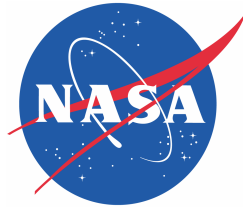


FUN3D Overview Part IV: Turbulence Model Discretization

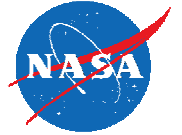


Eric J. Nielsen and James L. Thomas
FUN3D Development Team
NASA Langley Research Center

Technical Interchange with Sandia National Labs
March 8-9, 2011

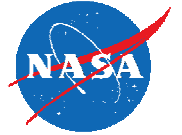
<http://fun3d.larc.nasa.gov>

General Background



- The models considered here take the form of convection-diffusion equation(s) with source terms
- The equations are discretized in a manner similar to the mean flow in both space and time, except:
 - First-order convective terms (therefore nearest neighbor stencil)
 - Source terms use Green-Gauss gradients
- Solved in a loosely-coupled fashion with the mean flow equations at each time step
 - Tend to use CFL numbers roughly one order of magnitude smaller than mean flow
- Primary difference from the mean flow implementation is the special care taken to ensure positive eddy viscosity during the solution process
 - Follows directly from the approach outlined in Spalart and Allmaras (1992) and Baldwin and Barth (1991)
- The outline shown here is the current approach; this is a very active area of research

Positive Operators and M-Type Matrices



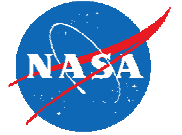
A positive operator is one that when applied to a vector of non-negative elements produces a vector whose elements remain non-negative:

$$\text{i.e., if } \mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{Bmatrix} \quad \text{where } v_i \geq 0$$

$$\text{then if } \mathbf{A} \text{ is a positive operator, then } \mathbf{s} = \begin{Bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{Bmatrix} = \mathbf{A}\mathbf{v} \quad \text{where } s_i \geq 0$$

An M-type matrix is a matrix that is diagonally-dominant with positive diagonal elements and negative off-diagonal elements. The primary property of an M-type matrix is that its inverse contains only non-negative elements and is therefore a positive operator.

Form of Implicit Equations



Consider the one equation Spalart-Allmaras model written in the following general form:

$$\frac{\partial \tilde{v}}{\partial t} = \underbrace{\mathbf{M}(\tilde{v})\tilde{v}}_{\text{Convection/Diffusion}} + \underbrace{\mathbf{P}(\tilde{v})\tilde{v}}_{\text{Production}} - \underbrace{\mathbf{D}(\tilde{v})\tilde{v}}_{\text{Destruction}}$$

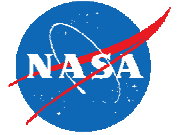
With Jacobian matrices denoted by an overbar, our implicit scheme can then be written as

$$\left[\frac{A}{\Delta t} \mathbf{I} - (\bar{\mathbf{M}} + \bar{\mathbf{P}} - \bar{\mathbf{D}}) \right] \Delta \tilde{v}^n = [\mathbf{M} + \mathbf{P} - \mathbf{D}] \tilde{v}^n$$

or

$$\underbrace{\left[\frac{A}{\Delta t} \mathbf{I} - (\bar{\mathbf{M}} + \bar{\mathbf{P}} - \bar{\mathbf{D}}) \right]}_{\text{Require this to be an M-type matrix}} \tilde{v}^{n+1} = \underbrace{\left[\frac{A}{\Delta t} \mathbf{I} + (\mathbf{M} - \bar{\mathbf{M}}) + (\mathbf{P} - \bar{\mathbf{P}}) - (\mathbf{D} - \bar{\mathbf{D}}) \right]}_{\text{Require this to be a positive operator}} \tilde{v}^n$$

Requirements for Positivity



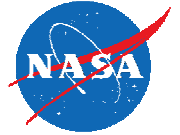
$$\underbrace{\left[\frac{A}{\Delta t} \mathbf{I} - (\bar{\mathbf{M}} + \bar{\mathbf{P}} - \bar{\mathbf{D}}) \right]}_{\text{Require this to be an M-type matrix}} \tilde{v}^{n+1} = \underbrace{\left[\frac{A}{\Delta t} \mathbf{I} + (\mathbf{M} - \bar{\mathbf{M}}) + (\mathbf{P} - \bar{\mathbf{P}}) - (\mathbf{D} - \bar{\mathbf{D}}) \right]}_{\text{Require this to be a positive operator}} \tilde{v}^n$$

In order to achieve this, it is sufficient to require the following:

$-\bar{\mathbf{M}}, -\bar{\mathbf{P}}, +\bar{\mathbf{D}}$ are M-type

$$\left. \begin{array}{l} [\mathbf{M} - \bar{\mathbf{M}}] \tilde{v} \geq 0 \\ [\mathbf{P} - \bar{\mathbf{P}}] \tilde{v} \geq 0 \\ [\mathbf{D} - \bar{\mathbf{D}}] \tilde{v} \leq 0 \end{array} \right\} \text{ for all } \tilde{v} \geq 0$$

Positivity for Convective Terms



Assuming a constant velocity within the control volume, the convective terms can be converted from area integrals to line integrals as:

$$\oiint \left(u \frac{\partial \tilde{v}}{\partial x} + v \frac{\partial \tilde{v}}{\partial y} \right) \partial A = u \oint \tilde{v} \hat{n}_x \partial \Gamma + v \oint \tilde{v} \hat{n}_y \partial \Gamma$$

First-order upwind scheme:

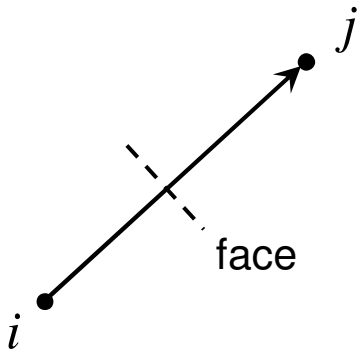
$$\text{If } \bar{U}_i > 0, \tilde{v}_{face} = \tilde{v}_i$$

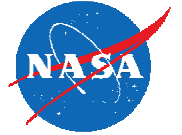
$$\text{If } \bar{U}_i < 0, \tilde{v}_{face} = \tilde{v}_j$$

So the convective flux on the face can be written as:

$$F_{conv} = (\bar{U}_i^+ \tilde{v}_i + \bar{U}_i^- \tilde{v}_j) l$$

$$\text{where } \bar{U}_i^\pm = \frac{\bar{U}_i \pm |\bar{U}_i|}{2} \text{ and } l \text{ is the length of the face}$$





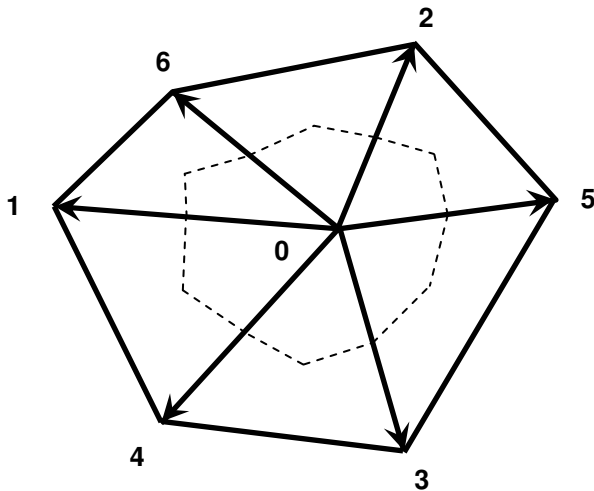
Positivity for Convective Terms

Recalling that $\frac{\partial \tilde{v}}{\partial t} = \mathbf{M}(\tilde{v})\tilde{v} + \mathbf{P}(\tilde{v})\tilde{v} - \mathbf{D}(\tilde{v})\tilde{v}$, the convective terms become

$$\mathbf{M}\tilde{v} = -\left(\bar{U}_i^+ \tilde{v}_i + \bar{U}_i^- \tilde{v}_j\right)l$$

Note $\mathbf{M}$ is independent of $\tilde{v}$, and since the residual contribution is defined as $\mathbf{M}\tilde{v}$, then the Jacobian $\bar{\mathbf{M}} = \mathbf{M}$, and we have satisfied the condition that $[\mathbf{M} - \bar{\mathbf{M}}]\tilde{v} \geq 0$.

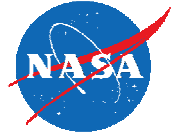
Now recall our condition that $-\bar{\mathbf{M}} = -\mathbf{M} = \left(\bar{U}_i^+ \tilde{v}_i + \bar{U}_i^- \tilde{v}_j\right)l$ be an M-type matrix.



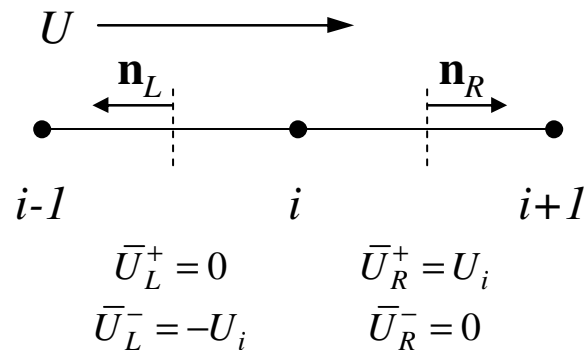
The diagonal when updating node 0 is $\sum_{i=1}^6 \bar{U}_i^+ l_i$

Note that all contributions will either be positive or zero. A similar situation exists for the off-diagonals, which will all be either negative or zero.

Positivity for Convective Terms



To see whether the matrix is diagonally dominant, it is easiest to consider a 1-D example.



Consider the residual at node i :

$$\begin{aligned}
 R_i = F_R + F_L &= \left(\bar{U}_i^+ \tilde{v}_i + \bar{U}_i^- \tilde{v}_{i+1} \right)_R + \left(\bar{U}_i^+ \tilde{v}_i + \bar{U}_i^- \tilde{v}_{i-1} \right)_L \\
 &= U_i \tilde{v}_i - U_i \tilde{v}_{i-1}
 \end{aligned}$$

The linearization will not have off-diagonals larger than the diagonal, so we have satisfied the requirements for positivity of the convective terms.

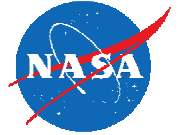
The diffusion and source terms are a bit more involved, but the approach is the same and is covered in the notes.

Other Items of Interest



- Coupling: Loose or tight?
- Enforce positivity or “take what you get”?
- Full linearization or approximate?

Some Relevant Publications



Rumsey, C.L., Smith, B.R., and Huang, G.P., “Description of a Website Resource for Turbulence Modeling Verification and Validation,” AIAA-2010-4742, 2010.

Rumsey, C.L., “Consistency, Verification, and Validation of Turbulence Models for Reynolds-Averaged Navier-Stokes Applications,” EUCASS 2009-7, Presented at the 3rd European Conference for Aerospace Sciences, 2009.

Lynch, C.E. and Smith, M.J., “Hybrid RANS-LES Turbulence Models on Unstructured Grids,” AIAA-2008-3854, 2008.

Rumsey, C.L. and Thomas, J.L., “Application of FUN3D and CFL3D to the Third Workshop on CFD Uncertainty Analysis,” NASA TM-2008-215537, 2008.

Turbulence Modeling Resource Website, sponsored by the AIAA Fluid Dynamics TC and maintained by Chris Rumsey at NASA LaRC:

<http://turbmodels.larc.nasa.gov>