}_\xi \mathbf{u}$$

$$\mathbf{u}^T \mathbf{P}[\mathcal{G}] \frac{d\mathbf{u}}{dt} = -\frac{1}{2} \mathbf{u}^T \mathbf{E}_\xi \mathbf{u} - \mathbf{u}^T \mathbf{R}_L^T (\mathbf{R}_L \mathbf{u} - \mathcal{G})$$

Semidiscrete analysis of the linear advection equation

Continuous:

3) Use the product rule and Leibniz' rule

$$\frac{1}{2} \frac{d\|u\|^2}{dt} = -\frac{1}{2} (u_R^2 - u_L^2), \quad \|u\|^2 \equiv \int u^2 g d\xi$$

Semidiscrete:

3) Use product rule

$$\frac{1}{2} \frac{d\|u\|_{P[g]}^2}{dt} = -\frac{1}{2} (u^T R_R^T R_R u + u^T R_L^T R_L u - 2u^T R_L^T G)$$

$$\|u\|_{P[g]}^2 \equiv u^T P[g] u$$

$$E_\xi = R_R^T R_R - R_L^T R_L$$

Semidiscrete analysis of the linear advection equation

Continuous:

4) Integrate in time and apply the data

$$\|\mathcal{U}(t)\|^2 \leq \|\mathcal{I}\|^2 + \int_0^t \mathcal{G}^2 d\tau$$

Semidiscrete:

4) Integrate in time, apply the initial condition, and complete the squares

$$\begin{aligned}\|\mathbf{u}\|_{\mathbb{P}[\mathcal{I}]}^2 &\leq \|\mathcal{I}\|_{\mathbb{P}[\mathcal{I}]}^2 - \int_0^t (\mathbf{u}^T \mathbf{R}_L^T \mathbf{R}_L \mathbf{u} - 2\mathbf{u}^T \mathbf{R}_L^T \mathcal{G}) d\tau \\ &\leq \|\mathcal{I}\|_{\mathbb{P}[\mathcal{I}]}^2 - \int_0^t \left\{ (\tilde{u} - \mathcal{G})^2 + \mathcal{G}^2 \right\} d\tau, \quad \tilde{u} \equiv \mathbf{R}_L \mathbf{u} \\ &\leq \|\mathcal{I}\|_{\mathbb{P}[\mathcal{I}]}^2 + \int_0^t \mathcal{G}^2 d\tau\end{aligned}$$

For nonlinear PDEs we need to go beyond the energy method

Consider the following hyperbolic PDE:

$$\mathcal{G} \frac{\partial \mathcal{U}}{\partial t} + \frac{\partial \mathcal{F}}{\partial \xi} = 0$$

Equipped with a convex entropy function $\mathcal{S} = \text{fnc}(\mathcal{U})$ that satisfies

$$\frac{\partial \mathcal{S}}{\partial \mathcal{U}} \frac{\partial \mathcal{F}}{\partial \mathcal{U}} = \mathcal{W}^T \frac{\partial \mathcal{F}}{\partial \mathcal{U}} = \frac{\partial f}{\partial \mathcal{U}} \rightarrow \mathcal{W}^T \frac{\partial \mathcal{F}}{\partial \mathcal{U}} \frac{\partial \mathcal{U}}{\partial \xi} = \frac{\partial f}{\partial \xi}$$

Integrebility condition

where f is the entropy flux and $\mathcal{W}$ are the entropy variables

Entropy stability analysis

1) Multiply the PDE by $\mathcal{W}^T$ and integrate in space

$$\int \mathcal{W}^T \frac{\partial \mathcal{U}}{\partial t} \mathcal{G} d\xi + \int \mathcal{W}^T \frac{\partial \mathcal{F}}{\partial \xi} d\xi = 0$$

2) Use the integration by parts property induced by the integrability condition

$$\int \mathcal{W}^T \frac{\partial \mathcal{F}}{\partial \xi} d\xi = \int \frac{\partial f}{\partial \xi} d\xi = \oint f n_\xi d\Gamma$$

$$\int \mathcal{W}^T \frac{\partial \mathcal{U}}{\partial t} \mathcal{G} d\xi + \oint f n_\xi d\xi \leq 0$$

3) Use the definition of the entropy variables and Leibniz' rule

$$\frac{d}{dt} \int \mathcal{S} \mathcal{G} d\xi + \oint f n_\xi d\xi \leq 0$$

4) Integrate in time and apply data

$$\int \mathcal{S}(t) \mathcal{G} d\xi \leq \int \mathcal{S}(0) \mathcal{G} d\xi - \int_0^t \oint f n_\xi d\xi d\tau$$

5) Convert the entropy statement into a stability statement on the solution

Will see this shortly

Entropy stability analysis Burgers' equation

$$\mathcal{G} \frac{\partial \mathcal{U}}{\partial t} + \frac{\partial}{\partial \xi} \left(\frac{\mathcal{U}^2}{2} \right) = 0,$$

$$\frac{(\mathcal{U}_L + |\mathcal{U}_L|)}{3} \mathcal{U}_L = \mathcal{G}_L, \quad \frac{(\mathcal{U}_R - |\mathcal{U}_R|)}{3} \mathcal{U}_R = \mathcal{G}_R,$$

$$\mathcal{U}(\xi, 0) = \mathcal{G}$$

Entropy function: $\mathcal{S} = \frac{\mathcal{U}^2}{2}$

Flux: $\mathcal{F} = \frac{\mathcal{U}^2}{2}$

Entropy variable: $\mathcal{W} = \frac{\partial \mathcal{S}}{\partial \mathcal{U}} = \mathcal{U}$

Entropy flux: $f = \frac{\mathcal{U}^3}{3}$

Entropy stability analysis Burgers' equation

1) Multiply the PDE by $\mathcal{W}^T$ and integrate in space

$$\int \mathcal{W}^T \frac{\partial \mathcal{U}}{\partial t} \mathcal{G} d\xi + \int \mathcal{W}^T \frac{\partial \mathcal{F}}{\partial \xi} d\xi = 0$$

2) Use the integration by parts property induced by the integrability condition

$$\int \mathcal{W}^T \frac{\partial \mathcal{U}}{\partial t} d\xi + \oint f n_\xi d\xi \leq 0$$

3) Use the definition of the entropy variables and Leibniz' rule

$$\frac{d}{dt} \int \mathcal{S} \mathcal{G} d\xi + \oint f n_\xi d\xi \leq 0$$

4) Integrate in time and apply data

$$\begin{aligned}\int \mathcal{S}(t) \mathcal{G} d\xi &\leq \int \mathcal{S}(0) \mathcal{G} d\xi - \int_0^t \oint f n_\xi d\xi d\tau \\ &\leq \frac{1}{2} \int \mathcal{G}^2 \mathcal{G} d\xi + \int_0^t \left\{ -\frac{1}{3} (3u_R \mathcal{G}_R + |u_R| u_R^2) \right\} d\tau \\ &\quad + \int_0^t \left\{ \frac{1}{3} (3u_L \mathcal{G}_L - |u_L| u_L^2) \right\} d\tau\end{aligned}$$

5) Convert the entropy statement into a stability statement on the solution

$$\int u^2(t) \mathcal{G} d\xi \leq \int \mathcal{G}^2 \mathcal{G} d\xi + \int_0^t \left(\frac{3\mathcal{G}_L^2}{2|u_L|} + \frac{3\mathcal{G}_R^2}{2|u_R|} \right) d\tau$$

Objective: construct operators that mimic the following IBP rule

$$\int \boldsymbol{w}^T \frac{\partial \mathcal{F}}{\partial \boldsymbol{\xi}} d\boldsymbol{\xi} = \int \frac{\partial f}{\partial \boldsymbol{\xi}} d\boldsymbol{\xi} = \oint f n_{\boldsymbol{\xi}} d\Gamma$$

Approximate inviscid terms using Hadamard formalism

$$2D_{\boldsymbol{\xi}} \circ F_x(\boldsymbol{u}, \boldsymbol{u}) \mathbf{1} \approx \frac{\partial \mathcal{F}}{\partial \boldsymbol{x}}(\boldsymbol{x})$$

The two-point flux matrix is constructed from a two-point flux function:

$$\boldsymbol{f}^{sc}(\boldsymbol{u}^{(i)}, \boldsymbol{u}^{(j)}) = \frac{\left\{ (\boldsymbol{u}^{(i)})^2 + \boldsymbol{u}^{(i)} \boldsymbol{u}^{(j)} + (\boldsymbol{u}^{(j)})^2 \right\}}{6}$$

An example

$$D_{\xi} = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}.$$

The two argument Hadamard matrix flux function, $F_x(\mathbf{u}, \mathbf{u})$ is given as

$$F_x(\mathbf{u}, \mathbf{u}) = \begin{bmatrix} \frac{(\mathbf{u}^{(1)})^2}{2} & \frac{(\mathbf{u}^{(1)})^2 + \mathbf{u}^{(1)}\mathbf{u}^{(2)} + (\mathbf{u}^{(2)})^2}{6} & \frac{(\mathbf{u}^{(1)})^2 + \mathbf{u}^{(1)}\mathbf{u}^{(3)} + (\mathbf{u}^{(3)})^2}{6} \\ \frac{(\mathbf{u}^{(2)})^2 + \mathbf{u}^{(2)}\mathbf{u}^{(1)} + (\mathbf{u}^{(1)})^2}{6} & \frac{(\mathbf{u}^{(2)})^2}{2} & \frac{(\mathbf{u}^{(2)})^2 + \mathbf{u}^{(2)}\mathbf{u}^{(3)} + (\mathbf{u}^{(3)})^2}{6} \\ \frac{(\mathbf{u}^{(3)})^2 + \mathbf{u}^{(3)}\mathbf{u}^{(1)} + (\mathbf{u}^{(1)})^2}{6} & \frac{(\mathbf{u}^{(3)})^2 + \mathbf{u}^{(3)}\mathbf{u}^{(2)} + (\mathbf{u}^{(2)})^2}{6} & \frac{(\mathbf{u}^{(3)})^2}{2} \end{bmatrix}.$$

The general case: Curvilinear coordinates, systems, etc.

Discretization: $2 (D_\xi \otimes I_5) \circ F_{x_m} (\mathbf{u}_\kappa, \mathbf{u}_\kappa) \mathbf{1} \approx \frac{\partial \mathcal{F}_{x_m}}{\partial \xi} :$

$$F_{x_m} (\mathbf{u}_\kappa, \mathbf{u}_\kappa) \equiv \begin{bmatrix} \left[\mathbf{f}_{x_m}^{SC} \left(\mathbf{u}_\kappa^{(1)}, \mathbf{u}_\kappa^{(1)} \right) \right] & \dots & \left[\mathbf{f}_{x_m}^{SC} \left(\mathbf{u}_\kappa^{(1)}, \mathbf{u}_\kappa^{(n)} \right) \right] \\ \vdots & & \vdots \\ \left[\mathbf{f}_{x_m}^{SC} \left(\mathbf{u}_\kappa^{(n)}, \mathbf{u}_\kappa^{(1)} \right) \right] & \dots & \left[\mathbf{f}_{x_m}^{SC} \left(\mathbf{u}_\kappa^{(n)}, \mathbf{u}_\kappa^{(n)} \right) \right] \end{bmatrix}$$

The general case: Curvilinear coordinates, systems, etc.

Two-point flux functions:

- Tadmor's shuffle condition:

$$\left(\mathbf{w}_{(i)}^\kappa - \mathbf{w}_{(j)}^\kappa \right)^T \mathbf{f}_{x_m}^{sc} \left(\mathbf{u}_\kappa^{(i)}, \mathbf{u}_\kappa^{(j)} \right) = \psi_{x_m}^{(i)} - \psi_{x_m}^{(j)}$$

- Symmetric: $\mathbf{f}_{x_m}^{sc} \left(\mathbf{u}_\kappa^{(i)}, \mathbf{u}_\kappa^{(j)} \right) = \mathbf{f}_{x_m}^{sc} \left(\mathbf{u}_\kappa^{(j)}, \mathbf{u}_\kappa^{(i)} \right)$
- Consistent: $\mathbf{f}_{x_m}^{sc} \left(\mathbf{u}_\kappa^{(i)}, \mathbf{u}_\kappa^{(i)} \right) = \mathcal{F}_{x_m} \left(\mathbf{u}_\kappa^{(i)} \right)$

Entropy potential flux $\Psi_{x_m} = \mathcal{W}^T \mathcal{F}_{x_m} - f_{x_m}$

Dyadics are expensive: 1) need to be careful, 2) gives a mechanism for communication hiding

Mimetic and telescoping properties

It can be shown that

$$\underbrace{\approx \int \mathbf{w}^T \frac{\partial \mathcal{F}_{x_m}}{\partial \xi} d\Omega}_{2\mathbf{w}^T \mathbf{P}(\mathbf{D}_\xi \otimes \mathbf{I}_5) \circ \mathbf{F}_{x_m}(\mathbf{u}, \mathbf{u}) \mathbf{1}} = \underbrace{\oint (\mathbf{w}^T \mathcal{F}_{x_m} - \psi_{x_m}) n_\xi d\Gamma}_{\mathbf{1}^T (\mathbf{E}_\xi \circ \mathbf{I}_5) \circ \mathbf{F}_{x_m}(\mathbf{u}, \mathbf{u}) \mathbf{w} - \mathbf{1}^T \mathbf{E}_\xi \psi_{x_m}} = \oint f_{x_m} n_\xi d\Gamma$$

Nonlinear SBP property

Mimetic term by term

Telescoping:

- The term $\mathbf{1}^T \mathbf{E}_\xi \psi_{x_m}$ is telescopic
- The term $\mathbf{1}^T (\mathbf{E}_\xi \circ \mathbf{I}_5) \circ \mathbf{F}_{x_m}(\mathbf{u}, \mathbf{u}) \mathbf{w}$
 - ▶ In the general case, it is not telescopic
 - ▶ For certain SBP operators, it is telescopic

Semidiscrete analysis: Burgers' equation

Discretization:

$$[g]_k \frac{d\mathbf{u}}{dt} + 2D_\xi \circ F_x(\mathbf{u}, \mathbf{u}) \mathbf{1} = -P^{-1}R_L^T \left\{ \frac{(R_L \mathbf{u} + |R_L \mathbf{u}|)}{3} R_L \mathbf{u} - \mathcal{G}_L \right\} \\ + P^{-1}R_R^T \left\{ \frac{(R_R \mathbf{u} - |R_R \mathbf{u}|)}{3} R_R \mathbf{u} - \mathcal{G}_R \right\}$$

Where:

- $-P^{-1}R_L^T \left\{ \frac{(R_L \mathbf{u} + |R_L \mathbf{u}|)}{3} R_L \mathbf{u} - \mathcal{G}_L \right\}$

weakly imposes $\frac{(u_L + |u_L|)}{3} u_L = \mathcal{G}_L$

- $P^{-1}R_R^T \left\{ \frac{(R_R \mathbf{u} - |R_R \mathbf{u}|)}{3} R_R \mathbf{u} - \mathcal{G}_R \right\}$

weakly imposes $\frac{(u_R - |u_R|)}{3} u_R = \mathcal{G}_R$

Semidiscrete analysis: Burgers' equation

Continuous:

1) Multiply the PDE by $\mathcal{W}^T$ and integrate in space

$$\int \mathcal{W}^T \frac{\partial \mathcal{U}}{\partial t} \mathcal{G} d\xi + \int \mathcal{W}^T \frac{\partial \mathcal{F}}{\partial \xi} d\xi = 0$$

Semidiscrete:

1) Multiply through by $\mathbf{w}^T \mathbf{P}$

$$\begin{aligned} \mathbf{w}^T \mathbf{P} [\mathcal{G}]_k \frac{d\mathbf{u}}{dt} + \mathbf{w}^T \mathbf{P} D_\xi \circ F_x(\mathbf{u}, \mathbf{u}) \mathbf{1} = \\ - \mathbf{w}^T \mathbf{R}_L^T \left\{ \frac{(\mathbf{R}_L \mathbf{u} + |\mathbf{R}_L \mathbf{u}|)}{3} \mathbf{R}_L \mathbf{u} - \mathcal{G}_L \right\} \\ + \mathbf{w}^T \mathbf{R}_R^T \left\{ \frac{(\mathbf{R}_R \mathbf{u} - |\mathbf{R}_R \mathbf{u}|)}{3} \mathbf{R}_R \mathbf{u} - \mathcal{G}_R \right\} \end{aligned}$$

Semidiscrete analysis: Burgers' equation

Continuous:

2) Use the integration by parts property induced by the integrability condition

$$\int \mathbf{w}^T \frac{\partial \mathcal{U}}{\partial t} \mathcal{G} d\xi + \oint f n_\xi d\xi \leq 0$$

Semidiscrete:

2) Use the nonlinear SBP property

$$\begin{aligned} & \mathbf{w}^T \mathbf{P}[\mathcal{G}]_k \frac{d\mathbf{u}}{dt} + \mathbf{1}^T \mathbf{E}_\xi \circ \mathbf{F}_x(\mathbf{u}, \mathbf{u}) \mathbf{w} + \mathbf{1}^T \mathbf{E}_\xi \psi_x = \\ & - \mathbf{w}^T \mathbf{R}_L^T \left\{ \frac{(\mathbf{R}_L \mathbf{u} + |\mathbf{R}_L \mathbf{u}|)}{3} \mathbf{R}_L \mathbf{u} - \mathcal{G}_L \right\} \\ & + \mathbf{w}^T \mathbf{R}_R^T \left\{ \frac{(\mathbf{R}_R \mathbf{u} - |\mathbf{R}_R \mathbf{u}|)}{3} \mathbf{R}_R \mathbf{u} - \mathcal{G}_R \right\} \end{aligned}$$

Semidiscrete analysis: Burgers' equation

Semidiscrete:

2) Use the nonlinear SBP property

For $R_R = [1, 0, \dots, 0]^T$ and $R_L = [1, 0, \dots, 0]^T$

$$\mathbf{1}^T E_\xi \circ F_x(\mathbf{u}, \mathbf{u}) \mathbf{w} + \mathbf{1}^T E_\xi \psi_x = \frac{\mathbf{u}(n)^3}{3} - \frac{\mathbf{u}(1)^3}{3}$$

Thus,

$$\mathbf{w}^T P[\mathcal{G}]_k \frac{d\mathbf{u}}{dt} = -\frac{|\mathbf{u}(1)|\mathbf{u}(1)^2}{3} + \mathbf{u}(1)\mathcal{G}_L - \frac{|\mathbf{u}(n)|\mathbf{u}(n)^2}{3} - \mathbf{u}(n)\mathcal{G}_R$$

Semidiscrete analysis: Burgers' equation

Continuous:

3) Use the definition of the entropy variables and Leibniz' rule

$$\frac{d}{dt} \int \mathcal{S} \mathcal{G} d\xi + \oint f n_{\xi} d\xi \leq 0$$

Semidiscrete:

3) Use the definition of the entropy variables

$$\mathbf{1}^T \mathbf{P} [\mathcal{G}]_k \frac{d\mathbf{s}}{dt} = -\frac{|\mathbf{u}(1)|\mathbf{u}(1)^2}{3} + \mathbf{u}(1)\mathcal{G}_L - \frac{|\mathbf{u}(n)|\mathbf{u}(n)^2}{3} - \mathbf{u}(n)\mathcal{G}_R$$

Semidiscrete analysis: Burgers' equation

Continuous:

4) Integrate in time and apply data

$$\begin{aligned} \int \mathcal{S}(t) \mathcal{G} d\xi \leq & \frac{1}{2} \int \mathcal{G}^2 \mathcal{G} d\xi + \int_0^t \left\{ -\frac{1}{3} (3u_R \mathcal{G}_R + |u_R| u_R^2) \right\} d\tau \\ & + \int_0^t \left\{ \frac{1}{3} (3u_L \mathcal{G}_L - |u_L| u_L^2) \right\} d\tau \end{aligned}$$

Semidiscrete:

4) Integrate in time and apply data

$$\begin{aligned} \mathbf{1}^T \mathbf{P} [\mathcal{G}]_k \mathbf{s}(t) = & \mathbf{1}^T \mathbf{P} [\mathcal{G}]_k \mathbf{s}(0) + \int_0^t \left\{ -\frac{1}{3} (3\mathbf{u}(n) \mathcal{G}_R) + |\mathbf{u}(n)| \mathbf{u}(n)^2 \right\} d\tau \\ & + \int_0^t \left\{ \frac{1}{3} (3\mathbf{u}(1) \mathcal{G}_L - |\mathbf{u}(1)| \mathbf{u}(1)^2) \right\} d\tau \end{aligned}$$

Semidiscrete analysis: Burgers' equation

Continuous:

5) Convert the entropy statement into a stability statement on the solution

Using $\mathcal{S} = \frac{u^2}{2}$

$$\begin{aligned}\|u(t)\|^2 &\leq \|g(0)\|^2 + 2 \int_0^t \left\{ -\frac{1}{3} (3u_R g_R + |u_R| u_R^2) \right\} d\tau \\ &\quad + 2 \int_0^t \left\{ \frac{1}{3} (3u_L g_L - |u_L| u_L^2) \right\} d\tau\end{aligned}$$

Completing the square

$$\|u(t)\|^2 \leq \|g(0)\|^2 + \int_0^t \left(\frac{3g_R}{2|u_R|} + \frac{3g_L}{2|u_L|} \right) d\tau$$

Semidiscrete analysis: Burgers' equation

Semidiscrete:

5) Convert the entropy statement into a stability statement on the solution

Using $s = \frac{1}{2} [\mathbf{u}] \mathbf{u}$

$$\begin{aligned} \|\mathbf{u}\|_{[g]P}^2 &= \|\mathbf{I}\|_{[g]P}^2 + 2 \int_0^t \left\{ -\frac{1}{3} (3\mathbf{u}(n)\mathcal{G}_R) + |\mathbf{u}(n)|\mathbf{u}(n)^2 \right\} d\tau \\ &\quad + 2 \int_0^t \left\{ \frac{1}{3} (3\mathbf{u}(1)\mathcal{G}_L - |\mathbf{u}(1)|\mathbf{u}(1)^2) \right\} d\tau \end{aligned}$$

Completing the square

$$\|\mathbf{u}\|_{[g]P}^2 \leq \|\mathbf{I}\|_{[g]P}^2 + \int_0^t \left(\frac{3\mathcal{G}_R}{2|\mathbf{u}(n)|} + \frac{3\mathcal{G}_L}{2|\mathbf{u}(1)|} \right) d\tau$$

Summary of entropy stability analysis

Procedure

- ① Perform entropy analysis at the continuous level
- ② Chose an entropy and find a a two-point flux function
- ③ Combine SBP operators with two argument flux function matrix to approximate volume terms
- ④ Construct appropriate coupling and boundary procedures that maintain entropy conservation/stability
 - ▶ This is where most of the technical challenge lies particularly for the BC

Recent work on entropy stability

ALE entropy stable scheme

Paper: Entropy stable spectral collocation schemes for the 3-D Navier-Stokes equations on dynamic unstructured grids <https://arxiv.org/pdf/1812.10185.pdf>

Authors: Nail K. Yamaleev, David C. Del Rey Fernández, Jialin Lou, and Mark H. Carpenter

- Skew-symmetric split form (splitting on the metric terms and metric Jacobian)

$$\frac{1}{2} \frac{g_\kappa U^\kappa}{\partial \tau} + \frac{1}{2} g_\kappa \frac{\partial U^\kappa}{\partial \tau} + \frac{1}{2} \sum_{l=1}^3 \left(\frac{\partial}{\partial \xi_l} \left(g_\kappa \frac{\partial \xi_l}{\partial t} U^\kappa \right) + g_\kappa \frac{\partial \xi_l}{\partial t} \frac{\partial U^\kappa}{\partial \xi_l} \right) + \text{space} = 0$$

- Geometric conservation laws (GCL)

$$\frac{\partial g_\kappa}{\partial \tau} + \sum_{l=1}^3 \frac{\partial}{\partial \xi_l} \left(g_\kappa \frac{\partial \xi_l}{\partial t} \right) = 0$$

$$\sum_{l=1}^3 \frac{\partial}{\partial \xi_l} \left(g_\kappa \frac{\partial \xi_l}{\partial x_m} \right) = 0, \quad m = 1, 2, 3$$

Entropy stable space-time discretizations

Paper: Entropy stable space-time discontinuous Galerkin schemes with summation-by-parts property for hyperbolic conservation laws <https://arxiv.org/pdf/1808.08218.pdf>

Authors: Lucas Friedrich, Gero Schnücke, Andrew R. Winters, David C. Del Rey Fernández, Gregor J. Gassner, and Mark H. Carpenter

- The critical term is the temporal term $\frac{\partial U^\kappa}{\partial t}$. Discretization

$$2\bar{D}_t \circ U^{sc}(\mathbf{u}_\kappa, \mathbf{u}_\kappa) \bar{\mathbf{1}}_{N_\kappa} + \textit{spatial} = \textit{SATs}$$

$$\bar{D}_t \circ U^{sc}(\mathbf{u}_\kappa, \mathbf{u}_\kappa) \bar{\mathbf{1}}_{N_\kappa} \approx \frac{1}{2} \frac{\partial \mathcal{U}}{\partial t} \Big|_{C_t}$$

- The nonlinear temporal derivative operator $\bar{D}_t \circ U^{sc}(\mathbf{q}_\kappa, \mathbf{q}_\kappa) \bar{\mathbf{1}}_{N_\kappa}$ telescopes when contracted with the entropy variables

$$\mathbf{w}_\kappa^T P \bar{D}_t \circ U^{sc}(\mathbf{u}_\kappa, \mathbf{u}_\kappa) \bar{\mathbf{1}}_{N_\kappa} = (\mathbf{w}_\kappa^T \mathbf{u}_\kappa - \mathbf{s})^T Q_\tau \mathbf{1} + \mathbf{1}^T Q_\tau (\mathbf{w}_\kappa^T \mathbf{u}_\kappa - \mathbf{s}_\kappa) = \mathbf{1} E_\tau (\mathbf{w}_\kappa^T \mathbf{u}_\kappa - \mathbf{s}_\kappa)$$

- Incredibly, the linear temporal upwind SAT leads to entropy stability!!!!

Efficient entropy stable Gauss collocation methods

Paper: Efficient entropy stable Gauss collocation methods

<https://arxiv.org/pdf/1809.01178.pdf>

Authors: Jesse Chan, David C. Del Rey Fernández, and Mark H. Carpenter

- Decoupled SBP operators
 - Extends SBP concept to modal based methods including numerical cubature
 - Can have separate volume and surface cubature nodes
- Motivation: collocated Gauss methods are very accurate (no mass lumping)
 - GSBP Gauss result in SATs that fully couple elements (really bad for entropy stable methods)
 - Improve the accuracy of GSBP Gauss for nonlinear problems
 - Construction of nonlinear boundary conditions for GSBP Gauss not clear
- For a general exposition on decoupled SBP, see

Discretely entropy stable weight-adjusted discontinuous Galerkin methods on curvilinear meshes

(JCP, 378(1), 2019, pg. 366-393)

Entropy conservative/stable wall bc

Paper: Conservative and entropy stable solid wall boundary conditions for the compressible Navier-Stokes equations: Adiabatic wall and heat entropy transfer
<https://arxiv.org/pdf/1812.11403.pdf>

Authors: Lisandro Dalcin, Diego B. Rojas, Stefano Zampini, David C. Del Rey Fernández, Mark H. Carpenter, and Matteo Parsani

- Continuous entropy analysis (smooth solutions) $(x_1, x_2, x_3) \in [0, 1]^3$

$$\frac{d}{dt} \int_{\Omega} \mathcal{S} dx_1 dx_2 dx_3 = -DT + \int_{x_1=0} \left(f_{x_1} - \mathcal{W}^T \mathcal{F}^{(v)} \right) dx_2 dx_3$$

- Entropy conservative wall-boundary condition:

$$\mathcal{U}_1 = 0, \mathcal{U}_2 = \mathcal{G}_2(x_2, x_3, t), \mathcal{U}_3 = \mathcal{G}_3(x_2, x_3, t), \text{ and } \kappa \frac{1}{\mathcal{J}} \frac{\partial \mathcal{F}}{\partial n} = 0$$

$$\frac{d}{dt} \int_{\Omega} \mathcal{S} dx_1 dx_2 dx_3 = 0$$

- Entropy stable wall-boundary condition: $\kappa \frac{1}{\mathcal{J}} \frac{\partial \mathcal{F}}{\partial n} = \mathcal{G}(x_2, x_3, t)$

$$\frac{d}{dt} \int_{\Omega} \mathcal{S} dx_1 dx_2 dx_3 = -DT + data + \widetilde{DT}$$

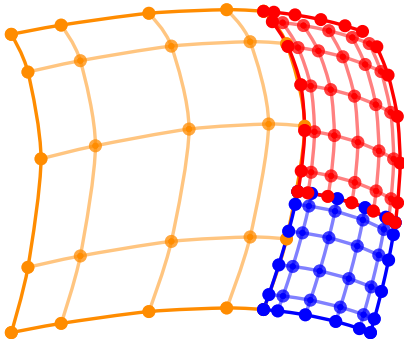
Entropy conservative/stable wall bc

- Objective: Construct numerical boundary procedures that
 - mimic the continuous entropy analysis
 - functional form identical to interface SATs
- Accomplished by penalizing against manufactured inviscid and viscous fluxes constructed from a combination of boundary data and the numerical solution

h/p entropy stable schemes

Collaborators: David C. Del Rey Fernández, Mark H. Carpenter, Lisandro Dalcin, Stefano Zampini, Lucas Friedrich, Andrew R. Winters, Gregor J. Gassner, and Matteo Parsani

h/p -adaptation



Objectives:

- Entropy conservation/stability
- Elementwise conservative
- Freestream preserving
- Simple

Concluding remarks

- Exciting time to be developing and using SBP
 - ▶ Mature and has been applied to many PDEs of interest
 - ▶ Lots of theoretical work to be done
 - ▶ Many extensions to come
- In general, provable properties should be at the core of all algorithms
 - ▶ Stability is the minimum
- What we should shoot for is provable convergence to the physically realizable solution

Thanks

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me

And to all of the wonderful people who I had the
opportunity to meet and discuss research with

Questions

- Entropy Stable p -Nonconforming Discretizations with the Summation-by-Parts Property for the Compressible Euler equations <https://arxiv.org/abs/1909.12536>
- Entropy Stable p -Nonconforming Discretizations with the Summation-by-Parts Property for the Compressible Navier–Stokes Equations <https://arxiv.org/abs/1909.12546>
- Entropy Stable h/p -Nonconforming Discretization with the Summation-by-Parts Property for the Compressible Euler and Navier–Stokes Equations <https://repository.kaust.edu.sa/handle/10754/656861>
- Entropy stable spectral collocation schemes for the 3-D Navier-Stokes equations on dynamic unstructured grids <https://arxiv.org/pdf/1812.10185.pdf>
- Entropy stable space-time discontinuous Galerkin schemes with summation-by-parts property for hyperbolic conservation laws <https://arxiv.org/pdf/1808.08218.pdf>
- Efficient entropy stable Gauss collocation methods <https://arxiv.org/pdf/1809.01178.pdf>
- Conservative and entropy stable solid wall boundary conditions for the compressible Navier-Stokes equations: Adiabatic wall and heat entropy transfer <https://arxiv.org/pdf/1812.11403.pdf>