

Self-consistency properties of elementary Reynolds stress closures

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Outline

- Series expansion of the correlation tensor in homogeneous anisotropic turbulence
- Substitute in mean flow coupling equations
- Truncated hierarchies
- Application to the Launder-Reece-Rodi model

Elementary properties of the correlation tensor $U_{ij}(\mathbf{k})$ in the Fourier representation:

- solenoidal property: $k_i U_{ij}(\mathbf{k}) = k_j U_{ij}(\mathbf{k}) = 0$.
- index symmetry: $U_{ij}(\mathbf{k}) = U_{ji}(\mathbf{k})$

Known representations of the correlation tensor

- isotropy (folk knowledge)

$$U_{ij}(\mathbf{k}) = U(k)P_{ij}(\mathbf{k})$$

$P_{ij}(\mathbf{k}) = \delta_{ij} - k^{-2}k_i k_j$ is the unique isotropic solenoidal tensor.

- axial symmetry (Batchelor, Chandrasekhar)

$$U_{ij}(\mathbf{k}) = U^{(1)}(\mathbf{k})P_{ij}(\mathbf{k}) + U^{(2)}(\mathbf{k})S_{ij}(\mathbf{k})$$

where

$$S_{ij}(\mathbf{k}) = k^2 a_i a_j - (\mathbf{a} \cdot \mathbf{k})(k_i a_j + k_j a_i) + \frac{1}{2}(\mathbf{a} \cdot \mathbf{k})^2 [\delta_{ij} + k^{-2}k_i k_j]$$

- general anisotropy

???

Directional-polarization decomposition (Cambon):

$$U_{ij}(\mathbf{k}) = U_{ij}^{dir}(\mathbf{k}) + U_{ij}^{pol}(\mathbf{k})$$

$$U_{ij}^{dir}(\mathbf{k}) = \frac{1}{2}(\mathbf{U}(\mathbf{k}) : \mathbf{P}(\mathbf{k}))P_{ij}(\mathbf{k}) \text{ and } U_{ij}^{pol}(\mathbf{k}) = U_{ij}(\mathbf{k}) - U_{ij}^{dir}(\mathbf{k}).$$

$$U_{ij}(\mathbf{k}) = \underbrace{U^{dir}(\mathbf{k})P_{ij}(\mathbf{k})}_{\text{tensorially isotropic}} + \underbrace{U_{ij}^{pol}(\mathbf{k})}_{\text{trace-free solenoidal}}$$

Expansion of $U^{dir}(\mathbf{k})$

Motivation: action of rotation group on homogeneous polynomials

$$\underbrace{\{1\}}_{1\text{D}} = \underbrace{\{1\}}_{1\text{D: spin } 0}$$

$$\underbrace{\{x, y, z\}}_{3\text{D}} = \underbrace{\{x, y, z\}}_{3\text{D: spin } 1}$$

$$\underbrace{\{x^2, xy, \dots\}}_{6\text{D}} = \underbrace{\{x^2 + y^2 + z^2\}}_{1\text{D: spin } 0} + \underbrace{\{x^2 - y^2, y^2 - z^2, \dots\}}_{5\text{D: spin } 2}$$

$$\underbrace{\{x^3, x^2y, \dots\}}_{10\text{D}} = \underbrace{(x^2 + y^2 + z^2)\{(x, y, z)\}}_{3\text{D: spin } 1} + \underbrace{\{(x^3 - xy^2, x^3 - xz^2, \dots\}}_{7\text{D: spin } 3}$$

The last sets: 1D, 3D, 5D, ... are homogeneous harmonic polynomials, the irreducible representations of the rotation group.

Expansion in symmetry type:

$$U^{dir}(\mathbf{k}) = U(k) + H_{mn}(k)k^{-2}k_mk_n + H_{mnrs}(k)k^{-4}k_mk_nk_rk_s + \dots$$

Note parametrization of the angular dependence.

The coefficient tensors are:

- trace-free
- functions of wavenumber k only
- angular *moments* of $U^{dir}(\mathbf{k})$.

Expansion of $U_{ij}^{pol}(\mathbf{k})$

Generate spin n tensors by operating on homogeneous harmonic polynomial of degree n with a rotation-invariant (spin 0) differential operator.

Zemach's operators:

$\mathbf{L} = \mathbf{k} \times \nabla$ (generators of the rotation group), $\mathbf{M} = \mathbf{k} \times \mathbf{L}$.

Index notation:

$$L_i = \epsilon^{imn} k_m \partial_n$$

$$M_i = \epsilon^{imn} k_m \epsilon^{nrs} k_r \partial_s.$$

The general even spin polarization tensor can be written using Zemach's operators as $\mathcal{L}_{ij}(\mathbf{k})[\phi(\mathbf{k})]$ where

$$\mathcal{L}_{ij}(\mathbf{k}) = (L_i L_j + L_j L_i) + k^{-2}(k_i M_j + k_j M_i) - L^2 P_{ij}(\mathbf{k})$$

where ϕ is homogeneous and harmonic[†] and even degree.[‡]

For odd spins, use $\mathcal{O}_{ij}(\mathbf{k})[\psi(\mathbf{k})]$ where

$$\mathcal{O}_{ij}(\mathbf{k}) = (k_i L_j + k_j L_i) - (M_i L_j + M_j L_i)$$

where ψ is homogeneous and harmonic[†] and odd degree.[‡]

†: representation theory

‡: parity

Examples:

$$\mathcal{L}_{ij}[J_{ij}(k)k_ik_j] \equiv J_{ij}^{pol}(\mathbf{k}) = J_{ij}(k) - k^{-2}k_m(k_iJ_{mj}(k) + k_jJ_{mi}(k)) \\ + \frac{1}{2}k^{-2}(\delta_{ij} + k^{-2}k_ik_j)(J(k) : \mathbf{k}\mathbf{k})$$

$$\mathcal{O}_{ij}[K_{ijm}(k)k_ik_jk_m] \equiv K_{ij}^{pol}(\mathbf{k}) = k^{-2}\epsilon^{iab}P_{jd}(\mathbf{k})K_{bcd}(k)k_ak_c \\ + k^{-2}\epsilon^{jab}P_{id}(\mathbf{k})K_{bcd}(k)k_ak_c$$

(Kassinos et al., *stropholysis*).

The complete series expansion is

$$\begin{aligned}
U_{ij}(\mathbf{k}) = & \left\{ U(k) + H_{ij}(k)k^{-2}k_i k_j + H_{ijmn}(k)k_k k_j k_m k_n + \dots \right\} P_{ij}(\mathbf{k}) \\
& + \mathcal{L}_{ij} \left[J_{\ell m}(k)k^{-2}k_\ell k_m + J_{\ell mnp}(k)k^{-4}k_\ell k_m k_n k_p + \dots \right] \\
& + \mathcal{O}_{ij} \left[K_{\ell mn}(k)k^{-3}k_\ell k_m k_n + K_{\ell mnpq}(k)k^{-5}k_\ell k_m k_n k_p k_q + \dots \right]
\end{aligned}$$

The coefficient tensors H, J, K are functions of wavenumber k alone: reduction of dimensionality from three ($\mathbf{k}$) to one.

Some truncations:

spin 2 only: $H_{ij}(k)$, $J_{ij}(k)$ 10D

spins 2,3 only: $H_{ij}(k)$, $J_{ij}(k)$, $K_{ijm}(k)$ 17D

spins 2,3,4 only: $H_{ij}(k)$, $J_{ij}(k)$, $K_{ijm}(k)$, $H_{ijmn}(k)$, $J_{ijmn}(k)$ 35D

Mean flow coupling equations

$$\begin{aligned} \dot{U}_{ij}(\mathbf{k}) = & - \underbrace{\left[U_{ip}(\mathbf{k}) \frac{\partial U_j}{\partial x_p} + U_{jp}(\mathbf{k}) \frac{\partial U_i}{\partial x_p} \right]}_{\text{production}} + \underbrace{k_m \frac{\partial}{\partial k_n} U_{ij}(\mathbf{k}) \frac{\partial U_m}{\partial x_n}}_{\text{mean-flow distortion}} \\ & + \underbrace{2k^{-2} \left[k_i k_m U_{pj}(\mathbf{k}) + k_j k_m U_{pi}(\mathbf{k}) \right]}_{\text{rapid pressure-strain}} \frac{\partial U_m}{\partial x_p} \end{aligned}$$

production: energy addition

mean-flow distortion: energy transfer among $\mathbf{k}$ modes

rapid pressure-strain: energy transfer among tensor components

First substitute directional-polarization decomposition:

$$\dot{U}^{dir}(\mathbf{k}) = U^{dir}(\mathbf{k})(S : \mathbf{k}\mathbf{k}) - U^{pol}(\mathbf{k}) : S + k_m \frac{\partial}{\partial k_n} U^{dir}(\mathbf{k}) \frac{\partial U_m}{\partial x_n}$$

where $S_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial k_j} + \frac{\partial U_j}{\partial k_i} \right)$.

$$\begin{aligned} & \dot{U}_{ij}^{pol}(\mathbf{k}) \\ &= -\frac{1}{2} U^{dir}(\mathbf{k}) \left(P_{im}(\mathbf{k}) P_{jn}(\mathbf{k}) + P_{in}(\mathbf{k}) P_{jm}(\mathbf{k}) - P_{ij}(\mathbf{k}) P_{mn}(\mathbf{k}) \right) \frac{\partial U_m}{\partial x_p} \\ & \quad - \left(U_{ip}^{pol}(\mathbf{k}) P_{jm}(\mathbf{k}) + U_{jp}^{pol}(\mathbf{k}) P_{im}(\mathbf{k}) - U_{mp}^{pol}(\mathbf{k}) P_{ij}(\mathbf{k}) \right) \frac{\partial U_m}{\partial x_p} \\ & \quad + k_m \frac{\partial}{\partial k_n} U_{ij}^{pol}(\mathbf{k}) \frac{\partial U_m}{\partial x_n} + k^{-2} \left(k_i k_m U_{pj}^{pol}(\mathbf{k}) + k_j k_m U_{pi}^{pol}(\mathbf{k}) \right) \frac{\partial U_m}{\partial x_p} \end{aligned}$$

Next substitute series expansion:

$$\dot{U}^{dir}(\mathbf{k}) = U^{dir}(\mathbf{k})(S : \mathbf{k}\mathbf{k}) - U^{pol}(\mathbf{k}) : S + k_m \frac{\partial}{\partial k_n} U^{dir}(\mathbf{k}) \frac{\partial U_m}{\partial x_n} \quad (\#)$$

In the term

$$\begin{aligned} & U^{dir}(\mathbf{k})(S : \mathbf{k}\mathbf{k}) \\ &= (H_{mn}(k)k^{-2}k_mk_n)(S_{pq}k_pk_q) + (H_{mnrs}(k)k^{-4}k_mk_nk_rk_s)(S_{pq}k_pk_q) + \dots \end{aligned}$$

$H_{mn}(k)$ generates terms of spins 0, 2, 4;

$H_{mnrs}(k)$ generates spins 2, 4, 6; etc.

Sorting it all out,

$$\dot{U}(k) = \text{spin 0 contributions to RHS of } (\#)$$

$$\dot{H}_{ij}(k) = \text{spin 2 contributions to RHS of } (\#)$$

$$\dot{H}_{ijmn}(k) = \text{spin 4 contributions to RHS of } (\#)$$

The structure of these spin contributions is

$$\|(H : \mathbf{k}\mathbf{k})(S : \mathbf{k}\mathbf{k})\|_4 \propto HS$$

$$\|(H : \mathbf{k}\mathbf{k})(S : \mathbf{k}\mathbf{k})\|_2 \propto H \cdot S + S \cdot H$$

$$\|(H : \mathbf{k}\mathbf{k})(S : \mathbf{k}\mathbf{k})\|_0 \propto H : S$$

Explicitly,

$$(H : \mathbf{k}\mathbf{k})(S : \mathbf{k}\mathbf{k}) = \|A\|_4 + \|A\|_2 + \|A\|_0$$

where

$$\|A\|_4 = (H : \mathbf{k}\mathbf{k})(S : \mathbf{k}\mathbf{k}) - \frac{2}{7}k^2(H \cdot S + S \cdot H) : \mathbf{k}\mathbf{k} + \frac{2}{35}k^4 H : S$$

$$\|A\|_2 = \frac{2}{7}k^2 \left(H \cdot S + S \cdot H - \frac{2}{3}(H : S)I \right) : \mathbf{k}\mathbf{k}$$

$$\|A\|_0 = \frac{2}{15}k^4 (H : S)$$

Polarization is similar.

We obtain a system coupling the spin components:

$$\begin{aligned}\dot{H}^n &\sim H^{n-2}S, H^n, H^{n+2} : S, J^{n-2}S, J^n, J^{n+2} : S, K^{n-1}\Omega, K^{n+1} \cdot \Omega \\ j^n &\sim H^{n-2}S, H^n, H^{n+2} : S, J^{n-2}S, J^n, J^{n+2} : S, K^{n-1}\Omega, K^{n+1} \cdot \Omega \\ \dot{K}^n &\sim H^{n-1}\Omega, H^{n+1} \cdot \Omega, J^{n-1}\Omega, J^{n+1} \cdot \Omega, K^{n-2}S, K^{n+2} : S\end{aligned}$$

where Ω is the mean vorticity (spin 1; couples odd and even spins).

The coefficient tensors depend on scalar wavenumber k instead of wavevector $\mathbf{k}$.

Nonlinear processes of ‘return to isotropy’ should permit usable truncations to low-order spins.

BHRZ is the truncation of this hierarchy to spin 2 (spherical averages).

Truncation at spin n incurs error terms due to spins $n + 1$ and $n + 2$.

Single-point modeling

Recall

$$\begin{aligned} \dot{U}_{ij}(\mathbf{k}) = & - \left[U_{ip}(\mathbf{k}) \frac{\partial U_j}{\partial x_p} + U_{jp}(\mathbf{k}) \frac{\partial U_i}{\partial x_p} \right] + k_m \frac{\partial}{\partial k_n} U_{ij}(\mathbf{k}) \frac{\partial U_m}{\partial x_n} \\ & + 2k^{-2} \left[k_i k_m U_{pj}(\mathbf{k}) + k_j k_m U_{pi}(\mathbf{k}) \right] \frac{\partial U_m}{\partial x_p} \end{aligned}$$

Integrate over $\mathbf{k}$ to suppress scale information (reduced dimension):

$$\int d\mathbf{k} \ U_{ij}(\mathbf{k}) = \frac{2}{3} k \delta_{ij} + R_{ij}$$

k: turbulence kinetic energy, R: Reynolds stress deviator.

Single-point modeling is a projected dynamics.

Limitations are unavoidable!

Single-point reduction:

$$\begin{aligned} \int d\mathbf{k} \dot{U}_{ij}(\mathbf{k}) &= - \underbrace{\int d\mathbf{k} U_{ip}(\mathbf{k}) \frac{\partial U_j}{\partial x_p}}_{\text{OK, 'closed'}} + \int d\mathbf{k} \underbrace{2k^{-2}k_i k_m U_{pj}(\mathbf{k})}_{\text{closure problem}} \frac{\partial U_m}{\partial x_p} + (ij) \\ &= -\frac{4}{3}\mathbf{k}S_{ij} - \left[R_{ip} \frac{\partial U_j}{\partial x_p} + R_{jp} \frac{\partial U_i}{\partial x_p} \right] + \Pi_{ij} \end{aligned}$$

Note no single-point mean-flow distortion effect.

Trace-deviator decomposition:

$$\begin{aligned} \dot{\mathbf{k}} &= -\mathbf{R} : \mathbf{S} \\ \dot{R}_{ij} &= -\frac{4}{3}\mathbf{k}S_{ij} - \left[R_{ip} \frac{\partial U_j}{\partial x_p} + R_{jp} \frac{\partial U_i}{\partial x_p} - \frac{2}{3}(\mathbf{R} : \mathbf{S})\delta_{ij} \right] + \Pi_{ij} \end{aligned}$$

Launder-Reece-Rodi model

Assume Π is linear in R and mean velocity gradient. Taking index symmetries into account,

$$\begin{aligned}\dot{R}_{ij} = & -\frac{8}{15}kS_{ij} + (C_b - 1) \left[R_{ip} \frac{\partial U_j}{\partial x_p} + R_{jp} \frac{\partial U_i}{\partial x_p} - \frac{2}{3}(R : S)\delta_{ij} \right] \\ & + (8C_b - 6) \left[R_{ip} \frac{\partial U_p}{\partial x_j} + R_{jp} \frac{\partial U_p}{\partial x_i} - \frac{2}{3}(R : S)\delta_{ij} \right]\end{aligned}$$

where C_b is an adjustable model constant.

Reformulation of the LRR model

The stress deviator has two parts (Cambon, Kassinos):

$R = R^{dir} + R^{pol}$ where

$$R_{ij}^{dir} = \int d\mathbf{k} (\mathbf{H} : \mathbf{k}\mathbf{k}) P_{ij}(\mathbf{k}) = \frac{2}{15} H_{ij} \quad R_{ij}^{pol} = \int d\mathbf{k} J_{ij}^{pol}(\mathbf{k}) = \frac{2}{5} J_{ij}$$

No other spin contributes to the stress.

What about stress evolution? Π_{ij} receives contributions from spins 2, 3, and 4. It is not 'closed' in terms of R^{dir} and R^{pol} , let alone in terms of R .

Truncate hierarchy at spin 2:

$$\begin{aligned}
\dot{R}_{ij}^{dir} &= \frac{4}{15}kS_{ij} - \frac{1}{7} \left(R_{im}^{dir}S_{mj} + R_{jm}^{dir}S_{mi} - \frac{2}{3}(R^{dir} : S)\delta_{ij} \right) \\
&\quad - \left(R_{im}^{dir}\Omega_{mj} + R_{jm}^{dir}\Omega_{mi} \right) - \frac{2}{7} \left(R_{im}^{pol}S_{mj} + R_{jm}^{pol}S_{mi} - \frac{2}{3}(R^{pol} : S)\delta_{ij} \right) \\
\dot{R}_{ij}^{pol} &= -\frac{12}{15}kS_{ij} - \frac{12}{7} \left(R_{im}^{dir}S_{mj} + R_{jm}^{dir}S_{mi} - \frac{2}{3}(R^{dir} : S)\delta_{ij} \right) \\
&\quad + \frac{1}{7} \left(R_{im}^{pol}S_{mj} + R_{jm}^{pol}S_{mi} - \frac{2}{3}(R^{pol} : S)\delta_{ij} \right) + \frac{1}{3} \left(R_{im}^{pol}\Omega_{mj} + R_{jm}^{pol}\Omega_{mi} \right)
\end{aligned}$$

where

$$\Omega_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right)$$

Even ignoring higher spins, the result is not a closed equation for the stress deviator R alone.

Such an equation can only be obtained by postulating a relation between R^{dir} and R^{pol} . Assume (Cambon) $R^{dir} = \lambda R^{pol}$. Then

$$\begin{aligned} \dot{R}_{ij} = & -\frac{4}{15}k \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \\ & - \frac{9\lambda + 5}{21(\lambda + 1)} \left(R_{im} \frac{\partial U_j}{\partial x_m} + R_{jm} \frac{\partial U_i}{\partial x_m} - \frac{2}{3}(R : S)\delta_{ij} \right) \\ & - \frac{30\lambda - 2}{21(\lambda + 1)} \left(R_{im} \frac{\partial U_m}{\partial x_j} + R_{jm} \frac{\partial U_m}{\partial x_i} - \frac{2}{3}(R : S)\delta_{ij} \right) \end{aligned}$$

LRR assumes the proportionality $R^{dir} \propto R^{pol}$.

Such proportionality is possible, but the constant is not unique.

No one LRR model can describe all cases.

(Cambon and Rubinstein, Phys. Fluids 18 (2006))

Example: for axisymmetric strain

$$S = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$R^{dir} : R^{pol} = 1 : 2$ and $-3 : 1$ are both possible depending on the sign of S . LRR cannot be consistent with both.

LRR ignores higher spins, in particular spin 3 (stropholysis) linked to antisymmetric contributions to the mean velocity gradient. Note conventional wisdom: LRR is least good for rotational flows.

Improved models?

Conventional modeling seeks improvement by increased model complexity, for example through nonlinearity in stresses. Success has been mixed, and the fundamentals of such models has been challenged. (Mishra and Girimaji, JFM **755** (2014))

Solution of a large number of transport equations for higher spins (17? 35?) is impractical. But rapid nonlinear relaxation of higher spin quantities may permit useful simplifications. Research is in progress.

Model derivation by truncation of the equation hierarchy permits systematic error analysis. This feature is completely absent in conventional modeling.

CONCLUSIONS

- Expansion of the correlation tensor was derived.
- Equation hierarchy derived from mean flow couplings.
- LRR model analyzed from this perspective.
- Advanced modeling is in progress.