

Enrichment Strategies in the Extended Stochastic FEM for Heat Transfer Analysis with Uncertain Material Interfaces

C. Lang^{1,2}, A. Doostan¹, and K. Maute¹

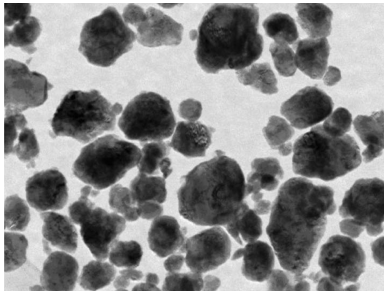
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¹Department of Aerospace Engineering Sciences
The University of Colorado, Boulder, CO

²Structural Mechanics and Concepts Branch
NASA Langley Research Center, Hampton, VA

- Goal: Predict stochastic material properties of heterogeneous materials to mitigate the need for complex experimental characterization
- Material properties of engineered composites depend on the size, shape, and distribution of the inclusions within the matrix
- The effects of geometric uncertainty must be incorporated for realistic material property predictions

Nanocomposite – $\text{Si}_{80}\text{Ge}_{20}$



[Joshi 2008]

Ceramic Matrix Composite – SiC/SiC

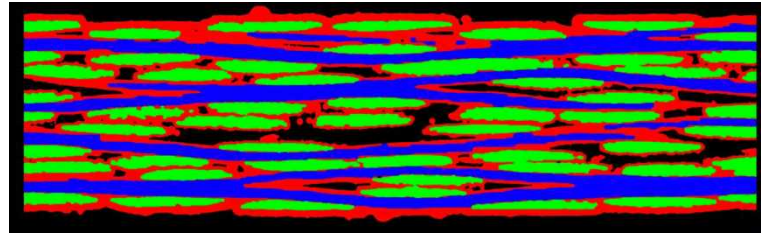
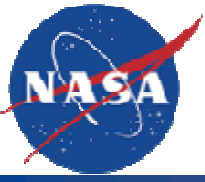


Image courtesy of Peter Bonacuse, GRC

- X-SFEM – combined Extended and Stochastic FEM approach for modeling heterogeneous materials with uncertain interfaces [Nouy 2008]



Challenges



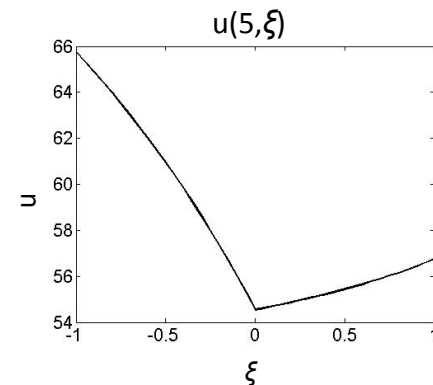
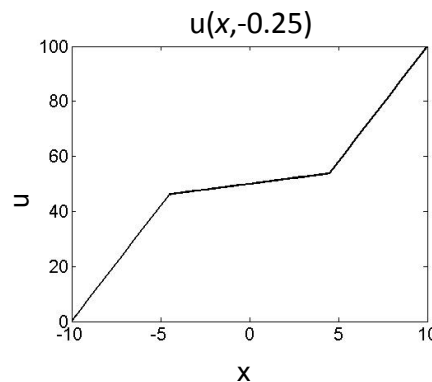
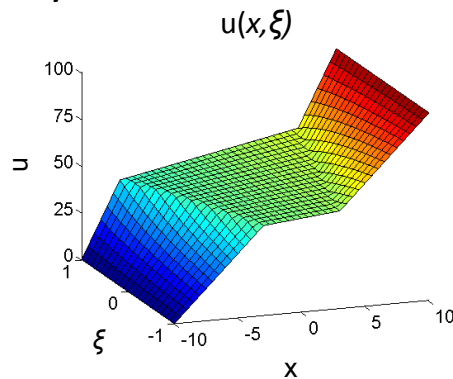
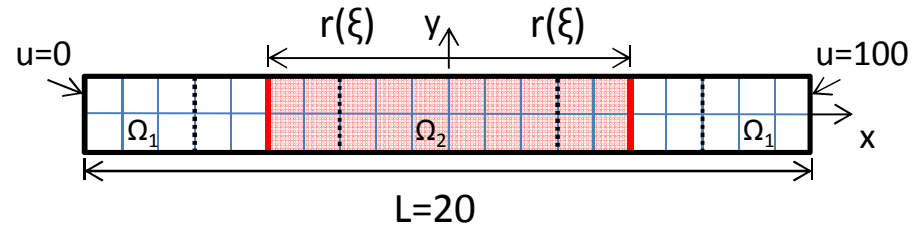
- Challenges for modeling heterogeneous materials with uncertain interfaces
 - Conforming mesh
 - Irregularity in the spatial and stochastic domains
 - Spectral approximation for stochastic space
 - Enrichment strategy

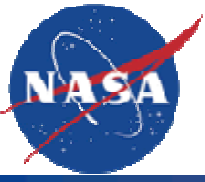
- Example – Bar with center inclusion
 - 1-D steady-state heat conduction

$$-\nabla^T(\kappa \nabla u) = 0 \quad \text{in } \Omega$$

$$\kappa = \begin{cases} 2 & \text{in } \Omega_1(\xi) \\ 20 & \text{in } \Omega_2(\xi) \end{cases} \quad r = 5 + 2\xi \quad \xi \in U(-1,1)$$

- Analytical Solution





Background: X-FEM



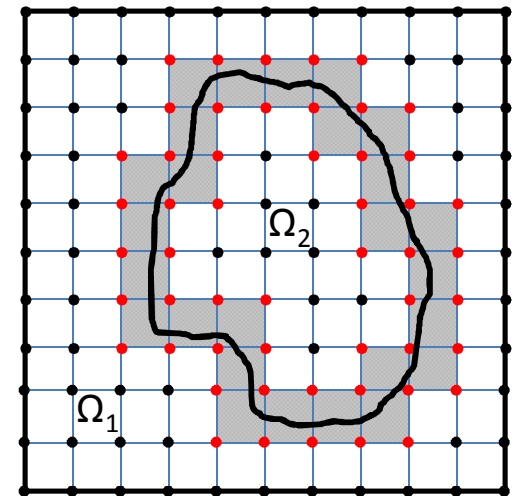
- Extended Finite Element Method (X-FEM) [Sukumar 2001, Moës 2003]
 - Useful for modeling inhomogeneities such as voids, cracks, or inclusions without requiring the mesh to conform to the interface
 - The finite element approximation is locally enriched

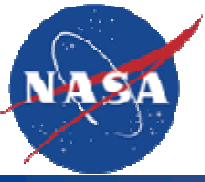
$$u = \sum_{i \in I} N_i(x) u_i + \sum_{i \in I^*} N_i(x) \Psi(x) a_i$$

- Standard elements – no enrichment 
- Reproducing elements – all nodes enriched 
- Blending elements – some nodes are enriched 
- The material interface is defined by a level set function (ϕ)
 - The iso-zero of ϕ defines the interface location
 - ϕ is negative in Ω_1 and positive in Ω_2
 - ϕ is discretized according to the finite element mesh

$$\phi(x) = \sum_{i \in I} N_i(x) \phi_i$$

- Regular nodes (I)
- Enriched nodes (I^*)





Background: SSFEM



- Spectral Stochastic Finite Element Method (SSFEM) [Ghanem 2003]
 - Useful for modeling partial differential equations with random inputs
 - For static heat conduction or structural mechanics problems, the discrete form of the governing equation is given by

$$\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) - \mathbf{f}(\boldsymbol{\xi}) = 0$$

where $\boldsymbol{\xi}$ is a vector of independent random parameters

- Stochastic approximation to represent each component of the unknown in the stochastic space

$$u_i(\boldsymbol{\xi}) \approx \sum_{j=1}^N \Gamma_j(\boldsymbol{\xi}) a_{i,j}$$

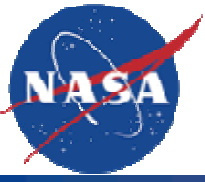
- Spectral PC – spectral approach using generalized Polynomial Chaos
 - Linear FE – finite element approximation using linear interpolation functions
- Following the Galerkin approach,

$$\mathbf{K}_s \mathbf{a} - \mathbf{f}_s = 0$$

where

$$\mathbf{K}_s = \langle \boldsymbol{\Gamma}(\boldsymbol{\xi})^T \mathbf{K}(\boldsymbol{\xi}) \boldsymbol{\Gamma}(\boldsymbol{\xi}) \rangle \quad \mathbf{f}_s = \langle \boldsymbol{\Gamma}(\boldsymbol{\xi})^T \mathbf{f}(\boldsymbol{\xi}) \rangle$$

- Requires a conforming mesh for each realization



X-SFEM



- Combined X-FEM and SFEM [Nouy 2008]
 - Approximation in spatial and stochastic domain

$$u(\mathbf{x}, \xi) = \sum_{i \in I} N_i(\mathbf{x}) u_i(\xi) + \sum_{i \in I^*} N_i(\mathbf{x}) \Psi(\mathbf{x}, \xi) a_i(\xi)$$

- Spectral PC or Linear FE approximation for $u_i(\xi)$ and $a_i(\xi)$
- I^* is the set of nodes for all possibly intersected elements

- Enrichment Strategies

- Absolute Value – approach in X-FEM, extended to the stochastic framework

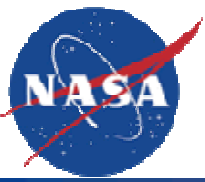
$$\Psi(x, \xi) = |\phi(x, \xi)| = \left| \sum_{i \in I} N_i(x) \phi_i(\xi) \right|$$

- Moës – optimal convergence in X-FEM, extended to the stochastic framework

$$\Psi(x, \xi) = \sum_{i \in I} N_i(x) |\phi_i(\xi)| - |\phi(x, \xi)|$$

- Nouy – proposed for X-SFEM, includes enriched degrees of freedom associated with blending elements

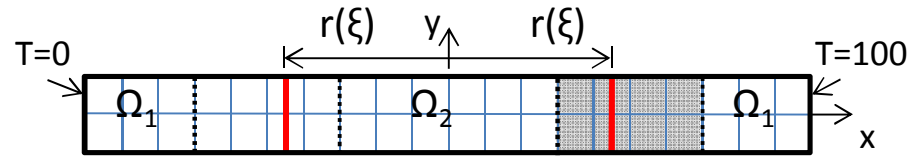
$$\Psi(x, \xi) = \sum_{i \in I_1^*} N_i(x) \beta - |\phi(x, \xi)| \quad \beta \approx \sup_{i \in I^*} |\phi_i(\xi)|$$



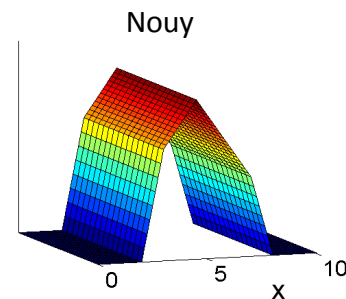
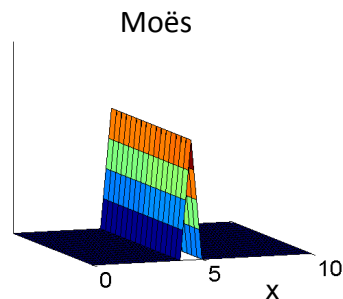
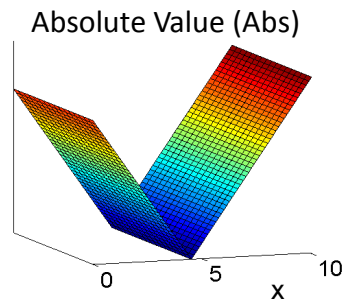
Numerical Example 1



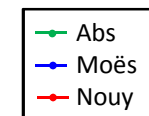
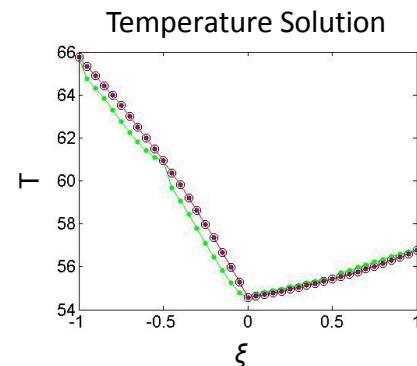
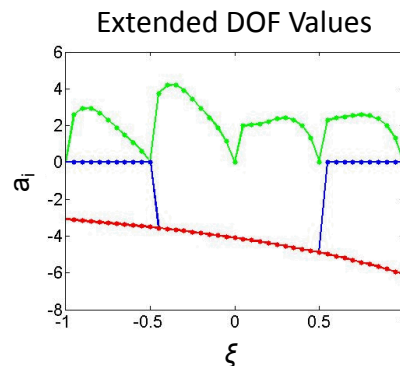
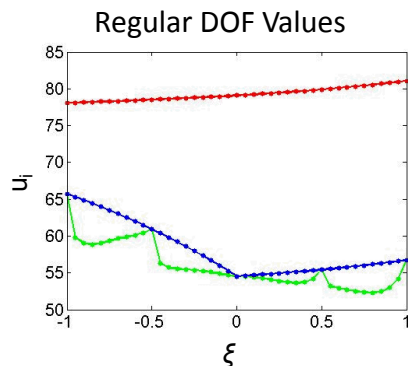
- Look at enrichment strategies
- Consider integration in stochastic domain
- Perform convergence analysis



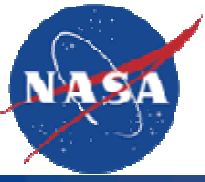
- Realization of $\Psi(x, \xi)$ for $r=4.5$



- X-FEM deterministic results for sample points
 - Degree of freedom values and the solution at $x=5$



$$u_i(\xi) \approx \sum_{j=1}^N \Gamma_j(\xi) a_{i,j}$$

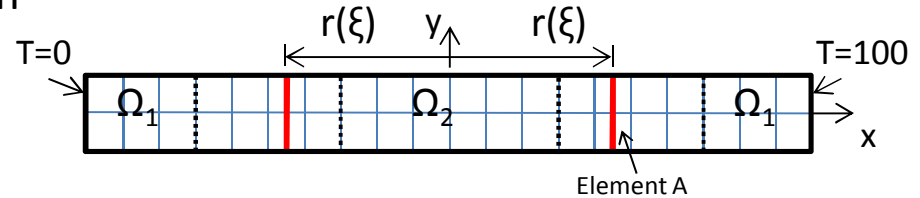


Numerical Integration



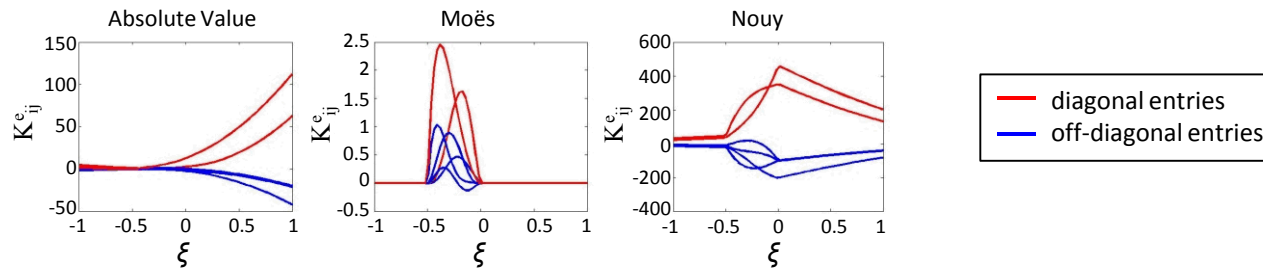
- Consider integration in stochastic domain

$$\mathbf{K}_s^e = \langle \mathbf{\Gamma}(\xi)^T \mathbf{K}^e(\xi) \mathbf{\Gamma}(\xi) \rangle$$

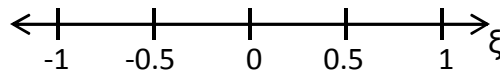


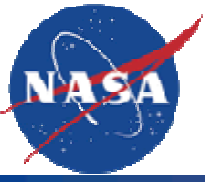
- Absolute Value and Nouy are defined over the entire enriched space, while Moës is restricted to the elements cut by the interface for each realization

$\mathbf{K}_{ij}^e(\xi)$ – Element A, enriched degrees of freedom



- An appropriate stochastic partition for specifying integration points would be 4 equal subdivisions for each element



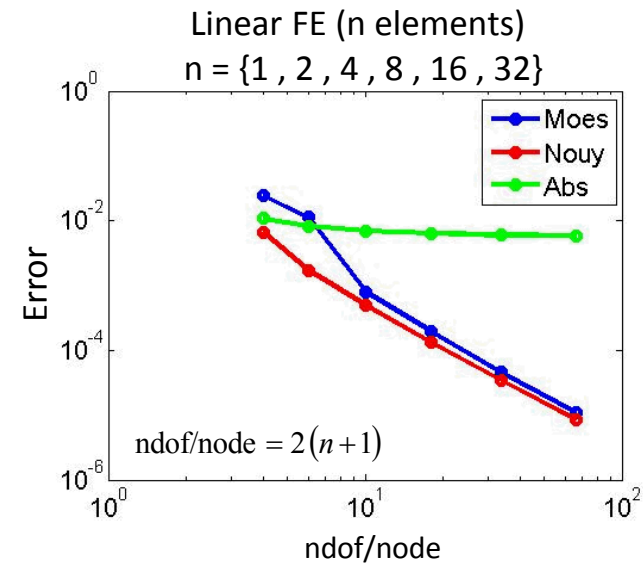
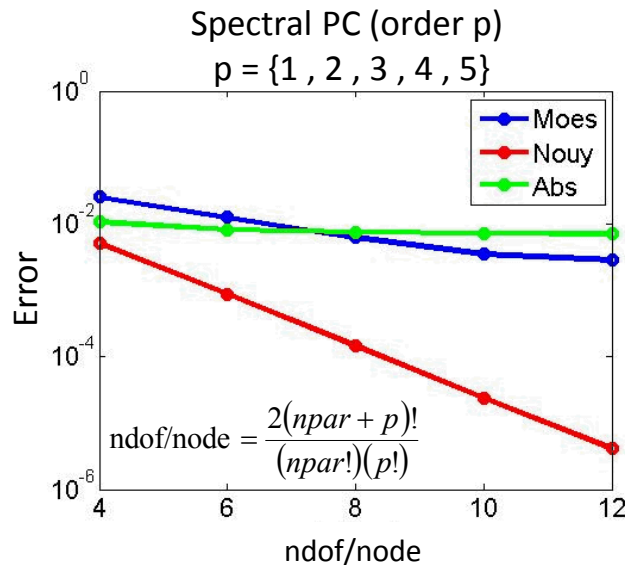


Error Convergence

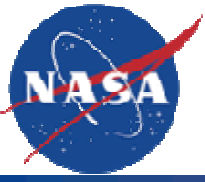


- Error convergence using the analytical solution (u) and the approximate solution (u^*) in the relative L^2 norm

$$\text{Error} = \frac{\|u - u^*\|_{L^2}}{\|u\|_{L^2}}$$



- The number of degrees of freedom per node (ndof/nod) depends on the number of random parameters (n_{par}) and the order (p) for spectral PC, and the number of nodes in the stochastic mesh for linear FE
- Nouy's enrichment has the highest rate of convergence
- Moës' enrichment has a similar rate of convergence as Nouy for linear FE
- For Nouy's enrichment, spectral PC achieves the same level of error as linear FE with fewer ndof/node



Numerical Example 2



- 2-D steady-state heat conduction

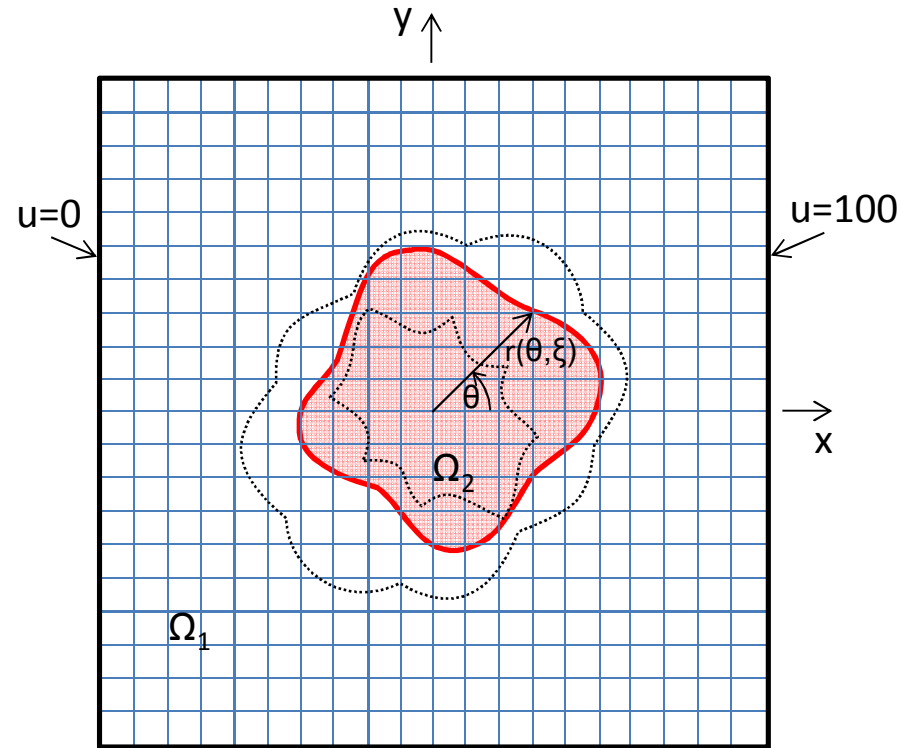
$$-\nabla^T(\kappa \nabla u) = 0 \quad \text{in } \Omega$$
$$\Omega = (-10,10) \times (-10,10)$$

$$\kappa = \begin{cases} 2 & \text{in } \Omega_1(\xi) \\ 20 & \text{in } \Omega_2(\xi) \end{cases}$$

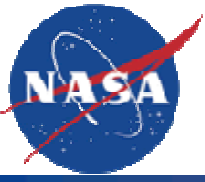
- Generate random 2-D inclusion

$$r(\theta, \xi) = \bar{r} + \sigma \sum_{k=1}^m \frac{1}{k} \xi_k [\cos(k^2 \theta) + \sin(k^2 \theta)]$$

$$h = 1, \bar{r} = 4, \sigma = 1, m = 2, \xi_k \in U(-1,1)$$



- Element integration in stochastic domain
- Compare convergence in the energy norm for the different enrichment strategies and stochastic approximations



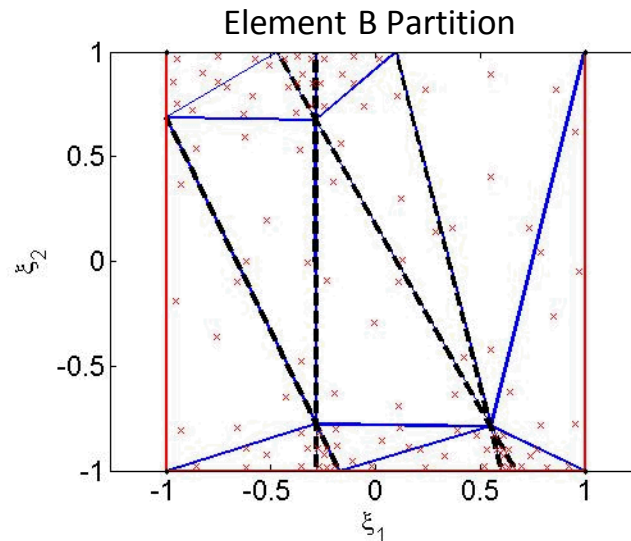
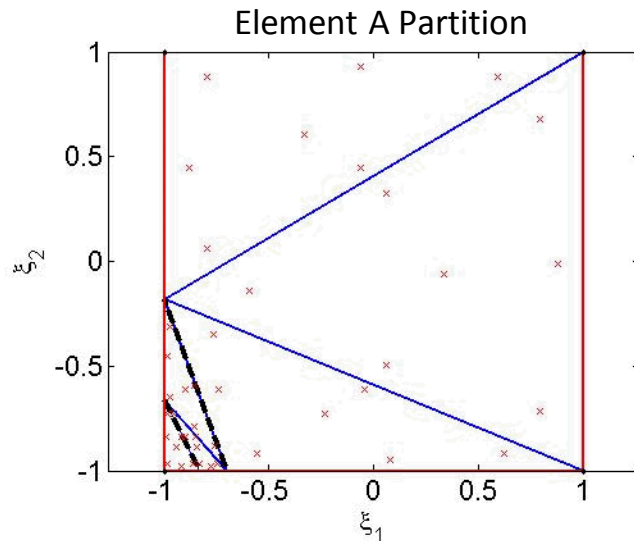
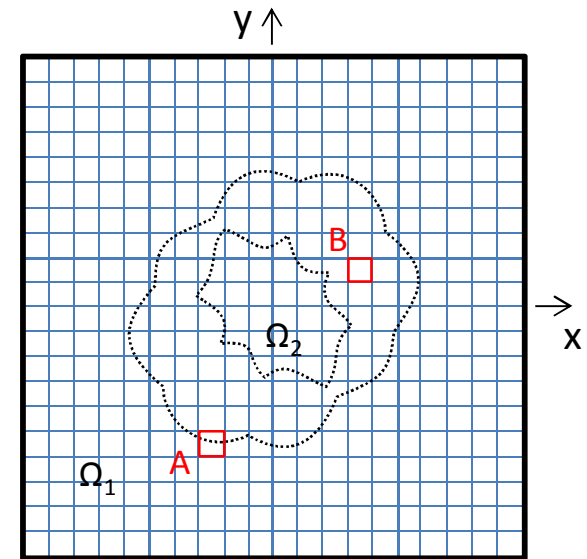
Numerical Integration

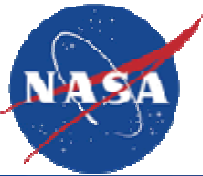


- Determine a partition on which $\mathbf{K}^e$ is piecewise regular

$$\mathbf{K}_s^e = \langle \boldsymbol{\Gamma}(\boldsymbol{\xi})^T \mathbf{K}^e(\boldsymbol{\xi}) \boldsymbol{\Gamma}(\boldsymbol{\xi}) \rangle$$

- Use nodal level set values, $\phi_i(\boldsymbol{\xi})$, to determine the zero level set curve for each node
- Apply a Delaunay triangulation to create partition
- Specify integration points per triangle



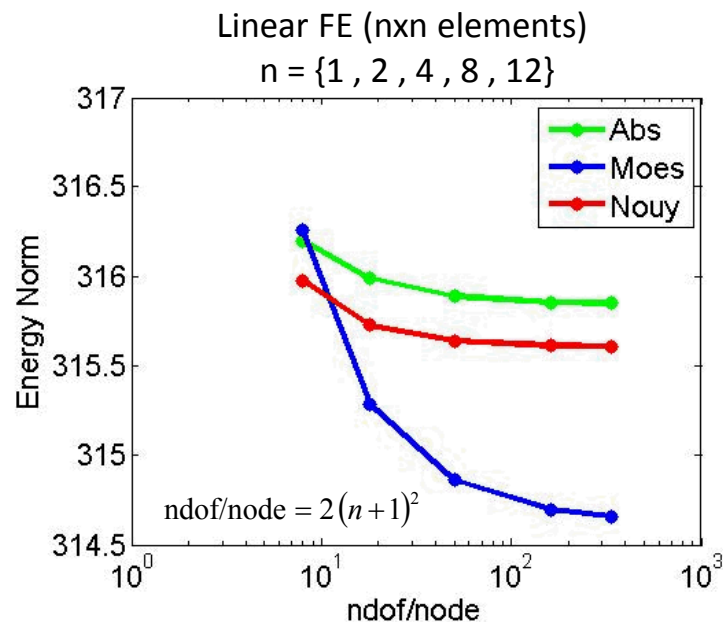
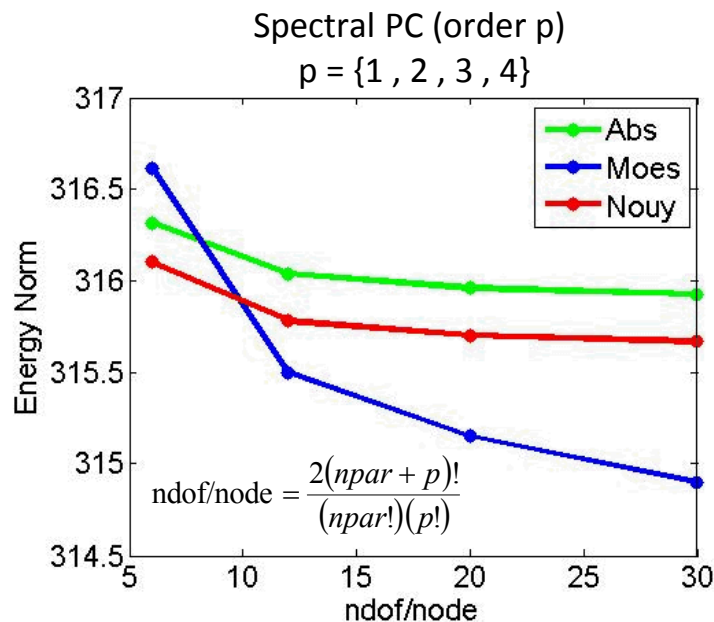


Solution Convergence

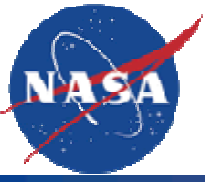


- Solution convergence in the energy norm

$$\text{Energy Norm} = \|u^*\|_E = \sqrt{u^T K u}$$



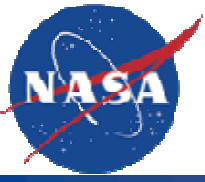
- Error versus the number of dof per node (ndof/node)
- Fixed spatial mesh at $h=1$, refinement for stochastic approximation
- Moës shows a slower convergence of the solution than Absolute Value and Nouy
- For Nouy's enrichment, the energy norm is approximately the same using the spectral PC (fewer dofs) or linear FE approximation



Summary



- Goal of this research is the stochastic prediction of material properties for heterogeneous materials
- X-SFEM – combined Extended FEM and Stochastic FEM approach for modeling heterogeneous materials with uncertain interfaces
- For element integration in the stochastic domain, a partition approach based on the nodal level set values and a Delaunay triangulation was implemented
- Enrichment strategy affects the accuracy and efficiency of the numerical solution
 - Three enrichment strategies were compared
 - For the 1-D example, Nouy's enrichment showed the highest convergence rate
 - For the 2-D example, Nouy's and Absolute Value enrichments showed the highest convergence rate
 - Spatial mesh refinement needs to be considered
- A spectral polynomial chaos and linear finite element stochastic approximation were compared
 - Spectral PC has fewer dofs, and Nouy's enrichment with spectral PC showed comparable results as linear FE for the examples studied



Thank you for your attention.



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