



Delft University of Technology: Aerospace Engineering
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Boundary Closures for ESWENO Schemes

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Applications with Shocks

Transonic flows
(BL stability, transition
and turbulence)

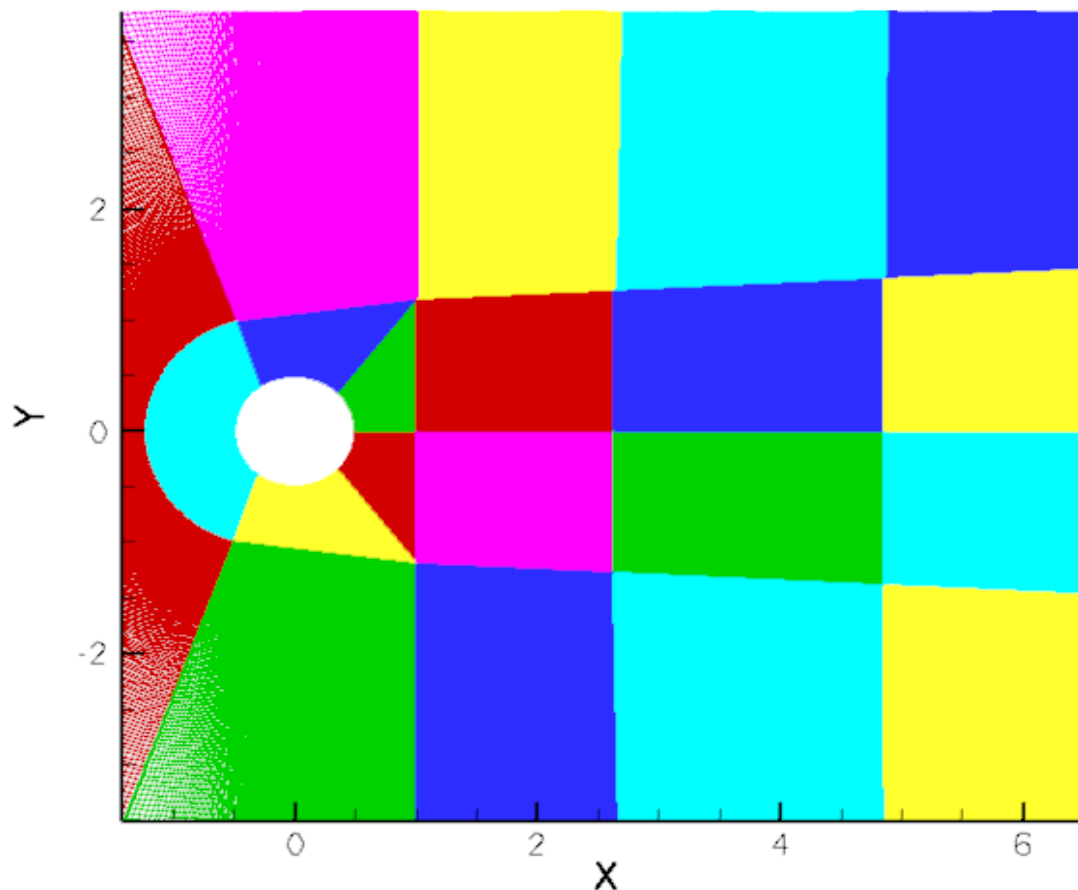
Supersonic/Hypersonic
(BL stab, trans, turb,
combustion, hypersonic
reentry, . . .)

Complex Geometries!





Multi-Block Finite-Difference





Big Picture: Complex Geometries



- WENO for Complex Geometries. Do we have a plan?
 - Block-Structure HO Finite-Difference
 - Requires:
 - C_0 interface connectivity (reasonably general)
 - Penalty interface coupling between blocks, penalty BC's
 - Essentially the FD equivalent of DG!
- Plan of attack.
 - Prove stability of WENO schemes on 1-D periodic case (ESWENO).
 - Prove stability of ESWENO including boundary closure operators.
 - Prove stability of shock crossing interface.
 - Nonlinear proofs
- To be completed by July 13, 2013



6th-order WENO Schemes:



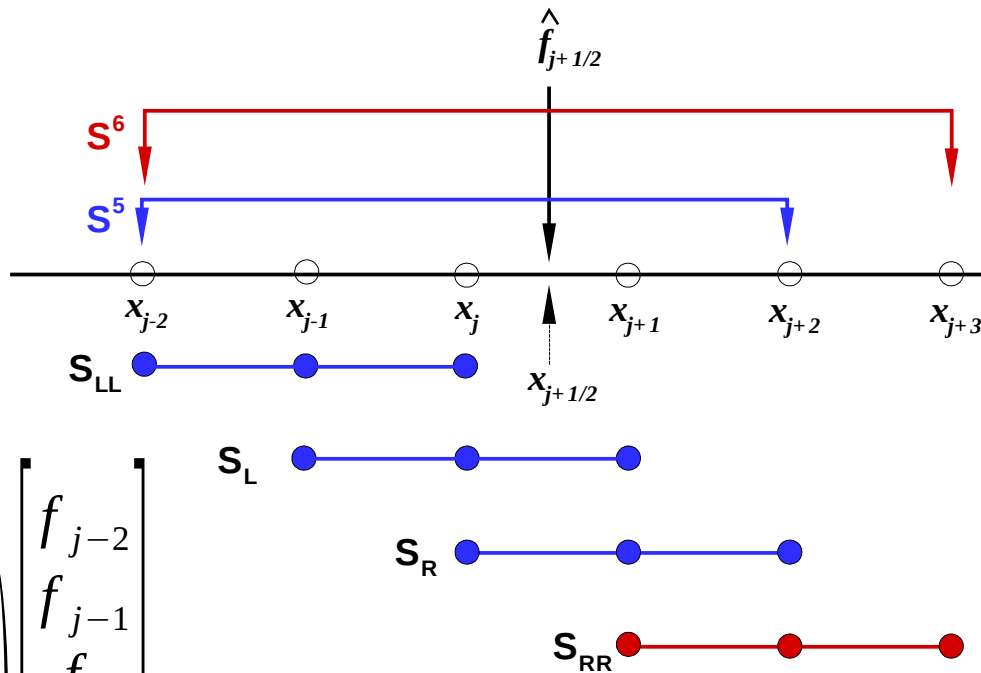
$$U_t + F_x = 0$$

$$\frac{du_j}{dt} + \frac{f_{j+1/2}^W - f_{j-1/2}^W}{\Delta x} = 0$$

The 6th-order WENO flux

$$f_{j+1/2}^W = \omega_{j+1/2}^{LL} f_{j+1/2}^{LL} + \omega_{j+1/2}^L f_{j+1/2}^L + \omega_{j+1/2}^R f_{j+1/2}^R + \omega_{j+1/2}^{RR} f_{j+1/2}^{RR}$$

$$\begin{bmatrix} f_{j+\frac{1}{2}}^{LL} \\ f_{j+\frac{1}{2}}^L \\ f_{j+\frac{1}{2}}^R \\ f_{j+\frac{1}{2}}^{RR} \end{bmatrix} = \frac{1}{6} \begin{pmatrix} 2 & -7 & 11 & 0 \\ -1 & 5 & 2 & 0 \\ 2 & 5 & -1 & 0 \\ 0 & 11 & -7 & 2 \end{pmatrix} \begin{bmatrix} f_{j-2} \\ f_{j-1} \\ f_j \\ f_{j+1} \\ f_{j+2} \\ f_{j+3} \end{bmatrix}$$





6th-order WENO Schemes



The weights should satisfy the following properties:

$$0 \leq \omega_{j+1/2}^r \leq 1, \quad \sum_r \omega_{j+1/2}^r = 1, \quad \forall j$$

$$\omega_{j+1/2}^r = d_r + O(\Delta x^{p-s+1})$$

Conventional weights of Jiang and Shu:

$$\omega_{j+1/2}^r = \frac{\alpha_r}{\sum_l \alpha_l}, \quad \alpha_r = \frac{d_r}{(\varepsilon + \beta_r)^2}, \quad \beta_r = \sum_{k=1}^{s-1} \Delta x^{2k-1} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\frac{d^k q_r(x)}{dx^k} \right)^2 dx \Rightarrow \omega^r = d^r + O(\Delta x^2)$$



Motivation (cont.)



ENO/WENO SCHEMES TO THE RESCUE

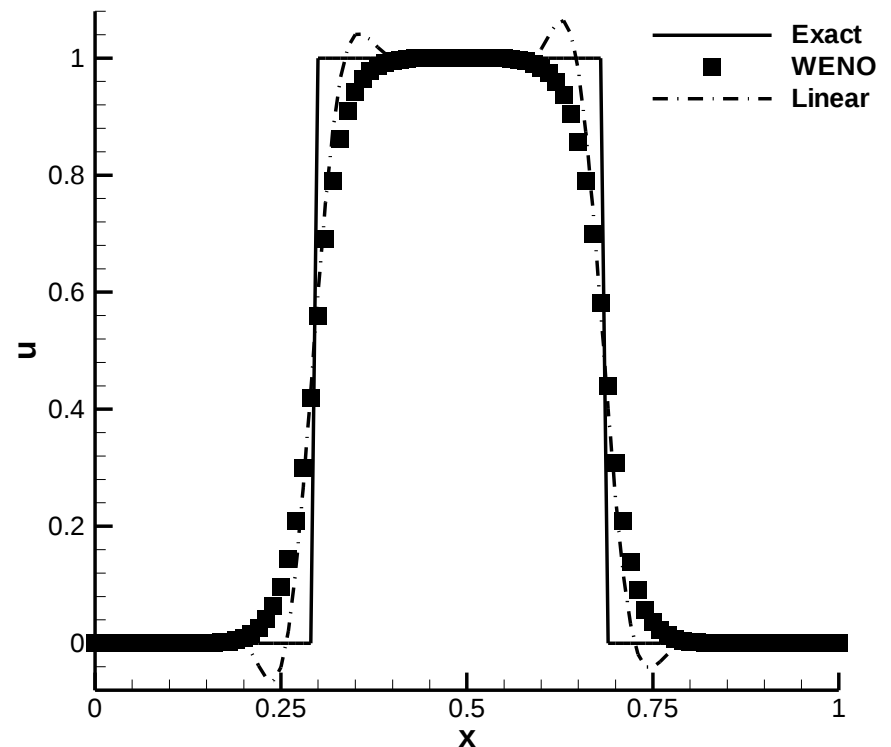
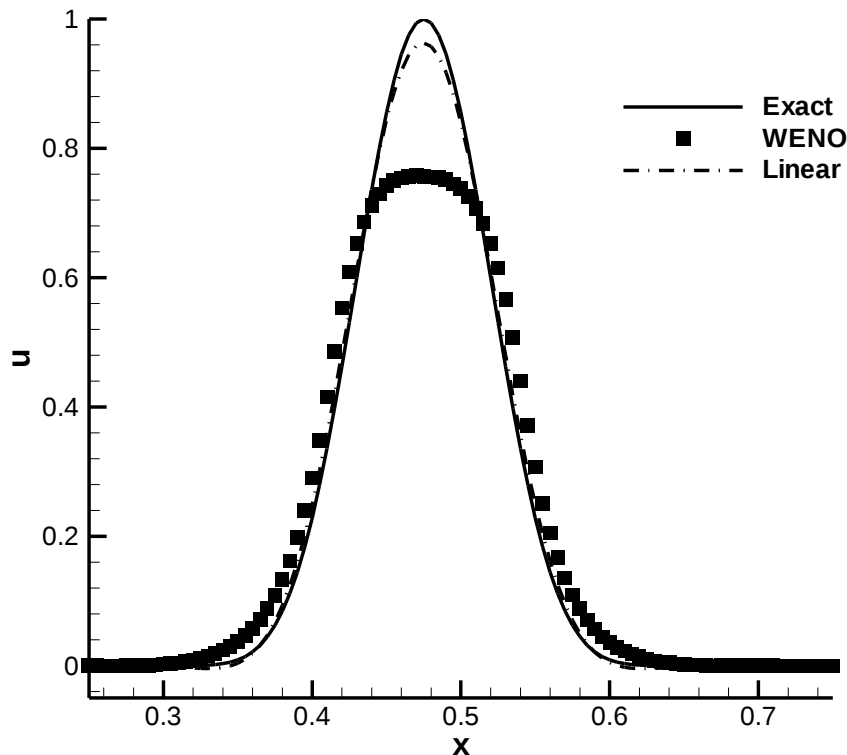
- A. Harten, B. Engquist, S. Osher, and S. Chakravarthy
“Uniformly high order essentially non-oscillatory schemes, III,” *J. of Computational Physics*, Vol. 71, pp. 231-303, 1987.
 - Adaptively chooses “smoothest” stencil from s candidates. Provides s^{th} -order accuracy.
- X.-D. Liu, S. Osher, and T. Chan
“Weighted Essentially Non-oscillatory Schemes,” *J. of Computational Physics*, Vol. 115, pp. 200-212, (1994).
 - Uses a weighted convex combination of all s candidate stencils. Provides $(s+1)^{th}$ -order accuracy.
- G. Jiang and C.-W. Shu
“Efficient implementation of weighted ENO schemes,” *J. of Computational Physics*, Vol. 126, pp. 202-228, (1996).
 - Generalization to the FD formulation. Provides $(2s-1)^{th}$ -order accuracy on sufficiently fine meshes.



Motivation



Achilles Heel of High-Order WENO Schemes



Moving pulse simulation using the 3rd-order WENO scheme



Motivation: Improvements



Improvements needed by conventional WENO:

- No stability proofs are available for discontinuous solutions of hyperbolic systems.
- The WENO schemes are too dissipative as compared with the corresponding target linear schemes
 - Typically based on upwind operators.
 - The order of the conventional WENO schemes deteriorates from $(2s-1)$ to s at critical points.
- For some steady state problems, the residual cannot be driven to the machine zero.
- Boundary stencils and extension to complex geometry



Energy Stable WENO



Energy Stable WENO (ESWENO)

- A WENO-type scheme that is stable in the L_2 norm by construction for **continuous and discontinuous solutions**

Developed New weights

- Faster convergence to the underlying linear scheme
- Improved shock-capturing capabilities

Boundary Closures for 4th-order ESWENO

- Design order accurate (3-4-3)
Full stencil biasing up to boundaries (almost)

L_2 Stability

Maintains original properties

- Design order of accuracy for smooth solutions/extrema
- Conservative (Lax-Wendroff theorem)
- Essentially nonoscillatory solutions



Mimetic Operators

- **Mimetic Operator**
 - Discrete equivalents of continuous operators
 - Could mimic “anything”
- **What should we mimic?**
 - Conventional H-O F-D suffer from instability
 - **CONTINUOUS: Conservative and (usually) stable**
 - **Energy Estimate Guarantees STABILITY**
 - **Integration-by-Parts** $\int_0^1 u v_x dx = (u v)|_0^1 - \int_0^1 v u_x dx$
 - **DISCRETE: Summation-by-parts operators**
 - H.-O. Kreiss, G. Scherer
 - **Mathematical Aspects of FE in PDE's, 1974**



Energy Estimates



Continuous problem

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \sum_{n=1}^N (-1)^{n-1} \frac{\partial^n}{\partial x^n} \left(\mu_n(u) \frac{\partial^n u}{\partial x^n} \right) \quad f = au, \quad a > 0, \quad \mu_n \geq 0$$

Multiplying Eq. (1) by u and integrating over $[0,1]$ yields

$$\frac{1}{2} \frac{d}{dt} \|u\|_{l_2}^2 + \frac{1}{2} au^2 \Big|_0^1 = \sum_{n=1}^N \int_0^1 (-1)^{n-1} u \frac{\partial^n}{\partial x^n} \left(\mu_n(u) \frac{\partial^n u}{\partial x^n} \right) dx$$

Integrating RHS by parts yields

$$\frac{d}{dt} \|u\|_{l_2}^2 + au^2 \Big|_0^1 = -2 \sum_{n=1}^N \int_0^1 \mu(u) \left(\frac{\partial^n u}{\partial x^n} \right)^2 dx \leq 0$$



Energy Estimates (cont.)



Discrete problem

$$\frac{\partial \mathbf{u}}{\partial t} + P^{-1} Q \mathbf{f} = - \sum_{n=0}^S P^{-1} D_1^n \Lambda_n [D_1^n]^T \mathbf{f}$$

Multiplying Eq. (2) by $\mathbf{u}^T P$ and adding it to its transpose yields

$$\frac{d}{dt} \|\mathbf{u}\|_P^2 + a \mathbf{u}^T (Q + Q^T) \mathbf{u} = -a \sum_{n=0}^S \left([D_1^n]^T \mathbf{u} \right)^T \left(\Lambda_n + \Lambda_n^T \right) D_1^n]^T \mathbf{u}$$

What if:

$$P = P^T ; \psi^T P \psi > 0 ; Q + Q^T = \text{Diag}[-1, 0, \dots, 0, 1] ; \psi^T (\Lambda + \Lambda^T) \psi \geq 0$$

$$\frac{\partial f}{\partial x} = P^{-1} Q \mathbf{f} + \sum_{n=0}^S P^{-1} D_1^n \Lambda_n [D_1^n]^T \mathbf{f} + O(\Delta x)^{2s}$$

Then

$$\frac{d}{dt} \|\mathbf{u}\|_P^2 + a \mathbf{u}^2|_0^1 = -2 \sum_{n=0}^N \left([D_1^n]^T \mathbf{u} \right)^T \Lambda_n \left([D_1^n]^T \mathbf{u} \right) \leq 0$$



6th-order WENO Schemes: Are They Stable?



Can WENO be represented as $\frac{\partial \mathbf{u}}{\partial t} + P^{-1} Q \mathbf{f} = - \sum_{n=0}^S P^{-1} D_1^n \Lambda_n [D_1^n]^T \mathbf{f}$?

The 6th-order WENO operator: $[D_W \mathbf{f}]_j = \frac{f_{j+1/2}^W - f_{j-1/2}^W}{\Delta x}$

$$D_W = \frac{1}{2} (D_W - D_W^T) + \frac{1}{2} (D_W + D_W^T) = P^{-1} Q + P^{-1} D_W^{\text{sym}}$$

Is D_W^{sym} Stable? Find a constructive stability test????

$$D_W^{\text{sym}} = P^{-1} M (A + A^T) M^T$$

$$D_W^{\text{sym}} = P^{-1} \left(D_1^3 \Lambda_3^W [D_1^3]^T + D_1^2 \Lambda_2^W [D_1^2]^T + D_1^1 \Lambda_1^W [D_1^1]^T + D_1^0 \Lambda_0^W [D_1^0]^T \right)$$

$$\Lambda_3^W = \frac{1}{6} \text{diag} [\omega_{j+5/2}^{LL} - \omega_{j+5/2}^{RR}]$$

$$\Lambda_2^W = \frac{1}{12} \text{diag} [\omega_{j+3/2}^{LL} - 4\omega_{j+5/2}^{LL} + \omega_{j+3/2}^L - \omega_{j+1/2}^R + 4\omega_{j-1/2}^{RR} - \omega_{j+1/2}^{RR}]$$

$$\Lambda_1^W = \frac{1}{12} \text{diag} [3\omega_{j+1/2}^{LL} - 5\omega_{j+3/2}^{LL} + 2\omega_{j+5/2}^{LL} + \omega_{j+1/2}^L - \omega_{j+3/2}^L + \omega_{j-1/2}^R - \omega_{j+1/2}^R - 2\omega_{j-3/2}^{RR} + 5\omega_{j-1/2}^{RR} - 3\omega_{j+1/2}^{RR}]$$

$$\Lambda_0^W = \frac{1}{2} \text{diag} [-\omega_{j-1/2}^{LL} - \omega_{j-1/2}^L - \omega_{j-1/2}^R - \omega_{j-1/2}^{RR} + \omega_{j+1/2}^{LL} + \omega_{j+1/2}^L + \omega_{j+1/2}^R + \omega_{j+1/2}^{RR}] = 0$$



6th-order WENO Schemes



Conventional WENO schemes:

- No energy estimate/stability proof (that we could find)
- The WENO dissipation operator is **not positive semi-definite**, because for unresolved solutions $\Lambda_n^W < 0, n = \overline{1,3}$!
- For discontinuous or unresolved solutions, the WENO artificial dissipation operator $-P^{-1} \sum_{n=1}^3 D_1^n \Lambda_n^W [D_1^n]^T$ could have positive eigenvalues.



6th-order ESWENO Schemes



Stability

ESWENO scheme has ***all the terms*** of the 6th-order WENO scheme and also includes ***an additional artificial dissipation term***

$$\frac{\partial \mathbf{u}}{\partial t} + P^{-1} Q \mathbf{f} = - \sum_{n=0}^3 P^{-1} D_1^n \Lambda_n^{ES} [D_1^n]^T \mathbf{f}$$

Central operator: $P = \Delta x I, \quad Q = P D_W^{skew}$

Additional artificial dissipation operator: $\bar{D}_{ad} = -P^{-1} \sum_{n=1}^3 D_1^n \bar{\Lambda}_n [D_1^n]^T$

$$(\bar{\lambda}_n)_{j,j} = \frac{1}{2} \left[\sqrt{(\lambda_n^W)_{j,j}^2 + \delta_n^2} - (\lambda_n^W)_{j,j} \right]$$

$$D^{ES} = D^W + \bar{D}_{ad}$$

$$(\lambda_n^{ES})_{j,j} = (\lambda_n^W)_{j,j} + (\bar{\lambda}_n)_{j,j} = \frac{1}{2} \left[\sqrt{(\lambda_n^W)_{j,j}^2 + \delta_n^2} + (\lambda_n^W)_{j,j} \right] > 0 \text{ for any } u_j \text{ and } \forall j$$



6th-order ESWENO Schemes



The weights should satisfy the following properties:

$$0 \leq \omega_{j+1/2}^r \leq 1, \quad \sum_r \omega_{j+1/2}^r = 1, \quad \forall j$$

$$\omega_{j+1/2}^r = d_r + O(\Delta x^{p-s+1})$$

Conventional weights of Jiang and Shu:

$$\omega_{j+1/2}^r = \frac{\alpha_r}{\sum_l \alpha_l}, \quad \alpha_r = \frac{d_r}{(\varepsilon + \beta_r)^2}, \quad \beta_r = \sum_{k=1}^{s-1} \Delta x^{2k-1} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\frac{d^k q_r(x)}{dx^k} \right)^2 dx \Rightarrow \omega^r = d^r + O(\Delta x^2)$$

New weights with Smooth Consistency

$$\omega_{j+1/2}^r = \frac{\alpha_r}{\sum_k \alpha_k}, \quad \alpha_r = d_r \left(1 + \frac{\tau_5}{\varepsilon + \beta_r} \right), \quad r = \{LL, L, R, RR\}$$

$$\tau_5 = \begin{cases} \left(-f_{j-2} + 5f_{j-1} - 10f_j + 10f_{j+1} - 5f_{j+2} + f_{j+3} \right)^2, & \text{for } \varphi \neq 0 \\ \left(f_{j-2} - 4f_{j-1} + 6f_j - 4f_{j+1} + f_{j+2} \right)^2, & \text{for } \varphi = 0 \end{cases} \Rightarrow \omega^r = d^r + O(\Delta x^8)$$
$$\Rightarrow \omega^r = d^r + O(\Delta x^6)$$



6th-order ESWENO Schemes



Consistency

Sufficient condition to attain p^{th} -order accuracy:

$$\omega^r = d_r + O\left(\Delta x^{p-s+1}\right)$$

For 5th-order schemes ($p=5$): $\omega^r = d_r + O\left(\Delta x^3\right)$

For 6th-order scheme ($p=6$): $\omega^r = d_r + O\left(\Delta x^4\right)$

This is a stiffer constraint than one obtained by Jiang and Shu!

$$\omega^r = d_r + O\left(\Delta x^2\right)$$

The 5th-order WENO scheme with the conventional weights deteriorates to 3rd-order near local extrema!



6th-order ESWENO Scheme



Consistency (cont.)

If the tuning parameters ε and δ_n of the ESWENO scheme

$$\omega_{j+1/2}^r = \frac{\alpha_r}{\sum_k \alpha_k}, \quad \alpha_r = d_r \left(1 + \frac{\tau_p}{\varepsilon + \beta_r} \right), \quad (\bar{\lambda}_n)_{j,j} = \frac{1}{2} \left[\sqrt{(\lambda_n^w)_{j,j}^2 + \delta_n^2} - (\lambda_n^w)_{j,j} \right]$$

satisfy the following conditions:

$$O(\Delta x^{p+s-1}) \leq \varepsilon \leq O(\Delta x^2) \quad \delta_n = O(\Delta x^{p-2n+1}), \quad 1 \leq n \leq s$$

then the ESWENO schemes

- are p th-order accurate **for smooth solutions with any number of vanishing derivatives.**
- provide good shock-capturing capabilities



How to choose the tuning parameters

For solutions containing spurious oscillations $f = f_0 + \delta f$

$$\beta_r = O(\delta f^2) \Rightarrow \alpha_r = d_r \left(1 + \frac{\tau_5}{\varepsilon + \beta_r} \right) = d_r \left(1 + \frac{\tau_5}{\varepsilon + O(\delta f^2)} \right)$$

If $\varepsilon \gg O(\delta f^2)$, then the spurious oscillations cannot be detected.

If $\varepsilon \leq O(\delta f^2)$, then ω^r deviates from $d_r \Rightarrow$ the dissipation increases at the r^{th} stencil.

ε and β_r have to be scaled consistently!

Based on these considerations, the tuning parameters are set to

$$\varepsilon = \begin{cases} C\Delta\xi^{3s-4}, & \text{for } \varphi = 0 \\ C\Delta\xi^{3s-3}, & \text{for } \varphi = \varphi_c \\ C\Delta\xi^{3s-2}, & \text{otherwise} \end{cases}$$

$$\delta_n = \Delta\xi^{p-2n+1}, 1 \leq n \leq s$$

$$\Delta\xi = \frac{1}{J}$$

$$C = \max_{\xi \neq \xi_d} \left(\|u_0^2\|, \|u_0'^2\|, \dots, \|(u_0^{(s-1)})^2\| \right)$$

This eliminates the ambiguity in determining ε and δ_n !



6th-order ESWENO Schemes



6th-ORDER ESWENO PROCEDURE

- Form the WENO flux:
$$f_{j+1/2}^W = \omega_{j+1/2}^{LL} f_{j+1/2}^{LL} + \omega_{j+1/2}^L f_{j+1/2}^L + \omega_{j+1/2}^R f_{j+1/2}^R + \omega_{j+1/2}^{RR} f_{j+1/2}^{RR}$$
- Compute the additional dissipation flux $e_{j+1/2}$:

$$e_{j+1/2} = P^{-1} \left(D_1^2 \Lambda_3 [D_1^3]^T + D_1^1 \Lambda_2 [D_1^2]^T + D_1^0 \Lambda_1 [D_1^1]^T \right) \mathbf{f}_{j+1/2}$$

$$\bar{\lambda}_n = \frac{1}{2} \left[\sqrt{\lambda_n^2 + \delta_n^2} - \lambda_n \right], \quad 1 \leq n \leq s$$

- Form the ESWENO flux:
$$f_{j+1/2}^{ES} = f_{j+1/2}^W + e_{j+1/2}$$

- Form the ESWENO scheme and update the solution:
$$\frac{du_j(t)}{dt} + \frac{f_{j+1/2}^{ES} - f_{j-1/2}^{ES}}{\Delta x} = 0$$



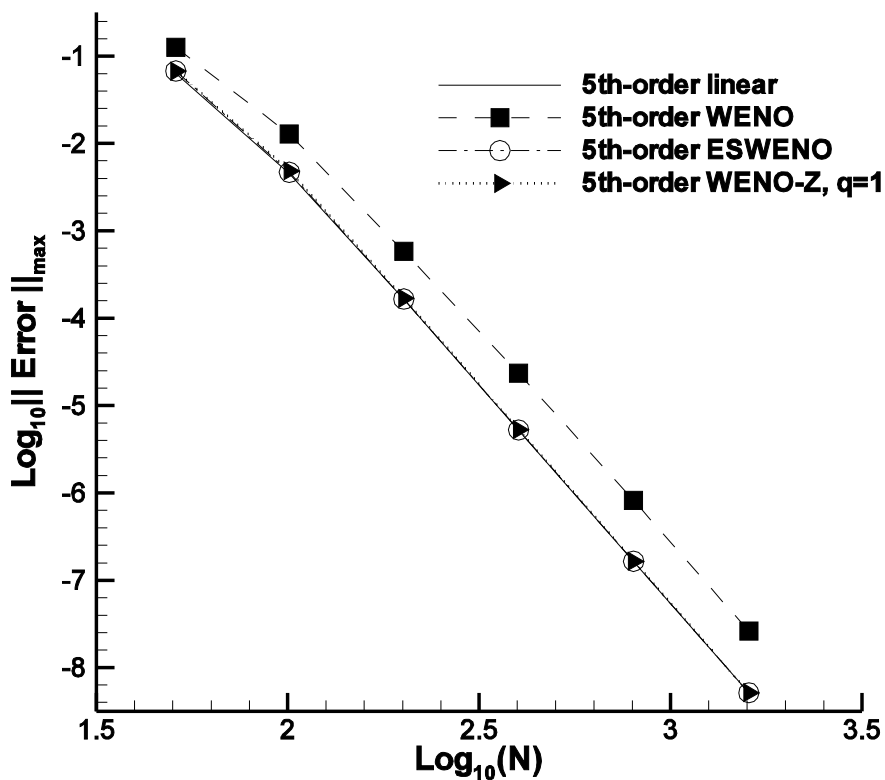
Numerical Results

Linear Wave Equation

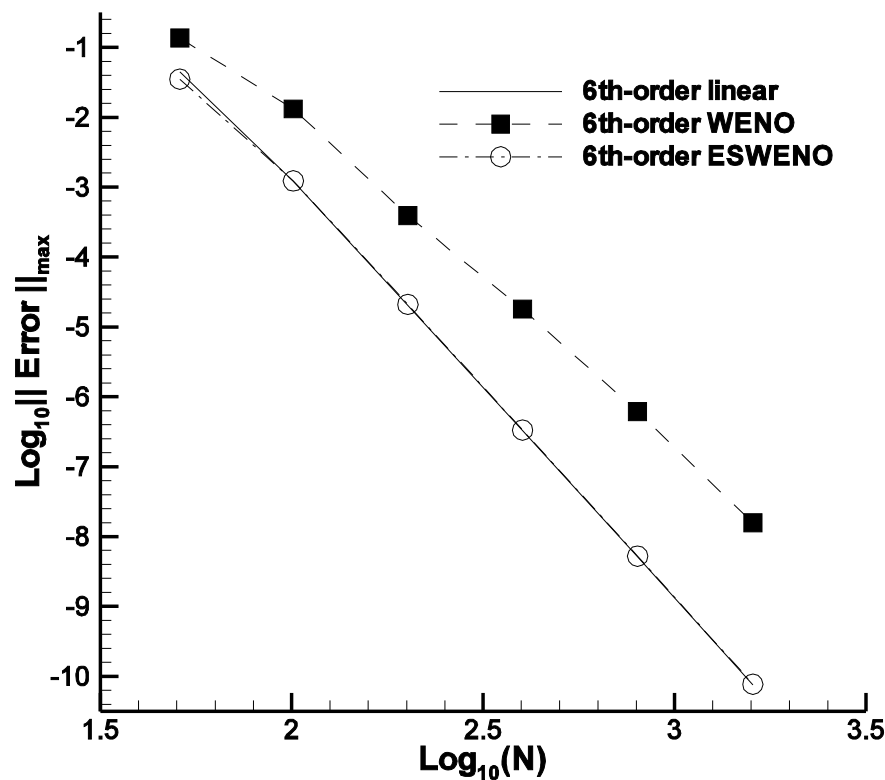


The 5th- and 6th-order WENO and ESWENO schemes (Gaussian pulse)

L_∞ error norm

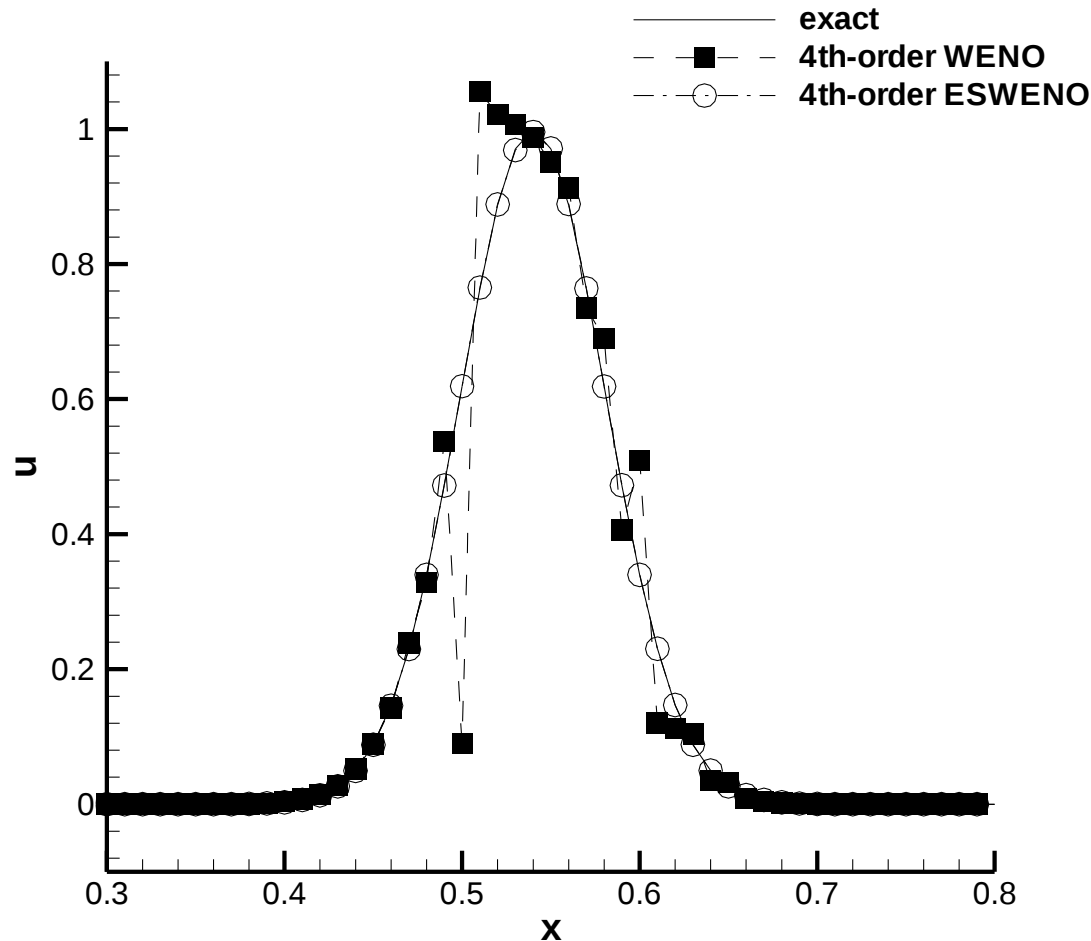


L_∞ error norm





6th-order WENO Schemes



Solution obtained with the 4th-order WENO scheme for the linear wave equation with initial condition $u_0(x) = e^{-300(x-0.5)^2}$



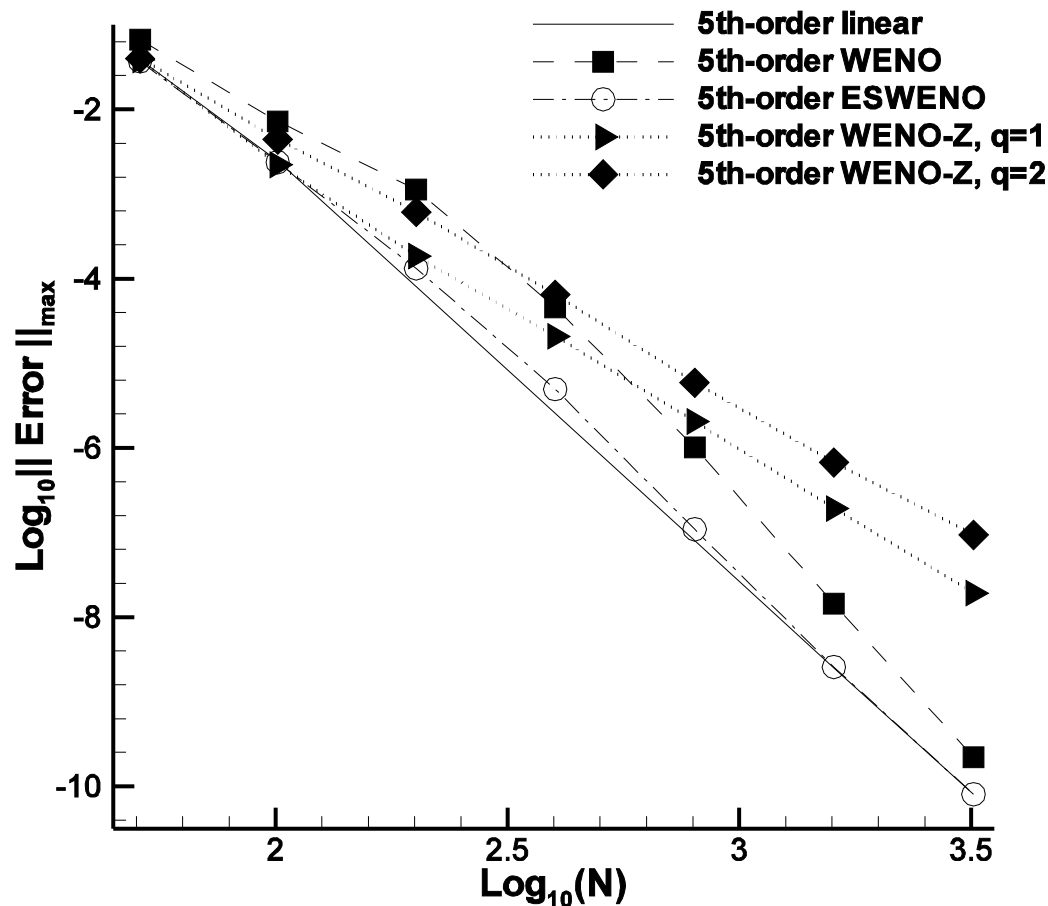
Numerical Results (cont.)



Linear Wave Equation

The 5th-order WENO and ESWENO schemes (solution with critical points)

L_{∞} error norm



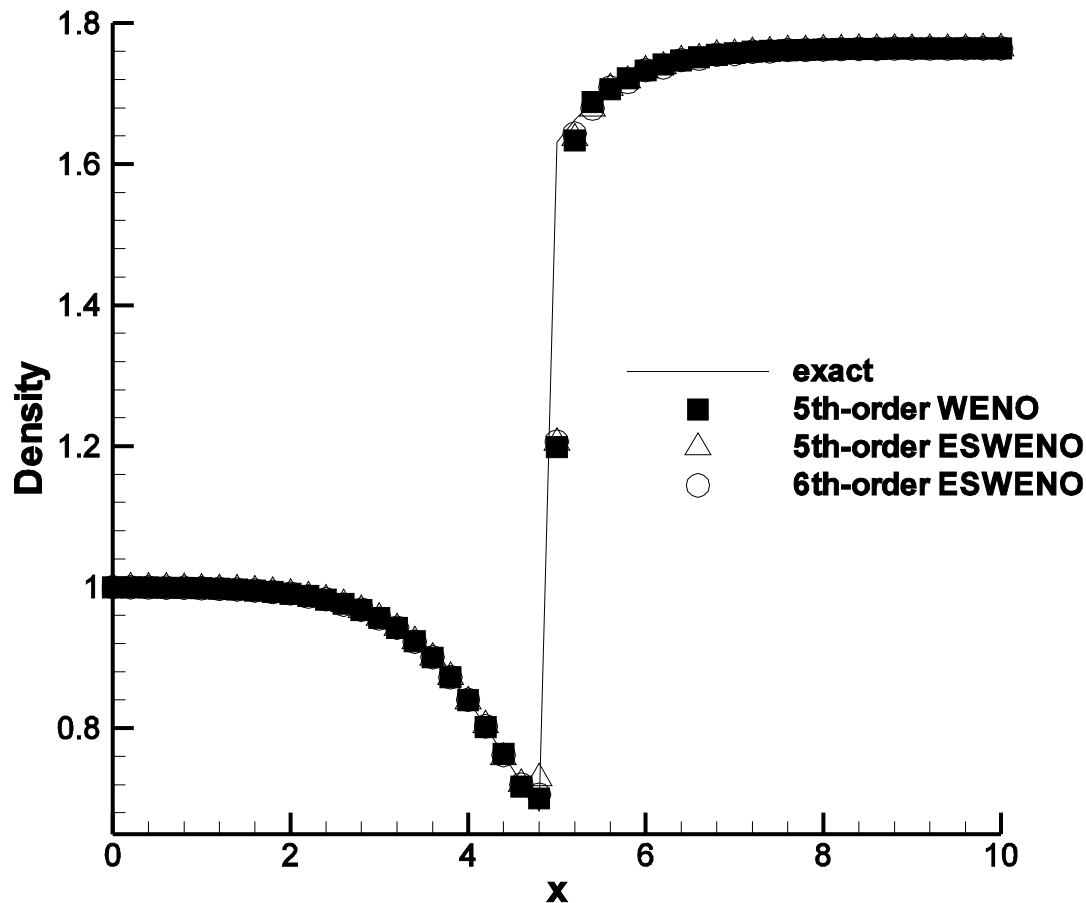


Numerical Results (cont.)

Steady Quasi-1-D Euler Equations



Quasi-1-D nozzle problem, 5th and 6th-order WENO and ESWENO schemes (Mach=1.5, $J=51$ grid points)





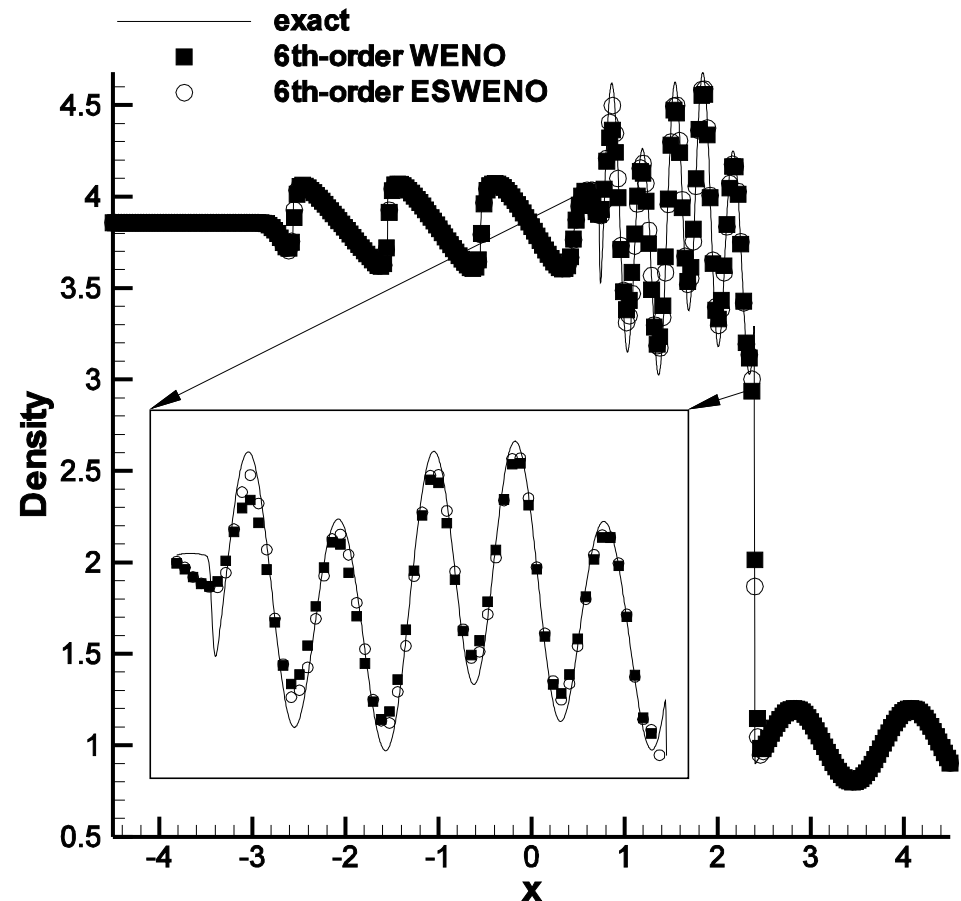
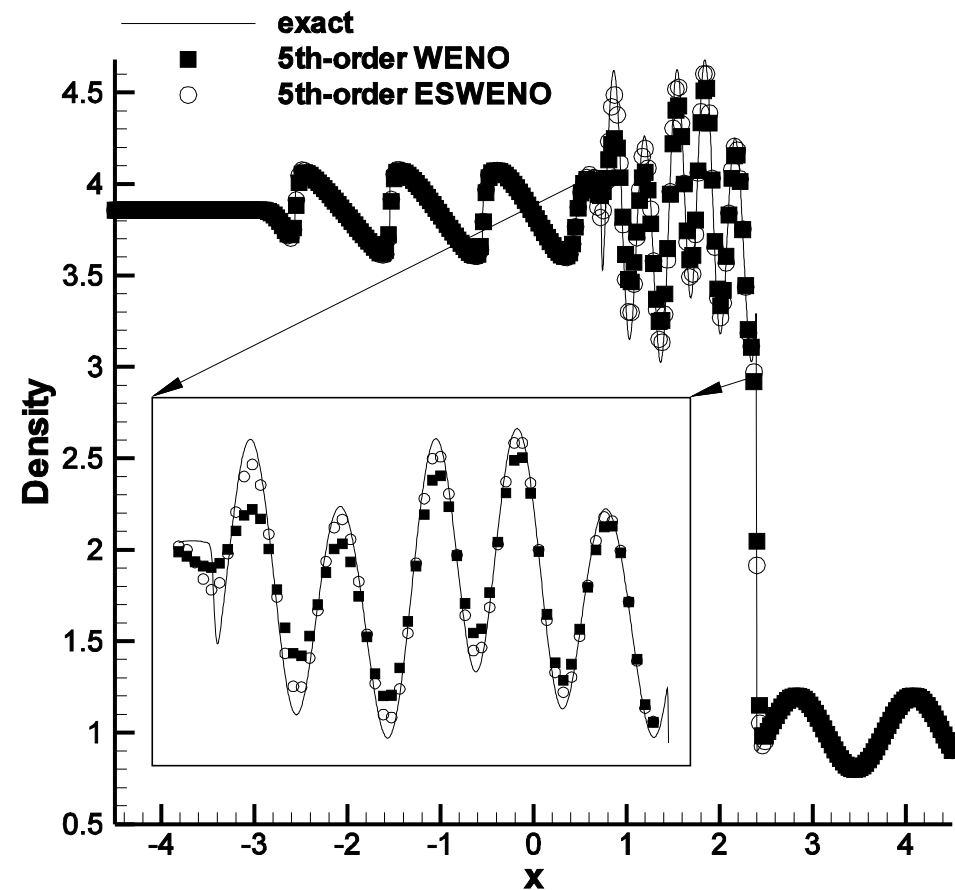
Numerical Results (cont.)

Unsteady 1-D Euler Equations

The



Shock/acoustic wave interaction problem , 5th and 6th-order WENO and ESWENO schemes ($J=300$ grid cells)





Boundary Closures: The Plan?



- **Goals**
 - **Extend conventional WENO stencil biasing to boundary**
 - **4th-order: only volume next to wall has reduced bias**
 - **Express WENO background target operator in SBP form**
 - **Resolved flow then operator reduces to SBP operator**
 - **Prove stability/accuracy of target operator**
- **Plan**
 - **Derive general “4th-order” SBP boundary closure**
 - **Express stencil biasing mechanics in Matrix form**
 - **Equate the two forms (matrices) and see what happens**



Boundary Closures: The Plan Summation-By-Parts (3-4-3)



- **Boundary stencils of $O(\Delta x^{p-1}) \Rightarrow O(\Delta x^p)$ Global accuracy**
 - **B. Gustafsson, Math. Comp. 29, 130, 1975**
- **Summation-by-parts operators**
 - **H.-O. Kreiss, G. Scherer, Math. Aspects FE PDEs 1974**
 - **B. Strand, J. Comp. Phy., 110, 1, 1994. For example**

$$P = P^T ; \psi^T P \psi > 0 ; Q + Q^T = \text{Diag}[-1, 0, \dots, 0, 1] ;$$

$$P = \begin{pmatrix} \frac{1}{4} & \frac{19}{144} & \frac{-11}{216} & \frac{-7}{216} & 0 & 0 \\ \frac{19}{144} & \frac{65}{54} & \frac{5}{108} & \frac{23}{216} & 0 & 0 \\ \frac{-11}{216} & \frac{5}{108} & \frac{91}{108} & \frac{-7}{432} & 0 & 0 \\ \frac{-7}{216} & \frac{23}{216} & \frac{-7}{432} & \frac{55}{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; Q = \begin{pmatrix} \frac{-1}{2} & \frac{593}{864} & \frac{-37}{216} & \frac{-13}{864} & 0 & 0 \\ \frac{-593}{864} & 0 & \frac{191}{288} & \frac{5}{216} & 0 & 0 \\ \frac{37}{216} & \frac{-191}{288} & 0 & \frac{497}{864} & \frac{-1}{12} & 0 \\ \frac{13}{864} & \frac{-5}{216} & \frac{-497}{864} & 0 & \frac{2}{3} & \frac{-1}{12} \\ 0 & 0 & \frac{1}{12} & \frac{-2}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{12} & \frac{-2}{3} & 0 \end{pmatrix}$$



Boundary Closures: SBP (3-4-3) General Parameterization



- Pentadiagonal 4th-order:** $Q_i = \text{Pentadiagonal}[\frac{1}{12}, \frac{-8}{12}, 0, \frac{8}{12}, \frac{-1}{12}]$;

$$P = \Delta_X \begin{pmatrix} P_0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_0^{PT} \end{pmatrix} ; \quad Q = \begin{pmatrix} Q_0 & Q_d & 0 \\ -Q_d & Q_i & Q_d \\ 0 & -Q_d^T & -Q_0^{PT} \end{pmatrix}$$

$$P_0 = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{pmatrix} ; \quad Q_0 = \begin{pmatrix} \frac{-1}{2} & q_{12} & q_{13} & q_{14} \\ q_{12} & 0 & p_{23} & q_{24} \\ q_{13} & q_{23} & 0 & q_{34} \\ q_{14} & q_{24} & q_{34} & 0 \end{pmatrix} ; \quad Q_d = \begin{pmatrix} \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \frac{-1}{12} & 0 & \dots & 0 \\ \frac{8}{12} & \frac{-1}{12} & 0 & \dots \end{pmatrix} ;$$

- General (3-4-3) two parameter solution after satisfying constraints on matrix properties and accuracy conditions**

$$P = P^T ; \quad \psi^T P \psi > 0 ; \quad Q + Q^T = \text{Diag}[-1, 0, \dots, 0, 1] ;$$

$$P_{3-4-3} = P_{3-4-3}(\alpha_1, \alpha_2) ; \quad Q_{3-4-3} = Q_{3-4-3}(\alpha_1, \alpha_2)$$



WENO Stencil Biasing Mechanics



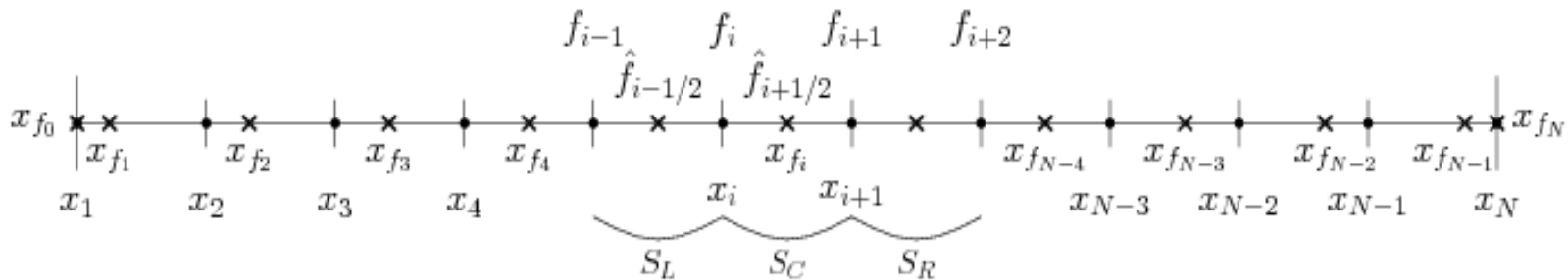
$$U_t + F_x = 0$$

$$\frac{du_j(t)}{dt} + \frac{f_{j+1/2}^{ES} - f_{j-1/2}^{ES}}{\Delta x} = 0$$

- Complementary Grids: Solution grid and Flux grid**

$$\mathbf{x}_N = [x_1, x_2, \dots, x_{N-1}, x_N]^T ; \quad \bar{\mathbf{x}} = [\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{N-1}, \bar{x}_N]^T$$

$f^+ \longrightarrow$
 $\longleftarrow f^-$



$$f_{j+1/2}^{ES} = \bar{\omega}_{j+1/2}^L \bar{f}_{j+1/2}^L + \bar{\omega}_{j+1/2}^C \bar{f}_{j+1/2}^C + \bar{\omega}_{j+1/2}^R \bar{f}_{j+1/2}^R$$

$$\bar{f}^L = I_{fs}^L \mathbf{f} ; \quad \bar{f}^C = I_{fs}^C \mathbf{f} ; \quad \bar{f}^R = I_{fs}^R \mathbf{f}$$



Expressing Flux Target operator in SBP Form



Flux form:

$$\frac{\mathbf{f}_{j+1/2}^{ES} - \mathbf{f}_{j-1/2}^{ES}}{\Delta x} = [\delta \bar{\mathbf{x}}]^{-1} D_1 \bar{\mathbf{f}}^{ES}$$

$$D_1 = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

SBP form:

$$P^{-1} Q \mathbf{f}$$

Equating the two forms yields

$$[\delta \bar{\mathbf{x}}]^{-1} D_1 \bar{\mathbf{f}}^{ES} = [\delta \bar{\mathbf{x}}]^{-1} D_1 [\bar{\omega}_t^L I_{fs}^L + \bar{\omega}_t^C I_{fs}^C + \bar{\omega}_t^R I_{fs}^R] \mathbf{f} = P^{-1} Q$$

Solve and you're done!

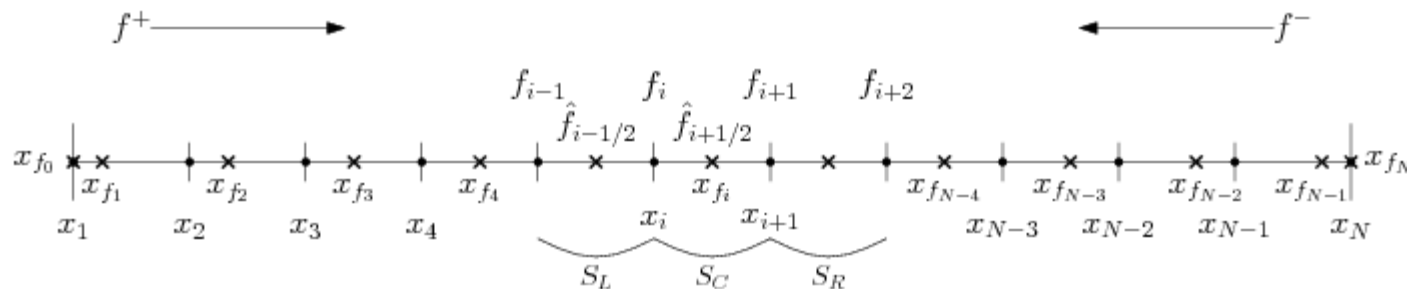


Expressing Flux Target operator in SBP Form



$$[\delta \bar{x}]^{-1} D_1 \bar{f}^{ES} = [\delta \bar{x}]^{-1} D_1 [\bar{\omega}_t^L I_{fs}^L + \bar{\omega}_t^C I_{fs}^C + \bar{\omega}_t^R I_{fs}^R] f = P^{-1} Q$$

- **Mathematica could not find solution for uniform meshes**
 - **Inconsistent constraints on system**
 - **Flux mesh must be consistent with P-norm used in proof**
- **Equating two matrices (requires??) nonuniform flux points**



$$\bar{x} = \Delta x \left[0, \frac{43}{144}, \frac{244}{144}, \frac{349}{144}, \frac{7}{2}, \dots, n-1 - \frac{349}{144}, n-1 - \frac{244}{144}, n-1 - \frac{43}{144}, n-1 \right]$$

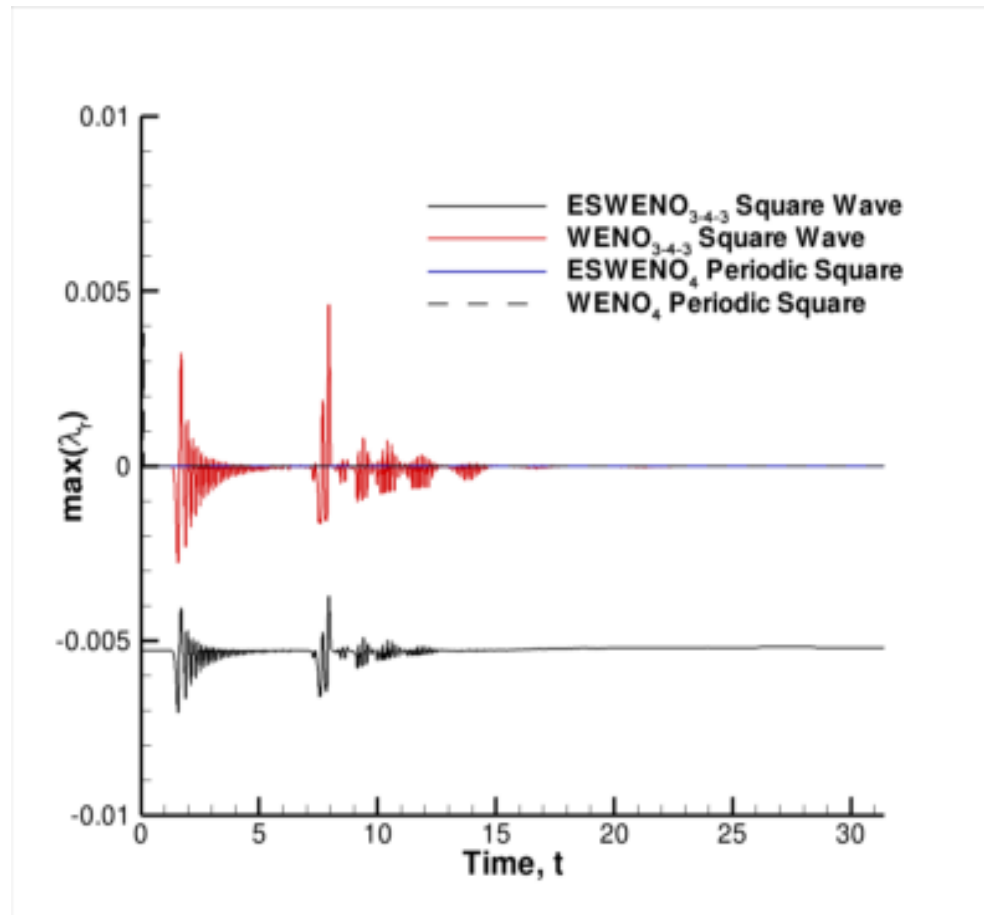


Numerical Results: Finite Domain

Eigenvalues WENO vs. ESWENO



Eigenvalues for finite domain 4th-order case (3-4-3).



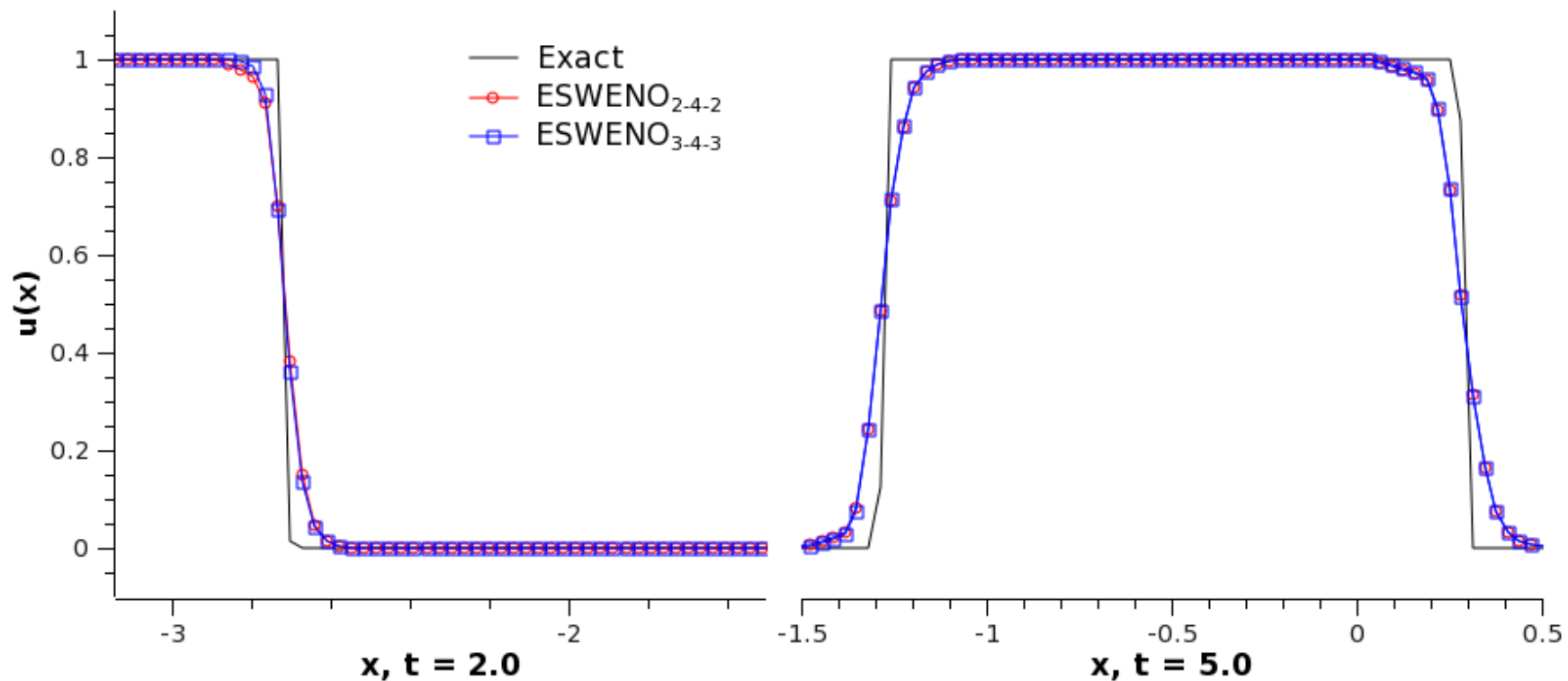


Numerical Results: Finite Domain

2nd and 3rd-order Boundary Closures



Square Wave coming into domain. ESWENO 4th:(3-4-3)



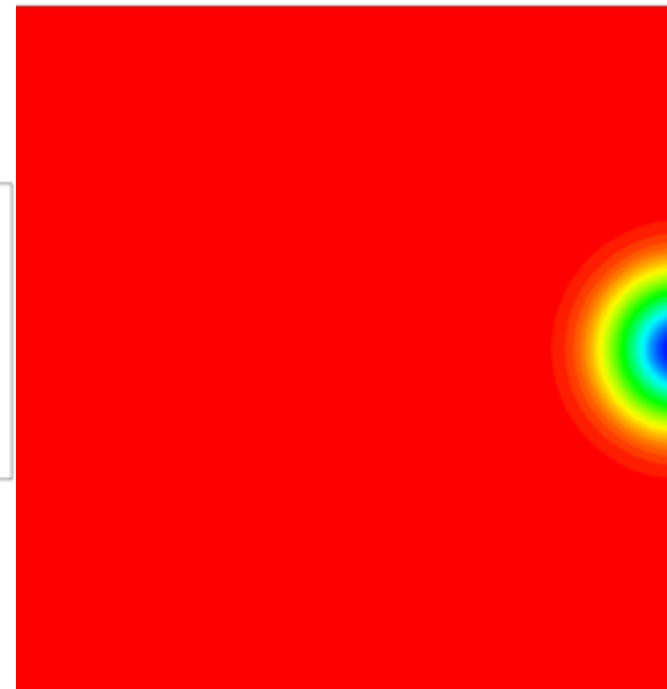
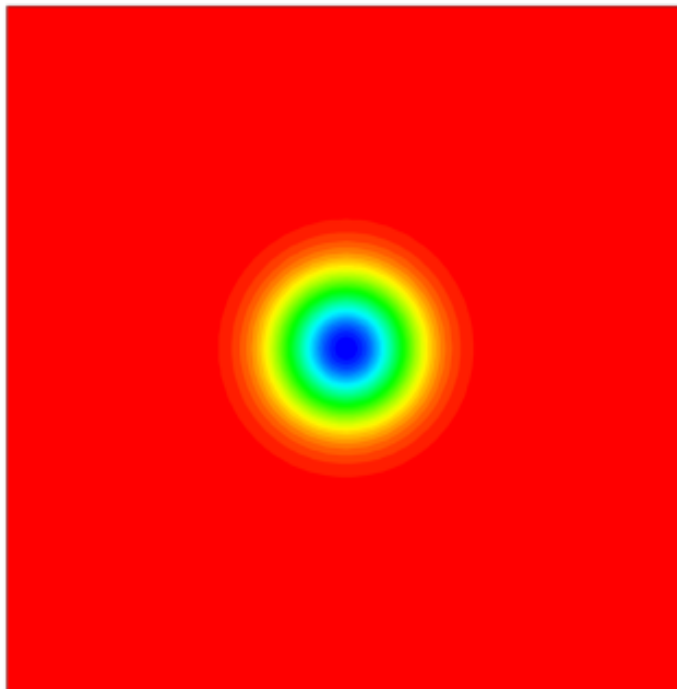


Numerical Results: Finite Domain

Euler isotropic vortex: ESWENO 4th(3-4-3)



Number of Cells	Linear Block Norm		ESWENO ₃₋₄₋₃	
	L ₂ Error	L ₂ Rate	L ₂ Error	L ₂ Rate
50 × 50	2.49E-05	-	5.32E-05	-
100 × 100	1.64E-06	3.92	2.25E-06	4.57
200 × 200	1.04E-07	3.98	1.08E-07	4.37
400 × 400	6.57E-09	3.99	6.62E-09	4.04



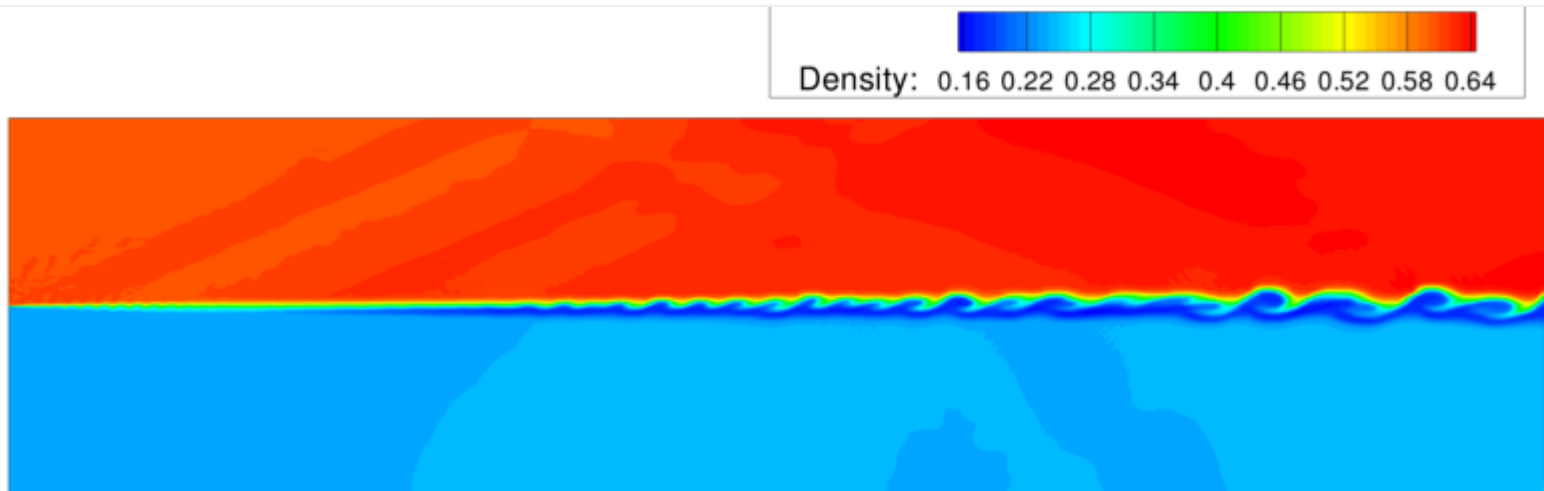
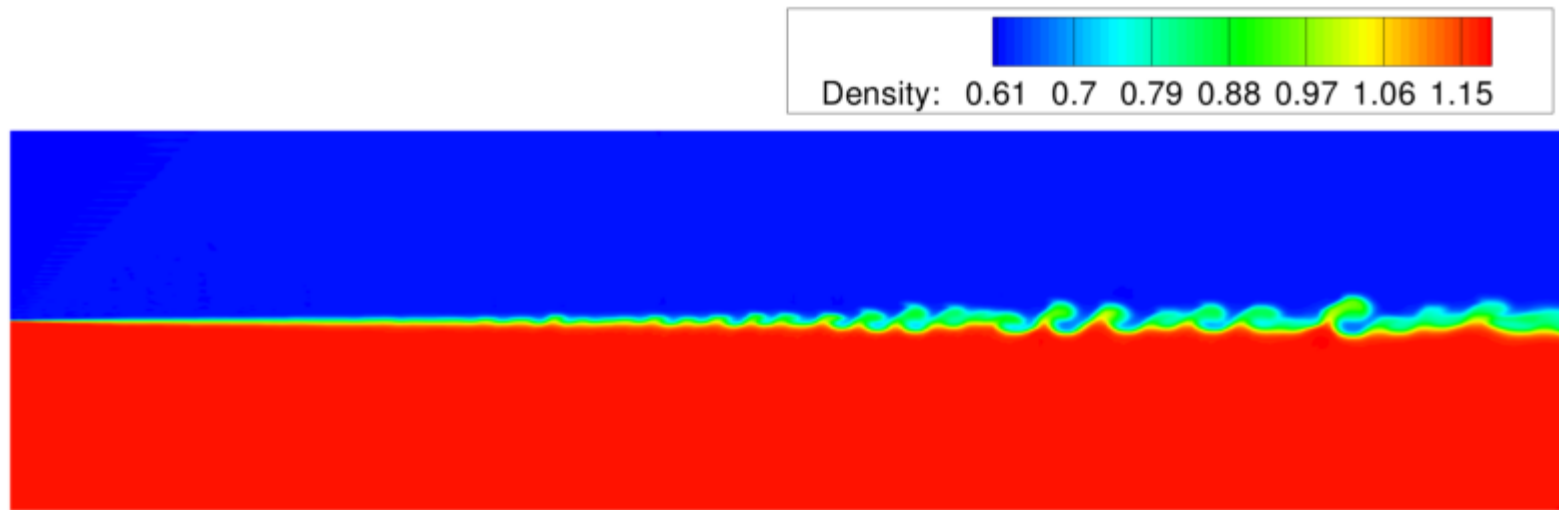


Numerical Results: Finite Domain

4th-order *ESWENO* for Chemistry



Compare Non-reacting and Reacting Mixing Layers



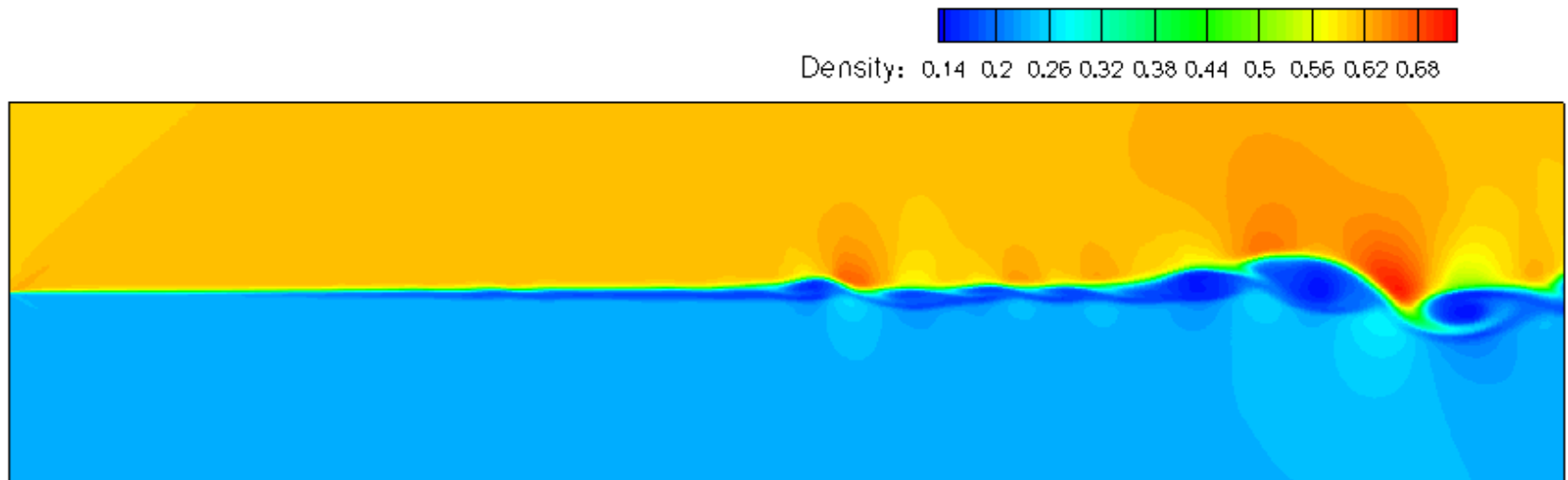
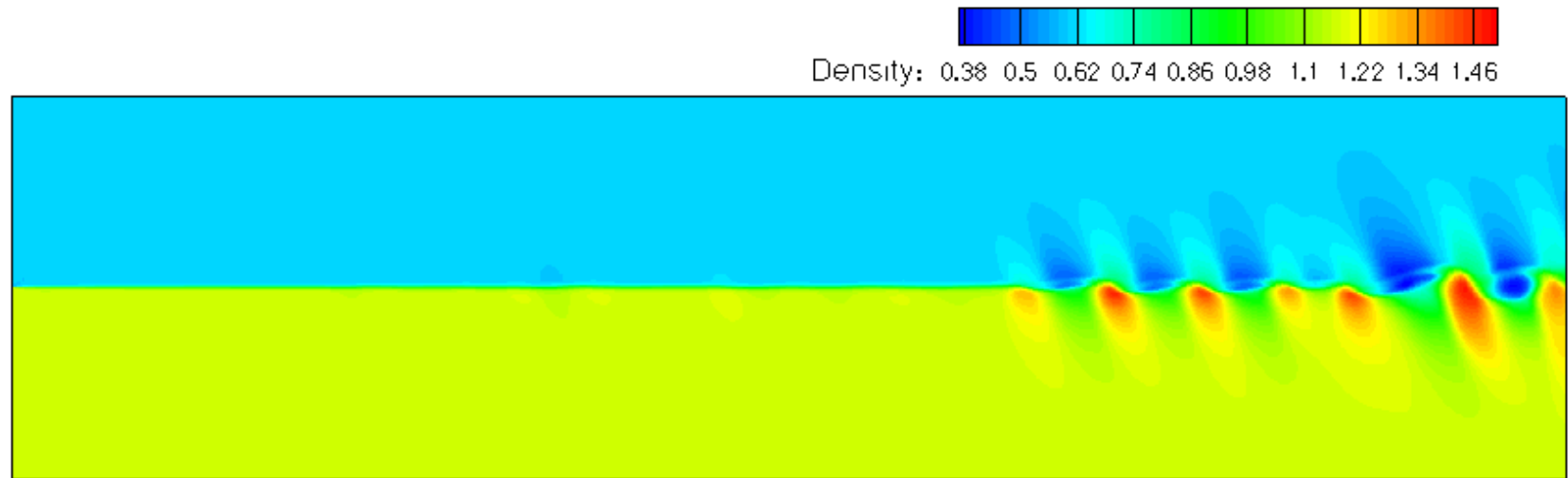


Numerical Results: Finite Domain

4th-order *ESWENO* for Chemistry



Compare Non-reacting and Reacting Mixing Layers





Conclusions



- The systematic methodology for constructing fourth-order finite domain **Energy Stable WENO schemes** is developed.
- We prove that for hyperbolic systems, the finite domain **ESWENO** scheme is **stable in the energy norm** for both continuous and **discontinuous** solutions.
- The eigenvalues of the finite domain ESWENO dissipation operator are located in the left-half plane.
- Based on the rigorous truncation error analysis, **the new weight functions** are developed, which **drastically improve the accuracy** of the ESWENO scheme and provide **excellent shock-capturing capabilities**
- Numerical experiments show that the new finite domain ESWENO scheme with the new weights outperform the conventional WENO schemes in terms of accuracy.