A second-order closure turbulence model:

new heat flux equations and no critical Richardson number

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Abstract

We formulate a new second-order closure turbulence model by employing a recent closure for the pressure-temperature correlation at the equation level. As a result, we obtain new heat flux equations that avoid the long-standing issue of a finite critical Richardson number. The new, structurally simpler model improves on the Mellor-Yamada 1982 and Galperin et al. 1988 models; key feature includes enhanced mixing under stable conditions facilitating agreement with observational, experimental and high-resolution numerical data sets. The model predicts a planetary boundary layer height deeper than predicted by models with low critical Richardson numbers, as demonstrated in single column model runs of the GISS ModelE general circulation model.
1. Introduction

Since the early 1970s, higher-order turbulence closure models in geophysics pioneered and developed by Mellor and Yamada and others (e.g., Mellor and Yamada 1974; Launder et al. 1975; Lumley 1978; André et al. 1978; Mellor and Yamada 1982 (MY82); Galperin et al. 1988 (GKHR88); Canuto 1992; Shih and Shabbir 1992; Nakanishi 2001; Cheng et al. 2002; Kantha and Clayson 1994, 2004 (KC94, KC04), Sukoriansky et al. 2005; Galperin et al. 2007; Canuto et al. 2008; Kantha and Carniel 2009 (KC09); Bretherton and Park 2009 (BP09); Zilitinkevich 2013) have been broadly successful. These models were derived from the basic dynamic equations which need to be closed, and there is room for improvement of their turbulence closures. The GKHR88 formulation, which significantly simplified the original MY82 model while preserving its important physics, was welcomed by the geophysics community and became widely used. However, MY82 and GKHR88 still have limitations, one of which is the presence of a finite critical Richardson number beyond which turbulence ceases to exist. The predicted value of \( \text{Ri}_{(cr)} = 0.19 \), used by both formulations, is also far too small since various data show turbulence does exist for \( \text{Ri} \gg 1 \) (see a detailed discussion in section 6).

Cheng et al. (2002), Canuto et al. (2008) and others (e.g., KC09, Zilitinkevich et al. 2013) addressed deficiencies in the MY model and its variants, including some noted in Mellor and Yamada (1974). The Cheng et al. (2002) model increased \( \text{Ri}_{(cr)} \) to O(1), while the Canuto et al. (2008) model enhanced \( \text{Ri}_{(cr)} \) to infinity and the KC09 model simplified Canuto et al. (2008). All changes resulted in improved agreement with available observational, experimental and high-resolution numerical data. A drawback was that these
models were more complicated than MY82 and none as efficient and convenient as
GKHR88. In the Canuto et al. (2008) model, while Ri(cr) is infinity and no longer appears
in the model, some of the originally-constant coefficients in the model become flow
dependent, leading to an increase in the complexity of the model. Bastak et al. (2014)
demonstrated that an alternative relation between the pressure-temperature time scale and
the dissipation time scale derived by Canuto et al. (2008) can be used to make the model
coefficients flow independent without a Ri(cr), a feature that deserves further exploration.
Despite these advances, the cause of a finite Ri(cr) was not yet evident. Since turbulence
exists at high Ri in geophysical flows, there is a need for clearer understanding of this
behavior and for an efficient model that reproduces it.

In this work, we present a turbulence closure model that leads to new horizontal and
vertical heat flux equations. A careful comparison between the old and new heat flux
equations reveals the cause of a finite (and unphysical) critical Richardson number Ri(cr),
while the new equations are without a finite Ri(cr), consistent with a variety of
meteorological observation, laboratory experimental, direct numerical simulation (DNS)
and large eddy simulation (LES) data. Furthermore, the new model’s stability functions are
structurally simpler than the GKHR88 model. The new model consists of a hierarchy of
levels based on MY82 and GKHR88’s terminology: non-equilibrium (levels 3 and 2.5),
quasi-equilibrium (levels 2.75 and 2.25) and equilibrium (level 2) with level 2.75 being our
addition to MY82 and GKHR88’s hierarchy (see Appendix B). All levels higher than 2 can
be combined with a non-local treatment of the turbulent kinetic energy and/or the turbulent
potential energy to enhance transport processes. Since the levels 3 and 2.75 are not the
main topic of the present study, we include them in an appendix.
In section 2 we derive the new heat flux equations using the new turbulence closure, leading to a non-equilibrium model (level 2.5) that improves upon the MY82 level 2.5 model while in section 3 we modify the GKHR88 procedure and derive a new quasi-equilibrium model (level 2.25). The realizability conditions of the new models are derived in section 4, and we derive the new equilibrium model (level 2) that improves upon the MY82 and GKHR88 level 2 models in section 5. In section 6 we compare the model results with various data. In section 7, we discuss the length scale parameterization appropriate to the new models. In section 8 we show improved predictions of PBL height and other variables in the single column model (SCM) of the GISS general circulation model (GCM), with additional discussion and overall conclusions provided in section 9. Appendix A provides the derivation of the new heat flux equations, Appendix B presents the new model at levels 3 and 2.75 and Appendix C summarizes the MY82 and GKHR88 models for easy reference and comparison.

**2. New heat flux equations and non-equilibrium model: improving on MY82 (level 2.5)**

We start by describing the component equations for the mean and second order turbulent variables with boundary layer approximations, neglecting the tendency and diffusion terms in the equations for the heat fluxes, temperature variance and the departure from isotropy tensor (except in the equation for kinetic energy):

\[ b_\gamma = \overline{u_\gamma u_\gamma} - \frac{1}{2} q^2 \delta_\gamma \]  

(1a)
where $\overline{u_j u_j}$ is the Reynolds stress tensor, $q^2$ is twice the turbulent kinetic energy $e$, and $\delta_{ij}$ is the Kronecker delta. For more detail, see Mellor (1973), MY82 and Cheng et al. (2002); we follow closely the notation of MY82 due to its wide use.

Equations for mean horizontal velocity components ($U, V$) and potential temperature ($\Theta$):

\[
\begin{align*}
\frac{\partial U}{\partial t} &= f_c (V - V_g) - \frac{\partial \overline{uw}}{\partial z} \\
\frac{\partial V}{\partial t} &= -f_c (U - U_g) - \frac{\partial \overline{vw}}{\partial z} \\
\frac{D \Theta}{Dt} &= -\frac{\partial \overline{w \theta}}{\partial z}
\end{align*}
\]  

where $f_c$ is the Coriolis parameter, $(U_g, V_g)$ is the geostrophic wind, $(u, v, w)$ is the turbulent velocity fluctuation, $\theta$ is the turbulent potential temperature fluctuation, $g$ is the gravitational acceleration, $\alpha$ is the coefficient of thermal expansion, $\overline{w \theta}$ is the vertical heat flux and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{U_i \frac{\partial}{\partial x_i}}$.

Equation for kinetic energy $e = \frac{q^2}{2}$:

\[
\frac{\partial e}{\partial t} = T_e + P - \varepsilon
\]  

where $T_e$ represents the vertical transport, $P = P_B + P_S$ where $P_B$ and $P_S$ denote buoyancy and shear production terms respectively, and $\varepsilon$ is the dissipation rate of the turbulent kinetic energy ($\varepsilon$),

$$P_B = g\alpha \theta_1, \quad P_S = -uw \frac{\partial U}{\partial z} - vw \frac{\partial V}{\partial z}, \quad \varepsilon = \frac{q^3}{\Lambda_i} \quad (1f)$$

Equations for the diagonal components of $b_{ij}$:

$$\overline{u^2} - \frac{1}{3} q^2 = -\frac{2l}{q} \left(2uw \frac{\partial U}{\partial z} - vw \frac{\partial V}{\partial z} + \beta g \alpha \theta_1 \right) \quad (2a)$$

$$\overline{v^2} - \frac{1}{3} q^2 = -\frac{2l}{q} \left(2vw \frac{\partial V}{\partial z} - uw \frac{\partial U}{\partial z} + \beta g \alpha \theta_1 \right) \quad (2b)$$

$$\overline{w^2} - \frac{1}{3} q^2 = \frac{2l}{q} \left(uw \frac{\partial U}{\partial z} + vw \frac{\partial V}{\partial z} + 2\beta g \alpha \theta_1 \right) \quad (2c)$$

Equations for the off-diagonal components of $b_{ij}$ (the momentum fluxes):

$$\overline{uv} = -\frac{3l}{q} \left(uw \frac{\partial V}{\partial z} + vw \frac{\partial U}{\partial z} \right) \quad (2d)$$
Equations for heat flux:

\[ \overline{u}\theta = -\frac{3l_1}{q} \left( \overline{uw}\frac{\partial \theta}{\partial z} + \overline{w}\frac{\partial u}{\partial z} \right) \]  

(2g)

\[ \overline{v}\theta = -\frac{3l_2}{q} \left( \overline{vw}\frac{\partial \theta}{\partial z} + \overline{w}\frac{\partial v}{\partial z} \right) \]  

(2h)

\[ \overline{w}\theta = -\frac{3l_2}{q} \left( \overline{w}^2\frac{\partial \theta}{\partial z} - (1-\gamma)g\alpha\overline{\theta} \right) \]  

(2i)

Equations for potential temperature variance \( \overline{\theta^2} \):

\[ \overline{\theta^2} = -\frac{\Lambda_2}{q} \overline{\theta} \overline{\frac{\partial \theta}{\partial z}} \]  

(2j)

(2a-j) above are the same as MY82 except that MY82 assumed \( \beta_s=1 \) and \( \gamma_1=0 \). In (1f) and (2a-j), \( l_i \) is the length scale associated with the return-to-isotropy term of the pressure-
velocity correlation, $C_1$ is the coefficient of a fast term associated with the shear tensor in the correlation, $l_2$ is the length scale associated with the return-to-isotropy term of the pressure-temperature correlation, and $\Lambda_1$ and $\Lambda_2$ are the dissipation length scales for $q^2$ and $\overline{\theta^2}$, respectively. MY82 and its variants parameterized these length scales as

$$(l_1, l_2, \Lambda_1, \Lambda_2) = (A_1, A_2, B_1, B_2)l$$

where $l$ is the dynamic length scale that approaches $\kappa z$ as $z \to 0$ (with $\kappa = 0.4$ being the von Kármán constant) and $A_1, A_2, B_1, B_2$ as well as $C_1, \beta_5, \gamma_1$ all being assumed to be constant (as listed in Table 1). $l$ can be prescribed or solved from a prognostic equation (see, e.g., MY82, Cheng et al. 2002). In section 7 we prescribe a modified turbulence length scale based on the one used by BP09.

In this study, we consider a new closure for the return-to-isotropy term of the pressure-temperature correlation that results in a different set of second moment equations. The closure was proposed by Canuto et al. (2008), when they examined the wavenumber spectra of the pressure-temperature relaxation time scale (see their Eq. 9c). Translated into the present notation, their formulation is re-written as

$$l_2/l = A'_2/(1+\sigma_l), \quad A'_2 \equiv A_2(1+\sigma_{l_0})$$
where the constant $A_2'$ has been determined such that in the neutral condition $l_2/l$ is still
equal to the constant $A_2$. $\sigma_{0}$ is the neutral value (listed in Table 1) of $\sigma$, as determined
by MY82, while $\sigma$ is the turbulent Prandtl number, defined as

$$\sigma = \frac{-P_sR_i}{P_b}$$

(3b)

where $P_b$ and $P_s$ have been defined in (1f) and $R_i$ is the gradient Richardson number:

$$R_i = \frac{N^2}{S^2}, \quad N^2 = g\alpha \frac{d\Theta}{dz}, \quad S^2 = \left(\frac{dU}{dz}\right)^2 + \left(\frac{dV}{dz}\right)^2$$

(3c)

with $N$ being the Brunt-Väisälä frequency (in the stable case), while shear is denoted by $S$.

Unlike previous models that parameterized the length scale ratio $l_2/l$ as a constant ($A_2$),
Eq. (3a) parameterizes it as a function of second moments, an idea motivated by MY82’s
comment: “The major weakness of all models probably relates to the turbulent master
length scale (or turbulent macroscale, or turbulent inertial scale) and, most important, to
the fact that one sets all process scales proportional to a single scale.” The pressure
correlations in turbulence closure models are non-linear in nature (e.g. Shih and Shabbir,
1992), but the existing non-linear pressure correlation parameterizations are too
complicated and not appropriate for geophysical applications at present. However, we can
show that the apparently non-linear pressure correlation closure (3a) can lead to new heat
flux equations that differ from the traditional ones with the new heat flux equations still linear in the second moments. More specifically, substituting (3a) into the heat flux equation (2g-i) thus allowing $l_2/l$ to vary, we obtain, after some algebra (see Appendix A for the derivation), a set of new heat flux equations as provided below.

New heat flux equations:

\[
\overline{u'\theta'} = -\frac{3A'_l}{q} \overline{w'\theta} \frac{\partial U}{\partial z} \quad (3d)
\]

\[
\overline{v'\theta'} = -\frac{3A'_l}{q} \overline{w'\theta} \frac{\partial V}{\partial z} \quad (3e)
\]

\[
\overline{w'\theta} = -\frac{3A'_l}{q} \left[ \overline{w'\frac{\partial \Theta}{\partial z}} (1 - \gamma_1) g \alpha \theta^2 \right] - \left( \overline{u'w'\frac{\partial U}{\partial z}} + \overline{v'w'\frac{\partial V}{\partial z}} \right) S^{-2} \frac{\partial \Theta}{\partial z} \quad (3f)
\]

We now compare the original heat flux equations (2g-i) with the new ones (3d-f) while recalling the long-standing issue of the critical Richardson number as $G_{II}$ approaches to negative infinity ($q^2$ goes to 0). Many previous models have no physically valid solution beyond a finite $Ri_{cr}$, but given that $Ri$ is a large-scale quantity that can assume any value, while the second moments are the responses of turbulence to the large-scale forcing, it would thus be desirable that a turbulence model yields physically sensible second moments for any $Ri$. We trace the finite $Ri_{cr}$ to the presence of the $\overline{uw}$ and $\overline{vw}$ terms in (2g,h). They are the sources of the terms containing $s_4$ and $d_4$ in (C1-2) for the MY82 model and
the source of the $d_4$ term in (C7) for the GKHR88 model (see appendix C), and we can show that these terms are responsible for the finite $\text{Ri}(cr)$. Since MY82 and GKHR88 are essentially identical in the equilibrium case, where $\text{Ri}$ is the only relevant independent variable, we will first discuss the simpler GKHR88 model. In the GKHR88 model, substituting $\overline{uw}$ in (2g) and $\overline{w^2}$ in (6c), where (6c) is introduced by the GKHR88 model described in section 3, into (2e) and formally solving for $\overline{uw}$ will result in a braking factor $(1-9A_4A_5G_{H})^{-1}$ for $\overline{uw}$, and similarly for $\overline{vw}$, halting the turbulent Prandtl number in (3b) from increasing as $|G_{H}|$ increases, thus making $\text{Ri}$ finite as $|G_{H}|$ goes to infinity, since $\text{Ri} = \sigma_f R_f$ with $R_f = -P_s / P_s < 1$ in the equilibrium state. This braking factor appears in (C7) as $(1 - d_4 G_{H})^{-1}$; if $\overline{uw}$ and $\overline{vw}$ were not in (2g-h), $d_4$ would be zero, and $\text{Ri}(cr)$ would be at infinity according to (C10). A similar argument can be made for the MY82 model: for $\overline{w^2}$ instead of (6c) we use (2c), and the appearance of $\overline{uw}$ and $\overline{vw}$ in (2g-h) is responsible for the non-zero $s_4$ and $d_4$ in (C1), and if they were zero then $\text{Ri}(cr)$ would be at infinity according to (C6). In contrast, the new equations (3d, e) are free from the $\overline{uw}$ and $\overline{vw}$ terms (they have been cancelled by the new closure (3a)), thus allowing $\text{Ri}$ to go as $|G_{H}|$ when $|G_{H}|$ goes to infinity rather than being cut-off. On the other hand, the new equation (3f) for $\overline{w\Theta}$ gains a new term containing $\overline{uw}$ and $\overline{vw}$, which is always positive for the stable case, thus acting to increase $\overline{w\Theta}$ and preventing $\overline{w\Theta}$ from vanishing at finite $\text{Ri}$. Now we solve the new set of linear algebraic equations (2a-f, j) and the new heat flux equations (3d-f) simultaneously to obtain:
\[ u_w = -K_M \frac{dU}{dz}, \quad v_w = -K_M \frac{dV}{dz}, \quad w_\Theta = -K_H \frac{d\Theta}{dz} \]  

(3g)

\[ K_H = l q S_H, \quad K_M = l q S_M, \quad \sigma_i = \frac{S_M}{S_H} \]  

(3h)

where \( K_H \) and \( K_M \) are the heat and momentum diffusivities respectively. The new results are contained in the turbulent structure functions \( S_H \) and \( S_M \):

\[ S_H = \frac{s_0 + s_1 G_M}{D}, \quad S_M = \frac{s_2 - s_3 G_H}{D}, \quad D = 1 - d_1 G_H + d_2 G_M - d_3 G_H G_M \]  

(4a)

\[ G_M = \frac{l^2 S^2}{q^2}, \quad G_H = -\frac{l^2 N^2}{q^2} \]  

(4b)

(4a) is structurally simpler than the MY82 model (Eq. C1 in Appendix C). The derived model constants (all \( s \) and \( d \) factors) are as follows:

\[ s_0 = A'_2 - A_i (1 - 3C_1), \quad s_1 = 18A'_3 A'_i C_1, \quad s_2 = A_i (1 - 3C_1) \]

\[ s_3 = 3A_i A'_i [B_2 (1 - \gamma_1) (1 - 3C_1) - 3(4A_i C_1 + A'_i) \beta_3] \]  

(4c)

\[ d_1 = 3A'_2 B_2 (1 - \gamma_1) - 3A_i (4A_i - A'_i) \beta_3, \quad d_2 = 6A'_3, \quad d_3 = 18A'_i A'_2 [B_2 (1 - \gamma_1) - 3A'_2 \beta_3] \]

where the constant \( A'_2 \) is in terms of \( A_2 \) and \( \sigma_{i0} \) according to (3a). All coefficients in (4c) depend only on the basic model constants listed in Table 1. These basic model constants were determined by MY82 except for \( \beta_5 \) and \( \gamma_1 \); MY82 assumed \( \beta_5=1 \) and \( \gamma_1=0 \)
corresponding to the neglect of the buoyancy contribution to the pressure-velocity correlation and of the temperature variance contribution to the pressure-temperature correlation respectively. We determine the parameters as follows:

\[ \beta_5 = 0.75, \quad \gamma_1 = 0.22 \] \hspace{1cm} (4d)

The values in (4d) are chosen for better agreement with the available data for the heat flux (Fig. 3b) which are not directly affected by the internal gravity waves. \( \beta_5 = 0.75 \) lies between the re-normalization group (RNG) value of 0.5 (Shih and Shabbir, 1992) and MY82 value of 1, while \( \gamma_1 = 0.22 \) lies between the RNG value of 1/3 and the MY value of 0. In comparison, KC94, which improved GKHR88, uses \( \beta_5 = 1 \) and \( \gamma_1 = 0.2 \). The derived constants in (4c) are listed in the first row of Table 2.

The turbulence length scale \( l \) can be prescribed or solved from a prognostic equation (e.g., MY82, Cheng et al. 2002); in section 7 we prescribe a modified turbulence length scale based on the one used by BP09.

In the present study, “non-equilibrium” refers to using the original MY82 equations for the diagonal components of \( b_y \) (i.e., Eqs. (2a-c)), in which the production of the turbulent kinetic energy \( P \) is not necessarily equal to its dissipation \( \varepsilon \), while “quasi-equilibrium” refers to switching to the simplified equations for the diagonal components of \( b_y \) (i.e., Eqs. (6a-c)), in which \( P = \varepsilon \) has been imposed in some fashion as described in section 3, but not imposed in the prognostic equation for the turbulent kinetic energy (1e), thus the term “quasi-equilibrium” arises following GKHR88. As such, the quasi-
equilibrium level is 0.25 levels lower than the corresponding non-equilibrium level, as proposed by GKHR88.

3. New quasi-equilibrium model: improving on GKHR88 (level 2.25)

GKHR88 significantly simplified the MY82 stability functions $S_H$ and $S_M$ by imposing the condition that production of the turbulent kinetic energy equals dissipation ($P=\varepsilon$) (see Eq.5a below), in the equations for the diagonal components of the departure-from-isotropy tensor $b_{ij}$ in (2a-c). GKHR88’s procedure is a novel way to achieve simplicity while keeping important physics, although the diagonal components of $b_{ij}$ from the thus modified (2a-c) would not sum up exactly to zero (i.e., the components of kinetic energy would not sum up exactly to the kinetic energy itself), due to higher order terms in GKHR88’s scale analysis. In this section, we modify their procedure to avoid that conceptual inconsistency without additional complications. To the equations for the traceless tensor $b_{ij}$, GKHR88 added the tensor:

$$2\delta_{ij}(P-\varepsilon) \quad (5a)$$

whose trace $6(P-\varepsilon)$ is non-zero unless $P=\varepsilon$. Instead of (5a), we propose to add the following traceless tensor to the equations for $b_{ij}$:

$$-(3\delta_{ij}\delta_{ij} - \delta_{ij})(P-\varepsilon) \quad (5b)$$
which has zero trace even if $P \neq \varepsilon$. With $P$ and $\varepsilon$ defined by (1f), adding (5b) to (2a-c), we obtain the following new equations.

**Modified equations for the diagonal components of $b_{ij}$:**

\[ u^2 = -\frac{A}{q} \left[ 3(uw \frac{\partial U}{\partial z} - vw \frac{\partial V}{\partial z}) + (1 + 2 \beta_s) g \alpha w \bar{\theta} \right] + \frac{q^2}{3} \left( 1 + \frac{3 A}{B} \right) \]  

(6a)

\[ v^2 = -\frac{A}{q} \left[ 3(vw \frac{\partial V}{\partial z} - uw \frac{\partial U}{\partial z}) + (1 + 2 \beta_s) g \alpha w \bar{\theta} \right] + \frac{q^2}{3} \left( 1 + \frac{3 A}{B} \right) \]  

(6b)

\[ w^2 = \frac{2A}{q} (1 + 2 \beta_s) g \alpha w \bar{\theta} + \frac{q^2}{3} \left( 1 - \frac{6 A}{B} \right) \]  

(6c)

It is straightforward to check that the right-hand sides of (6a-c) now sum up exactly to twice the kinetic energy $q^2$ as expected. Thus, in the new model the GKHR88 energy decomposition issue (see their Eq.15) no longer exists.

Now, replacing (2a-c) with (6a-c) and using the new heat flux equations (3d-f), we solve the second moment equations simultaneously. The procedure is parallel to that in section 2, and (3g, h) are still obtained but with the new quasi-equilibrium stability functions $S_H$ and $S_M$:

\[ S_H = \frac{s_0}{D}, \quad S_M = \frac{s_2 - s_2 G_H}{D}, \quad D = 1 - d_s G_H \]  

(7a)
(7a) is structurally simpler than the GKHR88 model (Eq. C7 in Appendix C). (C7) has a quadratic denominator in the structure function $S_m$, while in the present model, (7a), the denominator is simplified to a linear function in $G_H$. This structural difference drastically changed the asymptotic behavior of the model and moved $Ri(\sigma)$ from finite to infinite, as will be shown more clearly in Section 5.

In (7a), the derived model constants are expressed in terms of the basic model constants listed in Table 1 as follows:

\begin{align*}
s_0 &= A'_1(1 - 6A / B_1) - A_4(1 - 6A / B_1 - 3C_1) \\
s_2 &= A_4(1 - 6A / B_1 - 3C_1) \\
s_3 &= 3A_4A'_2[B_2(1 - \gamma_1)(1 - 6A / B_1 - 3C_1) - 6A_4C_1(1+2\beta_2) - 3A'_2(1 - 6A / B_1)\beta_2] \\
d_1 &= 3A'_2[A(2 + \beta_2) + B_2(1 - \gamma_1)] - 6A'_2(1 + 2\beta_2) \\
(7b)
\end{align*}

where the constant $A_2'$ is given in terms of $A_2$ and $\sigma_0$ by (3a). The numerical values in (7b) are in the second row of Table 2.

At this point we outline how to solve the TKE equation (1e). The vertical transport $T_e$ in (1e) can be parameterized by non-local schemes such as Cheng et al. (2005) or BP09. With large scale model applications in mind, we follow BP09’s novel relaxation scheme for $T_e$. Neglecting the time tendency, (1e) becomes,

\begin{align*}
T_e + P - \varepsilon &\approx 0 \\
(7c)
\end{align*}

For the convective case $T_e$ is parameterized as

\begin{align*}
T_e &= a_e(<\varepsilon> - \varepsilon)\varepsilon^{1/2} / l \\
(7d)
\end{align*}
where $a_e$ is a parameter taken as 1, $<>$ is a weighted average within the convective layers (see BP09 for more details), and thus $(1e)$ is reduced to the algebraic equation

$$a_e(< e > -e)e^{1/2} / l + P - e \approx 0 \quad (7e)$$

Applying $<>$ to $(7e)$ yields $< e >$ (see BP09 for details) in terms of convective layer averaged $S_m$ and $S_h$, thus $(7e)$ provides an expression for $e$.

For the stable case, $T_e$ in $(7c)$ is small and may be negligible, and thus $e$ can be solved from a second order algebraic equation; this constitutes Eq. (9a) described in section 5.

The new quasi-equilibrium model (level 2.25) is used in the current GISS moist atmospheric turbulence scheme for the convective case.

4. Realizability conditions for (4a) and (7a)

As with other second order closure models, we need to consider the realizability conditions for (4a) and (7a). For $G_H$ in the unstable case, since shear production must be non-negative, one has, with $G_M \to 0$ and using $P = e$ (using Eq.9a below):

$$S_{H} G_{H} - \frac{1}{B_i} \leq 0 \quad (8a)$$

$$G_{H} \leq \frac{1}{B_1 s_0 + d_1} \quad (8b)$$

Using (4a) or (7a), (8a) becomes:
Relation (8b) applies to both models (4a) and (7a). For the validity region of $G_M$, we follow Hassid and Galperin (1983) who argue that an increase of shear should not result in a decrease of the normalized momentum flux. We thus derive a limitation on $G_M$ from the following condition:

$$\frac{\partial}{\partial G_M} \left( \frac{u w^2 + v w^2}{q^2} \right)^{1/2} = \frac{\partial}{\partial G_M} \left( G_M^{3/2} S_M \right) \geq 0$$  \hfill (8c)

For the non-equilibrium model (4a), (8c) leads to the following realizability condition for $G_M$:

$$G_M \leq \frac{1 - d_1 G_H}{d_2 - d_3 G_H}$$  \hfill (8d)

For the quasi-equilibrium model (7a), however, no realizability condition for $G_M$ is needed since $S_M$ does not depend on $G_M$.

### 5. New equilibrium model: improving on MY82 and GKHR88 (level 2)

Under the equilibrium condition, the storage and transport terms are negligible, and therefore the equation for the TKE (1e) reduces to $P = \varepsilon$, which in turn can be written as:
where $G_M$ and $G_H$, defined in (4b), are related by the Richardson number $\text{Ri}$:

$$\text{Ri} = -\frac{G_H}{G_M}$$  \hspace{1cm} (9b)

$G_M$ or $G_H$ can be solved from the quadratic algebraic equation (9a) as a function of the single parameter $\text{Ri}$. The turbulent kinetic energy $q^2/2$ can then be expressed in terms of $G_M$ or $G_H$ according to (4b), as long as the large scale variables $S^2$ or $N^2$ and the length $l$ are known. Substituting the new models (4a) or (7a) in (9a), we obtain:

$$c_2G_H^2 + c_1G_H + c_0 = 0$$  \hspace{1cm} (10a)

For (4a):

$$c_2 = B_i(s_3-s_i) - d_3, \quad c_1 = (B_i s_0 + d_i)\text{Ri} - B_i s_2 + d_2, \quad c_0 = -\text{Ri}$$  \hspace{1cm} (10b)

For (7a):

$$c_2 = B_i s_3, \quad c_1 = (B_i s_0 + d_i)\text{Ri} - B_i s_2, \quad c_0 = -\text{Ri}$$  \hspace{1cm} (10c)

For both models (4a) and (7a), the asymptotic solution of (10a) for $q^2 \to 0$, $G_H \to -\infty$ is
Thus $R_i$ can go to infinity and there is no critical value. By contrast, Appendix C shows that the corresponding asymptotic solution for MY82 and GKHR88 models is of finite value: $R_i(\text{cr})=0.19$.

The new equilibrium model consists of (4a) or (7a) with (10a). In Fig. 1 we plot $G_M$, $G_H$, $S_M$ and $S_H$ as functions of $R_i$ under both unstable ($R_i<0$) and stable ($R_i>0$) conditions and compare the new model with the MY82 model. For the unstable case, the differences between the new model and MY82 are small. Important differences however occur in the stable case, as can be seen from Fig. 1: for the MY82 model there is a critical $R_i=0.19$ at which $G_M$ and $G_H$ diverge and $S_M$ and $S_H$ plunge to zero, whereas in the new model there is no such critical $R_i$ (in other words, the critical value is infinity in the new model).

The equilibrium model (level 2) is used in the current GISS moist atmospheric turbulence scheme for the stable case.

6. Comparison with data and previous models

Similar to Zilitinkevich et al. (2007, 2008), Canuto et al. (2008) and KC09, we compare the new model with diverse data and previous models. In Figs. 2 and 3, we plot the following quantities ($R_f$ being the flux Richardson number):
\[
\sigma_t(Ri) = \frac{S_M}{S_H}, \quad R_f = \frac{R_i}{\sigma_t(Ri)}, \quad \frac{(uw)^2 + (vw)^2}{(q^2/2)^2}, \quad \frac{(w\vartheta)^2}{(q^2/2)(\theta^2/2)}, \quad \frac{w^2}{q^2}
\]  

against a variety of meteorological observational, experimental, DNS and LES data as cited in Zilitinkevich et al. (2007, 2008), using the new equilibrium model (Eq. 10a-c, red, dashed line) and MY82 or GKHR88 equilibrium model (blue, dotted line) as summarized in appendix C.

It is discernible and well-perceived that, in the weak turbulence regime, \(\sigma_t, R_f\) and the momentum flux are largely dominated by internal waves that efficiently transport momentum but not heat (e.g. Stewart 1969; Jacobitz et al. 2002; Nappo, 2002; Zilitinkevich et al., 2007, 2008; Anderson 2009). The data presented in Figs 2 and 3 are consistent with this interpretation, with momentum flux, \(\sigma_t\) and \(R_f\) (the latter two being directly related to the momentum flux) exhibiting more scatter, especially at large \(Ri\), while the heat flux data exhibit much less variability. The present model, as well as MY82, GKHR88 and variants (e.g., KC09) that we are comparing with, do not account for the vertical transport due to internal gravity waves. Formulation of these turbulence closure models do not rely on the specifics of the wave-generated data. Therefore, it is important to note that the model is unable to reproduce the wave-generated data without the physics of internal waves in the models’ closures. While the extensive data presented in Figs 2 and 3 have been widely used by many researchers, it is recognized that the inclusion of the wave-driven transports of momentum is still needed for a more complete picture (e.g., Zilitinkevich et al., 2007, 2008, 2013; Canuto et al. 2008; KC04; KC09).

Initial effort is also made to parameterize the internal wave contribution in the context of
the $k$-$\varepsilon$ model (e.g., Baumert and Peters, 2004; Kantha, 2005). Accounting for internal waves in a higher order turbulence closure model is a topic for our future research.

A goal of our present model centers on derivation of new heat flux equations without need of the critical Richardson number. To this end, we compare the model results carefully with the heat flux data (in Fig. 3b) that are not directly affected by the waves. The heat flux data in Fig. 3b show that the critical Richardson number (if any), beyond which there would be no turbulence, far exceeds unity with the present model reproducing the data reasonably well in contrast to the MY82 and GKHR88 models with $Ri(cr)$ much less than unity. It is also useful to cautiously compare the models’ $\sigma_t$, $R_f$ and momentum flux against the data with wave effects. For small $Ri$, when the wave effects are relatively small, one can see that the model and data are in broad agreement. For larger $Ri$, the wave-driven data are so scattered that agreement becomes more and more unlikely. Nonetheless, it is a reasonable result that the model transitions smoothly from the small $Ri$ regime to the large $Ri$ regime, passing through the clusters of data points. It is also important to keep in mind that the model closure still lacks the physics of internal waves and future research should account for the wave structure.

In Fig 4 we plot the normalized inverse Prandtl number (left panel) and the normalized ratio of the horizontal heat flux and the vertical heat flux (right panel) vs. the data of Webster (1964); significant improvements are seen in comparison with MY82 and GKHR88.

Now we show how the new model recovers the similarity laws near the surface that empirically parameterize the non-dimensional shear and potential temperature gradients defined as
\[ \Phi_M = \frac{\kappa z}{u_*} S, \quad \Phi_H = \frac{\kappa z u_*}{w\theta_s} \frac{d\Theta}{dz} \]  

(11b)

where \( S \) is the mean shear of wind defined in (3c), \( u_* \) and \( w\theta_s \) are the surface friction velocity and the surface potential temperature flux respectively, \( \Theta \) is the mean potential temperature and \( \kappa (=0.4) \) is the von Kármán constant. Businger et al. (1971) analyzed Kansas data in the constant flux surface layer and parameterized \( \Phi_M \) and \( \Phi_H \) in terms of height \( z \) and \( \zeta \) which is the ratio between the height \( z \) and the Monin–Obukhov length \( L \),

\[ \zeta = z / L, \quad L = -\frac{\Theta u_*^3}{\kappa g w\theta_s} \]  

(11c)


In order to see how the new equilibrium model can recover the similarity law, we relax the condition of constant surface fluxes implied in (11b) and replace these constant fluxes with those determined by the turbulence model (3g, 3h and 7a). The resulting surface momentum and heat fluxes allow \( \text{Ri} \) and \( l \) (the dynamic length scale) dependences. After some algebra, \( \Phi_M, \Phi_H \) and \( \zeta \) are expressed as

\[ \Phi_M = \frac{G_{M}^{1/4} \kappa z}{S_M^{1/2}} \frac{1}{l}, \quad \zeta = \Phi_M R_f, \quad \Phi_H = \Phi_M \frac{S_M}{S_H} \]  

(11d)
where $R_f$ is the flux Richardson number defined in (11a). It has been known that $l = \kappa z$ is valid only in the neutral condition; in stably stratified flows, $l$ is less than $\kappa z$ and could be parameterized, e.g., by Nakanishi (2001), using the LES data, as

$$l = \frac{\kappa z}{1 + 2.7 \zeta}, \quad 0 \leq \zeta < 1$$  \hspace{1cm} (11e)

From (11d) and (11e) we derive the relation:

$$\Phi_M = \frac{1}{S_m^{1/2} G_M^{-1/4} - 2.7 \max(R_f, 0)}$$  \hspace{1cm} (11f)

In Fig. 5 we plot $\Phi_M$ and $\Phi_H$ vs $\zeta$ according to (11d-f) with the current model, in comparison with Kansas data as parameterized by Högström (1988). By relaxing the constant flux assumption near the surface to allow $R_i$ variation, we have shown that the closure model result is consistent with the similarity law representing the surface data; while the similarity law is limited to the surface, the closure model is designed for the whole PBL and beyond.

7. The length scale parameterization appropriate to the new model

We start from BP09’s length scale prescription in which the well-known Blackadar (1962) turbulent master length scale is employed:
where the asymptotic length scale $l_\infty$ is chosen to be proportional to the turbulent layer thickness $h$. Grenier and Bretherton (2001) chose the proportionality constant $\eta = 0.085$. BP09 further parametrized $\eta$ to include the effect of the bulk Richardson number $Ri^{\text{CL}}$ for a convective layer (CL, a convective region that may contain multiple model layers):

For unstable case:
\[
\eta = 0.085\{2 - \exp[\min (Ri^{\text{CL}}, 0)]\} \quad (12b)
\]

For neutral and stable cases:
\[
\eta = 0.085 \quad (12c)
\]

For fully convective layers (corresponding to large, negative $Ri^{\text{CL}}$), (12b) doubles the neutral value of $\eta$ (0.085), but for the neutral and stable layers ($Ri \geq 0$), (12c) yields the neutral value (0.085). We modify (12c) as follows:

\[
\eta = \max[0.015, 0.085\exp(-Ri)], \quad Ri \geq 0 \quad (12d)
\]

which takes into account the physical consideration that stable stratification makes the length scale smaller due to the smaller eddy size, and $\eta$ crosses $Ri=0$ smoothly when using (12b) and (12d).
8. SCM simulations for the stable case GABLS1 and the unstable case DCBL

In the present study we simulate examples of the stable and unstable cases considered by BP09: the first GEWEX Atmospheric Boundary Layer Study inter-comparison case (GABLS1; Beare et al. 2006; Cuxart et al. 2006) and the surface-forced dry convective boundary layers (DCBL), using the GKHR88 model and the new model in comparison with large-eddy simulation (LES) results. A neutral experiment SCM was also considered by BP09 using the GKHR88 model; in the neutral case, the new model is identical to the MY82/GKHR88 model (having identical stability functions $S_m$ and $S_h$ at $R_i=0$, also as seen in Fig. 1) and therefore the new model results would be indistinguishable from those using the GKHR88.

In a development version of the GISS GCM (ModelE), which is an improved version of the GISS ModelE2 (Schmidt et al. 2014), we recently implemented a moist atmospheric turbulence model based on BP09 that employs the GKHR88 quasi-equilibrium, second-order turbulence model. We toggled between the GKHR88 model (C7) and the new quasi-equilibrium model (7a) and simulated the stable GABLS1 case and the unstable DCBL case in single-column model (SCM) mode. Since we are implementing the turbulence model in the GISS atmospheric GCM with coarse resolutions, we chose a typical operational vertical resolution of 62 layers (10 mb resolution beneath 900 mb so that the grid spacing is about 100 m), and a time step of 1800 s. For the GABLS1 case we also tested the new model with 184 vertical layers (1 mb or 10 m resolution beneath 900 mb, with 300 s time step) and other vertical resolutions, and the results are consistent with those of the 62-layer configuration. We noticed that in the stable case (GABLS1), the new model
allows for longer time steps to be taken while maintaining numerical stability for 1mb resolution, from 10 s used by BP09's L175 run to 300 s used by the present model's L184 run, due to the combined effects of the shape of the stability functions near the PBL top and the suppression of the length scale by small eddy sizes in the new model.

Both the GABLS1 and the DCBL cases are described by BP09. The GABLS1 case is based on an Arctic study from the Beaufort Sea Arctic Stratus Experiment (BASE) as developed by Kosovic and Curry (2000), and the DCBL is an idealized case. Our SCM results are compared with results from an LES (Stevens et al. 2002) with a dynamic Smagorinsky subgrid-scale model (Kirkpatrick et al., 2006) that for GABLS1 we obtained using a 400 x 400 x 400 m grid with a uniform 64 x 64 x 64 mesh (grid spacing of 6.25 m; results on a 128 x 128 x 128 mesh are indistinguishable), and for DCBL a 4.8 x 4.8 x 3.5 km grid with a 96 x 96 x 175 mesh (horizontal and vertical grid spacings of 50 and 20 m). Both use a sponge layer to dampen any gravity waves reflecting off the model top, which respectively start at 300 m and 3 km for GABLS1 and DCBL.

For the stable GABLS1 case in Fig 6 we compared the PBL height for the operational resolution of L62 (upper panel) and the fine resolution L184 (lower panel). In addition to the present model (red dashed lines) and the GKHR88 model (blue dotted lines) we also show the KC94 model (an improved version of GKHR88; green, dot-dashed lines). GKHR88, KC94 and the present models have progressively increasing Ri(cr): 0.19, 0.24 and infinity, and the PBL heights simulated are progressively higher, with the present model result closest to the LES. The stable PBL height is defined for the LES based on the momentum flux profile (see Eq. 13 below) and for the SCM where TKE diminishes to a specified low level; the momentum flux and the TKE vanish completely at Ri(cr) and
diminish to the specified low level close to $R_i(c(r))$. Thus, larger $R_i(c(r))$ corresponds to larger PBL height since $R_i$ increases with height near and beyond the top of the PBL. Fig. 6 shows progressively the dependence of the PBL height on $R_i(c(r))$.

Fig. 7 shows, for the GABLS1 case, the vertical profiles of the potential temperature $\Theta$, the zonal and meridional velocity components ($U$ and $V$), the wind stress defined in Eq. (13) below, the turbulent kinetic energy TKE and its vertical component $\overline{w^2}$, the Richardson number $R_i$, the TKE buoyancy production $P_B$ and the shear $P_S$ as in (1f), and the TKE dissipation $\varepsilon$. Following BP09, the storage and transport terms are neglected in the stable case. The wind stress is defined as

$$\text{wind stress} = \left( \overline{u^2} + \overline{v^2} \right)^{1/2} \quad (13)$$

and for the LES the PBL height is defined following Beare et al. (2006) as the height at which the stress falls to 5% of its surface value, divided by 0.95.

Although with the operational GCM L62 (vertical grid spacing of 100 m) the sharp gradients in potential temperature (near PBL top) and wind speed (near the surface) as seen in LES results cannot be reproduced, L184 (vertical grid spacing of 10m) shows some improvement. Both the GHKR88 model and the new model have discrepancies with the LES in these mean quantities and the new model results show improvements with respect to GHKR88. The discrepancies may be partially attributed to other GCM parameterizations, given that the mean quantities are also updated by other physics in the host GCM. In addition, LES may be less reliable near the surface where the eddy size is small (~ $z$), especially in the stable case. For example, in Fig. 7, both TKE (LES) and
\( \overline{w^2} \) (LES) decrease in an unrealistically sharp manner when \( z \) approaches 0. In reality, TKE and \( \overline{w^2} \) are nearly constant with \( z \) in the neutral surface layer (e.g., see Tables 1 and 2 in MY82 for a detailed discussion). We also note that there is considerable spread in LES results for this stable case; so while the TKE from the old and new models is substantially less than in this LES, the TKE from the ensemble mean of Beare et al. (2006) provided in Fig. 11 of BP09 is a much better match. Furthermore, the main purpose of our SCM test is to assess how well the new turbulence model behaves relative to GKHR88 instead of determining how well the host GCM with BP09 scheme performs.

With GCM host inputs and most aspects of the BP09 scheme unchanged, what are the effects of toggling between GKHR88 and the present turbulence models? Figs. 6, 7 and 8 demonstrate that the new model results are generally in closer agreement with the LES results relative to the GKHR88 model.

For the unstable DCBL experiment, Fig. 8 shows the profiles of potential temperature, TKE, and the TKE buoyancy production, transport and dissipation. Despite some discrepancy of the TKE between the models and the LES that is also seen in Fig.7 (a) of BP09, both GKHR88 and the new model are in reasonable agreement with the LES. The computation of transport and dissipation terms from the LES diagnostics are problematic near the surface (not shown), but the model results (both GKHR88 and new model) are close to the LES transport and dissipation presented by Figs. 7 (c) and 7(d) in BP09. The differences between using the GKHR88 model (C7) and the present model (7a) are relatively small although visible, as expected, since (7a) differs with (C7) only slightly in the unstable case.
We have presented a new second order turbulence closure model that improves on previous models, e.g., MY82, GKHR88, Cheng et al. 2002 and Canuto et al. 2008 models. The new model consists of new heat flux equations derived from a new turbulence closure. Using the new heat flux equations we identified and removed the cause of a finite critical Richardson number. The new model extends the Ri(cr) to infinity and results compare favorably with meteorological, experimental, DNS and LES data; at the same time, the new model is structurally simpler than MY82 and GKHR88, a feature that facilitates its use in large scale models such as GCMs. The new non-equilibrium model, Eq. (4a), or its quasi-equilibrium version, Eq. (7a), can be combined with non-local treatments of the turbulent kinetic energy such as Eq. (7e) to enhance the transport processes, as in the non-local scheme in the GISS ModelE based on BP09. For the equilibrium version of (4a) or (7a), the turbulent kinetic energy can be found from $G_{ii}$ by solving algebraically the quadratic equation (10a).

SCM simulations of both the stable GABLS1 case and the convective DCBL case show that the new model results are in reasonable agreement with LES and indicate improvements in the stable case in particular, in which the new model predicts a deeper PBL than previous models. This and other features of the new model in a global framework will be evaluated using long-term observations in future work. Since Martin (1985) and others have shown that Ri(cr) must be larger than unity in the oceanic boundary layer, and since more recent models that improved GKHR88 (e.g. KC94, KC04) showed reasonable
results in ocean mixed layers, we will generalize the new model to include salinity for
implementation in an oceanic turbulence mixing scheme.

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Flight Center.
Appendix A. Derivation of the new heat flux equations (3d)-(3f)

The new pressure-temperature correlation (3a) implies a length scale ratio $l_2/l$ that depends on the Prandtl number, which we can re-write using the definitions in (1f) and (3b,c) as follows (after some re-arrangement):

$$l_2/l = A_2 \frac{\overline{w\theta}}{w\theta + (uw\frac{\partial U}{\partial z} + vw\frac{\partial V}{\partial z})S^2 \frac{\partial \Theta}{\partial z}} \quad (A1)$$

To aid the following derivation we identify a useful relation from (2d-i)

$$\frac{\partial V}{uw\frac{\partial U}{\partial z}} = \frac{\partial U}{vw\frac{\partial V}{\partial z}} \quad (A2)$$

(A2) can be easily verified using the first 2 equations in (3g). Furthermore, (A2) leads us to another useful relation:

$$\frac{(uw\frac{\partial U}{\partial z} + vw\frac{\partial V}{\partial z})S^2}{(uw\frac{\partial U}{\partial z} + vw\frac{\partial V}{\partial z})S^2} = \frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} \quad (A3)$$

Using (A3), (A1) can be alternatively written as:
\[ l_2/l = A_1 \frac{\overline{w\theta}}{\overline{w\theta} \frac{\partial U}{\partial z} + u\overline{w\theta} \frac{\partial \Theta}{\partial z}} \]  

(A4)

Substituting \( l_2/l \) in (A4) into the original zonal heat flux equation (2g) we obtain the new heat flux equation (3d). The derivation of the new meridional heat flux equation (3e) is entirely parallel to the derivation of (3d).

To derive the new vertical heat flux equation (3f), simply substitute \( l_2/l \) in (A1) into the original vertical flux equation (2i), and notice there is a cancellation of \( \overline{w\theta} \) on both sides of the equation; after some re-arrangement, we obtain (3f).

Appendix B. New non-local model: improving on MY82 (levels 3 and 2.75)

For completeness we present the version of the new model in which one solves the prognostic equations for both the turbulent kinetic energy \( \overline{q^2}/2 \) and the potential energy represented by \( \overline{\Theta}/2 \). This provides a way to take into account non-local vertical transports.

In MY82’s terminology, this is called the level 3 or level 2.75 model, depending on whether it is non-equilibrium or quasi-equilibrium; level 2.75 being our new addition to MY82 and GKH88’s hierarchy (see section 2 for a detailed discussion). The formulation is similar to those in sections 2 and 3. The new model at level 3 or 2.75 not only offers new stability functions, but also yields a counter-gradient component \( \gamma_c \) for the heat flux. More
specifically, in addition to solving the prognostic equation (1d) for $q^2/2$, we solve the following prognostic equation for $\theta^2$:

$$\frac{\partial \theta^2}{\partial t} + \frac{\partial \theta^2 \theta}{\partial z} = -2\theta \frac{\partial \theta}{\partial z} - 2\epsilon_\theta, \quad \epsilon_\theta = \frac{q\theta^2}{\Lambda_2} \quad (B1)$$

where the third order moment $\theta^2$ could be parameterized by down-gradient approximation (e.g., MY82 for details) or by non-local models (e.g., Cheng et al. 2005), which is a separate topic outside the scope of the present study.

The equations for the remaining second moments are solved simultaneously with the following results:

a) New non-equilibrium model (level 3):

$$u_\theta = -l q S_M \frac{dU}{dz}, \quad w = -l q S_M \frac{dV}{dz}, \quad \theta = -l q S_M \left( \frac{d\Theta}{dz} - \gamma' \right) \quad (B2)$$

$$S_H = \frac{s_0 + s_1 G_M}{D}, \quad S_M = \frac{s_2 + s_3 G_M + s_4 (\gamma \alpha \theta^2)^2}{D}, \quad \gamma' = \frac{s_5 + s_6 G_M}{s_0 + s_1 G_M}, \quad \gamma'' = \frac{g \alpha \theta^2}{q^2} \quad (B3)$$

where

$$D = 1 - d_1 G_M + d_2 G_M - d_3 G_H G_M \quad (B4)$$
The constants in (B3) and (B4) (other than $s_4$, $s_5$ and $s_6$) can be calculated using Eq. (4c) with $B_2$ formally set as zero. All these values are summarized in Table 3.

b) New Quasi-equilibrium model (level 2.75):

\[ s_4 = 9A_iA'_i(4A_i + 3A'_i)\beta_5(1 - \gamma_1), \quad s_5 = 3A'_i(1 - \gamma_1), \quad s_6 = 18A_iA'_i(1 - \gamma_1) \]  \hspace{1cm} (B5)

The constants in (B6) (other than $s_4$ and $s_5$) can be calculated using Eq. (7b) with $B_2$ formally set as zero. All these values are summarized in Table 3.

Appendix C. The stability functions of the MY82 and GKHR88 models
For reference and comparison, we summarize the structure functions of the MY82 and GKHR88 models, and for their level 2 equilibrium models in the form of Eq. 10a, we derive the asymptotic solutions for Ri as \( q^2 \to 0, \ G_H \to -\infty \).

a) For the MY82 model at levels 2.5 and 2:

\[
S_H = \frac{s_0 + s_1 G_M - s_4 G_H}{D}, \quad S_M = \frac{s_2 - s_4 G_H}{D}
\]  \hspace{1cm} (C1)

\[
D = 1 - d_1 G_H + d_2 G_M - d_3 G_H G_M + d_4 G_H^2
\]  \hspace{1cm} (C2)

\[
s_0 = A_2, \quad s_1 = 18A_2^2 A_4 C_1, \quad s_4 = 9A_4 A_2^2
\]
\[
s_2 = A_4 (1 - 3C_1), \quad s_3 = 3A_4 A_2 [B_2 - 3A_2 - 3(4A_4 + B_2)C_1]
\]  \hspace{1cm} (C3)

\[
d_1 = 3A_2 (7A_4 + B_2), \quad d_2 = 6A_2^2
\]
\[
d_3 = 18A_2^2 A_2 (B_2 - 3A_2), \quad d_4 = 3s_4 (4A_4 + B_2)
\]  \hspace{1cm} (C4)

The numerical values in (C3-4) are listed in Table 4. The coefficients in the level 2 equilibrium model (10a) are:

\[
c_2 = (B_1s_4 + d_4) Ri - B_1s_2 + d_3, \quad c_1 = -(B_1s_0 + d_1) Ri + B_1s_2 - d_2, \quad c_0 = Ri
\]  \hspace{1cm} (C5)

The asymptotic solution of (10a) for Ri as \( q^2 \to 0, \ G_H \to -\infty \) is
Ri \rightarrow \frac{B_i(s_i - s_0) - d_i}{B_is_0 + d_4} = 0.19, \quad \text{as} \ G_H \rightarrow -\infty \quad (C6)

b) For the GKHR88 model at levels 2.5 and 2:

\[ S_{\mu} = \frac{s_0}{1 - d_i G_{\mu}}, \quad S_{\nu} = \frac{s_2 - s_{\mu} G_{\mu}}{(1 - d_i G_{\mu})(1 - d_4 G_{\mu})} \quad (C7) \]

\[ d_i = 3A_2(6A_i + B_i) \]
\[ s_0 = A_i(1 - 6A_i / B_i) \]
\[ s_2 = A_i(1 - 6A_i / B_i - 3C_i) \]
\[ s_3 = 3A_i [(B_2 - 3A_2)s_0 - C_i d_i] \]
\[ d_4 = 9A_i A_2 \quad (C8) \]

The numerical values in (C8) are listed in Table 4. The coefficients in the level 2 equilibrium model (10a) are:

\[ c_2 = (B_i s_0 + d_i) d_4 R_i - B_is_3, \quad c_1 = -(B_i s_0 + d_i + d_4) R_i + B_is_2, \quad c_0 = R_i \quad (C9) \]

The asymptotic solution of (10a) for Ri as \( q^2 \rightarrow 0, \ G_H \rightarrow -\infty \) is

\[ Ri \rightarrow \frac{B_i s_3}{(B_i s_0 + d_i)d_4} = 0.19, \quad \text{as} \ G_H \rightarrow -\infty \quad (C10) \]


Sci., 59, 3285-3301.


Table 1. The numerical values for the basic model constants. These constants are identical to those of MY82 except $\beta_5$ and $\gamma_1$; in MY82, $\beta_5 = 1$ and $\gamma_1 = 0$.

Table 2. The numerical values for the derived model constants in (4a) and (7a). The expressions used to calculate these constants are in (4c-d) and (7b).

Table 3. The numerical values for the derived model constants in the new model at levels 3 and 2.75. The calculation of these constants are discussed in appendix B.

Table 4. The numerical values for the derived constants in the level 2.5 MY82 and level 2.25 GKHR88 models. The calculation of these constants are discussed in appendix C.
Table 1. The basic model constants

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Table 1. The numerical values for the basic model constants. These constants are identical to those of MY82 except $\beta_5$ and $\gamma_1$; in MY82, $\beta_5 = 1$ and $\gamma_1 = 0$. 
Table 2. The derived constants for (4a) and (7a)

<table>
<thead>
<tr>
<th></th>
<th>$A'_2$</th>
<th>$s_0$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$G_{H_{\text{Max}}}$</th>
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<tbody>
<tr>
<td>Eq.(4a)</td>
<td>1.332</td>
<td>0.6328</td>
<td>1.623</td>
<td>0.6992</td>
<td>8.558</td>
<td>26.62</td>
<td>5.078</td>
<td>99.05</td>
<td>0.0269</td>
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<tr>
<td>Eq.(7a)</td>
<td>1.332</td>
<td>0.4958</td>
<td>0</td>
<td>0.3933</td>
<td>0.9676</td>
<td>28.89</td>
<td>0</td>
<td>0</td>
<td>0.0269</td>
</tr>
</tbody>
</table>

Table 2. The numerical values for the derived model constants in (4a) and (7a). The expressions used to calculate these constants are in (4c-d) and (7b).
Table 3. The derived constants for new model at levels 3 and 2.75

<table>
<thead>
<tr>
<th>Eqs.</th>
<th>$A'_2$</th>
<th>$s_0$</th>
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<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B3)-(B5)</td>
<td>1.332</td>
<td>0.6328</td>
<td>1.623</td>
<td>0.6992</td>
<td>-13.45</td>
<td>49.53</td>
</tr>
<tr>
<td></td>
<td>$s_5$</td>
<td>$s_6$</td>
<td>$d_1$</td>
<td>$d_2$</td>
<td>$d_3$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.117</td>
<td>15.83</td>
<td>-4.860</td>
<td>5.078</td>
<td>-60.82</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Eqs.</th>
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<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$d_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B6)-(B7)</td>
<td>1.332</td>
<td>0.4958</td>
<td>0.3933</td>
<td>-11.41</td>
<td>65.35</td>
<td>3.117</td>
<td>-2.586</td>
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</table>

Table 3. The numerical values for the derived model constants in the new model at levels 3 and 2.75. The calculation of these constants are discussed in appendix B.
Table 4. The derived constants for MY82 (level 2.5) and GKHR88 (level 2.25)

<table>
<thead>
<tr>
<th></th>
<th>$s_0$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
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<tr>
<td>MY82, Eqs. (C3)-(C4)</td>
<td>0.74</td>
<td>0.9019</td>
<td>0.6992</td>
<td>9.339</td>
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<td>$d_4$</td>
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<tr>
<td></td>
<td>36.72</td>
<td>5.078</td>
<td>88.84</td>
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<tr>
<td>GKHR88 Eqs. (C8)</td>
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<td>$s_3$</td>
<td>$d_1$</td>
<td>$d_4$</td>
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<tr>
<td></td>
<td>0.4939</td>
<td>0.3933</td>
<td>3.086</td>
<td>34.68</td>
<td>6.127</td>
</tr>
</tbody>
</table>

Table 4. The numerical values for the derived constants in the level 2.5 MY82 and level 2.25 GKHR88 models. The calculation of these constants are discussed in appendix C.
**Figure caption**

**Fig. 1.** Dimensionless shear number $G_M$, dimensionless buoyancy number $G_H$, and stability functions $S_M$ and $S_H$ versus Richardson number $R_i$ in the MY82 (or GKHR88) model (blue dotted line) and the new model (red dashed line).

**Fig. 2.** Turbulent Prandtl number $\sigma_t$ and flux Richardson number $R_f$ in Eq. (11a) versus Richardson number $R_i$ in the equilibrium version of the new model (4a) or (7a) with (10a-c) (red dashed line), the MY82 or GKHR88 model (blue dotted line, overlapping the red dashed line for $R_i < 0.19$), meteorological observations (Kondo et al. 1978, slanting black triangles; Bertin et al. 1997, snowflakes), laboratory experiments (Strang and Fernando 2001, black circles; Rehmann and Koseff 2004, slanting crosses; Ohya 2001, diamonds), LES (Zilitinkevich et al. 2007, 2008, triangles), and DNS (Stretch et al. 2001, five-pointed stars).

**Fig. 3.** a) As in Fig. 2 but for the squared dimensionless turbulent momentum flux (normalized to its value at $R_i=0$) defined in Eq.(11a), compared with lab experiments (Ohya, 2001, diamonds), LES (Zilitinkevich et al., 2007, 2008, triangles), and meteorological observations (Mahrt and Vickers 2005, squares; Uttal et al. 2002, circles; Poulos et al. 2002, Banta et al. 2002, overturned triangles). b) As in a) but for the squared dimensionless heat fluxes as defined in Eq.(11a). c) As in a) but for the dimensionless $w$-variance defined in Eq.(11a), with the DNS data of Stretch et al. (2001) shown by five-pointed stars.
Fig. 4. The inverse turbulent Prandtl number (normalized by its value for neutral stratification) and the ratio of the horizontal heat flux and the vertical heat flux number (normalized by its value for neutral stratification) as a function of Ri from the new model (4a) or (7a) at level 2 (red dashed line), the MY82 or GKHR88 models (blue dotted line), and the Webster (1964) experimental data (filled circles).

Fig. 5. The non-dimensional shear $\Phi_M$ and the potential temperature gradient $\Phi_H$ as a function of $\zeta = z/L$ from the MY82 or GKHR88 model (blue dotted line), the new model (4a) or (7a) (red dashed line), Businger et al. (1971)’s formula as modified by Högström (1988) (triangles).

Fig. 6. PBL height evolution from simulations of the stable atmospheric boundary layer following the GABLS1 case study (see e.g. BP09) using an LES model (solid black line), the GKHR88 model (blue dotted line), the KC94 model (green dash-dotted line) and the new quasi-equilibrium model (red dashed line). For the model results, the upper panel uses 62 layers and the lower panel uses 184 layers.

Fig. 7. From simulations of the atmospheric stable boundary layer for GABLS1 following Beare et al. (2006), vertical profiles of potential temperature $\Theta$, zonal and meridional velocity components $U$ and $V$, the stress defined in Eq. (13), the turbulence kinetic energy TKE and (twice of) its vertical component $\overline{w^2}$, the Richardson number $Ri$, the TKE buoyancy production $P_B$ and shear production $P_S$ as in (1f), and the TKE
Fig. 8. From simulations of the atmospheric convective boundary layer (DCBL case, following BP09), vertical profiles of potential temperature $\Theta$, turbulence kinetic energy TKE, and TKE buoyancy production $P_B$, transport $T_e$ and dissipation $\varepsilon$. Profiles are horizontal means averaged over hours 4-5 from the LES (black solid line), the GKHR88 model with 62 layers (blue dotted line), and the new quasi-equilibrium model with 62 layers (red dashed line).
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