

AN INTRODUCTION TO MODELING AND SIMULATION



NASA / JPL-Cal Tech

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MARS SCIENCE LABORATORY ROVER “CURIOSITY”



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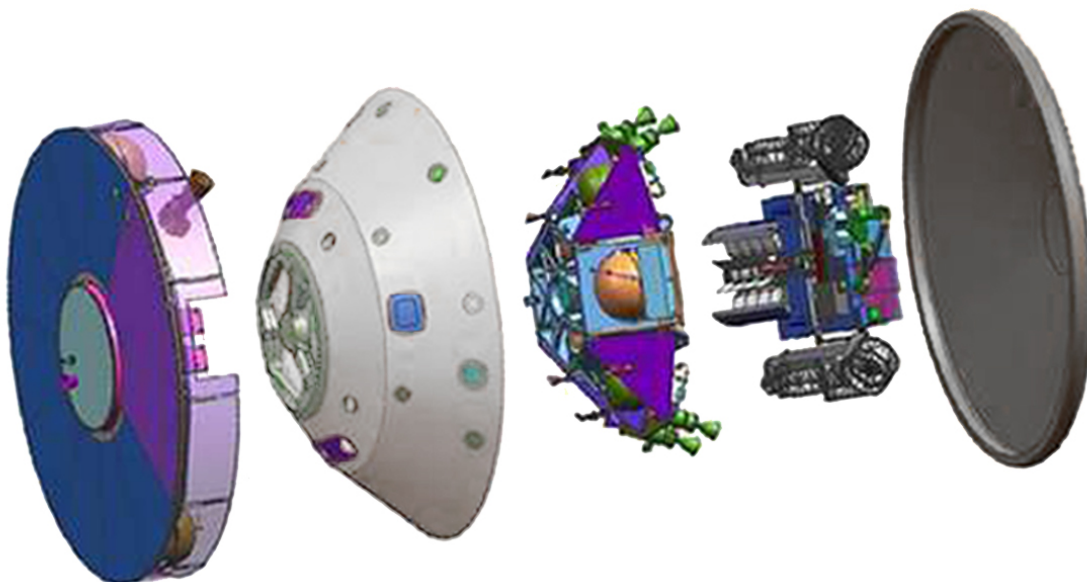
CRUISE STAGE AND AEROSHELL



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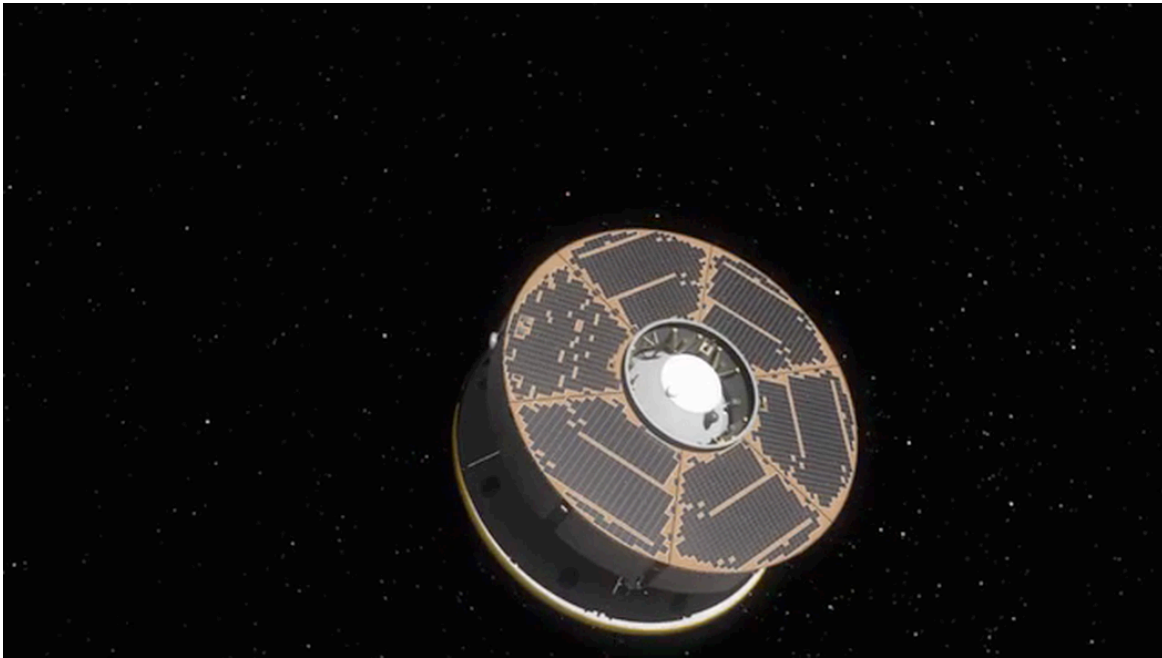
MARS SCIENCE LABORATORY FLIGHT SYSTEM

Cruise Stage, Backshell, Descent Stage, Rover, Heatshield



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ENTRY, DESCENT, AND LANDING



Animation: NASA / JPL-Cal Tech

MODEL – A mathematical description of a system's or environment's, physical properties

Examples:

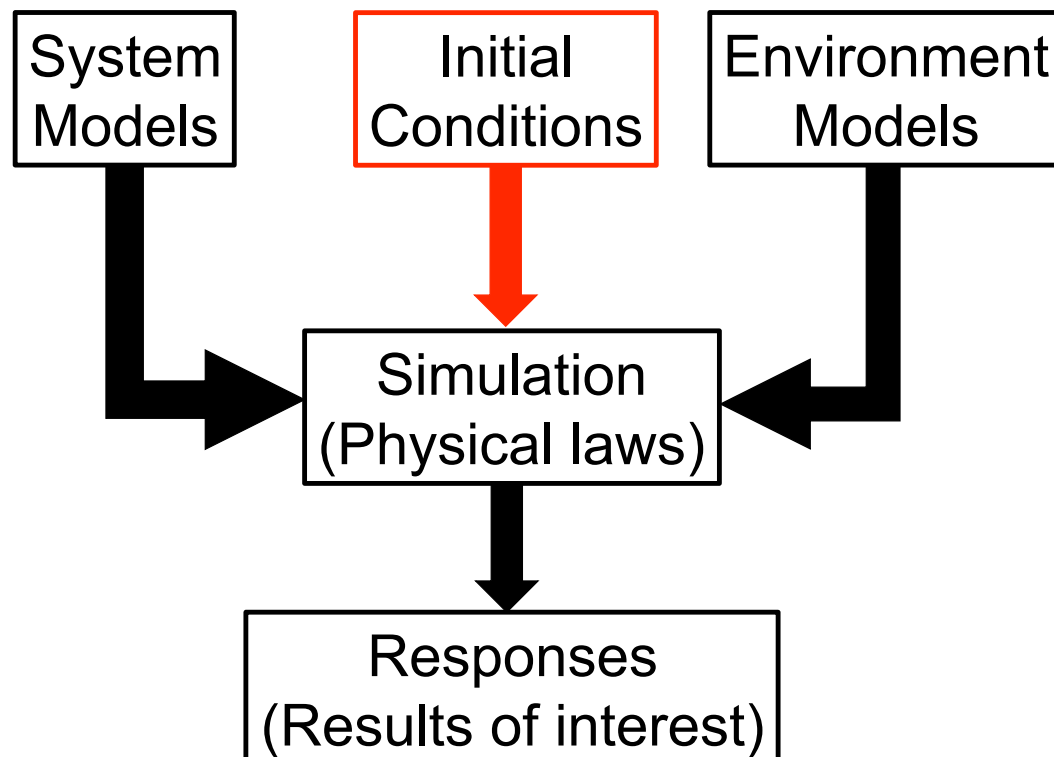
- Parachute area (system)
- Aeroshell drag coefficient (system)
- Rover mass (system)
- Acceleration of gravity (environment)
- Atmospheric density (environment)

SIMULATION – The combination of models and physical laws that allows us to calculate the responses of a system

Example: Parachute & Aeroshell (system)

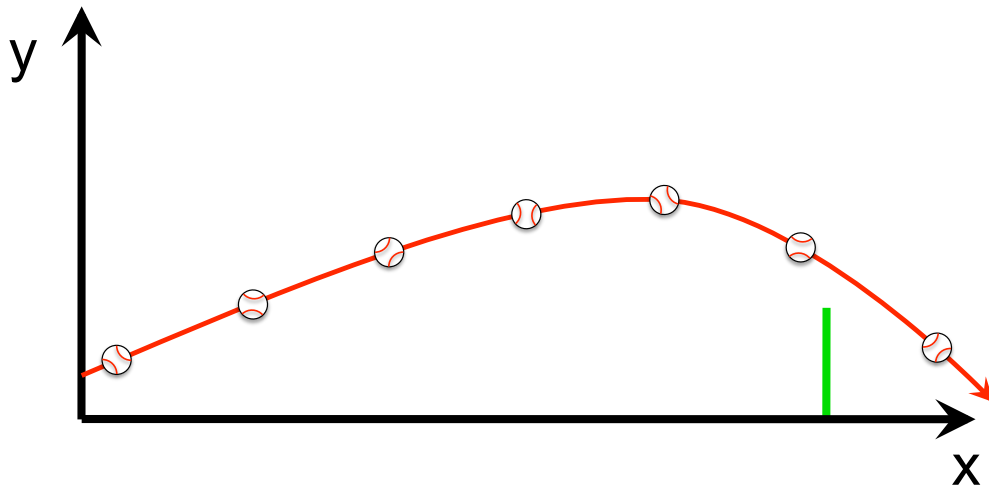
Parachute area (model)
+ Parachute drag coefficient (model)
+ Atmospheric density (model)
+ Aeroshell mass (model)
+ Newton's 2nd Law (physical law)

↳ Aeroshell deceleration (response)



Example: I want to hit a Home Run...

What is the velocity and angle at which the ball must be hit so that it will go over the center field fence?



SYSTEM

Baseball

SYSTEM MODELS

Baseball mass = 0.0097 slug = 0.14 kg

Baseball diameter = 0.24 ft = 0.073 m

Baseball aero drag coefficient = 0.47

ENVIRONMENT MODELS

$$\begin{aligned}\text{Acceleration of gravity} &= 32.2 \text{ ft/s}^2 \\ &= 9.81 \text{ m/s}^2\end{aligned}$$

$$\begin{aligned}\text{Atmospheric density} &= 0.002377 \text{ slug/ft}^3 \\ &= 1.225 \text{ kg/m}^3\end{aligned}$$

Location of center field fence

$$x_{\text{FENCE}} = 400 \text{ ft} = 122 \text{ m}$$

$$y_{\text{FENCE}} = 20 \text{ ft} = 6 \text{ m}$$

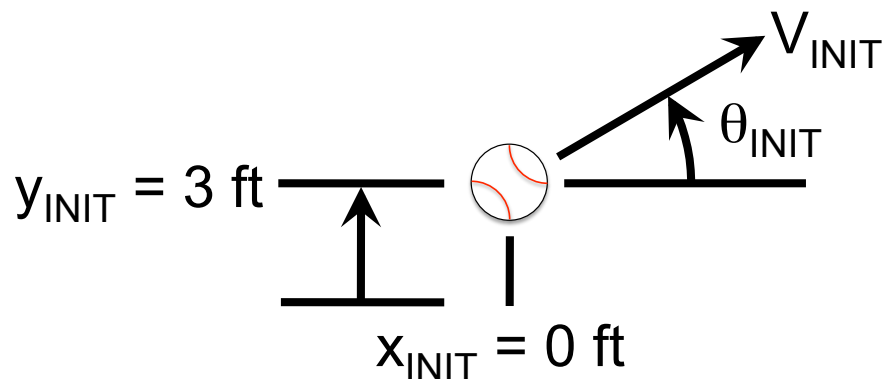
INITIAL CONDITIONS (at time $t = 0 \text{ s}$)

$$x_{\text{INIT}} = 0 \text{ ft} = 0 \text{ m (home plate)}$$

$$y_{\text{INIT}} = 3 \text{ ft} = 0.91 \text{ m (waist high)}$$

$$V_{\text{INIT}} = ?? \text{ ft/s} = ?? \text{ m/s (initial velocity)}$$

$$\theta_{\text{INIT}} = ?? \text{ degrees (initial angle)}$$



SIMULATION

Newton's 2nd Law

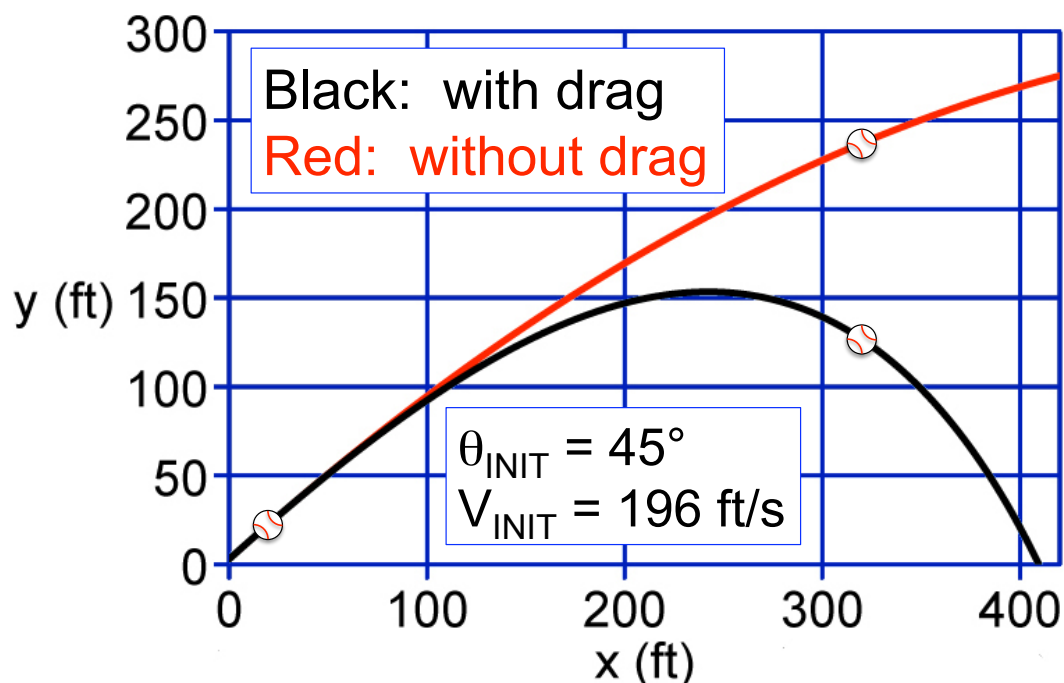
RESPONSES OF INTEREST

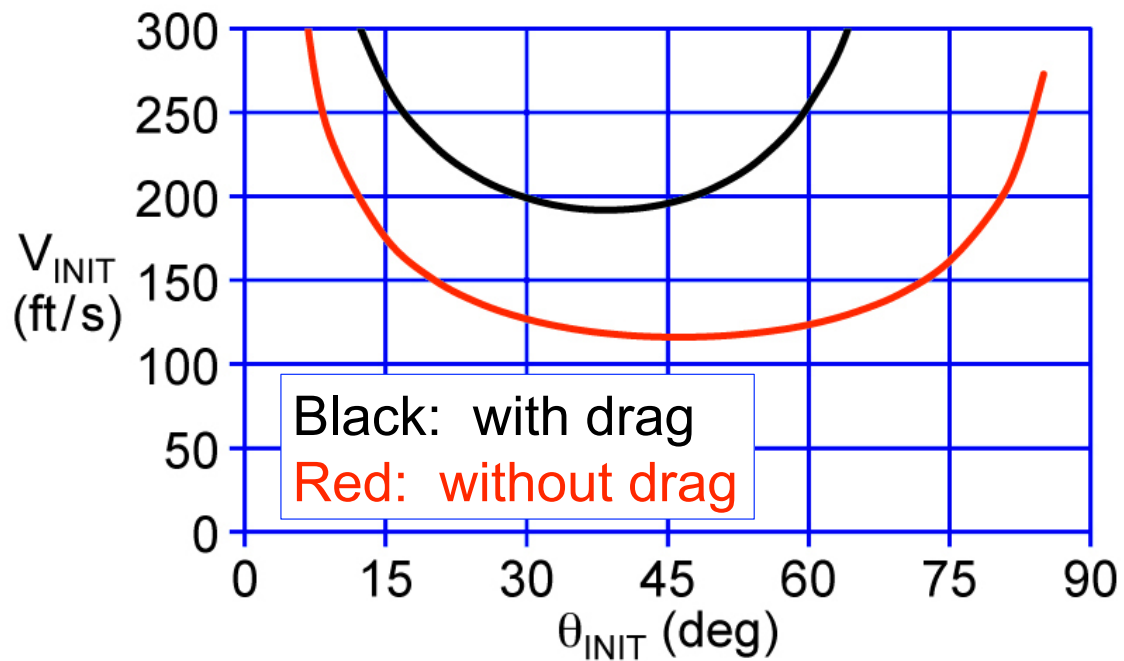
What is y when $x = x_{\text{FENCE}}$?

It is a home run if...

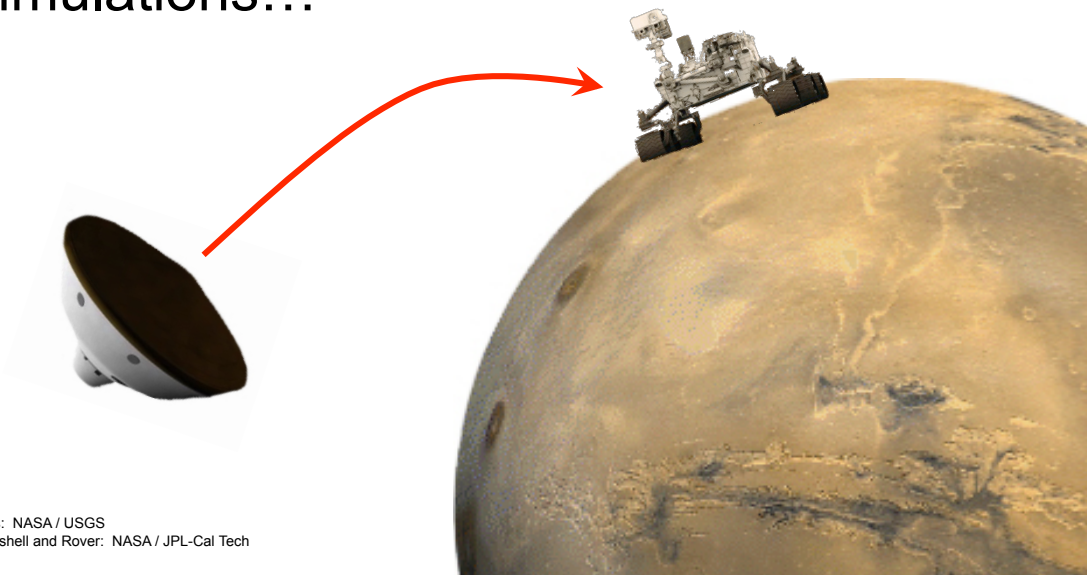
$$(y \text{ when } x = x_{\text{FENCE}}) > y_{\text{FENCE}}$$

What combinations of V_{INIT} and θ_{INIT} result in a home run?



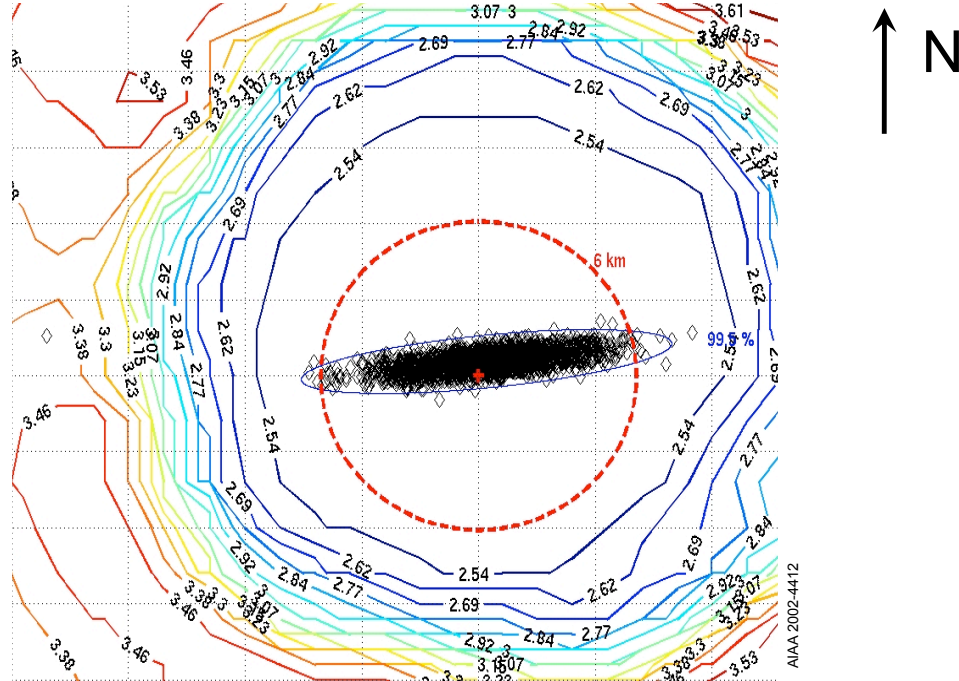


The basic modeling and physics of the baseball simulation are the same as those used for entry, descent, and landing simulations...



Mars: NASA / USGS
Aeroshell and Rover: NASA / JPL-Cal Tech

MARS LANDING FOOTPRINT SIMULATION

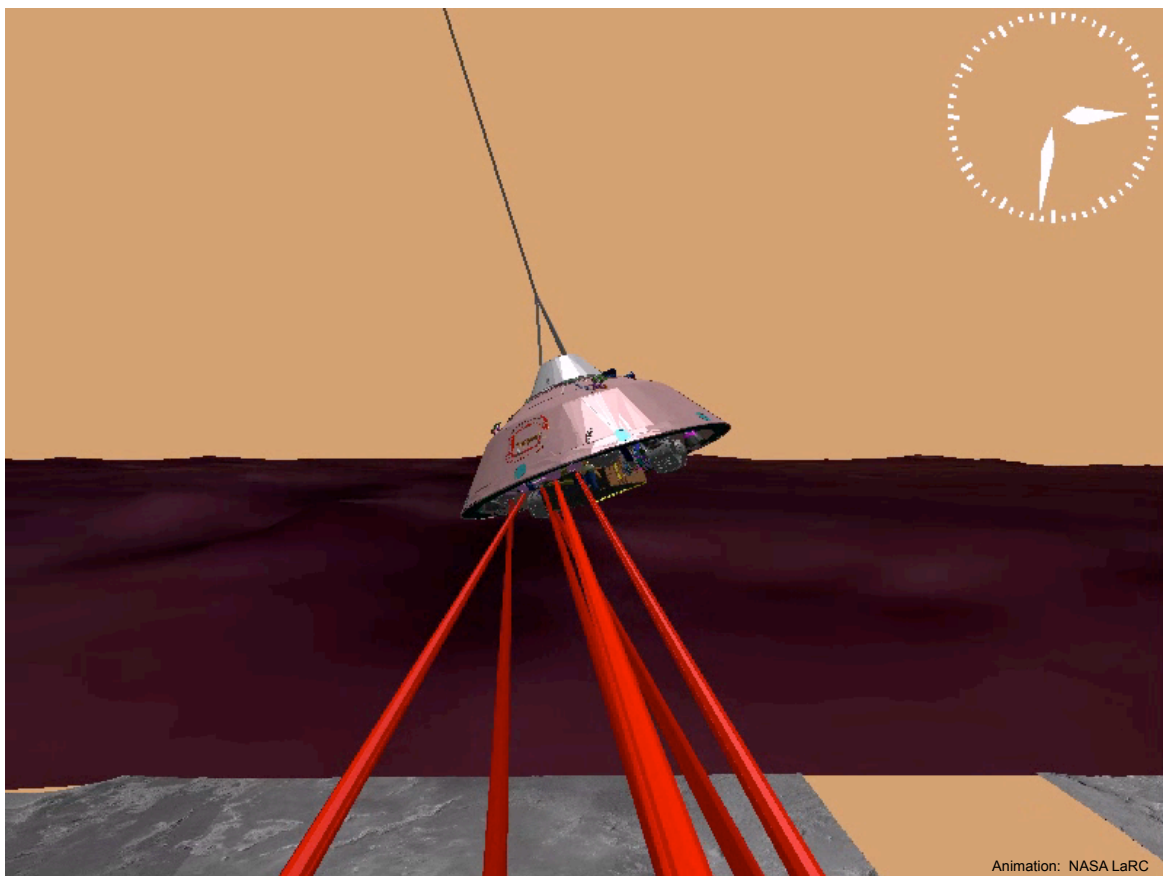


BASICS OF MODELING AND SIMULATION

- Define **SYSTEM**
- Create the needed **MODELS**
- Assemble the **MODELS** and **PHYSICAL LAWS** into a **SIMULATION**
- Specify the **INITIAL CONDITIONS**
- Decide which are the relevant **RESPONSES**
- Critically evaluate the **RESULTS** of the **SIMULATION**
- Update and repeat as necessary

ADDITIONAL MATERIALS



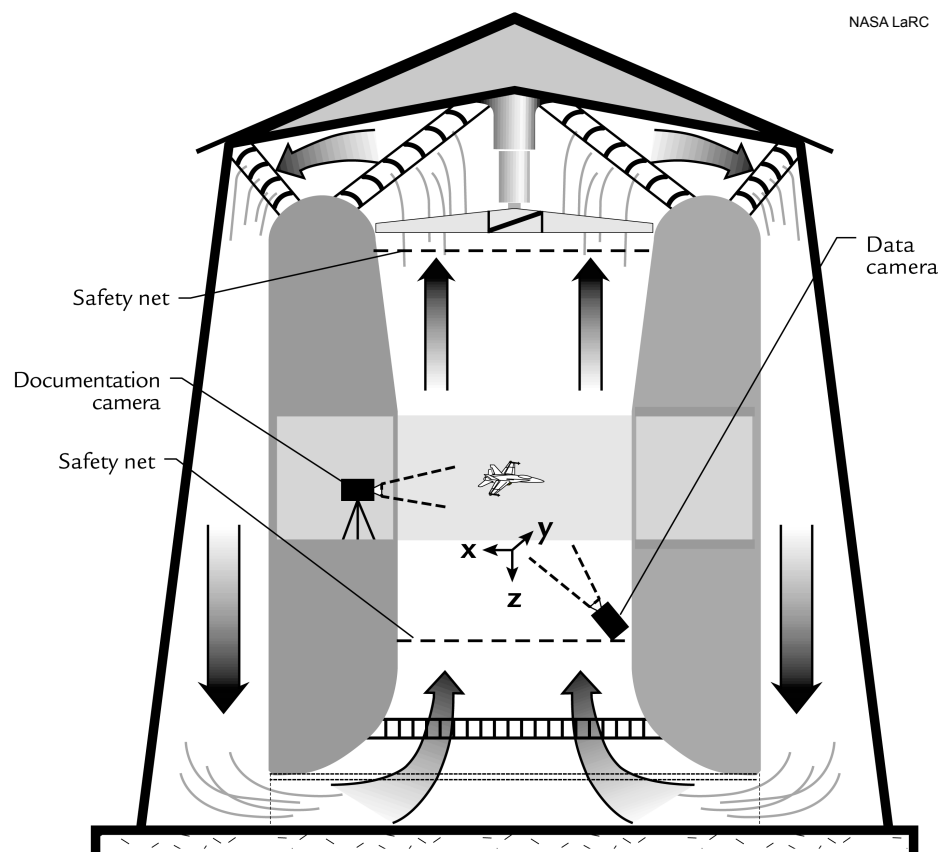


VENUS Lander Model



NASA LaRC

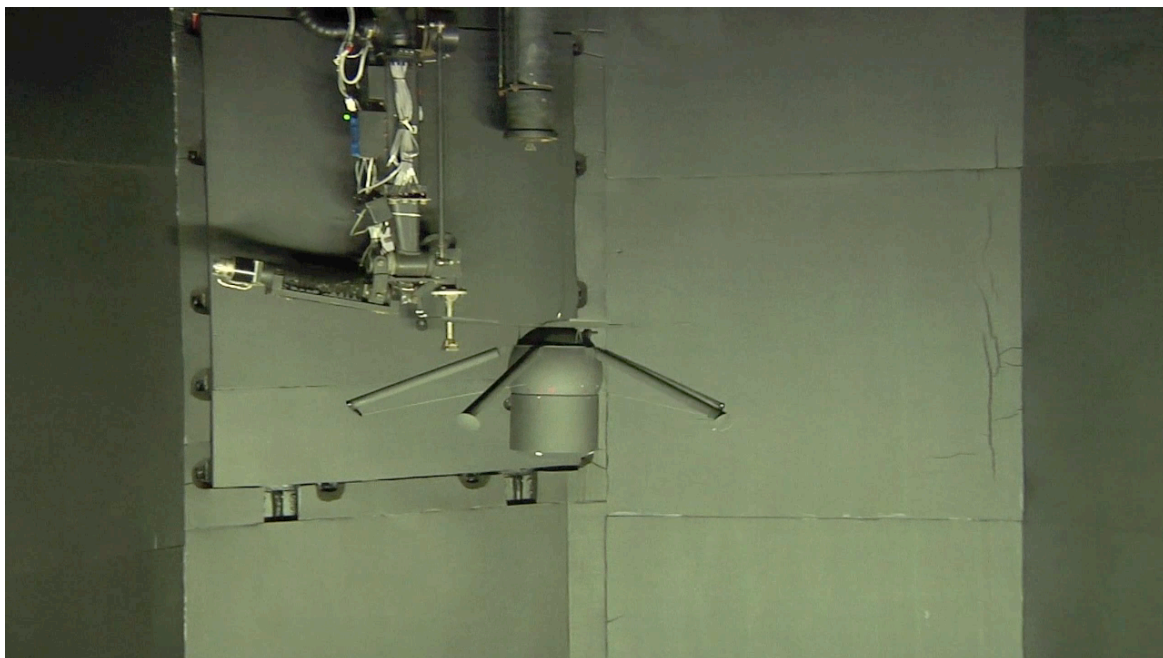
Vertical (Spin) Wind Tunnel



NASA LaRC



NASA LaRC



Video: NASA LaRC

AN INTRODUCTION TO MODELING AND SIMULATION

BASEBALL TRAJECTORY EXAMPLES

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Introduction

This document provides the details for the baseball trajectory examples in the presentation *An Introduction to Modeling and Simulation*^[1] and coded in the Excel spreadsheet *Baseball ModSim V1.xls*^[2]. Two baseball trajectory simulation cases are described here: with and without aerodynamic drag. Each case is described separately. The model and simulation is defined using the taxonomy of reference 1, the equations are derived, and the numerical algorithm used to solve the problem in reference 2 are presented.

Statement of Problem

This problem involves the calculation of the trajectory of a baseball being hit at home plate. A sketch of the trajectory is shown in figure 1. Home plate is at $x = 0$ ft.[†] The coordinate y is the height above the playing field. The center field fence is at $x = x_{Fence}$, and has a height of $y = y_{Fence}$. The ball is hit over home plate at waist height (i.e., $x = x_{Init} = 0$ ft, $y = y_{Init} = 3$ ft, at $t = 0$ s). The velocity, V , and angle, θ , defining the baseball's trajectory are shown in figure 2. The initial (i.e., at $t = 0$ s) velocity and angle at which the ball is hit are $V = V_{Init}$ and $\theta = \theta_{Init}$. The question to be answered is: what combination of V_{Init} and θ_{Init} will yield a home run over the center field fence [i.e., meet the condition $(y \text{ when } x = x_{Fence}) > y_{Fence}$].

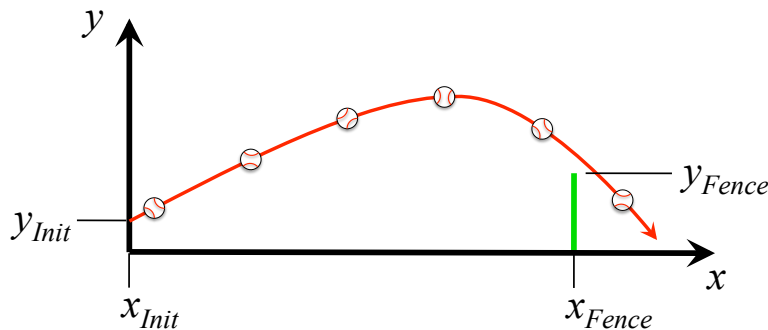


Figure 1. Coordinate system.

[†] A list of symbols is presented at the end of this document.

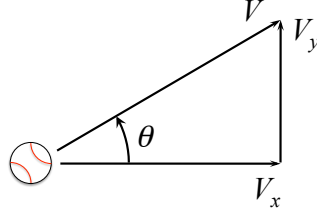


Figure 2. Velocity and angle definitions.

Case I: Baseball trajectory without aerodynamic drag

This is a simplified version of the case described in reference 1. The modeling and simulation for this case are described as follows:

<u>System:</u>	Baseball
<u>System Models:</u>	None required
<u>Environment Models:</u>	Acceleration of gravity, $g = 32.2 \text{ ft/s}^2$ Location and height of the center field fence, $x_{Fence} = 400 \text{ ft}$, $y_{Fence} = 20 \text{ ft}$
<u>Initial Conditions:</u>	Simulation starts at $t = 0 \text{ s}$ $x_{Init} = 0 \text{ ft}$, $y_{Init} = 3 \text{ ft}$ $V_{Init} = \text{TBD ft/s}$, $\theta_{Init} = \text{TBD deg}$
<u>Simulation:</u>	Newton's 2 nd Law, $\mathbf{F} = m\mathbf{a}$
<u>Response of Interest:</u>	y when $x = x_{Fence}$ The trajectory yields a home run if: $(y \text{ when } x = x_{Fence}) > y_{Fence}$
<u>Question to be Answered:</u>	What combinations of V_{Init} and θ_{Init} result in a home run?

For this case the equations defining the trajectory, $x(t)$ and $y(t)$, can be written explicitly. Here these trajectory equations are derived from first principles. Newton's 2nd Law in scalar form for the x -direction is

$$F_x = ma_x = m \frac{dV_x}{dt} \quad (1)$$

where F_x , a_x , and V_x are, respectively, the force, acceleration, and velocity in the x -direction, and m is the mass of the baseball. Note that

$$a_x = \frac{dV_x}{dt} \quad (2)$$

Because the only force being considered is gravity, which acts only in the y -direction (i.e., it is perpendicular to the x -direction), $F_x = 0$. Thus, equation (1) becomes

$$\frac{dV_x}{dt} = 0 \quad (3)$$

Integrating equation (3) yields

$$V_x = C_1 \quad (4)$$

where C_1 is a constant of integration. Equation (3) is telling us that V_x is a constant throughout the trajectory (within the assumptions used for this case). Thus, V_x remains equal to its initial value. With the use of figure 2 to determine the x -component of the velocity we can write

$$C_1 = V_x = V_{init} \cos(\theta_{init}) \quad (5)$$

Using the fact that

$$V_x = \frac{dx}{dt} \quad (6)$$

equation (5) can be written as

$$\frac{dx}{dt} = V_{init} \cos(\theta_{init}) \quad (7)$$

Integrating equation (7) yields

$$x = [V_{init} \cos(\theta_{init})]t + C_2 \quad (8)$$

where C_2 is a constant of integration. Substituting the initial condition previously specified

$$x|_{t=0} = x_{init} \quad (9)$$

into equation (8) we can determine that

$$C_2 = x_{init} \quad (10)$$

Substituting equation (10) into equation (8) yields the equation for the x -position of the baseball vs. time:

$$x = [V_{init} \cos(\theta_{init})]t + x_{init} \quad (11)$$

We now proceed to determine $y(t)$ in a similar manner. Newton's 2nd Law in scalar form for the y -direction is

$$F_y = ma_y = m \frac{dV_y}{dt} \quad (12)$$

where F_y , a_y , and V_y are, respectively, the force, acceleration, and velocity in the y -direction, and m is the mass of the baseball. Note that

$$a_y = \frac{dV_y}{dt} \quad (13)$$

Gravity acts in the $-y$ direction (i.e., down). Thus,

$$F_y = -mg \quad (14)$$

where g is the acceleration of gravity. Combining equations (14) and (12) yields

$$\frac{dV_y}{dt} = -g \quad (15)$$

Note that the baseball mass does not appear in equation (15). Integrating equation (15) yields

$$V_y = -gt + C_3 \quad (16)$$

where C_3 is a constant of integration. Using figure 2 and the velocity initial condition previously specified we can determine that

$$V_y|_{t=0} = V_{init} \sin(\theta_{init}) \quad (17)$$

Combining this initial condition with equation (16) allows us to determine that

$$C_3 = V_{init} \sin(\theta_{init}) \quad (18)$$

Using the fact that

$$V_y = \frac{dy}{dt} \quad (19)$$

and equation (18) allows us to rewrite equation (16) as

$$\frac{dy}{dt} = -gt + V_{init} \sin(\theta_{init}) \quad (20)$$

Integrating equation (20) yields

$$y = -\frac{1}{2}gt^2 + [V_{init} \sin(\theta_{init})]t + C_4 \quad (21)$$

where C_4 is a constant of integration. Substituting the initial condition previously specified,

$$y|_{t=0} = y_{init} \quad (22)$$

into equation (21) we can determine that

$$C_4 = y_{init} \quad (23)$$

Combining equations (23) and (21) allows us to determine the y -position of the baseball with respect to time

$$y = -\frac{1}{2}gt^2 + [V_{init} \sin(\theta_{init})]t + y_{init} \quad (24)$$

Equations (11) and (24) represent the x - and y -position (i.e., trajectory) of the baseball as a function of time. Note that mass is not one of the models needed for this simulation.

By plotting $y(t)$ vs. $x(t)$ the trajectory can be visualized. Using the Excel spreadsheet (reference 2), various trajectories that satisfy the home run requirement have been obtained and plotted in figure 3.[‡] The initial velocity and angle conditions required to yield a home run were determined by trial and error. A plot of the initial velocities required to hit a home run for specific initial angles are shown in figure 4.

[‡] Note that equations (11) and (24) can be combined to eliminate t and yield an equation for the trajectory in the form $y(x)$. This equation is:

$$y = -\frac{g}{2[V_{init} \cos(\theta_{init})]^2}(x - x_{init})^2 + \tan(\theta_{init})(x - x_{init}) + y_{init}$$

which can be recognized as the equation of a parabola. For this simple case deriving an explicit equation for $y(x)$ is possible. However, for more complex cases doing so will not be possible. In the spirit of having the examples be a representation of more complex problems, the spreadsheet in reference 2 for this case retains t as a parameter and calculates the trajectory as $[x(t), y(t)]$.

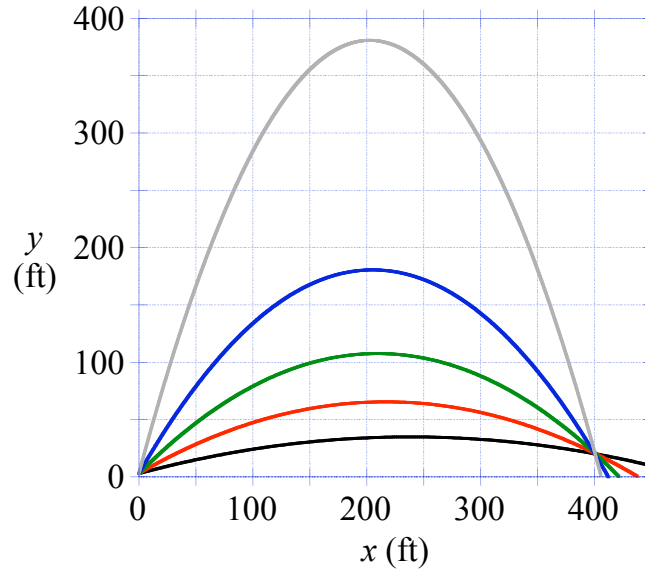


Figure 3. Example trajectories for the case without aerodynamic drag. Initial angles are 15° (black), 30° (red), 45° (green), 60° (blue), and 75° (gray).

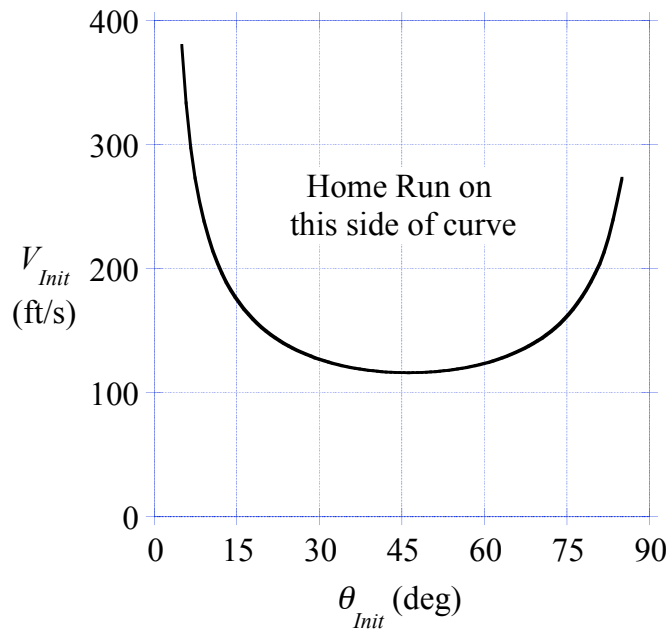


Figure 4. Combinations of V_{Init} and θ_{Init} required for a home run for the case without aerodynamic drag. For a given value of θ_{Init} any value of V_{Init} greater than or equal to that shown by the curve will result in a home run.

Case II: Baseball trajectory with aerodynamic drag

This is the baseball modeling and simulation example used in reference 1. The modeling and simulation for this case are described as follows:

<u>System:</u>	Baseball
<u>System Models:</u>	Baseball mass, $m = 0.0097$ slug Baseball diameter, $d = 0.24$ ft Baseball aerodynamic drag coefficient, $C_D = 0.47$
<u>Environment Models:</u>	Acceleration of gravity, $g = 32.2$ ft/s ² Atmospheric density, $\rho = 0.002377$ slug/ft ³ Location and height of the center field fence, $x_{Fence} = 400$ ft, $y_{Fence} = 20$ ft
<u>Initial Conditions:</u>	Simulation starts at $t = 0$ s $x_{Init} = 0$ ft, $y_{Init} = 3$ ft $V_{Init} = \text{TBD}$ ft/s, $\theta_{Init} = \text{TBD}$ deg
<u>Simulation:</u>	Newton's 2 nd Law, $\mathbf{F} = m\mathbf{a}$
<u>Response of Interest:</u>	y when $x = x_{Fence}$ The trajectory yields a home run if: $(y \text{ when } x = x_{Fence}) > y_{Fence}$
<u>Question to be Answered:</u>	What combinations of V_{Init} and θ_{Init} result in a home run?

In setting up this simulation it is useful to realize that there are four *state variables* needed to describe the trajectory of the baseball: two position variables (x, y) , and two velocity variables (V_x, V_y) . We will need four *first-order ordinary differential equations* (ODEs) to solve for the trajectory. The two ODEs for the position variables are obtained from the definition of the velocities V_x and V_y

$$\frac{dx}{dt} = V_x \quad (25)$$

$$\frac{dy}{dt} = V_y \quad (26)$$

The two ODEs for the velocity variables are obtained from Newton's 2nd Law in the x - and y -directions

$$F_x = -D_x = m \frac{dV_x}{dt} = ma_x \quad (27)$$

$$F_y = -mg - D_y = m \frac{dV_y}{dt} = ma_y \quad (28)$$

where D_x and D_y are the components of the aerodynamic drag force (subsequently known just as “drag”) in the x - and y -directions, respectively. Positive drag forces act in the opposite direction to their corresponding axes. The drag forces will be defined in a moment. Note that the only force acting in the x -direction is the drag (D_x), whereas in the y -direction the total force is due to both drag and gravity (D_y and mg , respectively). Note that the gravity force acts in the negative y -direction (i.e., down). It will be convenient to rewrite equations (27) and (28) as follows:

$$\frac{dV_x}{dt} = -\frac{D_x}{m} \quad (29)$$

$$\frac{dV_y}{dt} = -\frac{D_y}{m} - g \quad (30)$$

Drag, D , acts in a direction opposite to the velocity, V , as shown in figure 5. It can be calculated from

$$D = \frac{1}{2} \rho V^2 C_D S \quad (31)$$

where ρ is the atmospheric density, C_D is the drag coefficient, and S is the reference area for the drag calculation. Values for ρ and C_D were given earlier as part of the environmental and system models. The reference area for this case is the frontal area of the baseball

$$S = \frac{\pi}{4} d^2 \quad (32)$$

where d is the diameter of the baseball.[§] This diameter was given earlier as part of the system models.

[§] A different reference area can be defined. However, the drag coefficient is related to how the reference area is defined. If a change is made to the definition of the reference area, the numerical value of the drag coefficient will have to change. In other words, the definitions for the reference area and the drag coefficient must be consistent.

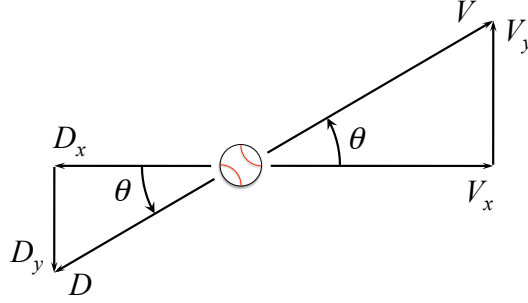


Figure 5. Definition of drag and its components.

From figure 5 it can be determined that the drag components in the x - and y -directions are

$$D_x = D \cos(\theta) = \frac{1}{2} \rho V^2 C_D S \cos(\theta) \quad (33)$$

$$D_y = D \sin(\theta) = \frac{1}{2} \rho V^2 C_D S \sin(\theta) \quad (34)$$

But,

$$\cos(\theta) = \frac{V_x}{V} \quad (35)$$

$$\sin(\theta) = \frac{V_y}{V} \quad (36)$$

Substituting equation (35) into (33) to eliminate $\cos(\theta)$, substituting equation (36) into (34) to eliminate $\sin(\theta)$, and making use of the fact that

$$V = \sqrt{V_x^2 + V_y^2} \quad (37)$$

allows us to rewrite equations (33) and (34) as

$$D_x = \frac{1}{2} \rho V_x \sqrt{V_x^2 + V_y^2} C_D S \quad (38)$$

$$D_y = \frac{1}{2} \rho V_y \sqrt{V_x^2 + V_y^2} C_D S \quad (39)$$

Substituting equation (38) into (29) to eliminate D_x , and substituting equation (39) into (30) to eliminate D_y yields the other two ODEs we need

$$\frac{dV_x}{dt} = - \frac{\frac{1}{2} \rho V_x \sqrt{V_x^2 + V_y^2} C_D S}{m} \quad (40)$$

$$\frac{dV_y}{dt} = -\frac{\frac{1}{2}\rho V_y \sqrt{V_x^2 + V_y^2} C_D S}{m} - g \quad (41)$$

At this point is worth restating the four first-order ODEs we have derived for the four state variables x , y , V_x , and V_y

$$\frac{dx}{dt} = V_x \quad (25)$$

$$\frac{dy}{dt} = V_y \quad (26)$$

$$\frac{dV_x}{dt} = -\frac{\frac{1}{2}\rho V_x \sqrt{V_x^2 + V_y^2} C_D S}{m} \quad (40)$$

$$\frac{dV_y}{dt} = -\frac{\frac{1}{2}\rho V_y \sqrt{V_x^2 + V_y^2} C_D S}{m} - g \quad (41)$$

They are a set of coupled first-order ODEs. Although they cannot be solved explicitly, they are in a form suitable for approximate numerical integration. I will integrate these equations using Euler's Method.[¶] Let:

- Δt be a small integration time step
- i be an integer $i = 0, 1, 2, 3 \dots$ representing the time steps
- $(x_i, y_i, V_{x,i}, V_{y,i})$ be the values of the state variables at time $t = i\Delta t$
- $(x_{i+1}, y_{i+1}, V_{x,i+1}, V_{y,i+1})$ be the values of the state variables at time $t = (i+1)\Delta t$
- $\Delta x = (x_{i+1} - x_i)$; $\Delta y = (y_{i+1} - y_i)$; $\Delta V_x = (V_{x,i+1} - V_{x,i})$; $\Delta V_y = (V_{y,i+1} - V_{y,i})$ be the changes in the state variables from step i to $i+1$

Using these quantities, the ODEs in equations (25), (26), (40), and (41) can be approximated by the numerical equations

$$\frac{\Delta x}{\Delta t} = \frac{(x_{i+1} - x_i)}{\Delta t} = V_{x,i} \quad (42)$$

[¶] Euler's Method is very simple, but has accuracy and convergence problems. It is suitable for only the simplest of example problems such as this one, and even then a small time step is needed to obtain reasonably accurate results. More sophisticated, and accurate, methods are typically used to numerically integrate differential equations.

$$\frac{\Delta y}{\Delta t} = \frac{(y_{i+1} - y_i)}{\Delta t} = V_{y,i} \quad (43)$$

$$\frac{\Delta V_x}{\Delta t} = \frac{V_{x,i+1} - V_{x,i}}{\Delta t} = -\frac{\frac{1}{2}\rho V_{x,i}\sqrt{V_{x,i}^2 + V_{y,i}^2}C_D S}{m} \quad (44)$$

$$\frac{\Delta V_y}{\Delta t} = \frac{V_{y,i+1} - V_{y,i}}{\Delta t} = -\frac{\frac{1}{2}\rho V_{y,i}\sqrt{V_{x,i}^2 + V_{y,i}^2}C_D S}{m} - g \quad (45)$$

Equations (42) to (45) can be solved for the values of the state variables at time $t = (i+1)\Delta t$ to yield

$$x_{i+1} = x_i + V_{x,i}\Delta t \quad (46)$$

$$y_{i+1} = y_i + V_{y,i}\Delta t \quad (47)$$

$$V_{x,i+1} = V_{x,i} + \left(-\frac{\frac{1}{2}\rho V_{x,i}\sqrt{V_{x,i}^2 + V_{y,i}^2}C_D S}{m} \right) \Delta t \quad (48)$$

$$V_{y,i+1} = V_{y,i} + \left(-\frac{\frac{1}{2}\rho V_{y,i}\sqrt{V_{x,i}^2 + V_{y,i}^2}C_D S}{m} - g \right) \Delta t \quad (49)$$

Notice that on the right hand side of equations (46) to (49) all quantities are either constants (Δt , m , C_D , ρ , g) or values of the state variables at time step i (x_i , y_i , $V_{x,i}$, $V_{y,i}$). Thus, if we have a complete set of initial conditions, that is values of the state variables at $t = 0$ s (i.e., $i = 0$), we can start marching forward in time, applying equations (46) to (49) recursively to obtain solutions for $t > 0$ s at discrete time steps. The method just described is that implemented in reference 2 to solve this example problem.

By plotting $y(t)$ vs. $x(t)$ the trajectory can be visualized. Using the Excel spreadsheet (reference 2), various trajectories that satisfy the home run requirement have been obtained and plotted in figure 6. The initial conditions required to yield a home run were determined by trial and error. A plot of the initial velocities required to hit a home run for specific initial angles are shown in figure 7.

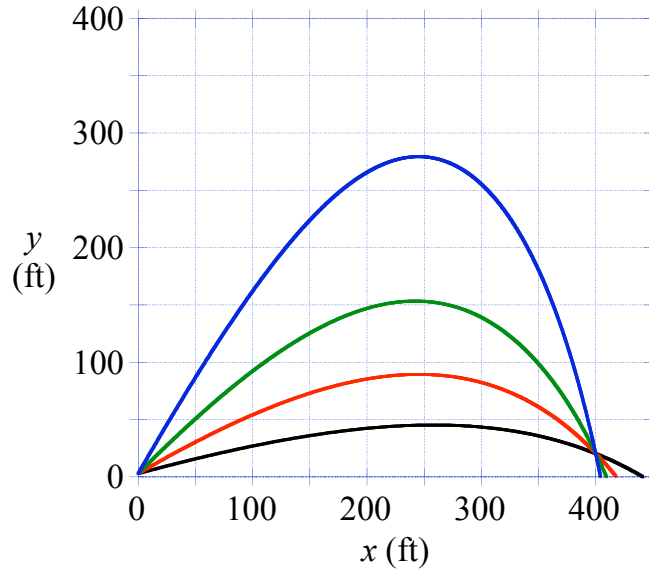


Figure 6. Example trajectories for the case with aerodynamic drag. Initial angles are 15° (black), 30° (red), 45° (green), 60° (blue).

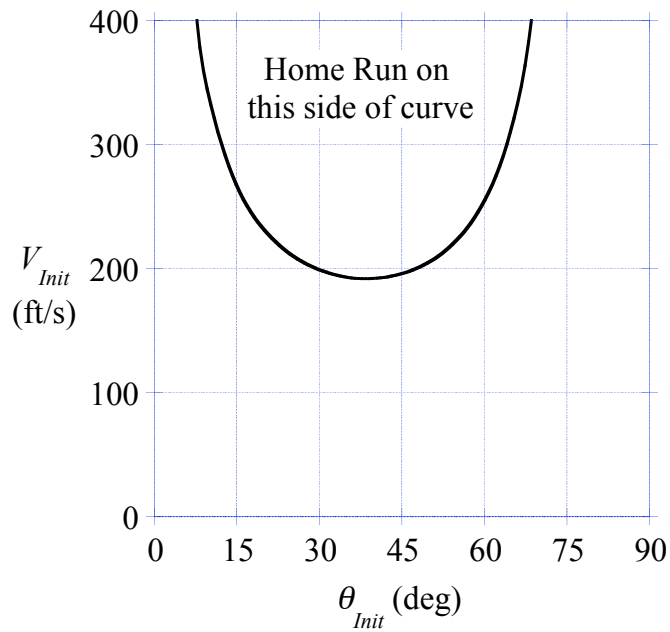


Figure 7. Combinations of V_{Init} and θ_{Init} required for a home run for the case with aerodynamic drag. For a given value of θ_{Init} any value of V_{Init} greater than or equal to that shown by the curve will result in a home run.

Concluding Remarks

Comparing figures 3 and 4 vs. 6 and 7 yields insight as to the accuracy of the “no drag” approximation used in Case I. Clearly aerodynamic drag has an important effect on the trajectory and on the initial conditions required to get a home run. Notice that Case II becomes Case I if the drag coefficient in Case II is set to zero (within the accuracy of the approximate numerical integration algorithm used in Case II).

References

- 1) Cruz, J. R.: An Introduction to Modeling and Simulation, presentation for the Modeling and Simulation Workshop at the NASA Langley Research Center, July 2011.
- 2) Cruz, J. R.: Baseball Trajectory Modeling and Simulation, Excel spreadsheet Baseball ModSim V1.xlsx, July 2011.

Symbols

a	acceleration of the baseball (vector)
a_x, a_y	acceleration of the baseball in the x - and y -directions, respectively
$C_1, C_2,$ C_3, C_4	constants of integration
D	aerodynamic drag
D_x, D_y	drag component in the x - and y -directions, respectively
d	baseball diameter
F	net force on the baseball (vector)
F_x, F_y	components of the force acting on the baseball in the x - and y -directions, respectively
g	acceleration of gravity
i	integer representing the time steps
m	baseball mass
S	reference area
t	time
V	velocity
V_{Init}	velocity initial condition
V_x, V_y	velocity of the baseball in the x - and y -directions, respectively
$V_{x,i}, V_{y,i}$	velocity of the baseball in the x - and y -directions, respectively, at time step i
x, y	position coordinate
x_{Fence}, y_{Fence}	location and height of the center field fence
x_{Init}, y_{Init}	position initial condition
x_i, y_i	position coordinate at time step i

Δt	integration time step
$\Delta V_x, \Delta V_y$	change in the velocity of the baseball in the x - and y -directions, respectively, from time step i to time step $i + 1$
$\Delta x, \Delta y$	change in the x and y position coordinates, respectively, from time step i to time step $i + 1$
θ	baseball trajectory angle with respect to the horizon
θ_{init}	initial baseball trajectory angle with respect to the horizon
ρ	atmospheric density