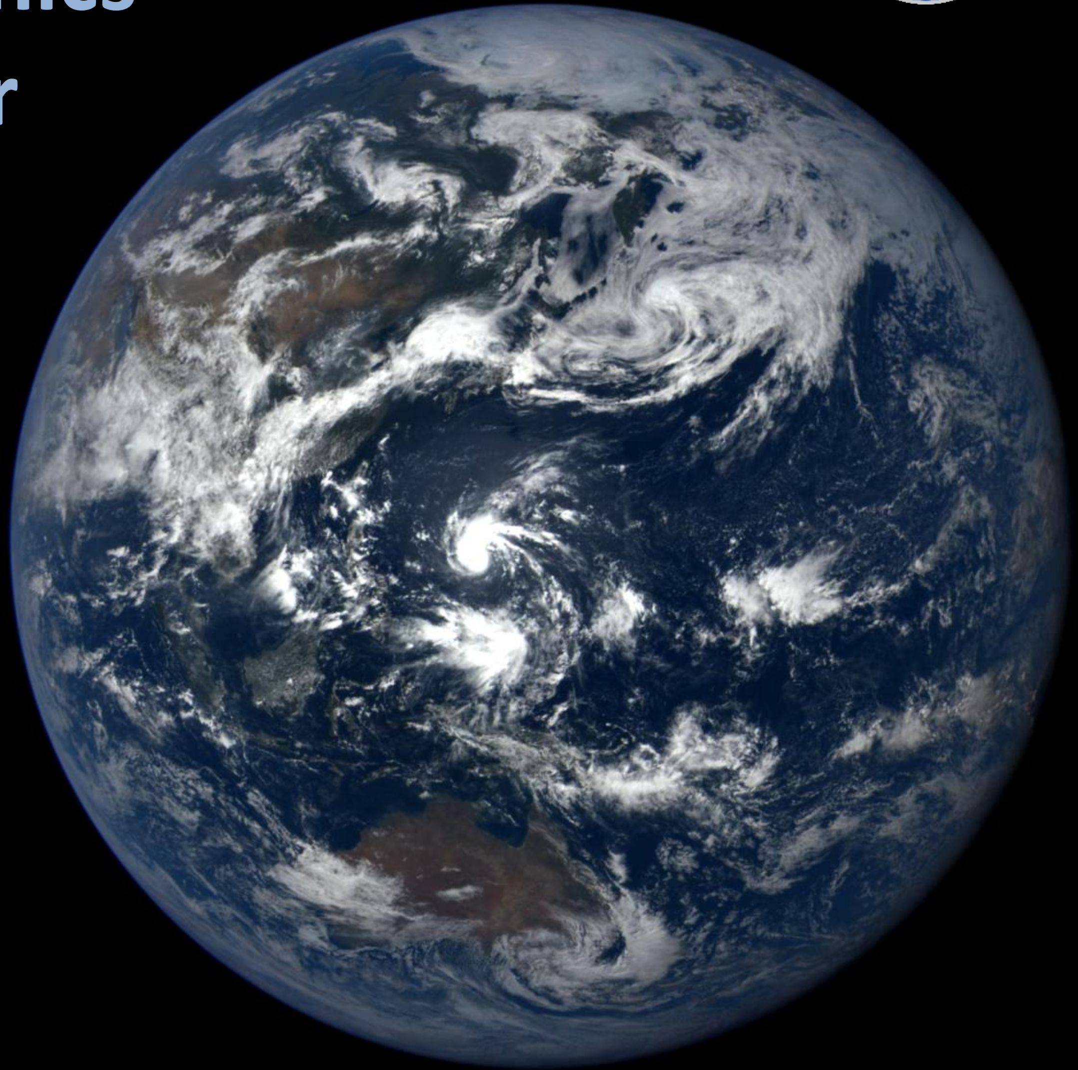
Introduction to Orbital Mechanics and Spacecraft Attitudes for Thermal Engineers



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Disclaimers

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Introduction

Orbiting spacecraft are subject to a variety of environments.

Knowledge of the orbit is required to quantify the solar, albedo and planetary (also called outgoing longwave radiation, or OLR) fluxes.

Some specific questions that might arise are:

- How close (or how far) does the planet/spacecraft pass from the sun?
- How close (or how far) does a spacecraft pass from a planet and how does it affect orbital heating to spacecraft surfaces?
- How long does the spacecraft spend in eclipse during each orbit?
- At what angle does the solar flux impinge on the orbit plane (β angle) and how does that affect the thermal environment?
- What path does a spacecraft take between planets and how does the solar flux change during that transfer?
- Why is one type of orbit used for some spacecraft and another type used for others (e.g., sun synchronous versus geostationary)?
- What factors can make an orbit change over time and how might that affect the thermal environment?
- What type of thermal environment extremes will the spacecraft experience?

Introduction

Orbit information alone is insufficient to determine how the environment affects the spacecraft.

Spacecraft orientation (or “attitude”) and orbit information is required to determine which spacecraft surfaces experience a given thermal environment.

Spacecraft attitude and orbit information are required to determine the view factor to the central body which is required for planetary and albedo flux calculations to a spacecraft surface.

What are the effects on the heating fluxes experienced by a spacecraft due to the attitude reference frame (e.g., celestial inertial versus local vertical – local horizontal reference frames)?

What spacecraft orientation(s) provide favorable thermal conditions for spacecraft components?

Orbits and spacecraft attitudes must be considered together for a successful spacecraft and mission design.

Scope of this Lesson

Orbits

Spacecraft attitudes

Governing differential equation.

Conservation of specific mechanical energy

Conservation of specific angular momentum

Kepler’s laws

Perturbations

Consequences for the thermal environment.

Lesson Contents (1 of 5)

Part 1 -- Review of Scalar, Vector and Matrix Operations

Lesson Contents (2 of 5)

Lesson Contents (3 of 5)

Lesson Contents (4 of 5)

Part 4 -- Advanced Orbit Concepts

Transfer Orbit
Orbit Plane Change
Aerobraking Orbit
Gravity Assists
The Restricted Three-Body Problem
Halo Orbits
Artemis I
Gateway (Near Rectilinear Halo Orbit)

Lesson Contents (5 of 5)

Part 1 -- Review of Scalar, Vector, and Matrix Operations

Part 1 -- Content

Part 1 of this lesson is a review of mathematical operations we will need in our study of orbital mechanics and spacecraft attitudes.

We will begin with a review of scalars and vectors.

After a brief review of Cartesian and Polar coordinates, we'll consider vector dot and cross products, units vectors, coordinate transformations with particular focus on the Euler angle sequence, forming transformation matrices and, finally, stacking transformations.

Scalars and Vectors

A scalar has a magnitude whereas a vector has, both, a magnitude and a direction.

As an example, speed is a scalar and has a magnitude (e.g., 30 m/s) but velocity is a vector and has a magnitude and direction (e.g., 30 m/s in the x-direction).

We will use, both, scalars and vectors in our study of orbital mechanics and attitudes.

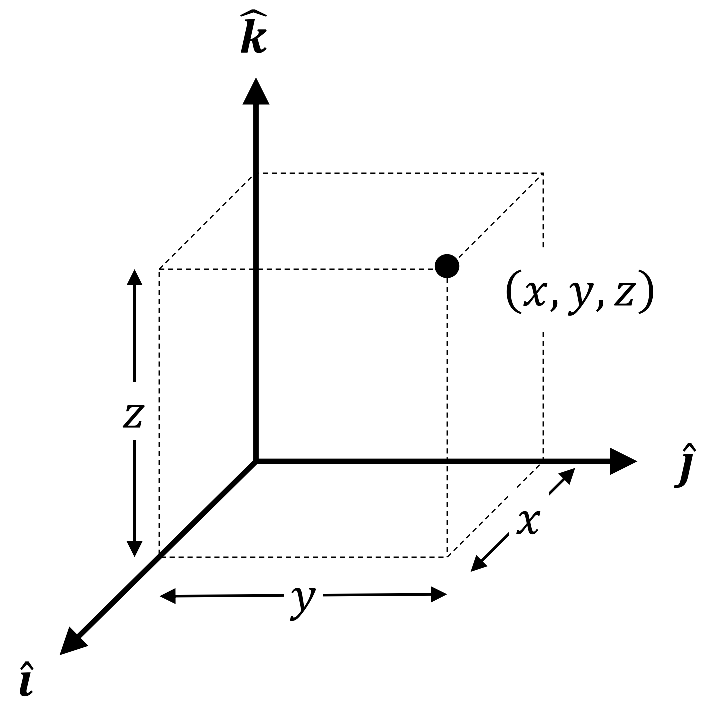
Cartesian Coordinates

Consider the Cartesian coordinate system.

Each axis is orthogonal to the others.

Any point in the coordinate system may be described by three coordinates (x, y, z).

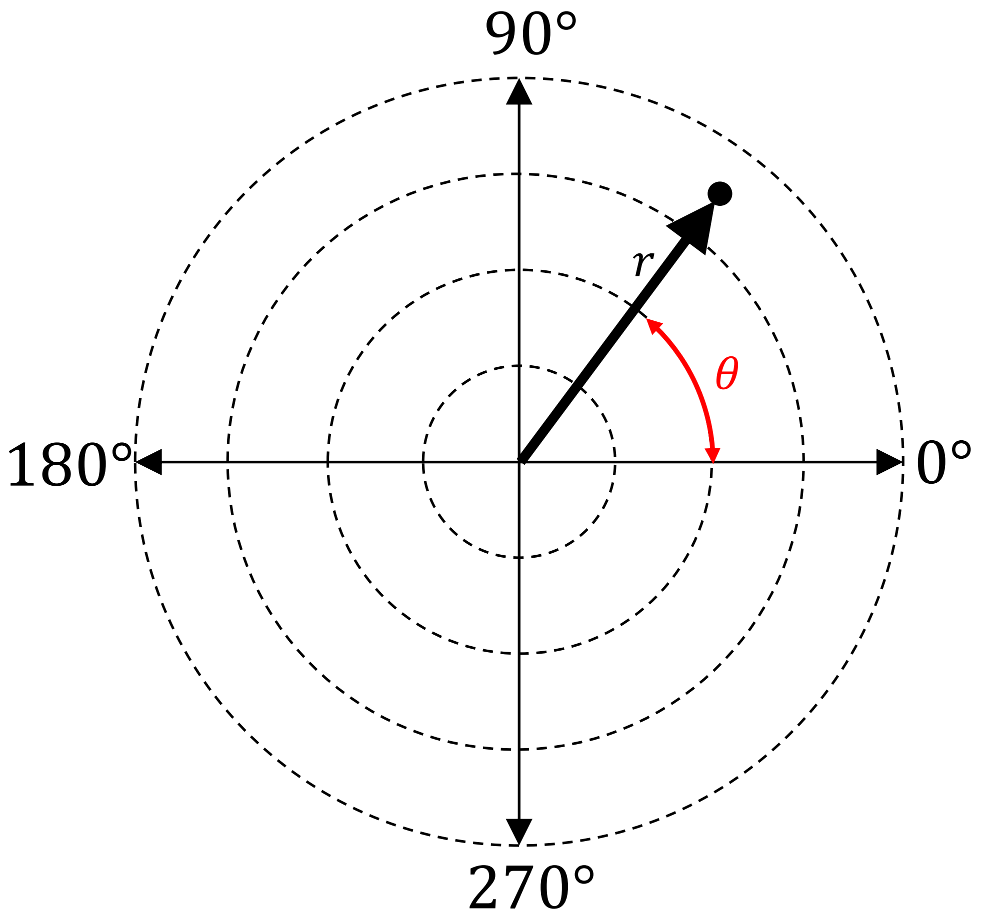
To aid in describing the amount of travel in each orthogonal direction, we specify unit vectors $(\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}})$.



Polar Coordinates

Polar coordinates specify the location of a point using two points, a distance from the origin, r and an angle, θ.

Polar coordinates will be especially useful in our discussion of orbits.



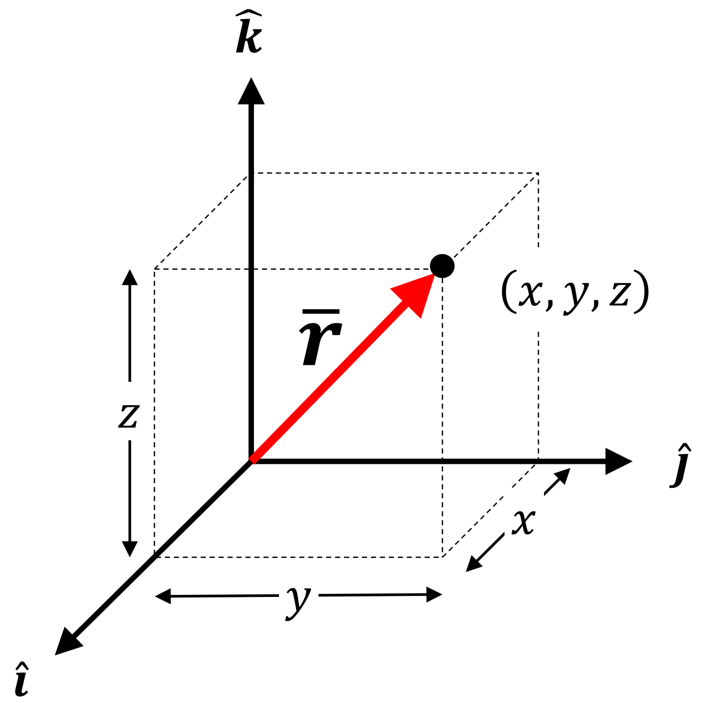
Vectors

The vector, $\bar{\boldsymbol{r}}$ can be expressed in Cartesian coordinates as:

$$\bar{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$

The magnitude of the vector, $\bar{\boldsymbol{r}}$ is given by:

$$|\bar{\boldsymbol{r}}| = \sqrt{x^2 + y^2 + z^2}$$

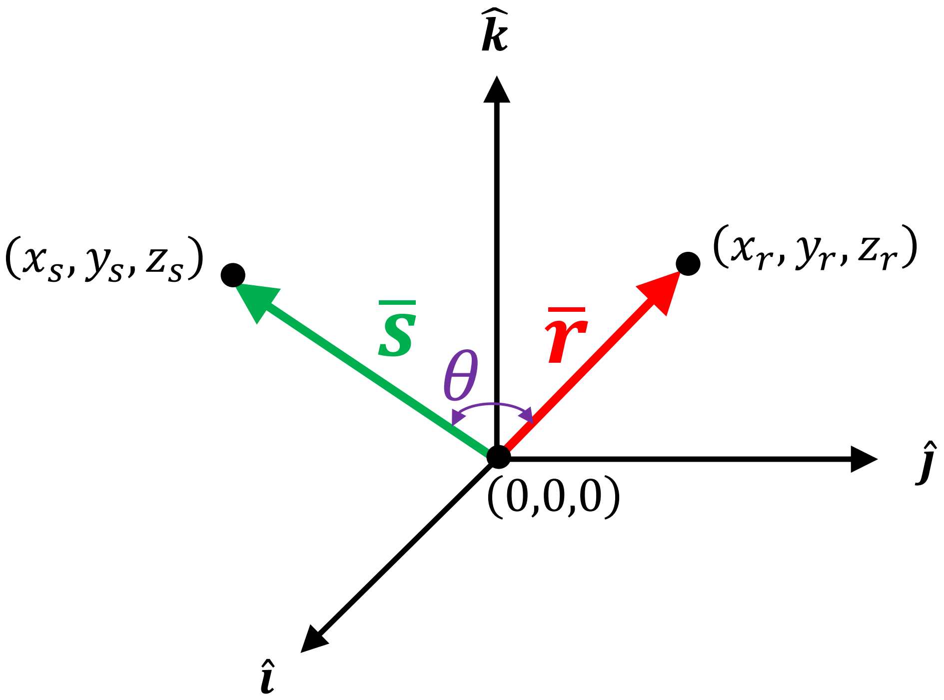


Useful Vector Operations

Consider the two vectors shown at the right…

$$\bar{r} = (x_r - 0)\hat{\boldsymbol{\imath}} + (y_r - 0)\hat{\boldsymbol{\jmath}} + (z_r - 0)\hat{\boldsymbol{k}}$$
$$= x_r\hat{\boldsymbol{\imath}} + y_r\hat{\boldsymbol{\jmath}} + z_r\hat{\boldsymbol{k}}$$

$$\bar{s} = (x_s - 0)\hat{\boldsymbol{\imath}} + (y_s - 0)\hat{\boldsymbol{\jmath}} + (z_s - 0)\hat{\boldsymbol{k}}$$
$$= x_s\hat{\boldsymbol{\imath}} + y_s\hat{\boldsymbol{\jmath}} + z_s\hat{\boldsymbol{k}}$$



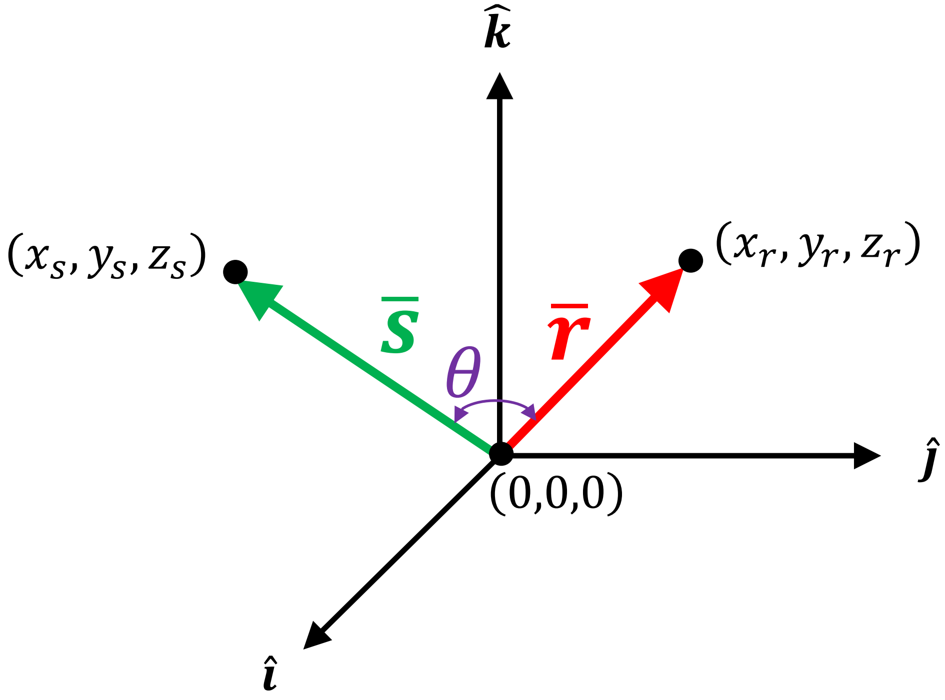
Vector Dot Product

The dot product of two vectors, $\bar{\boldsymbol{r}}$ and $\bar{\boldsymbol{s}}$, is a *scalar* given by…

$$\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{s}} = |\bar{\boldsymbol{r}}||\bar{\boldsymbol{s}}| \cos \theta$$

For the vectors shown at the right…

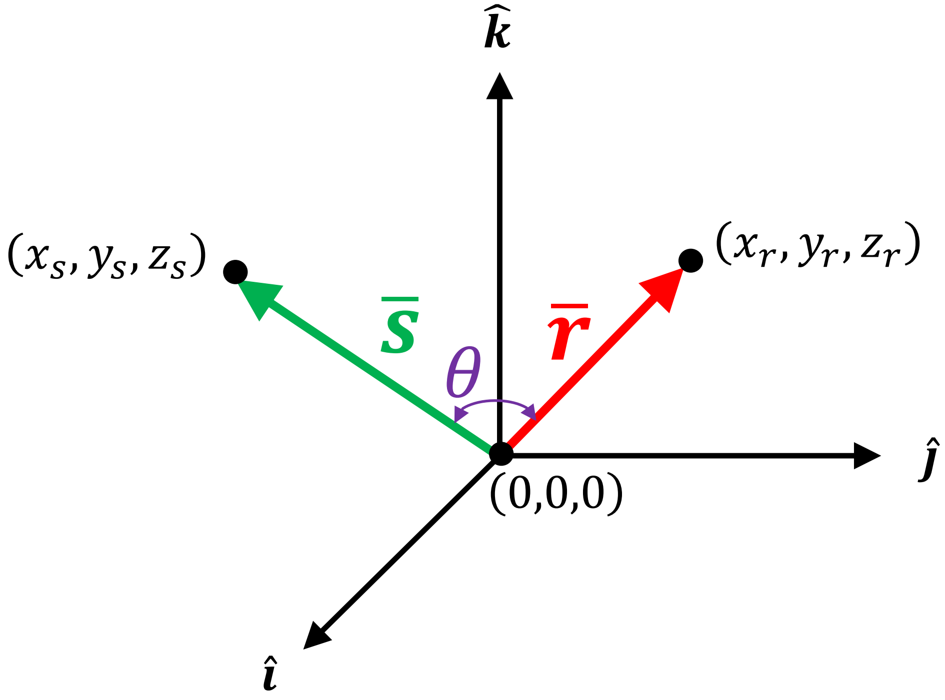
$$\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{s}} = x_r x_s + y_r y_s + z_r z_s$$



Vector Cross Product

The cross product of two vectors, $\bar{r}$ and $\bar{s}$, is a *vector* given by…

$$\bar{r} \times \bar{s} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_r & y_r & z_r \\ x_s & y_s & z_s \end{vmatrix}$$



For the vectors shown at the right…

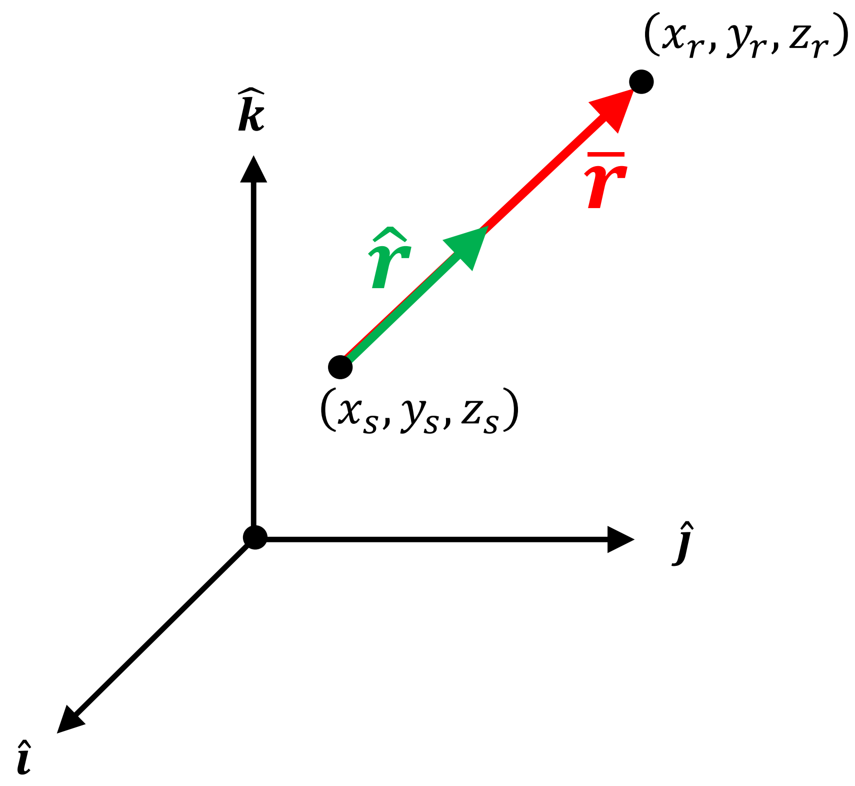
$$\bar{r} \times \bar{s} = (y_r z_s - z_r y_s)\hat{\imath} - (x_r z_s - z_r x_s)\hat{\jmath} + (x_r y_s - y_r x_s)\hat{k}$$

Unit Vectors

As the name implies, a unit vector is a vector with one unit of length;

To form a unit vector, $\hat{\boldsymbol{r}}$ in the direction of $\bar{\boldsymbol{r}}$...

$$\hat{\boldsymbol{r}} = \frac{\bar{\boldsymbol{r}}}{|\bar{\boldsymbol{r}}|} = \frac{\bar{\boldsymbol{r}}}{\sqrt{(x_r - x_s)^2 + (y_r - y_s)^2 + (z_r - z_s)^2}}$$



Coordinate Transformations

Analysis of spacecraft in orbit in a specified attitude requires an understanding of coordinate system transformations.

The position in orbit and the position with respect to heating sources and the eclipse is determined using coordinate system transformations.

Additional transformations are performed to orient the spacecraft as desired at any given point in orbit.

These transformations are performed as Euler angle sequences.

The Euler Angle Sequence

An Euler angle sequence is a sequence of rotations of a rigid body with respect to a fixed coordinate system.

The sequence is order dependent – that is, changing the order of the rotations will affect the resulting transformation.

We will rely on Euler angle transformations considerably during this lesson.

They are easily executed using multiplication of 3×3 matrices.

Some info from: https://en.wikipedia.org/wiki/Euler_angles

Rotation Sequences

However, we need to be specific about the type of rotation we seek – there are two possibilities:

Rotation of the axes, or
Rotation of an object relative to *fixed* axes.

We ultimately seek a rotation of an object relative to *fixed axes*.

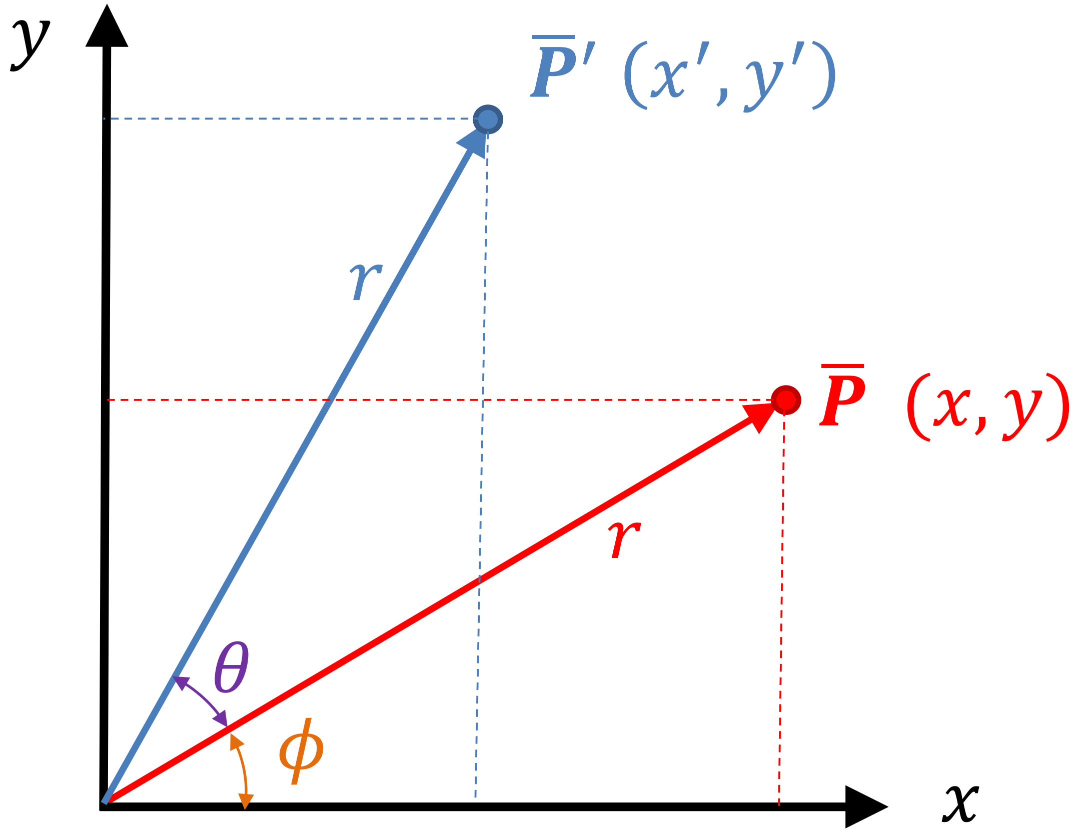
From: mathworld.wolfram.com/RotationMatrix.html

Rotation Sequences

Consider the vector $\overline{\boldsymbol{P}}$ which is at an angle, ϕ from the x-axis in the *fixed coordinate system*.

We wish to transform this vector into $\overline{\boldsymbol{P}}'$ by rotating it through angle, θ in the *same fixed coordinate system*.

What are the coordinates of the tip of $\overline{\boldsymbol{P}}'$, that is x', y', in terms of x and y?



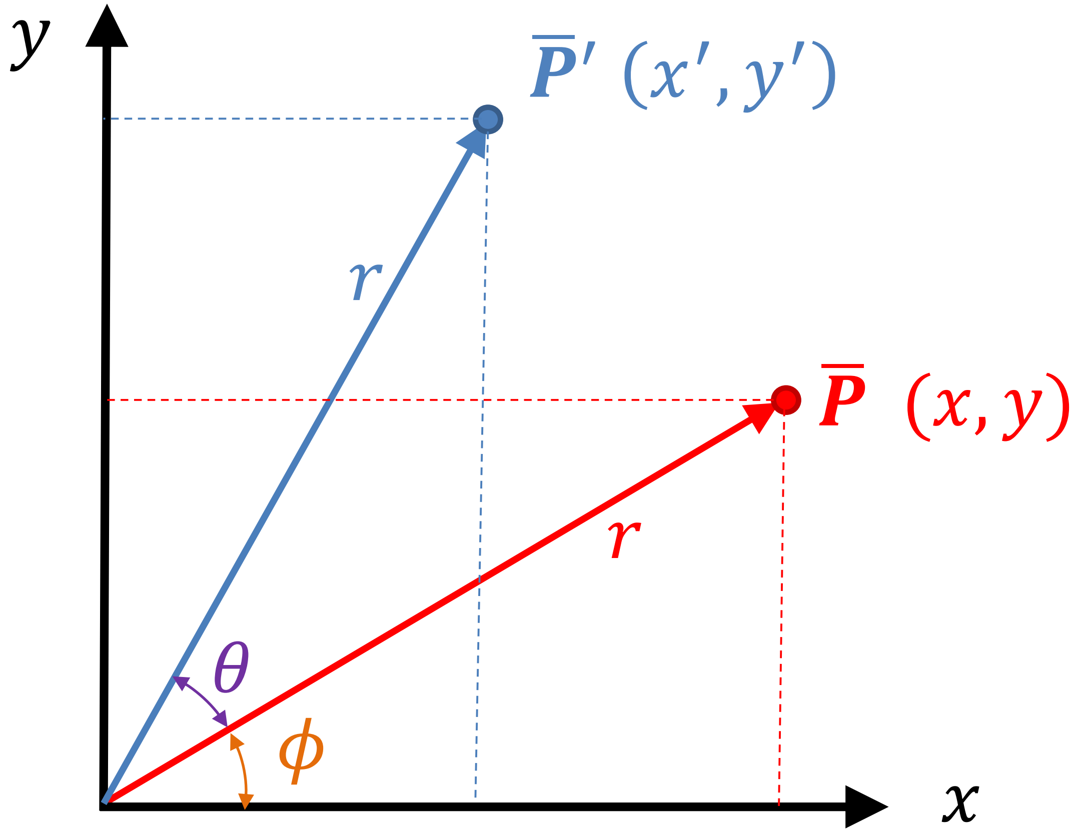
Rotation Sequences

From the figure, we see…

$$x = r\cos\phi$$
$$y = r\sin\phi$$

And…

$$x' = r\cos(\phi + \theta)$$
$$y' = r\sin(\phi + \theta)$$



From: https://www.youtube.com/watch?v=NNWeu3dNFWA

Rotation Sequences

But, using trigonometric identities, we see that…

$$x' = r\cos(\phi + \theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$y' = r\sin(\phi + \theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$

And since $x = r\cos\phi$ and $y = r\sin\phi$, we can substitute to obtain…

$$x' = r\cos(\phi + \theta) = x\cos\theta - y\sin\theta$$
$$y' = r\sin(\phi + \theta) = x\sin\theta + y\cos\theta$$

Or, in matrix form…

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

From: https://www.youtube.com/watch?v=NNWeu3dNFWA

Forming the Transformation Matrix

We rotated the vector $\overline{\boldsymbol{P}}$ in the xy plane about a vector coming out of the page.

This is a z-axis transformation and any z coordinate would remain unchanged. Hence, the 3×3 transformation matrix becomes…

$$\overset{2 \times 2}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \Rightarrow \overset{3 \times 3}{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Forming the Transformation Matrix

Similar operations allow formation of rotation matrices about the x- and y-axes. The resulting transformation matrices are…

x-axis:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

y-axis:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

z-axis:

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

Forming the Transformation Matrix

We will employ the following shorthand to represent transformation of a vector, in this case $\bar{\boldsymbol{r}}$ into $\bar{\boldsymbol{r}}'$, about the x −, y −, and z −axes, respectively…

$$\{\bar{\boldsymbol{r}}'\}=[X]\{\bar{\boldsymbol{r}}\}$$

$$\{\bar{\boldsymbol{r}}'\}=[Y]\{\bar{\boldsymbol{r}}\}$$

$$\{\bar{\boldsymbol{r}}'\}=[Z]\{\bar{\boldsymbol{r}}\}$$

Stacking the Transformations

A series of rotations may be formed through multiplication of the 3×3 transformation matrices *in the order which they are to occur.*

For example, if we wish to transform $\bar{\boldsymbol{r}}$ in to $\bar{\boldsymbol{r}}'$ through an Euler angle rotation sequence first about the x −axis, then about the y −axis and finally about the z −axis, the transformation is given by…

$$\{\bar{\boldsymbol{r}}'\}=[X][Y][Z]\{\bar{\boldsymbol{r}}\}$$

Part 1 Wrap Up

In Part 1, we established that many facets of orbital mechanics and spacecraft attitudes are of interest to thermal engineers;

We reviewed key vector and matrix operations including Euler angle transformations that will serve as a tool kit for our study of orbital mechanics and attitudes.

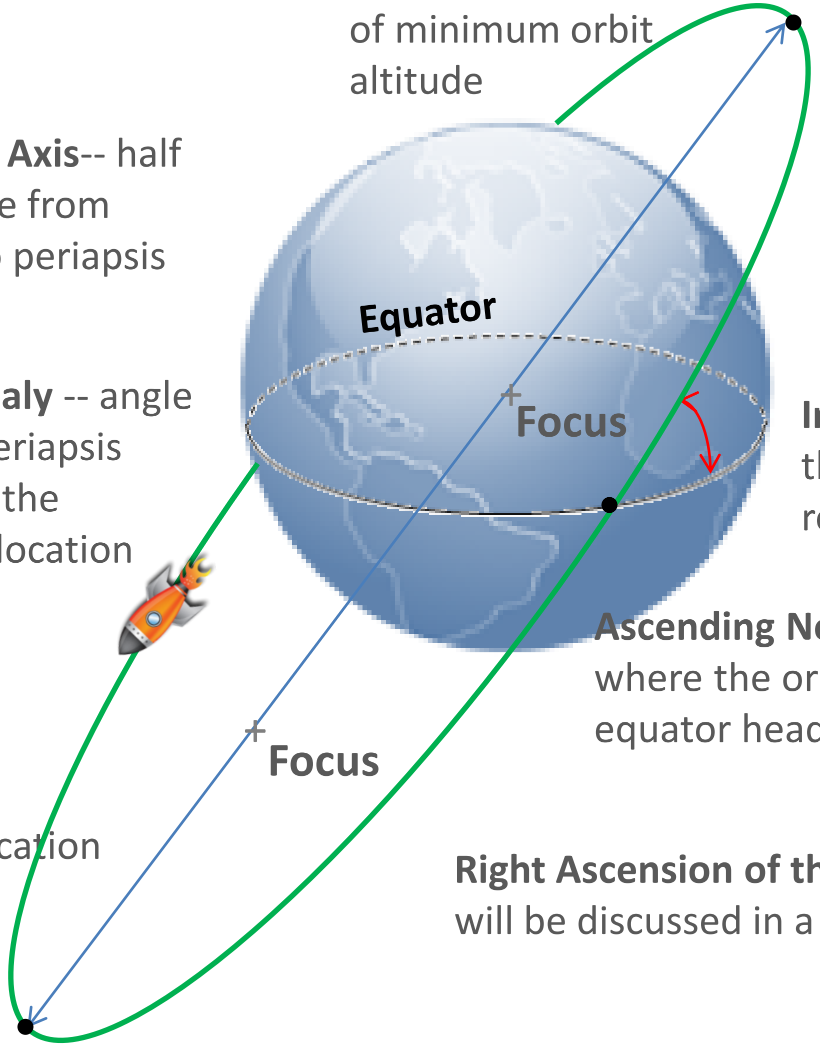
Part 2 -- The Two Body Problem

Aside: Anatomy of an Orbit

Periapsis -- the location of minimum orbit altitude

a **Semimajor Axis**-- half the distance from apoapsis to periapsis

ν **True Anomaly** -- angle from the periapsis location to the spacecraft location



Argument of Periapsis -- the angle, measured in the orbit plane, from the ascending node to the periapsis ω

Inclination -- the tilt of the orbit plane with respect to the equator i

Ascending Node -- the location where the orbit crosses the equator headed south to north

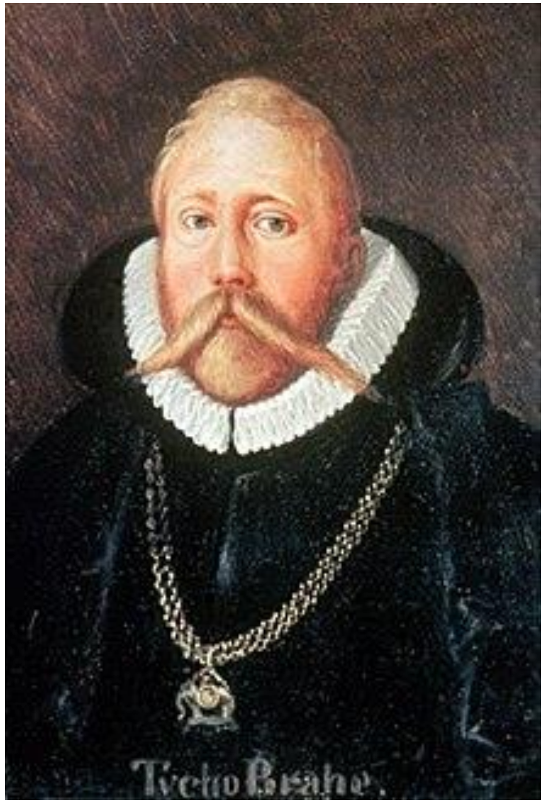
Apoapsis -- the location of maximum orbit altitude

Right Ascension of the Ascending Node will be discussed in a subsequent section. Ω

Aside: History

Tycho Brahe was an outstanding observational astronomer and meticulously recorded the positions of the planets.

Johannes Kepler used Brahe's observational data to fit geometrical curves to explain the position of Mars.



Brahe, 1546-1601

Kepler, 1571-1630

Image Credits: https://en.wikipedia.org/wiki/Johannes_Kepler and https://en.wikipedia.org/wiki/Tycho_Brahe
Other info from: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Aside: History

Kepler formulated his three laws of planetary motion:

Kepler's 1st Law: The orbit of each planet is an ellipse, with the sun as a focus.

Kepler's 2nd Law: The line joining the planet to the sun sweeps out equal areas in equal times.

Kepler's 3rd Law: The square of the period of a planet is proportional to the cube of its mean distance to the sun.

Kepler, 1571-1630

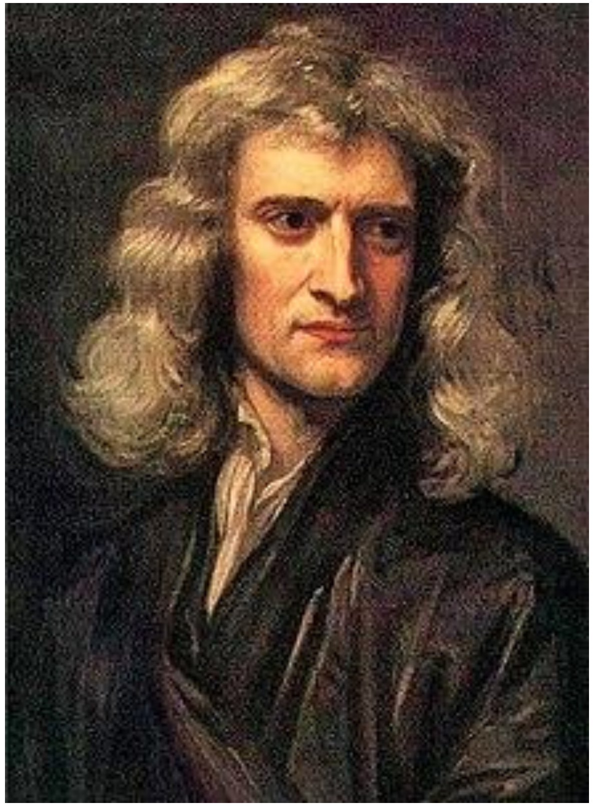
Image Credit: https://en.wikipedia.org/wiki/Johannes_Kepler
Other info from: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Aside: History

In the context of orbital mechanics, Newton's 2nd Law and his Law of Universal Gravitation are pertinent:

Newton's 2nd Law: The sum of the forces is equal to mass times acceleration.

Gravitation: Every particle attracts every other particle in the universe with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.



Newton, 1643 (1642 O.S.) - 1727

Image Credit: https://en.wikipedia.org/wiki/Isaac_Newton
Source: https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation

Strategy

We will derive the governing differential equation for two body motion for an unperturbed orbit.

We will also show that specific mechanical energy and specific angular momentum are conserved for the unperturbed orbit.

From this, we will derive Kepler's Laws and apply them in examples.

Newton's Laws

The governing differential equation for two body astrodynamics is derived from two laws originated by Sir Isaac Newton.

Newton's 2nd Law

$$\bar{\boldsymbol{F}} = m\bar{\boldsymbol{a}}$$

Newton's Law of Gravitation

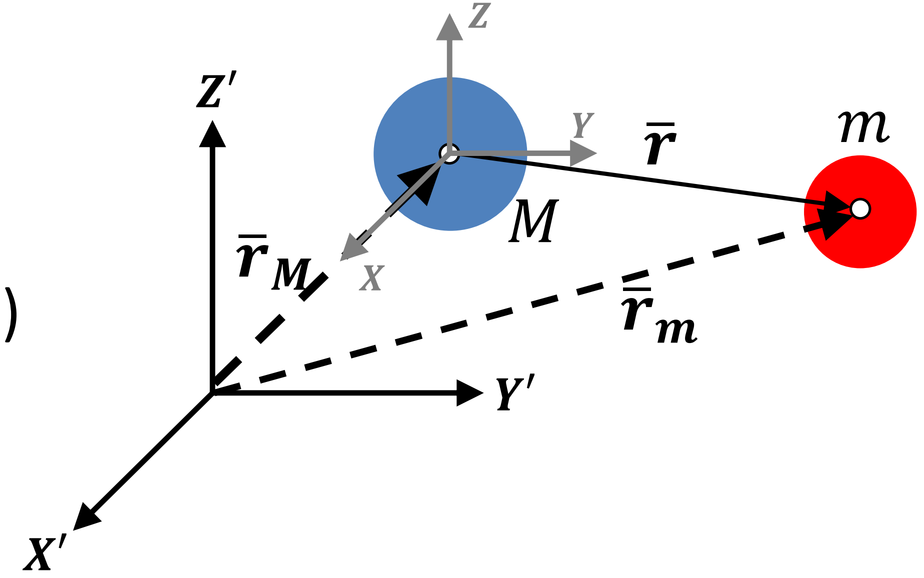
$$\bar{\boldsymbol{F}} = \frac{-GMm}{r^2}\left(\frac{\bar{\boldsymbol{r}}}{r}\right)$$

Relative Motion

We will need to define a reference frame for the calculations. Consider the coordinate systems with masses M and m at the right where…

M is the mass of the first body (assumed to be the larger mass)
m is the mass of the smaller body (assumed here $m \ll M$)
$\bar{\boldsymbol{r}}_{\boldsymbol{M}}$ is the vector from the origin of the reference coordinate system to the center of M
$\bar{\boldsymbol{r}}_{\boldsymbol{m}}$ is the vector from the origin of the reference coordinate system to the center of m
$\bar{\boldsymbol{r}}$ is the vector between M and m

$X'Y'Z'$ is inertial and XYZ is non-rotating.



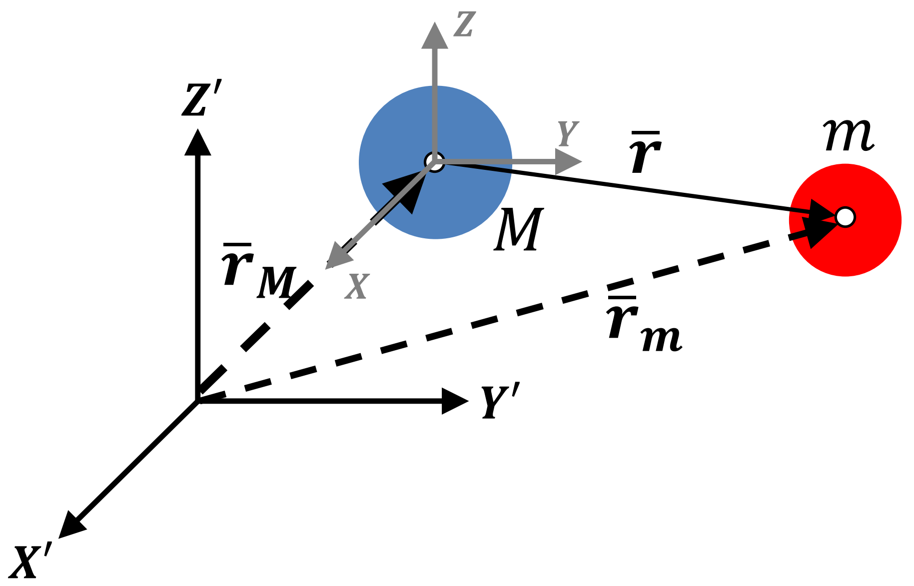
From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Relative Motion

We see that:

$$\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}_m - \bar{\boldsymbol{r}}_M$$

Recognize that since XYZ is non-rotating with respect to $X'Y'Z'$, the respective magnitudes of $\bar{\boldsymbol{r}}$ and $\ddot{\bar{\boldsymbol{r}}}$, will be equal in both systems.



$$\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}}_m - \bar{\boldsymbol{r}}_M \;\Rightarrow\; \ddot{\bar{\boldsymbol{r}}} = \ddot{\bar{\boldsymbol{r}}}_m - \ddot{\bar{\boldsymbol{r}}}_M$$

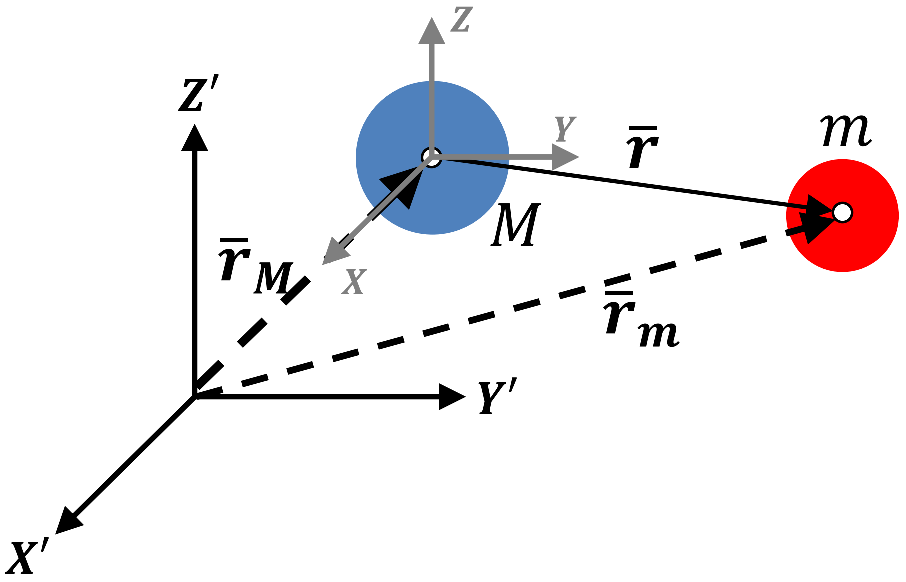
From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Relative Motion

Applying Newton's laws, we have:

$$m\ddot{\bar{\boldsymbol{r}}}_{\boldsymbol{m}} = -\frac{GMm}{r^2}\left(\frac{\bar{\boldsymbol{r}}}{r}\right)$$

$$M\ddot{\bar{\boldsymbol{r}}}_{\boldsymbol{M}} = \frac{GMm}{r^2}\left(\frac{\bar{\boldsymbol{r}}}{r}\right)$$

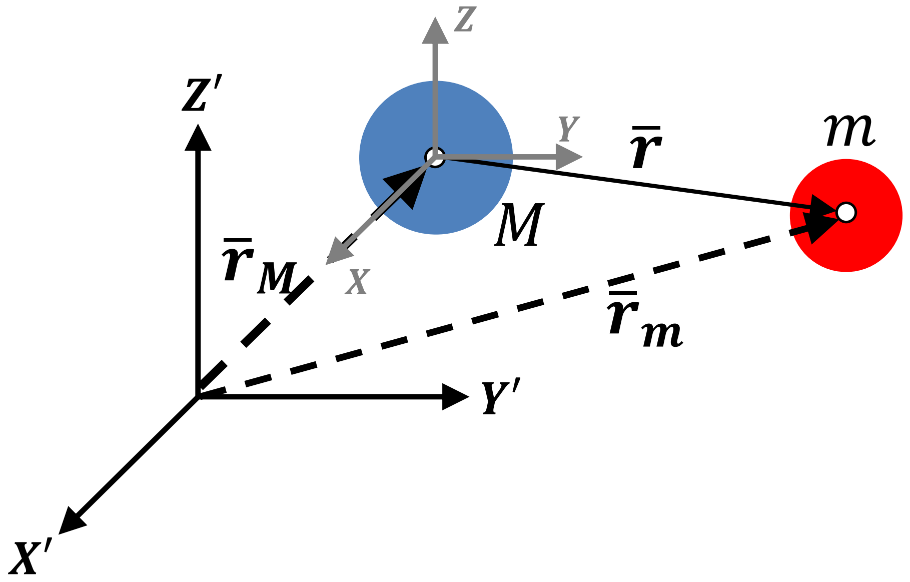


From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Relative Motion

Combining the two previous expressions, we arrive at:

$$\ddot{\bar{\boldsymbol{r}}} = -\frac{G(M+m)}{r^3}\bar{\boldsymbol{r}}$$



From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Relative Motion

For a spacecraft orbiting a planet or the sun (or even planets or other bodies orbiting the sun), $M \gg m$ so the expression becomes:

$$\ddot{\bar{\boldsymbol{r}}} + \frac{\mu}{r^3}\bar{\boldsymbol{r}} = \mathbf{0}$$

Where $\mu = GM$, M is the mass of the central body (i.e., the body being orbited) and $G = 6.67 \times 10^{-11} Nm^2kg^{-2}$.

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Aside: Some Useful Expansions of Terms

For an orbit, we have:

$$\bar{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$$

$$\bar{\boldsymbol{v}} = \dot{\bar{\boldsymbol{r}}} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}$$

$$\bar{\boldsymbol{a}} = \dot{\bar{\boldsymbol{v}}} = \ddot{\bar{\boldsymbol{r}}} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\boldsymbol{r}} + \left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\widehat{\boldsymbol{\theta}}$$

From: Battin, R. H., An Introduction to the Mathematics and Methods of Astrodynamics, New York, American Institute of Aeronautics and Astronautics, 1987.

Conservation of Specific Mechanical Energy

To show conservation of specific mechanical energy, form the dot product of the governing differential equation with $\dot{\bar{\boldsymbol{r}}}$:

$$\dot{\bar{\boldsymbol{r}}} \cdot \left(\ddot{\bar{\boldsymbol{r}}} + \frac{\mu}{r^3} \bar{\boldsymbol{r}} \right) = \dot{\bar{\boldsymbol{r}}} \cdot \mathbf{0}$$

$$\dot{\bar{\boldsymbol{r}}} \cdot \ddot{\bar{\boldsymbol{r}}} + \dot{\bar{\boldsymbol{r}}} \cdot \frac{\mu}{r^3} \bar{\boldsymbol{r}} = \mathbf{0}$$

Conservation of Specific Mechanical Energy

Rearranging…

$$\left(\dot{\bar{\boldsymbol{r}}} \cdot \ddot{\bar{\boldsymbol{r}}}\right) + \frac{\mu}{r^3}\left(\dot{\bar{\boldsymbol{r}}} \cdot \bar{\boldsymbol{r}}\right) = \mathbf{0}$$

We note that…

$$2\left(\dot{\bar{\boldsymbol{r}}} \cdot \ddot{\bar{\boldsymbol{r}}}\right) = \frac{d}{dt}\left(\dot{\bar{\boldsymbol{r}}} \cdot \dot{\bar{\boldsymbol{r}}}\right) \Rightarrow \dot{\bar{\boldsymbol{r}}} \cdot \ddot{\bar{\boldsymbol{r}}} = \frac{1}{2}\frac{d}{dt}\left(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}}\right)$$

$$2\left(\dot{\bar{\boldsymbol{r}}} \cdot \bar{\boldsymbol{r}}\right) = \frac{d}{dt}\left(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}}\right) = 2r\frac{dr}{dt}$$

Conservation of Specific Mechanical Energy

Substituting…

$$\frac{1}{2}\frac{d}{dt}(\bar{\boldsymbol{v}} \cdot \bar{\boldsymbol{v}}) + \frac{\mu}{r^3}\frac{1}{2}\frac{d}{dt}(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}}) = \mathbf{0}$$

Which becomes…

$$\frac{1}{2}\frac{d(v^2)}{dt} + \frac{\mu}{r^3}\frac{1}{2}\frac{d(r^2)}{dt} = \mathbf{0}$$

Conservation of Specific Mechanical Energy

This becomes…

$$\frac{1}{2}\frac{d(v^2)}{dt} + \frac{\mu}{r^3} r \frac{dr}{dt} = \mathbf{0}$$

Rearranging and simplifying…

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) + \frac{\mu}{r^2}\frac{dr}{dt} = \mathbf{0}$$

Conservation of Specific Mechanical Energy

Integrating with respect to time…

$$\frac{v^2}{2} - \frac{\mu}{r} = constant$$

The first term is recognized as the kinetic energy per unit mass and the second term is gravitational potential energy per unit mass. The quantity is constant and the specific mechanical energy is conserved.

Adapted from: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971 and http://docplayer.net/11209890-Orbital-mechanics-and-space-geometry.html

Conservation of Specific Angular Momentum

Conversation of specific angular momentum (i.e., momentum per unit mass) may be shown taking the cross product of $\bar{\boldsymbol{r}}$ and the governing differential equation…

$$\bar{\boldsymbol{r}} \times \ddot{\bar{\boldsymbol{r}}} + \bar{\boldsymbol{r}} \times \frac{\mu}{r^3}\bar{\boldsymbol{r}} = \mathbf{0}$$

We note that a vector crossed with itself it is $\mathbf{0}$. The equation simplifies to…

$$\bar{\boldsymbol{r}} \times \ddot{\bar{\boldsymbol{r}}} = \mathbf{0}$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Conservation of Specific Angular Momentum

We also note that…

$$\frac{d}{dt}(\bar{\boldsymbol{r}} \times \dot{\bar{\boldsymbol{r}}}) = (\dot{\bar{\boldsymbol{r}}} \times \dot{\bar{\boldsymbol{r}}}) + (\bar{\boldsymbol{r}} \times \ddot{\bar{\boldsymbol{r}}})$$

We note $\dot{\bar{\boldsymbol{r}}} \times \dot{\bar{\boldsymbol{r}}} = \boldsymbol{0}$. The equation simplifies to…

$$\frac{d}{dt}(\bar{\boldsymbol{r}} \times \dot{\bar{\boldsymbol{r}}}) = \bar{\boldsymbol{r}} \times \ddot{\bar{\boldsymbol{r}}} = \boldsymbol{0}$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Conservation of Specific Angular Momentum

Finally, we recognize that $\dot{\bar{\boldsymbol{r}}} = \bar{\boldsymbol{v}}$ so the equation becomes…

$$\frac{d}{dt}(\bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}) = \mathbf{0}$$

where $\bar{\boldsymbol{h}} = \bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}$ is recognized as the specific angular momentum and we have shown that this does not change with time…

$$\bar{\boldsymbol{h}} = \bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}} = \boldsymbol{constant}$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Derivation of Kepler’s Laws

Now that we have the governing differential equation, the conservation of, both, specific mechanical energy and specific angular momentum, we are ready to derive Kepler’s laws.

Kepler's First Law

The orbit of each planet is an ellipse*, with the sun as a focus.

*Actually, other orbit shapes are possible and are described by the "conic sections."

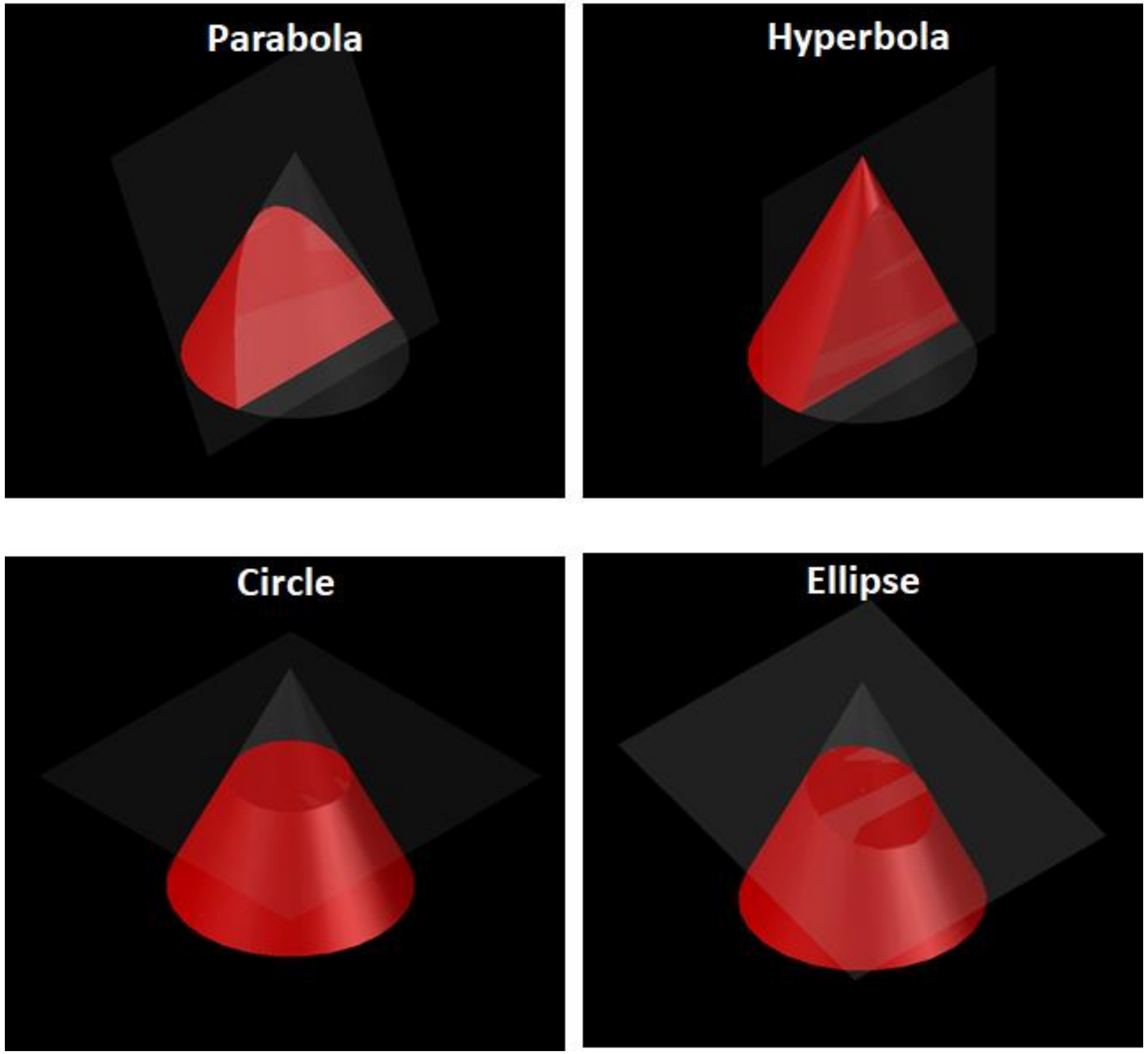
Orbits of the Inner Planets

Kepler’s First Law

But what is a “conic section”?

Take a cone and cut it with a plane at different angles

The shapes appearing at the cutting plane are also the shapes of the orbits.



Kepler's First Law

Starting with the governing differential equation:

$$\ddot{\bar{\boldsymbol{r}}} + \frac{\mu}{r^3}\bar{\boldsymbol{r}} = \boldsymbol{0}$$

Re-arrange to get:

$$\ddot{\bar{\boldsymbol{r}}} = -\frac{\mu}{r^3}\bar{\boldsymbol{r}}$$

Form the cross product with the angular momentum vector:

$$\ddot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}} = -\frac{\mu}{r^3}\left(\bar{\boldsymbol{r}} \times \bar{\boldsymbol{h}}\right) = \frac{\mu}{r^3}\left(\bar{\boldsymbol{h}} \times \bar{\boldsymbol{r}}\right)$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

Let's examine this equation in more detail:

$$\ddot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}} = \frac{\mu}{r^3}\left(\bar{\boldsymbol{h}} \times \bar{\boldsymbol{r}}\right)$$

We see that:

$$\ddot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}} = \ddot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}} + \dot{\bar{\boldsymbol{r}}} \times \overbrace{\dot{\bar{\boldsymbol{h}}}}^{0} = \frac{d}{dt}\left(\dot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}}\right)$$

$$\frac{\mu}{r^3}\left(\bar{\boldsymbol{h}} \times \bar{\boldsymbol{r}}\right) = \frac{\mu}{r^3}\left(\bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}\right) \times \bar{\boldsymbol{r}} = \frac{\mu}{r^3}\left[\bar{\boldsymbol{v}}\left(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}}\right) - \bar{\boldsymbol{r}}\left(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{v}}\right)\right]$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

And, within the triple vector triple product $(\bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}) \times \bar{\boldsymbol{r}}$:

$$(\bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}) \times \bar{\boldsymbol{r}} = [\bar{\boldsymbol{v}}(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}}) - \bar{\boldsymbol{r}}(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{v}})]$$

We note that:

$$(\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{v}}) = (\bar{\boldsymbol{r}} \cdot \dot{\bar{\boldsymbol{r}}}) = r\hat{\boldsymbol{r}} \cdot \left(\frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right) = r\dot{r}$$

We end up with:

$$\frac{\mu}{r^3}(\bar{\boldsymbol{h}} \times \bar{\boldsymbol{r}}) = \frac{\mu}{r}\bar{\boldsymbol{v}} - \frac{\mu\dot{r}}{r^2}\bar{\boldsymbol{r}}$$

From: https://en.wikipedia.org/wiki/Vector_algebra_relations
From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

Further simplification yields:

$$\frac{\mu}{r}\bar{\boldsymbol{v}} - \frac{\mu \dot{r}}{r^2}\bar{\boldsymbol{r}} = \mu \frac{d}{dt}\left(\frac{\bar{\boldsymbol{r}}}{r}\right)$$

Our equation becomes:

$$\frac{d}{dt}\left(\dot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}}\right) = \mu \frac{d}{dt}\left(\frac{\bar{\boldsymbol{r}}}{r}\right)$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

Repeating for convenience:

$$\frac{d}{dt}(\dot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}}) = \mu \frac{d}{dt}\left(\frac{\bar{\boldsymbol{r}}}{r}\right)$$

Integrating the above equation:

$$\dot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}} = \mu\left(\frac{\bar{\boldsymbol{r}}}{r}\right) + \bar{\boldsymbol{B}}$$

Where $\bar{\boldsymbol{B}}$ is a vector constant.

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

Dot both sides of the equation with $\bar{\boldsymbol{r}}$:

$$\bar{\boldsymbol{r}} \cdot (\dot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}}) = \bar{\boldsymbol{r}} \cdot \mu \left(\frac{\bar{\boldsymbol{r}}}{r}\right) + \bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{B}}$$

And since:

$$\bar{\boldsymbol{r}} \cdot (\dot{\bar{\boldsymbol{r}}} \times \bar{\boldsymbol{h}}) = (\bar{\boldsymbol{r}} \times \dot{\bar{\boldsymbol{r}}}) \cdot \bar{\boldsymbol{h}} = (\bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}) \cdot \bar{\boldsymbol{h}} = h^2$$

$$\bar{\boldsymbol{r}} \cdot \mu \left(\frac{\bar{\boldsymbol{r}}}{r}\right) = \bar{\boldsymbol{r}} \cdot \mu \hat{\boldsymbol{r}} = r \hat{\boldsymbol{r}} \cdot \mu \hat{\boldsymbol{r}} = \mu r$$

$$\bar{\boldsymbol{r}} \cdot \bar{\boldsymbol{B}} = r B \cos \nu$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

The equation simplifies to:

$$h^2 = \mu r + rB \cos \nu$$

Rearranging gives:

$$r = \frac{h^2/\mu}{1 + \left(B/\mu\right) \cos \nu}$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

We see that the equation is in the same form as the general equation for a conic section in polar coordinates:

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos\nu}$$

$$r = \frac{p}{1 + e\cos\nu} = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

The parameter, a, is the semimajor axis and e is the orbit eccentricity.

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's First Law

The form of the equation confirms that orbits derived under these assumptions take the shape of the conic sections and is dependent upon the orbit eccentricity, e:

Eccentricity	Orbit Shape
$e = 0$	Circle
$0 < e < 1$	Ellipse
$e = 1$	Parabola
$e > 1$	Hyperbola

Let's take a look at some orbits representing each orbit type.

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

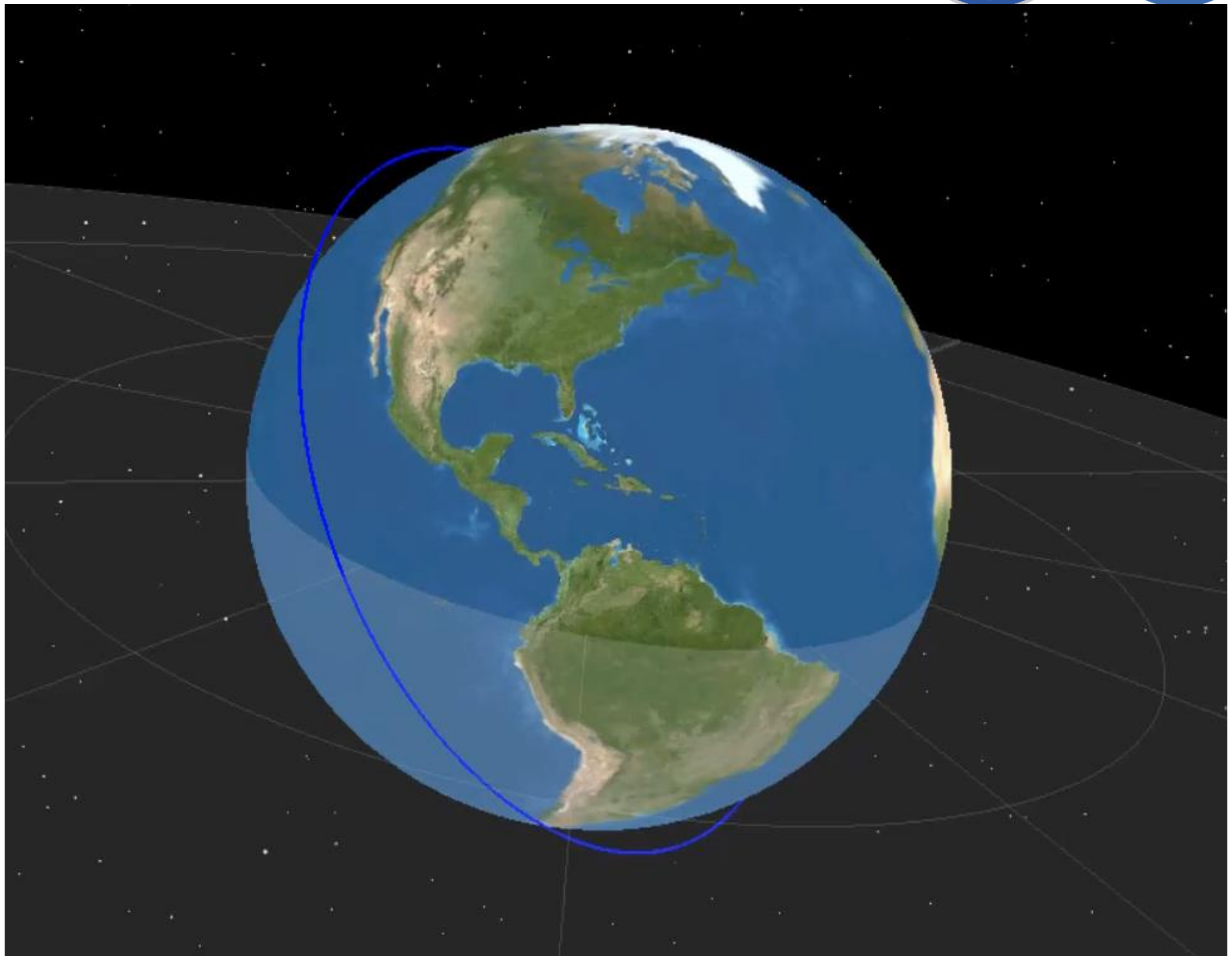
Circular Orbit

Circular orbits maintain a constant distance from their central body.

Orbit eccentricity, $e = 0$.

Many Earth satellites have circular orbits.

The International Space Station is in a circular orbit.



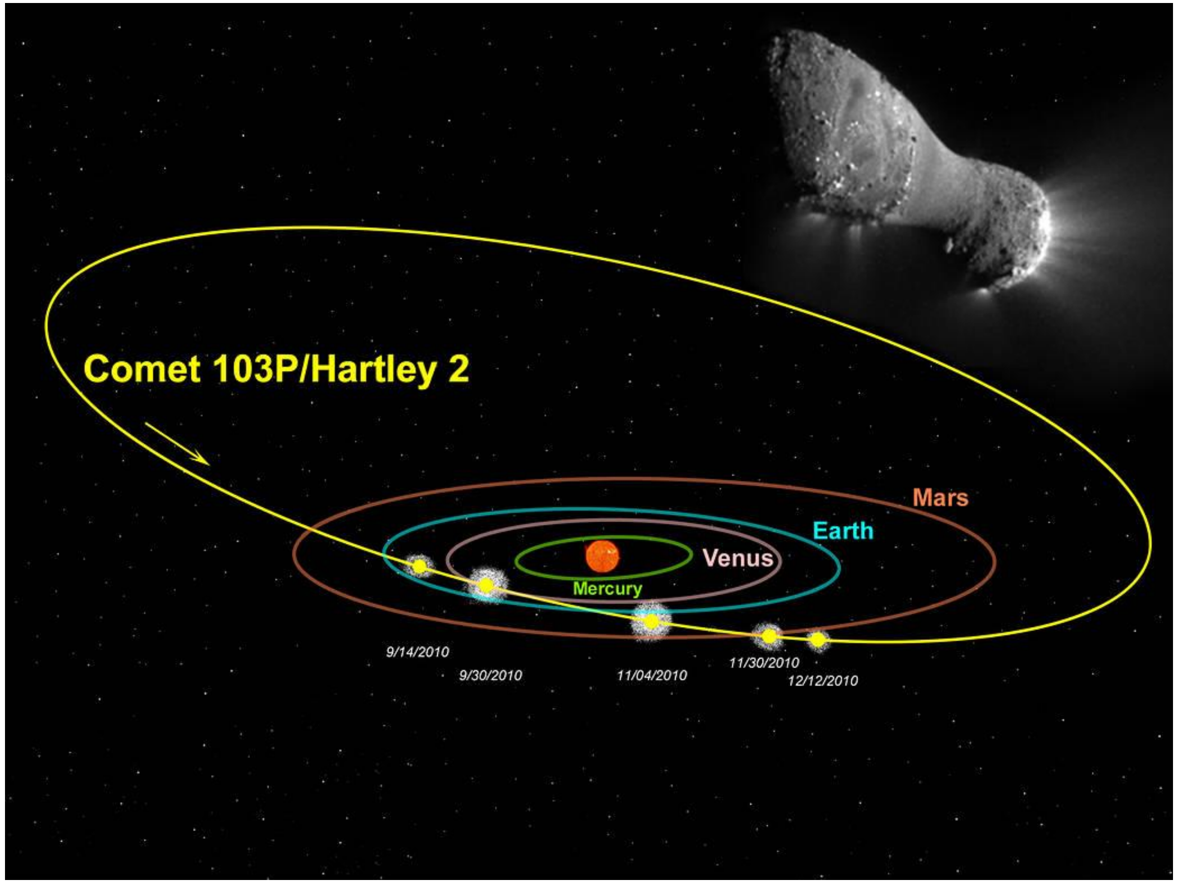
Example: International Space Station Orbit

Elliptical Orbit

Orbit eccentricity, $0 < e < 1$.

An elliptical orbit traces out an ellipse with the central body at one focus.

Comets such as 103P/Hartley 2 are in elliptical orbits with a period of 6.46 years ($e = 0.694$).



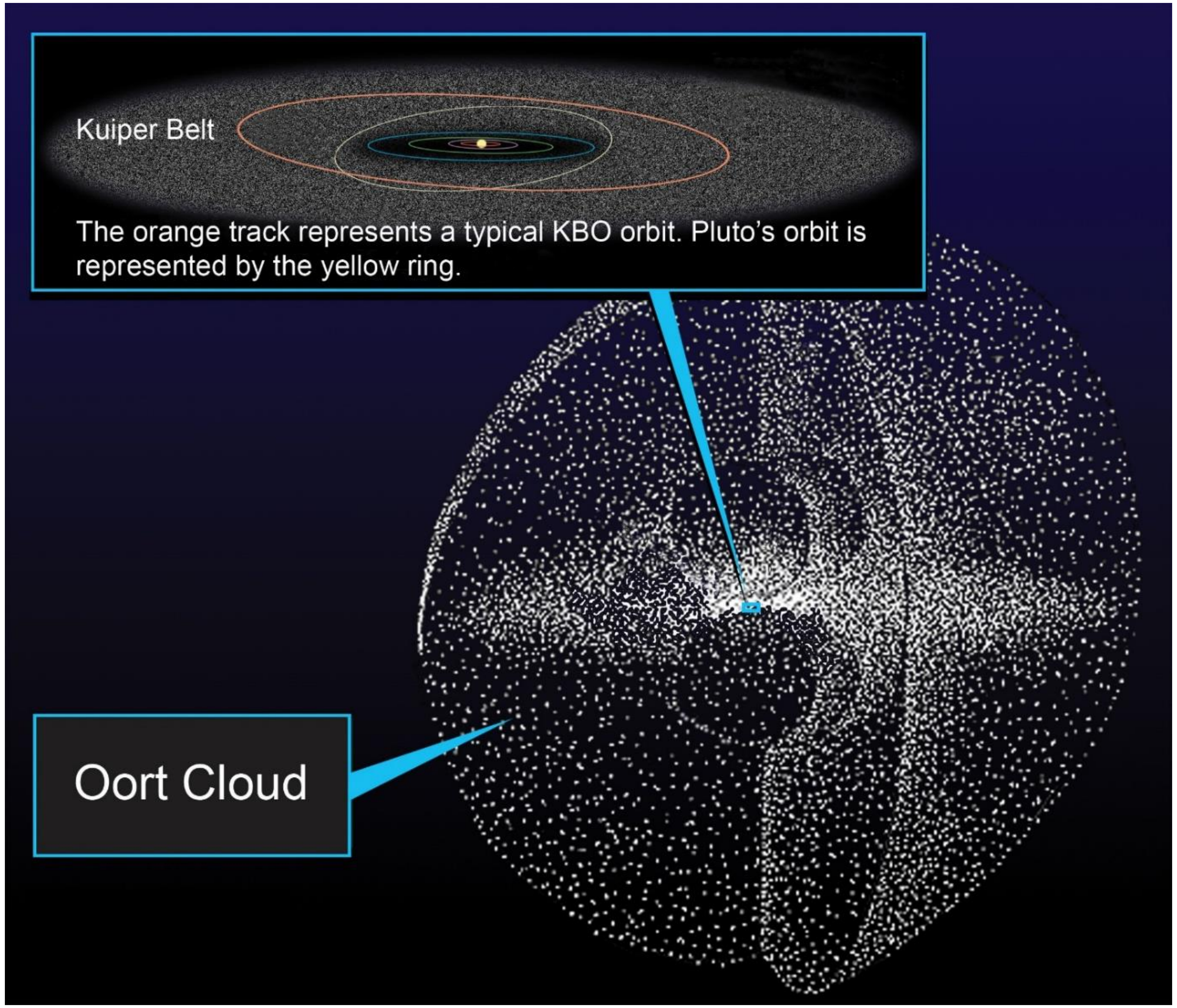
Example: Comet Hartley 2 Orbit

Image Credit: NASA/Steele Hill; Inset Image Credit: NASA/JPL-Caltech/UMD

Parabolic Orbit

When orbit eccentricity, $e = 1$, we have parabolic orbit.

"Within observational uncertainty, long term comets all seem to have parabolic orbits. That suggests they are not truly interstellar, but are loosely attached to the Sun. They are generally classified as belonging the *Oort cloud* on the fringes of the solar system, at distances estimated at 100,000 AU."



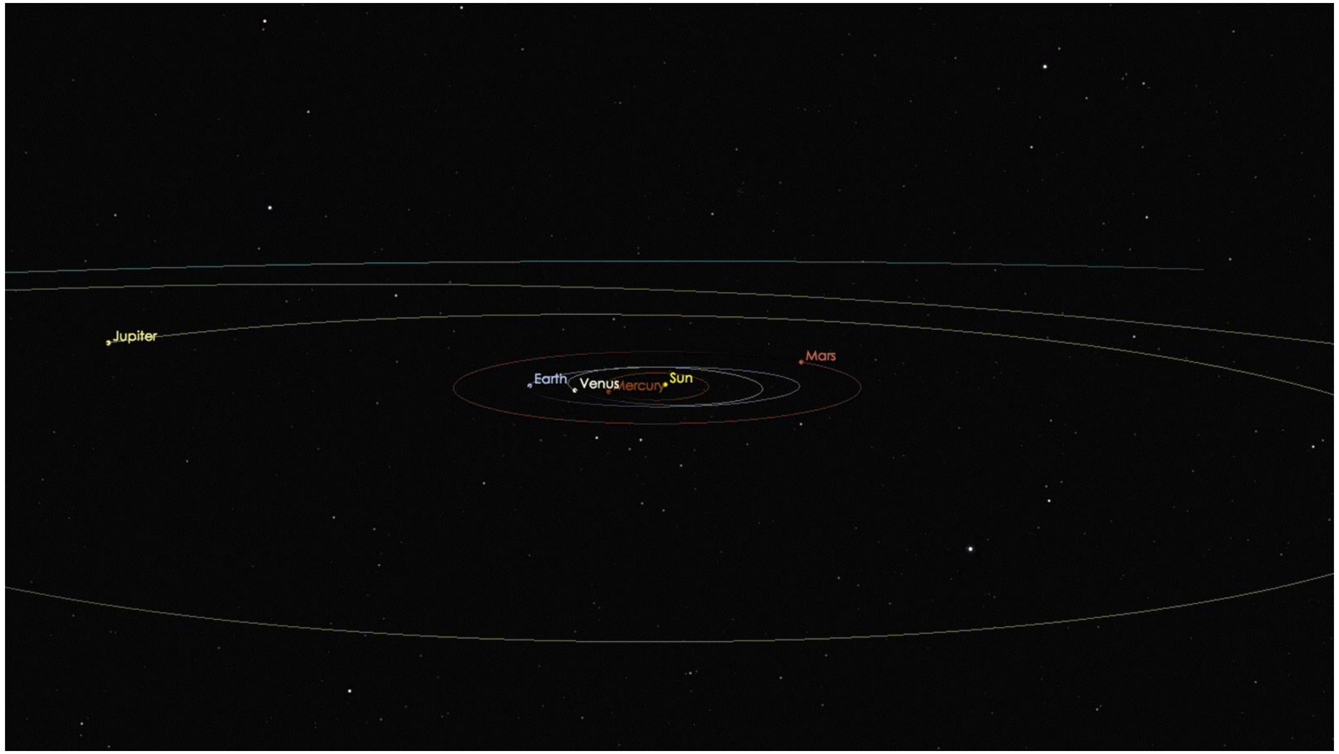
The Oort Cloud

Source: https://www-spof.gsfc.nasa.gov/stargaze/Scomets.htm
Image Credit: https://solarsystem.nasa.gov/resources/491/oort-cloud/

Hyperbolic Orbit

Orbit eccentricity, $e > 1$;

For objects passing through the solar system, a hyperbolic orbit suggests an interstellar origin -- Asteroid Oumuamua was discovered in 2017 and is first known object of this type ($e = 1.19951$).



Example: Asteroid Oumuamua

Video Credit: https://solarsystem.nasa.gov/asteroids-comets-and-meteors/comets/oumuamua/in-depth/
Source: https://en.wikipedia.org/wiki/%CA%BBOumuamua

Example: Determining Solar Flux Using Kepler's First Law

We saw that the equation is in the same form as the general equation for a conic section in polar coordinates:

$$r = \frac{p}{1 + e\cos\nu} = \frac{a(1 - e^2)}{1 + e\cos\nu}$$

where a and e are constants and ν is the true anomaly. For a planet orbiting the sun, r is a minimum (a.k.a, perihelion) when $\nu = 0°$ and r is maximum (a.k.a., aphelion) when $\nu = 180°$.

Example: Determining Solar Flux Using Kepler's First Law

At Earth's mean distance from the sun (i.e., $1\ au$), the measured solar flux is on the order of $1371\ W/m^2$.

We can determine the solar flux at any distance, r (measured in au) from the sun by noting:

$$\dot{q}_{solar}(r) = \frac{1371\ W/m^2}{r^2}$$

Example: Determining Solar Flux Using Kepler's First Law

Solar flux values for the planets are readily calculated:

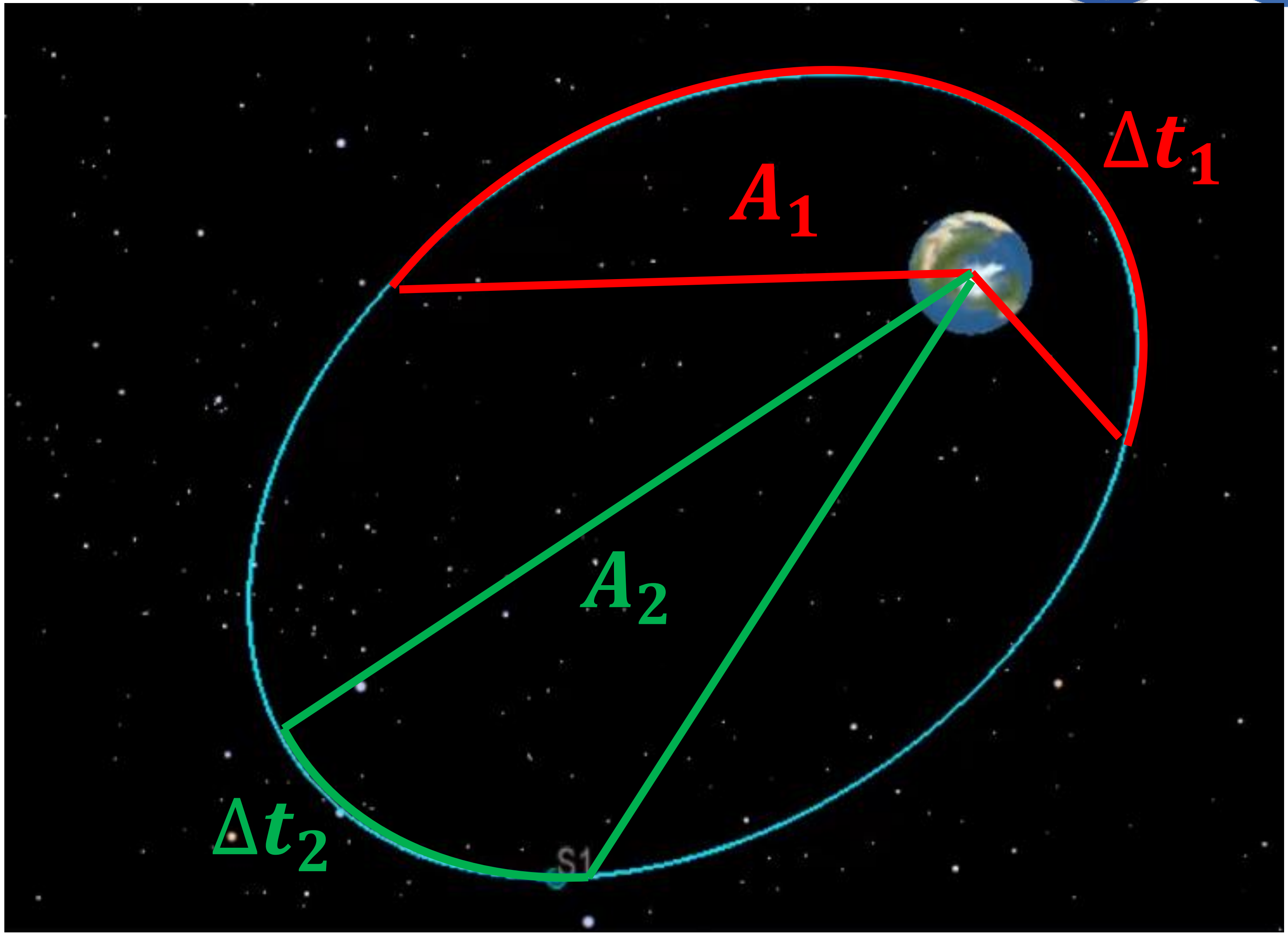
Planet	Semimajor Axis, a (au)	Orbit Eccentricity, e	Perihelion Distance (au)	Aphelion Distance (au)	Solar Flux at Perihelion (W/m^2)	Solar Flux at Aphelion (W/m^2)
Mercury	0.3871	0.2056	0.3075	0.4667	14498.23	6294.87
Venus	0.7233	0.0067	0.7185	0.7282	2655.86	2585.63
Earth	1.0000	0.0167	0.9833	1.0167	1417.96	1326.33
Mars	1.5235	0.0935	1.3811	1.6660	718.79	493.97
Jupiter	5.2043	0.0489	4.9499	5.4588	55.96	46.01
Saturn	9.5824	0.0565	9.0410	10.1238	16.77	13.38
Uranus	19.2009	0.0457	18.3235	20.0784	4.08	3.40
Neptune	30.0472	0.0113	29.7077	30.3867	1.55	1.48

Semimajor axis and eccentricity data from: nssdc.gsfc.nasa.gov

Kepler's Second Law

The line joining the planet to the sun sweeps out equal areas, A in equal times, Δt.

$$\Delta t_1 = \Delta t_2$$
$$A_1 = A_2$$



Demonstration of Constant "Areal" Velocity

Kepler's Second Law

We begin with our previously derived expression for the angular momentum vector, $\bar{\boldsymbol{h}}$:

$$\bar{\boldsymbol{h}} = \bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}$$

And recalling the expressions for vectors $\bar{\boldsymbol{r}}$ and $\bar{\boldsymbol{v}}$:

$$\bar{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$$

$$\bar{\boldsymbol{v}} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}$$

Kepler's Second Law

The expression for $\bar{\boldsymbol{h}}$ is, then:

$$\bar{\boldsymbol{h}} = \bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}} = \begin{vmatrix} \hat{\boldsymbol{r}} & \widehat{\boldsymbol{\theta}} & \widehat{\boldsymbol{k}} \\ r & 0 & 0 \\ \dfrac{dr}{dt} & r\dfrac{d\theta}{dt} & 0 \end{vmatrix} = r^2 \frac{d\theta}{dt} \widehat{\boldsymbol{k}}$$

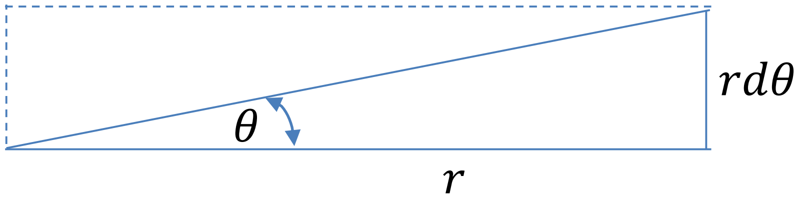
The magnitude of this vector is:

$$h = |\bar{\boldsymbol{h}}| = r^2 \frac{d\theta}{dt}$$

Kepler's Second Law

We showed previously that the specific angular momentum is constant…

$$h = r^2 \frac{d\theta}{dt} = constant$$



We also recognize that the area swept out over time is simply one half of the specific angular momentum…

$$\frac{dA}{dt} = \frac{1}{2} h = \frac{1}{2} r^2 \frac{d\theta}{dt} = constant$$

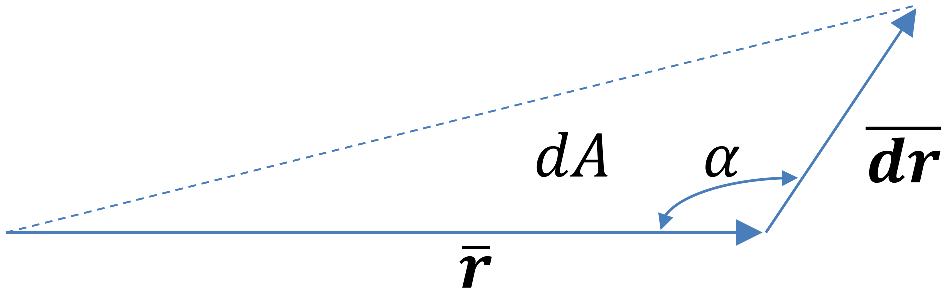
Kepler's Second Law

Consider another approach…

$$dA = \frac{1}{2} r\, dr\ \sin\alpha$$

If we let the differential area, dA be represented as a vector, $\overline{\boldsymbol{dA}}$ …

$$\overline{\boldsymbol{dA}} = \frac{1}{2}\bar{\boldsymbol{r}} \times \overline{\boldsymbol{dr}}$$

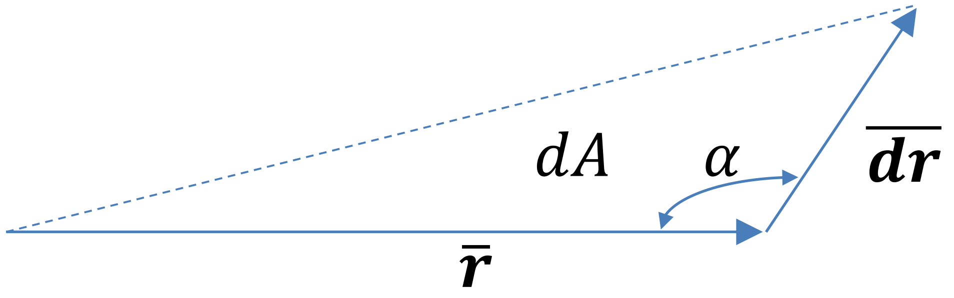


From: https://radio.astro.gla.ac.uk/a1dynamics/keplerproofs.pdf

Kepler's Second Law

Differentiate with respect to time…

$$\dot{\boldsymbol{A}} = \frac{\overline{\boldsymbol{dA}}}{\boldsymbol{dt}} = \frac{1}{2}\bar{\boldsymbol{r}} \times \dot{\bar{\boldsymbol{r}}}$$



Differentiate again…

Vector crossed with itself

Vectors pointed in same direction

$$\ddot{\boldsymbol{A}} = \frac{\dot{\overline{\boldsymbol{dA}}}}{\boldsymbol{dt}} = \frac{1}{2}\bar{\boldsymbol{r}} \times \dot{\bar{\boldsymbol{r}}} = \frac{1}{2}\left(\cancelto{0}{\dot{\bar{\boldsymbol{r}}} \times \dot{\bar{\boldsymbol{r}}}} + \cancelto{0}{\bar{\boldsymbol{r}} \times \ddot{\bar{\boldsymbol{r}}}}\right) = \mathbf{0}$$

So $\dfrac{\overline{\boldsymbol{dA}}}{\boldsymbol{dt}} = \boldsymbol{constant}$

From: https://radio.astro.gla.ac.uk/a1dynamics/keplerproofs.pdf

Example: Using Kepler's Second Law to Determine How Solar Flux Varies with Time

We saw that knowing the shape of a planet's orbit (aphelion and perihelion distances) and the solar flux at $1\ au$ could be used to determine the minimum and maximum solar flux.

In this example, we'll calculate how the solar flux for Earth varies with time throughout the year.

In doing so, we'll compare a simplified model with a more accurate representation accounting for Kepler's Second Law.

Example: Using Kepler's Second Law to Determine How Solar Flux Varies with Time

A consequence of Kepler's Second Law is that to sweep out equal areas in equal times, a planet (or moon or spacecraft) orbiting a central body (i.e., the sun, a planet, moon, etc.) must move through its orbit faster at some locations and slower at others.

In other words, the angular rate at which the orbiting body moves around its orbit of the central body changes depending on where it is in its orbit.

Example: Using Kepler's Second Law to Determine How Solar Flux Varies with Time

Consider Earth's orbit around the sun. We know Earth makes one circuit of the sun in ~365.25 days.

If Earth's orbit about the sun were circular, the angular rate would be:

$$\dot{v} = \frac{360°}{365.25\ days} \approx 0.986\ °/day$$

Example: Using Kepler's Second Law to Determine How Solar Flux Varies with Time

But, Earth's orbit about the sun isn't circular, it is slightly elliptical with an $e = 0.0167$.

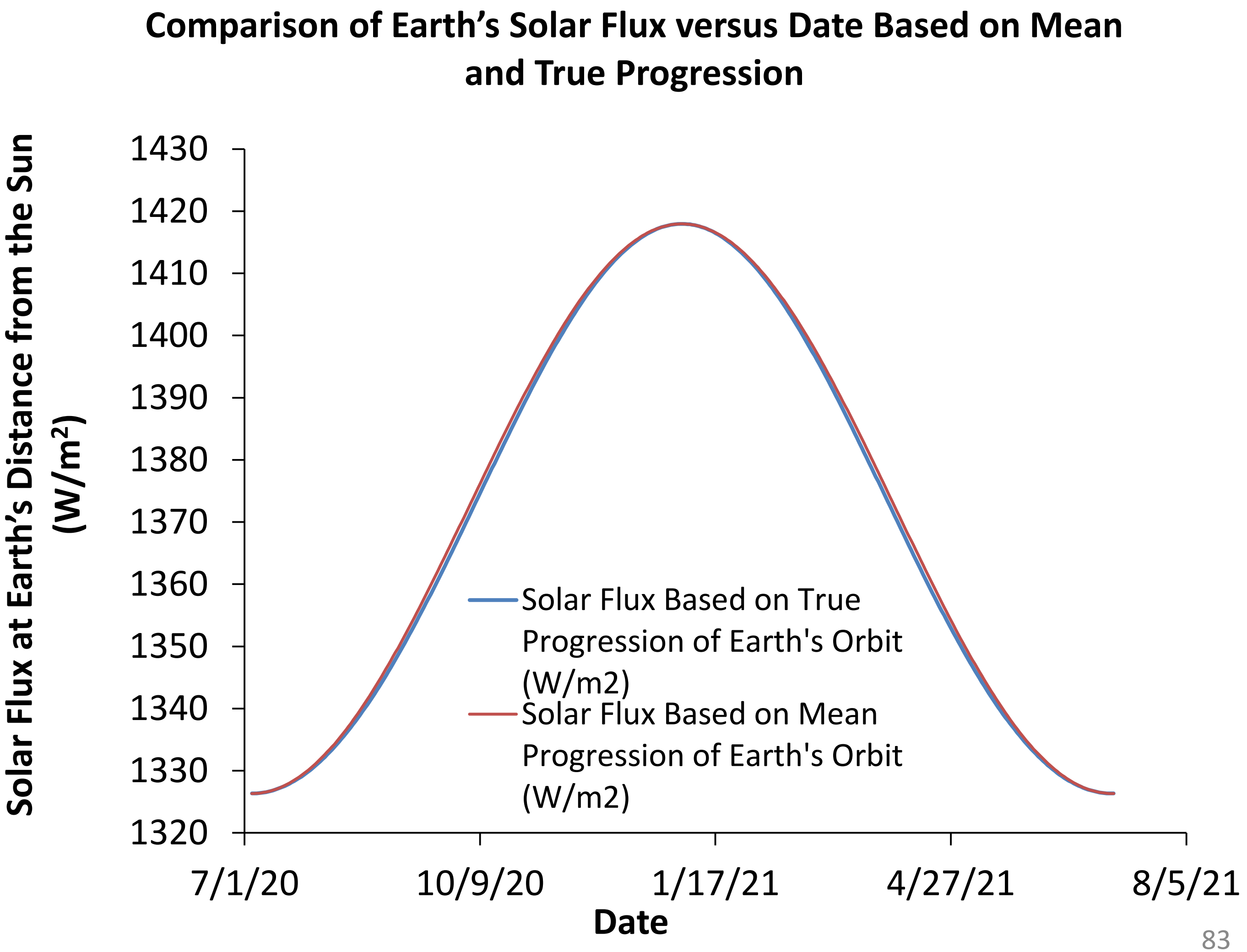
This elliptical shape is what gives rise to the aphelion and perihelion distances and, hence, the variation in solar flux.

But because of Kepler's Second Law, the angular rate will vary depending on Earth's distance from the sun.

Example: Using Kepler's Second Law to Determine How Solar Flux Varies with Time

We see that the assuming the mean motion for Earth's orbit ($e = 0.0167$) about the sun is a reasonable approximation to the slightly elliptical orbit. This due to the very low eccentricity of Earth's orbit.

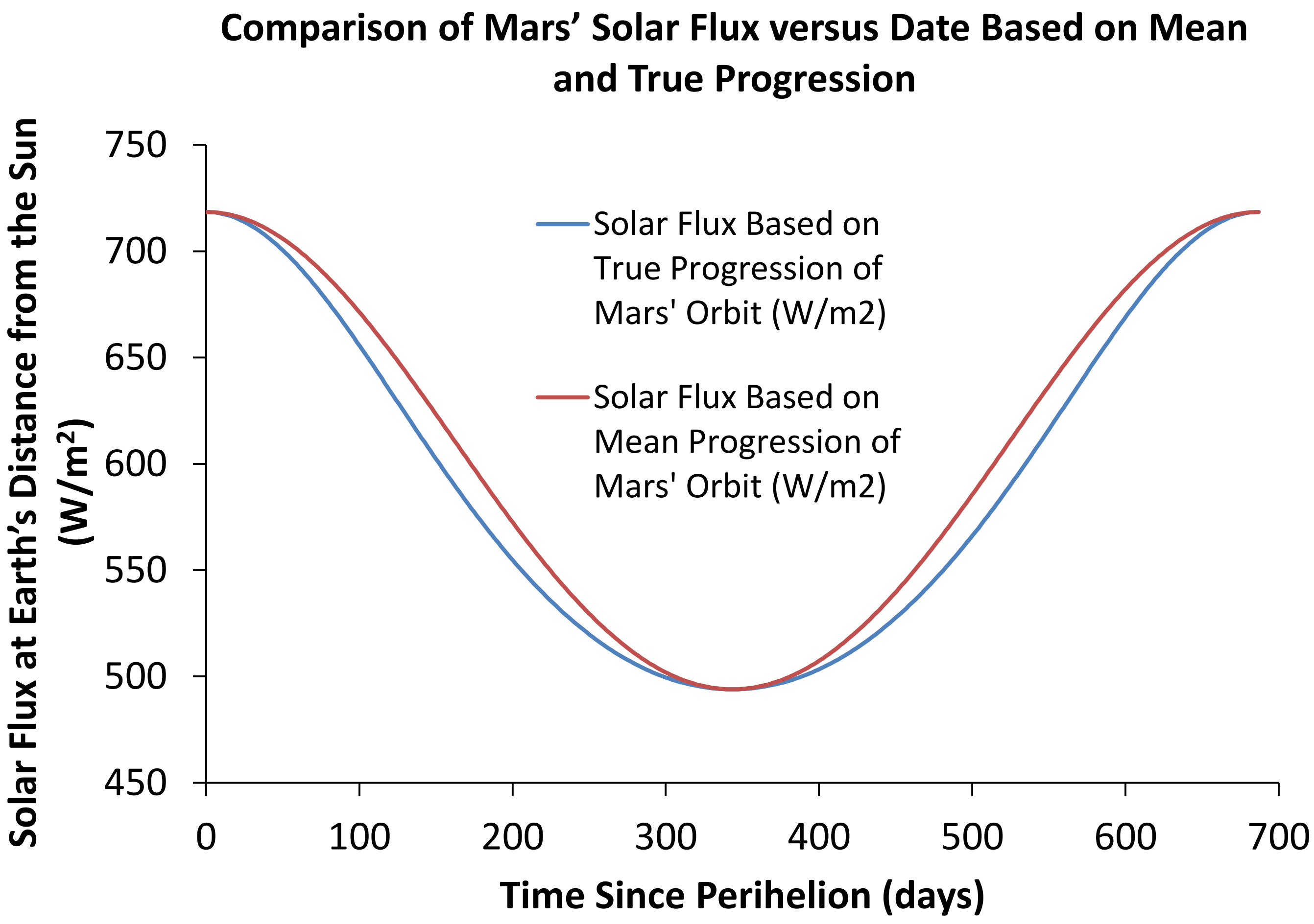
Such an approximation will *not* work as well for planets with more eccentric orbits.



Example: Using Kepler's Second Law to Determine How Solar Flux Varies with Time

Consider Mars with an eccentricity, $e = 0.09339$.

The time variation of flux is more pronounced due to the effect of Kepler's Second Law.



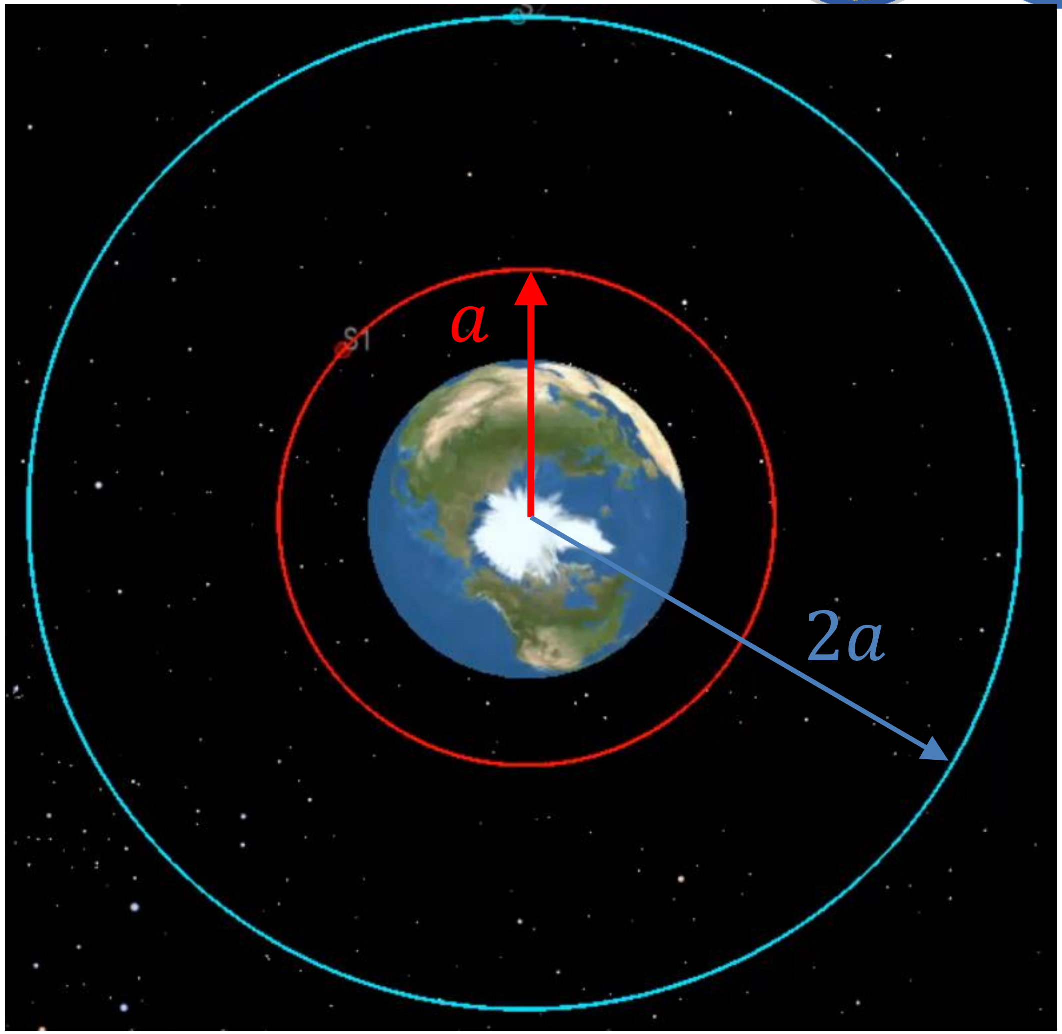
Kepler's Third Law

The square of the period, T of a planet is proportional to the cube of its mean distance, a to the sun (or its central body).

$$T^2 \propto a^3$$

For the orbits at the right:

$$T_{outer\ orbit} = \sqrt{2^3}\, T_{inner\ orbit}$$



Orbits with Different Semimajor Axes

Kepler's Third Law

For an ellipse:

$$a^2 = b^2 + c^2$$

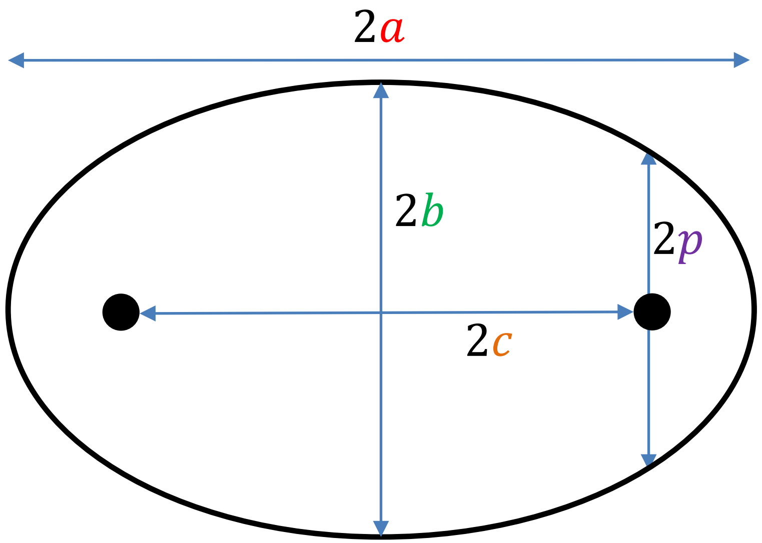
$$e = \frac{c}{a}$$

$$p = a(1 - e^2)$$

$$b = \sqrt{ap}$$

$$A = \pi ab = 2\pi a^{3/2}\sqrt{p}$$

$$p = \frac{h^2}{\mu}$$



From: Bate, Mueller and White, Fundamentals of Astrodynamics

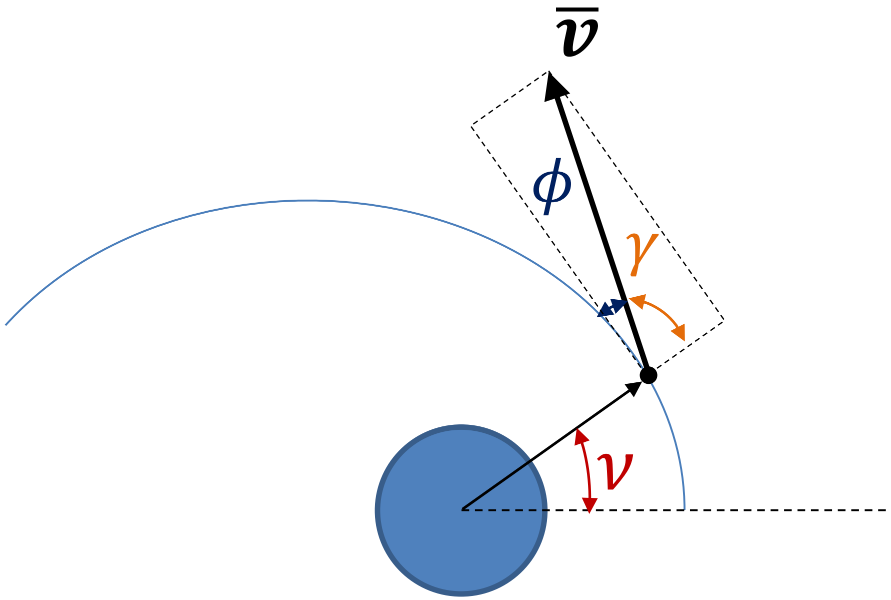
Kepler's Third Law

We start with conservation of specific angular momentum:

$$\bar{\boldsymbol{h}} = \bar{\boldsymbol{r}} \times \bar{\boldsymbol{v}}$$

The magnitude of $\bar{\boldsymbol{h}}$ is given by:

$$h = rv\sin\gamma = rv\cos\phi = r r\dot{\nu} = r^2\frac{d\nu}{dt}$$



Note: v represents the velocity and ν is an angle – the true anomaly

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's Third Law

So the magnitude of the angular momentum becomes:

$$h = r^2 \frac{d\nu}{dt}$$

Rearranging:

$$dt = \frac{r^2}{h} d\nu$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's Third Law

A differential area element in the ellipse is given by:

$$dA = \frac{1}{2} r^2 d\nu$$

So the expression becomes:

$$dt = \frac{2}{h} dA$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's Third Law

Integrating and simplifying, we arrive at a mathematical expression for Kepler's Third Law:

$$T = \frac{2\pi ab}{h} = \frac{2\pi a^{3/2}\sqrt{p}}{\sqrt{\mu p}} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

This law states: *The square of the period, T of a planet (or spacecraft) is proportional to the cube of its mean distance, a to the sun (or its central body).*

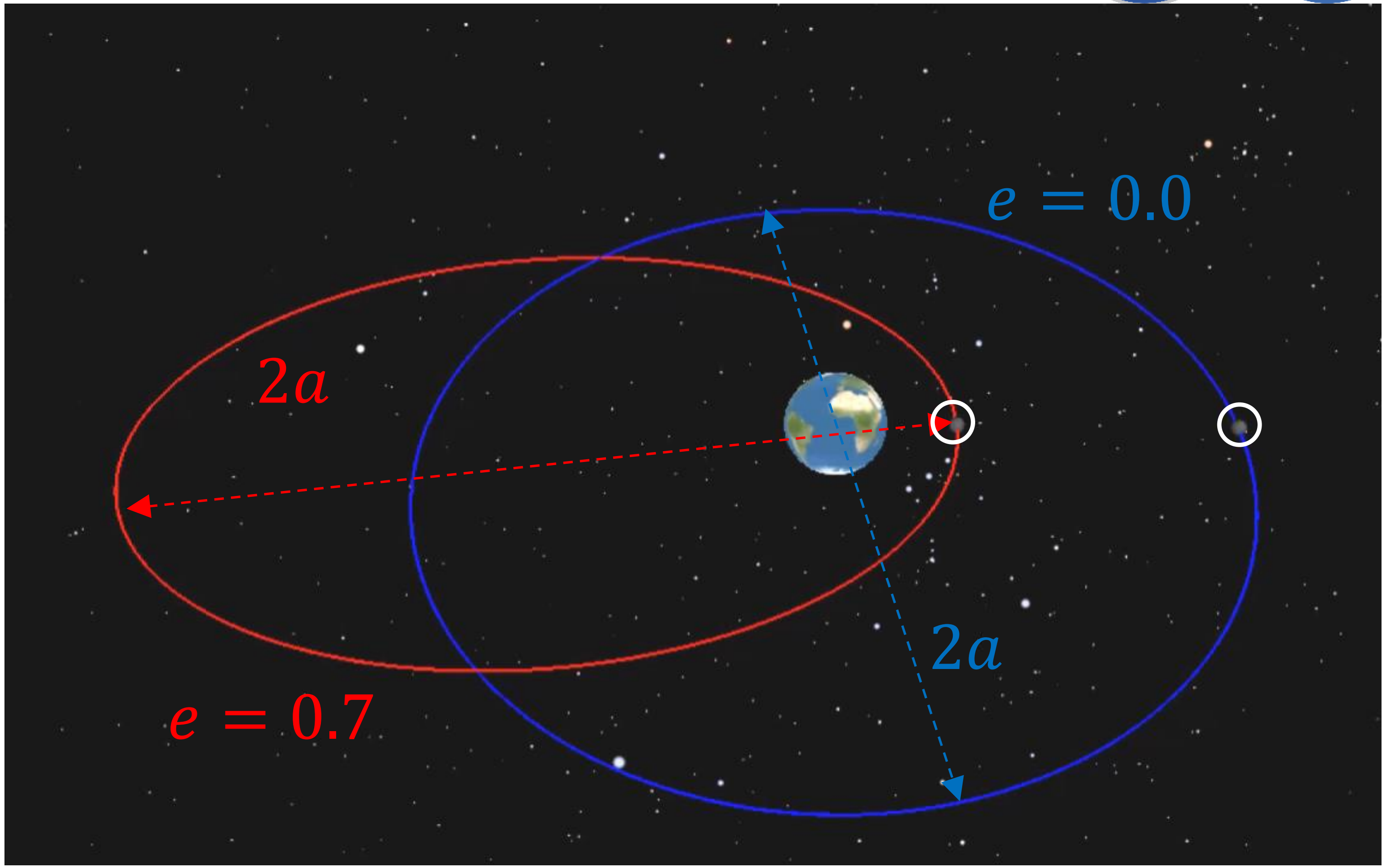
$$T^2 = \left(\frac{4\pi^2}{\mu}\right)a^3$$

From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Kepler's Third Law

As long as the semimajor axis, a is the same, the orbit period will be the same;

At the right, each orbit has a different eccentricity, e but both orbits have the same a.



Two Orbits with the Same Period

Example: Determining Planet Orbital Periods Using Kepler's Third Law

We see the orbital period is a function only of M and a:

$$T^2 = \left(\frac{4\pi^2}{\mu}\right)a^3 = \left(\frac{4\pi^2}{GM}\right)a^3$$

Where a is the orbit semimajor axis, G is Newton's constant of gravitation, and M is the mass of the central body, in this case, the Sun:

$$G = 6.67430 \times 10^{-11}\ m^3/kg\ s^2$$
$$M = 1.988500 \times 10^{30} kg$$

Newton's constant of gravitation from NIST. Mass of Sun from NSSDC.

Example: Determining Planet Orbital Periods Using Kepler's Third Law

Calculating the orbit periods yields:

Planet	Semimajor Axis, $a\ (au)$	Orbital Period * $(Years)$
Mercury	0.3871	0.24
Venus	0.7233	0.62
Earth	1.0000	1.00
Mars	1.5235	1.88
Jupiter	5.2043	11.88
Saturn	9.5824	29.68
Uranus	19.2009	84.20
Neptune	30.0472	164.82

*Actual orbit period may differ slightly

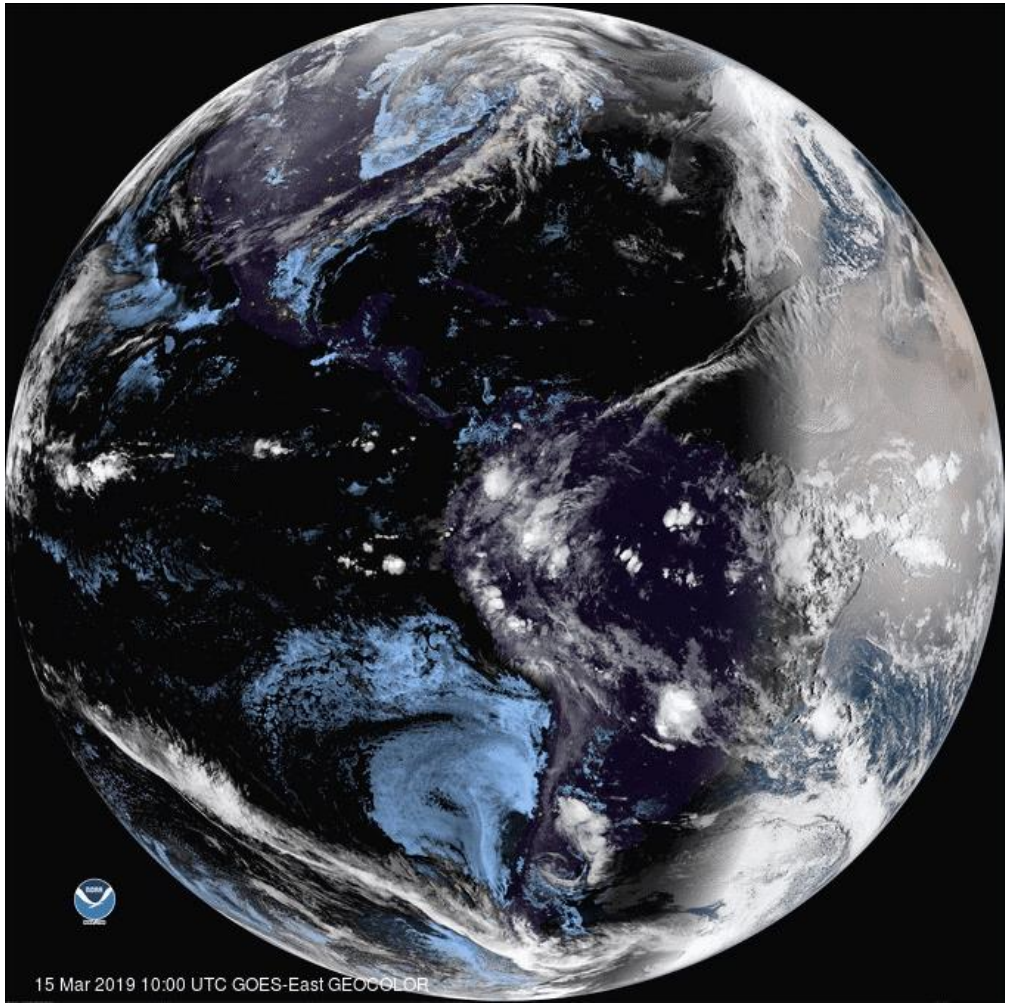
From: https://gea.esac.esa.int/archive/documentation/GDR2/Data_processing/chap_cu3ast/sec_cu3ast_prop/ssec_cu3ast_prop_ss.html
Semimajor axis data from nssdc.gsfc.nasa.gov

Example: Geostationary Orbit

A geostationary orbit has a period of 24 hours with an orbit inclination of 0 degrees;

In this orbit, the spacecraft remains stationary over a specific location on Earth's equator;

Geostationary orbits are used for communications satellites and weather satellites.



Earth as Seen from GOES 16

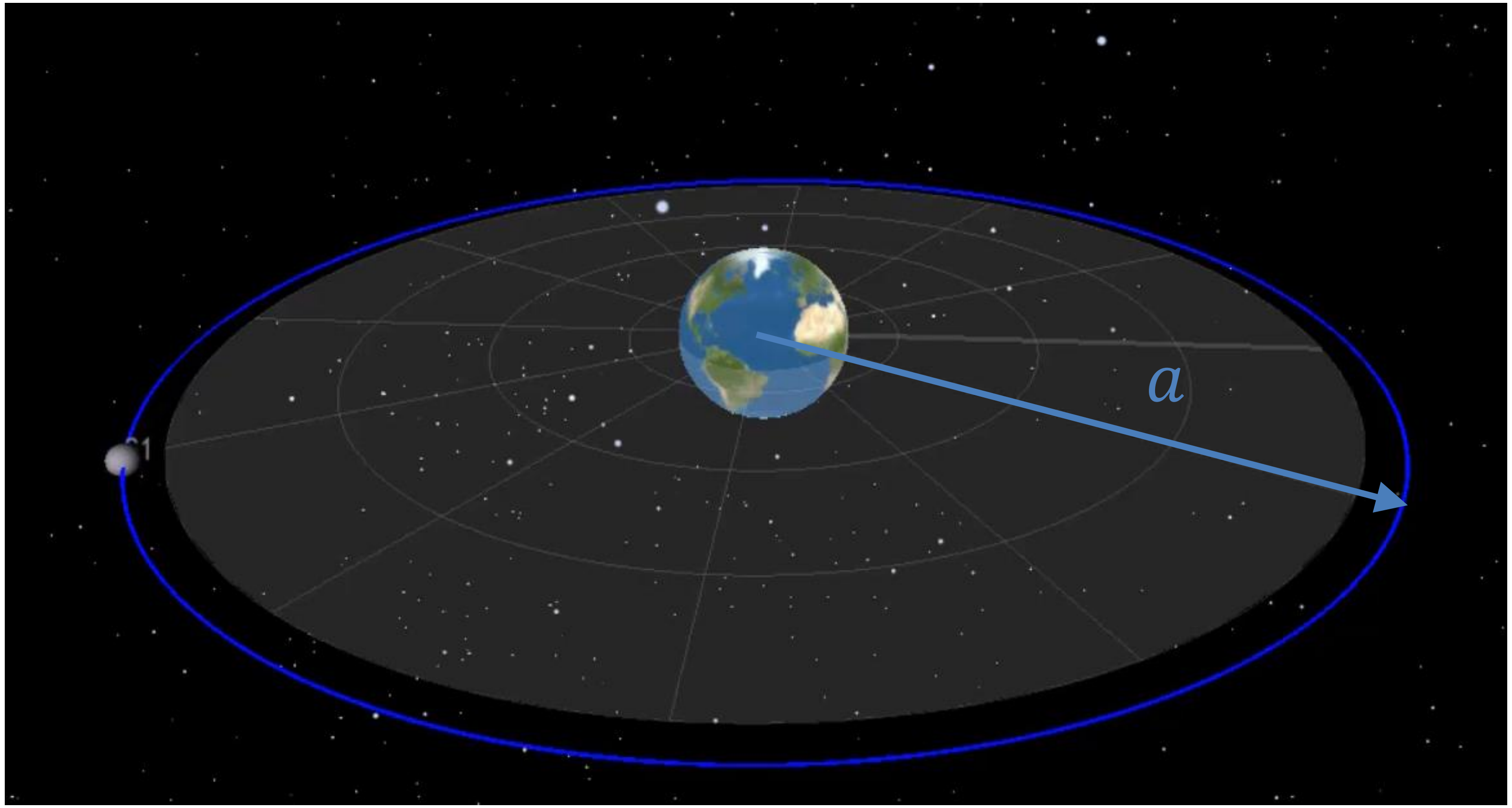
Animation Source: https://www.star.nesdis.noaa.gov/GOES/fulldisk_band.php?sat=G16&band=GEOCOLOR&length=12

Example: Geostationary Orbit

At what altitude, d must the satellite be positioned to be geostationary?

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

$$a = \sqrt[3]{\frac{\mu T^2}{4\pi^2}}$$



Geostationary Orbit

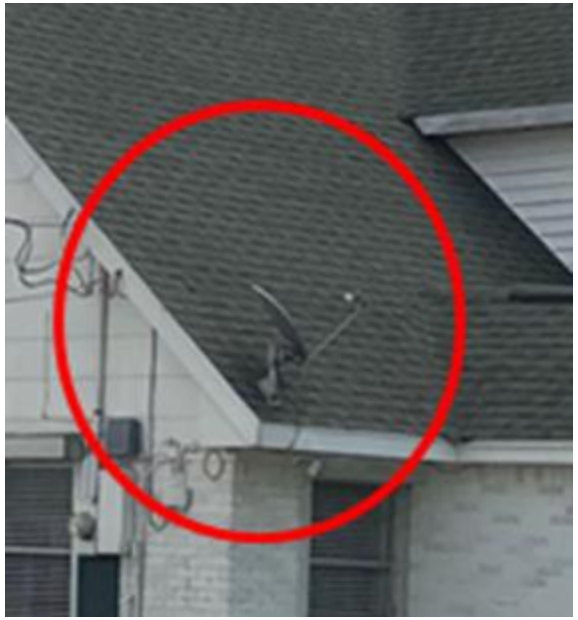
$$d = a - r_e = \sim 35786\, km\ (22{,}236\, miles)$$

Note: Earth's radius, $r_e = \sim 6378.14\ km$

Geostationary Orbit

Since the location of a geostationary satellite is fixed, as the name implies, antennas on the ground need only point at a fixed point in space.

However, since the location of the spacecraft is somewhere in the equatorial plane, the angle at which the antenna points is dependent on the location on the ground as well as the spacecraft location.



~29.5 ° N Latitude

~59.9° N Latitude

Photos by author

Part 2 – Wrap-Up

In Part 2, we introduced the unperturbed two body problem and derived the governing differential equation.

We showed that the unperturbed two body problem obeys, both, conservation of specific mechanical energy as well as conservation of specific angular momentum.

We derived Kepler's three laws of planetary motion and applied the laws to problems of interest to thermal engineers.

Part 3 -- Perturbed Orbits

Part 3 -- Contents

In this section, we will consider the case of orbits where we account for effects of other forces acting on the orbit with focus on those arising from a non-spherical earth and we'll see how these forces give rise to some effects that can be exploited to give the desired orbit.

We'll also see how these perturbations affect the thermal environment – specifically the effect on the orbit beta angle and fraction of orbit in eclipse. Numerous examples will be presented.

Revisiting the Governing Differential Equation

Recall our previous equation was derived for a body moving under the influence of *only* the gravity of a central body:

$$\ddot{\bar{\boldsymbol{r}}} + \frac{\mu}{r^3}\bar{\boldsymbol{r}} = \mathbf{0}$$

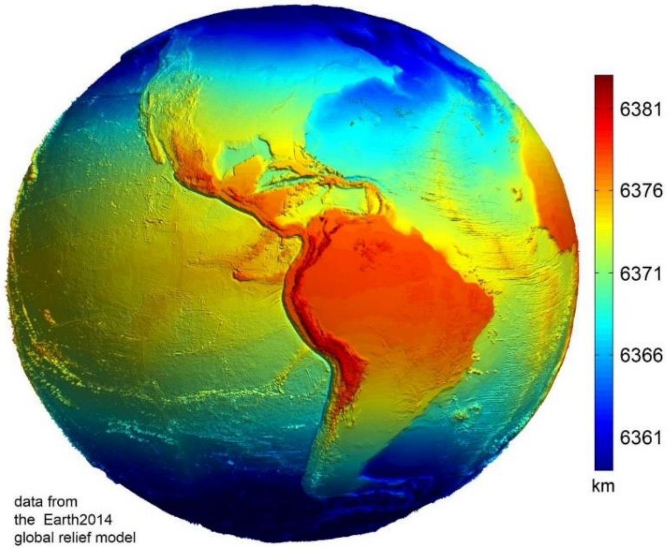
Some interesting things happen when there is a perturbing force such that:

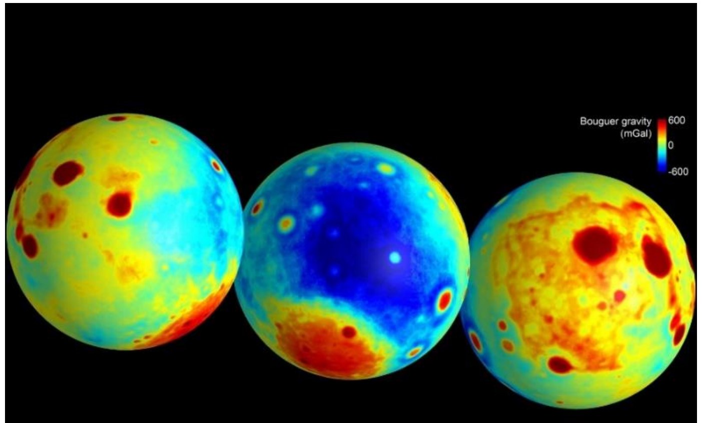
$$\ddot{\bar{\boldsymbol{r}}} + \frac{\mu}{r^3}\bar{\boldsymbol{r}} \neq \mathbf{0}$$

Perturbations

There are many forces that can perturb an orbit including:

- Spherical harmonics
- Drag
- Radiation pressure
- Other celestial bodies
- Tidal forces
- Mass concentrations
- etc.





Lunar Gravity Anomalies Measured by NASA's GRAIL Mission

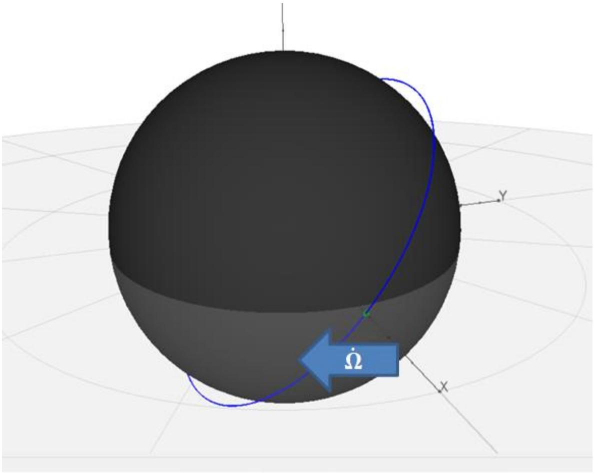
Image Credits: Earth 2014 Global Relief Model, C. Hirt, used with permission
http://www.ngs.noaa.gov/PUBS_LIB/Geodesy4Layman/80003051.GIF
Image credit: NASA/JPL-Caltech/CSM

Perturbations

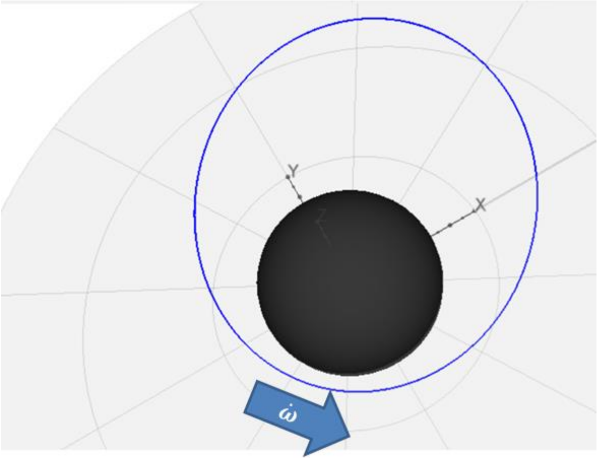
We will focus on two perturbations in this lesson, both arising from the non-spherical shape of the central body:

<u>Precession of the Ascending Node</u> – the orbit ascending node moves westward for orbits where $i < 90°$ and eastward for orbits where $i > 90°$ retrograde orbits.

<u>Precession of the Periapsis</u> – the orbit periapsis (i.e., the low point) moves in the direction of the orbiting spacecraft up to $i \approx 63.4°$and in the direction opposite the orbiting spacecraft for inclinations above this value.



Precession of the Orbit Ascending Node



Precession of the Orbit Periapsis

Perturbations

Earth is not a perfect sphere.

The gravitational potential may be expressed as the summation of a number of terms more representative of the actual gravitational potential.

Each of these terms (harmonics) has an associated coefficient, J_N which multiplies a Legendre polynomial.

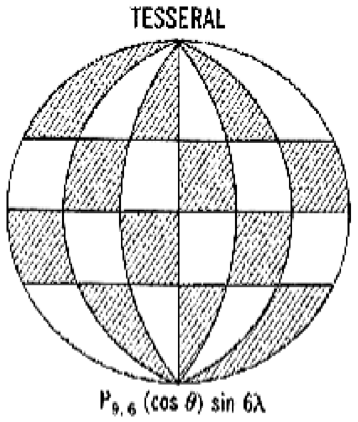
J_N are determined through experimental observation.

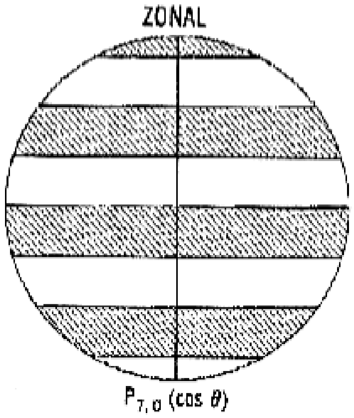
Even numbered harmonics are symmetric about the equator.

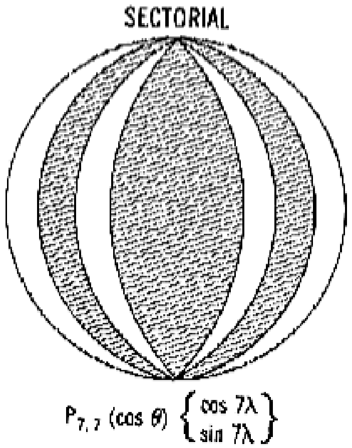
Odd numbered harmonics are antisymmetric.

Sectorial harmonics depend only on longitude.

Tesseral harmonics depend on, both, latitude and longitude.



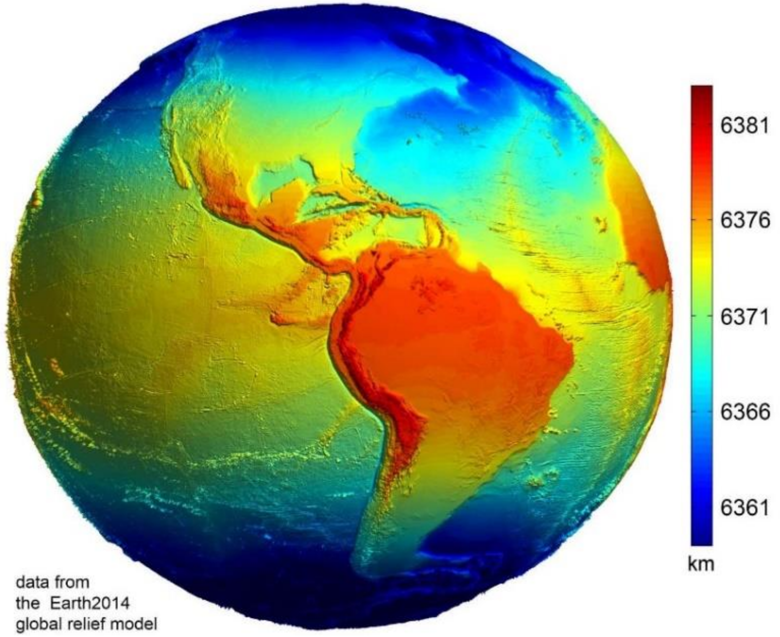




From: https://www.ngs.noaa.gov/PUBS_LIB/Geodesy4Layman/TR80003F.HTM
Other material adapted from: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Perturbations

Earth is not a perfect sphere -- it is oblate and has a slight bulge in the equatorial region and this imperfection gives rise to some major orbit perturbations;



Precession of the Ascending Node:

$$\frac{d\Omega}{dt} = \dot{\Omega} = \frac{-3J_2 n r_e^2 \cos i}{2a^2(1-e^2)^2}$$

Precession of the Periapsis:

$$\frac{d\omega}{dt} = \dot{\omega} = \frac{3J_2 n r_e^2}{4a^2(1-e^2)^2}(4 - 5\sin^2 i)$$

where $n = 2\pi/T$ and $J_2 = 1.082626683 \times 10^{-3}$ (for Earth)

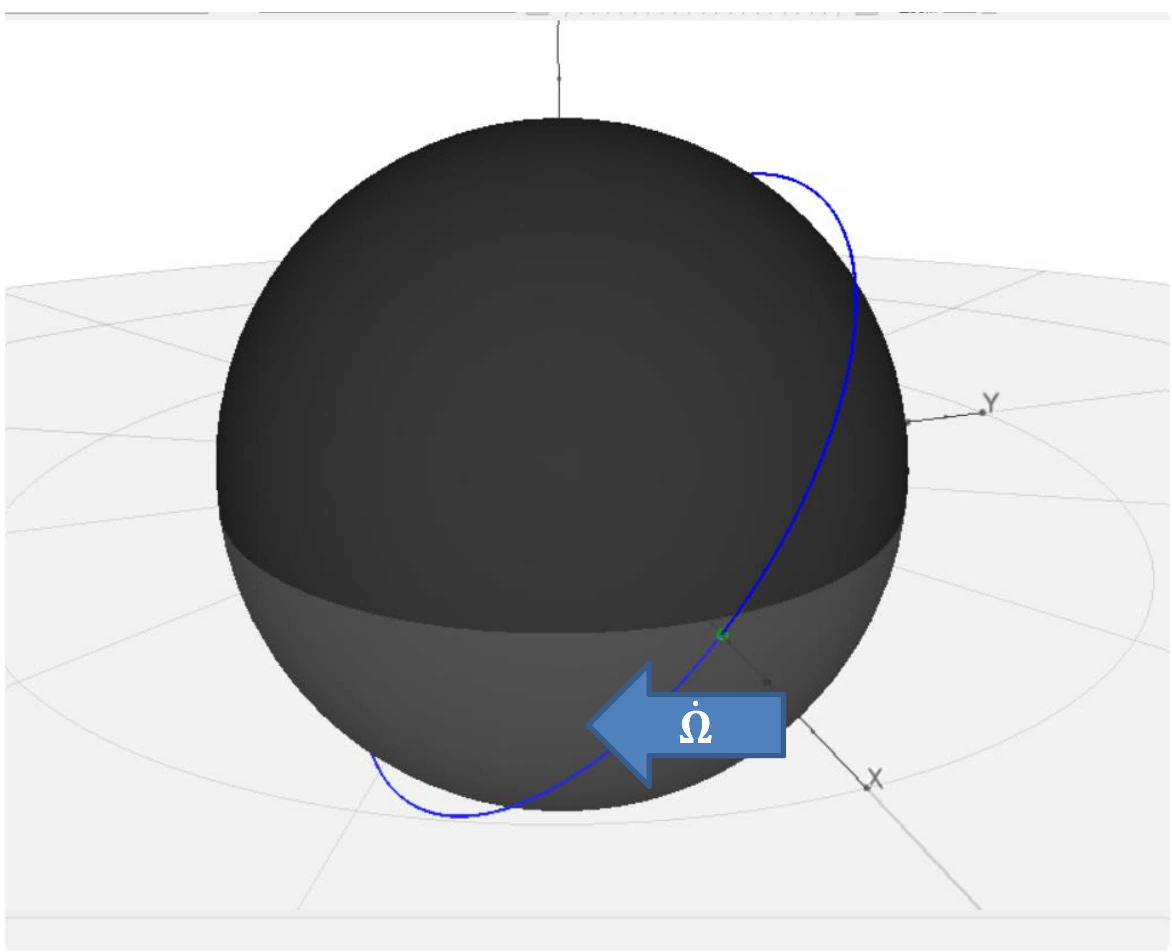
Image Credit: Earth 2014 Global Relief Model, C. Hirt, used with permission

Precession of the Ascending Node

The oblateness perturbation causes the orbit ascending node to precess at the rate:

$$\frac{d\Omega}{dt} = \dot{\Omega} = \frac{-3J_2 n r_e^2 \cos i}{2a^2(1-e^2)^2}$$

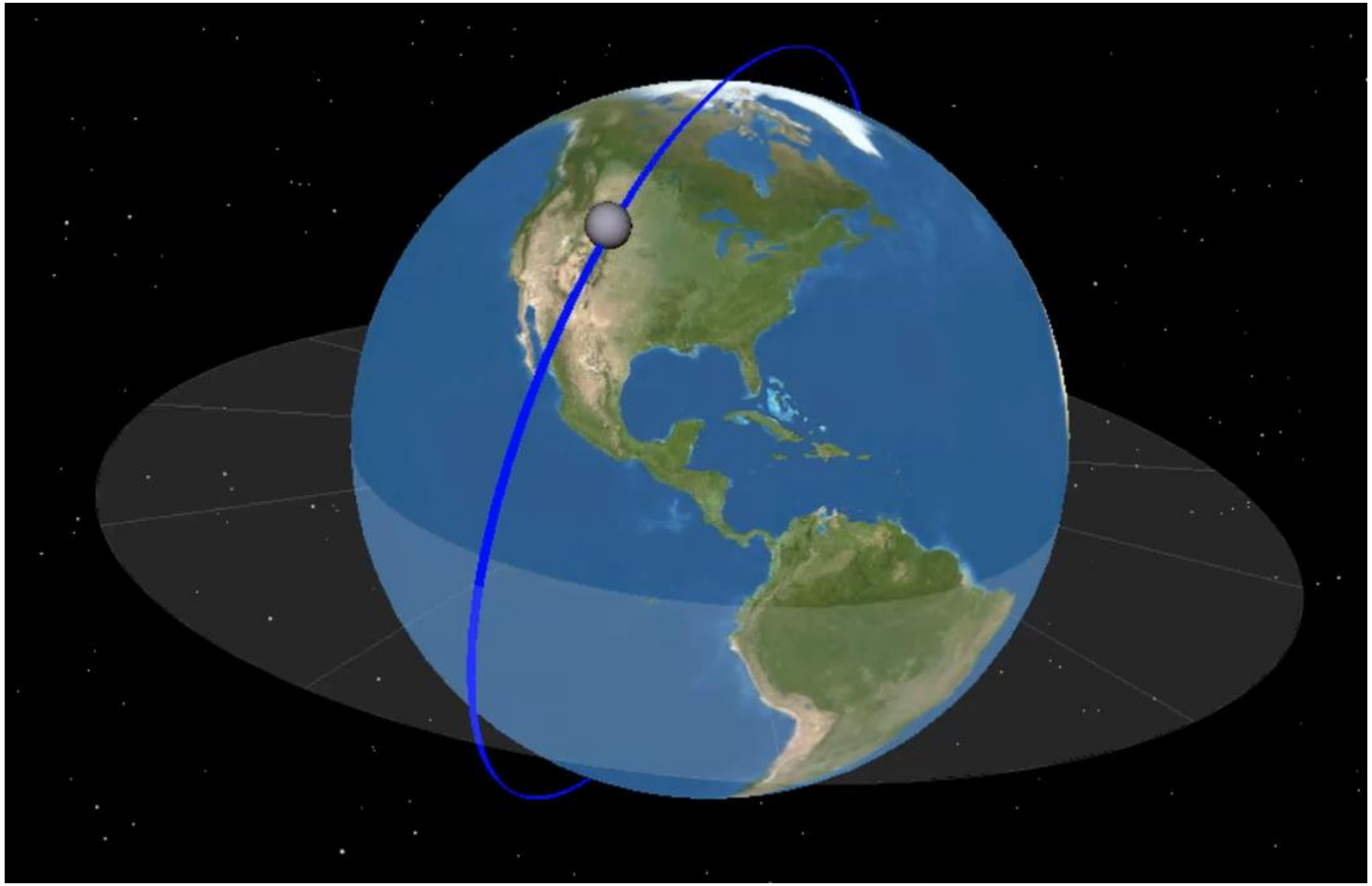
For orbit inclinations, $i < 90°$, precession is westward – when $i > 90°$, precession is eastward.



Precession of the Orbit Ascending Node

Example: Sun Synchronous Orbit

Sun synchronous orbits are useful for Earth observation spacecraft because they are designed to pass over sunlit portions of the planet at the same “local solar” time – this results in consistent illumination conditions for observations;

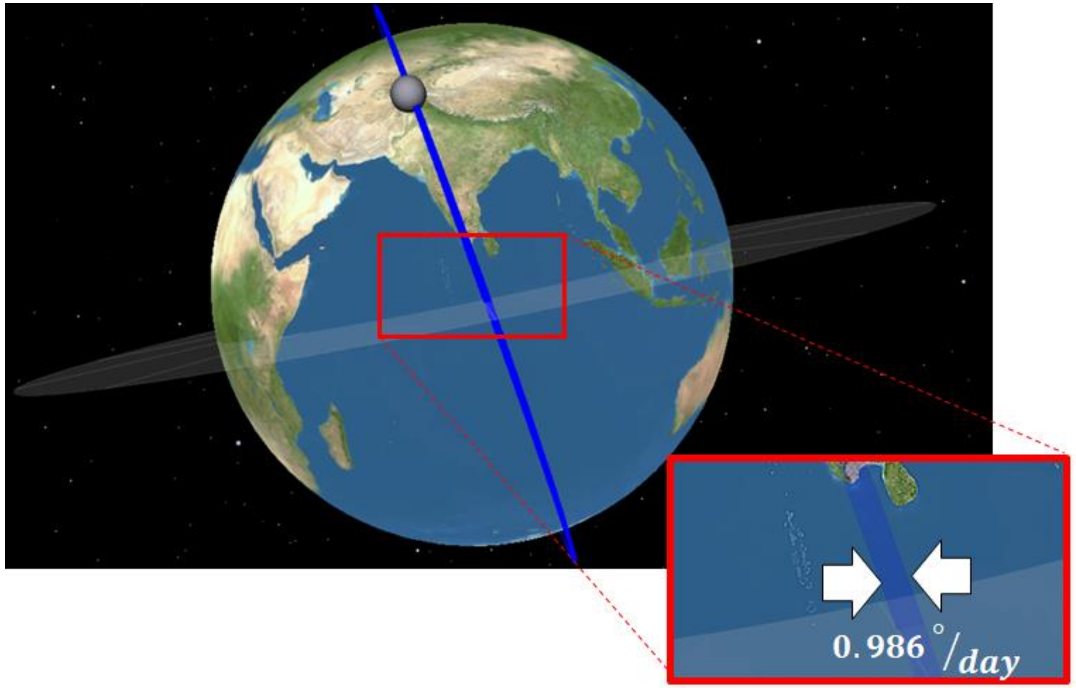


Sun Synchronous Orbit

Example: Sun Synchronous Orbit

To achieve this, the orbit ascending node must maintain a consistent offset from the orbit subsolar point – this is accomplished by moving the orbit ascending node at the same rate the sun appears to move around the celestial sphere --to meet this condition :

$$\dot{\Omega} \approx 0.986^{\circ}/_{day} \ EASTWARD$$



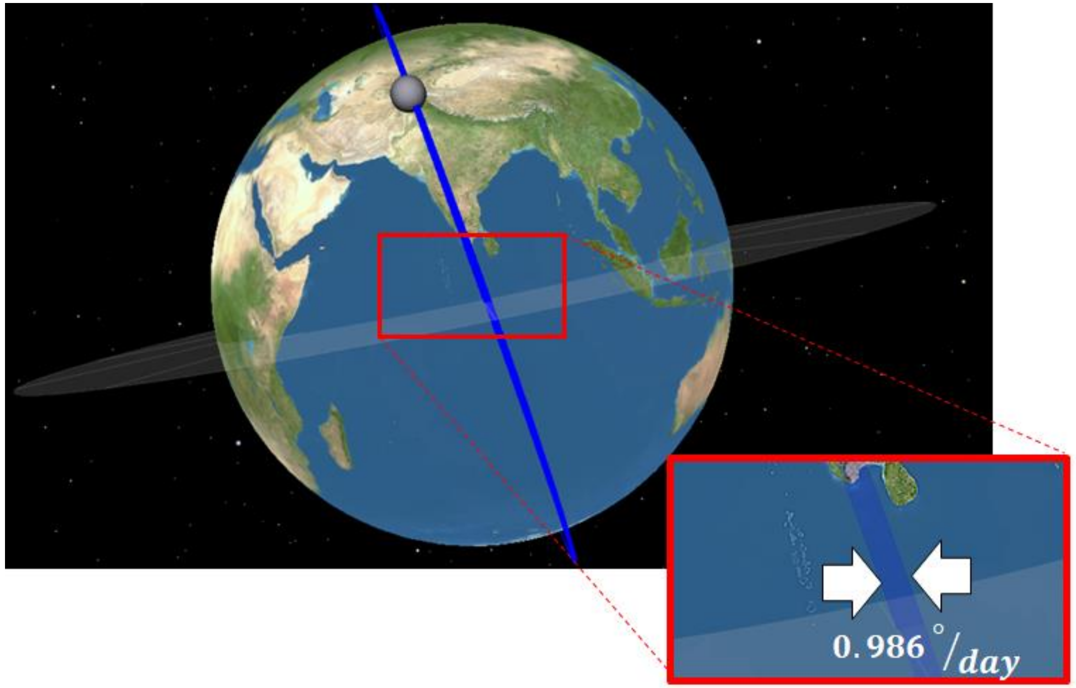
Daily Precession of the Ascending Node

Example: Sun Synchronous Orbit

Assuming a circular orbit ($e = 0$), we see that combinations of i and a may be used to specify the desired orbit.

$$\dot{\Omega} = \frac{-3J_2 n r_e^2 \cos i}{2a^2(1 - e^2)^2}$$

One such combination is $i = 98.2°$ and $a = 7083\ km\ (altitude = 705\ km)$



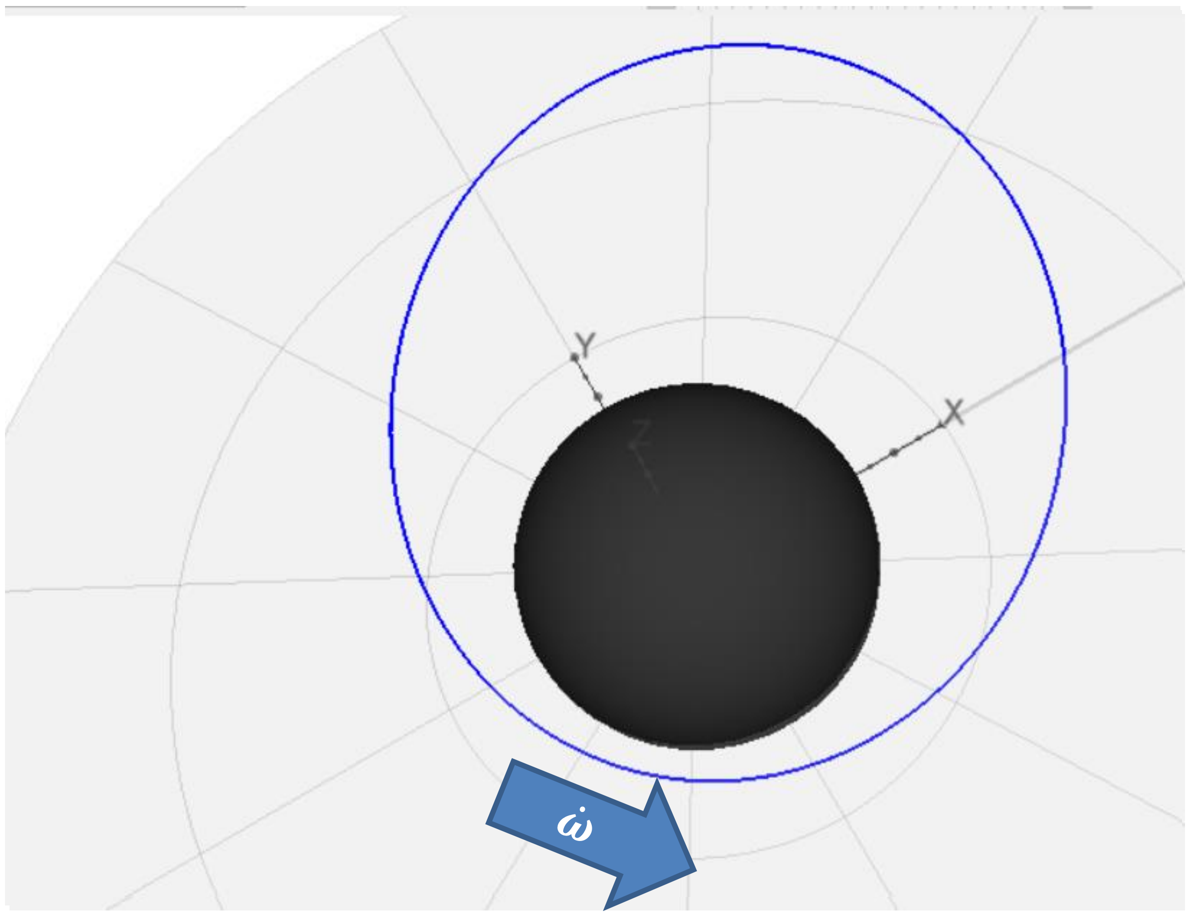
Daily Precession of the Ascending Node

Precession of the Periapsis

The oblateness perturbation also causes the periapsis and apoapsis to precess at the rate:

$$\dot{\omega} = \frac{3J_2 n r_e^2}{4a^2(1-e^2)^2}(4 - 5\sin^2 i)$$

Precession is positive when $(4 - 5\sin^2 i) > 0$ and negative when $(4 - 5\sin^2 i) < 0$.



Precession of the Orbit Periapsis

Example: Molniya Orbit

Communication satellites in geostationary orbits over the equator are of little use to those living at higher latitudes because they appear low in the sky;

A satellite orbiting at a higher inclination is desired;

However, it won't appear to remain over the same point on the ground;

A Molniya orbit may be used to cause the spacecraft to dwell at nearly the same point for long periods of time.

Example: Molniya Orbit

In order to “lock” the location of the apoapsis and periapsis in place, we desire an orbit where the rate of movement of the periapsis goes to zero:

$$\frac{d\omega}{dt} = \dot{\omega} = \frac{3J_2 n r_e^2}{4a^2(1-e^2)^2}(4 - 5\sin^2 i) = 0$$

We see from the equation that this happens when:

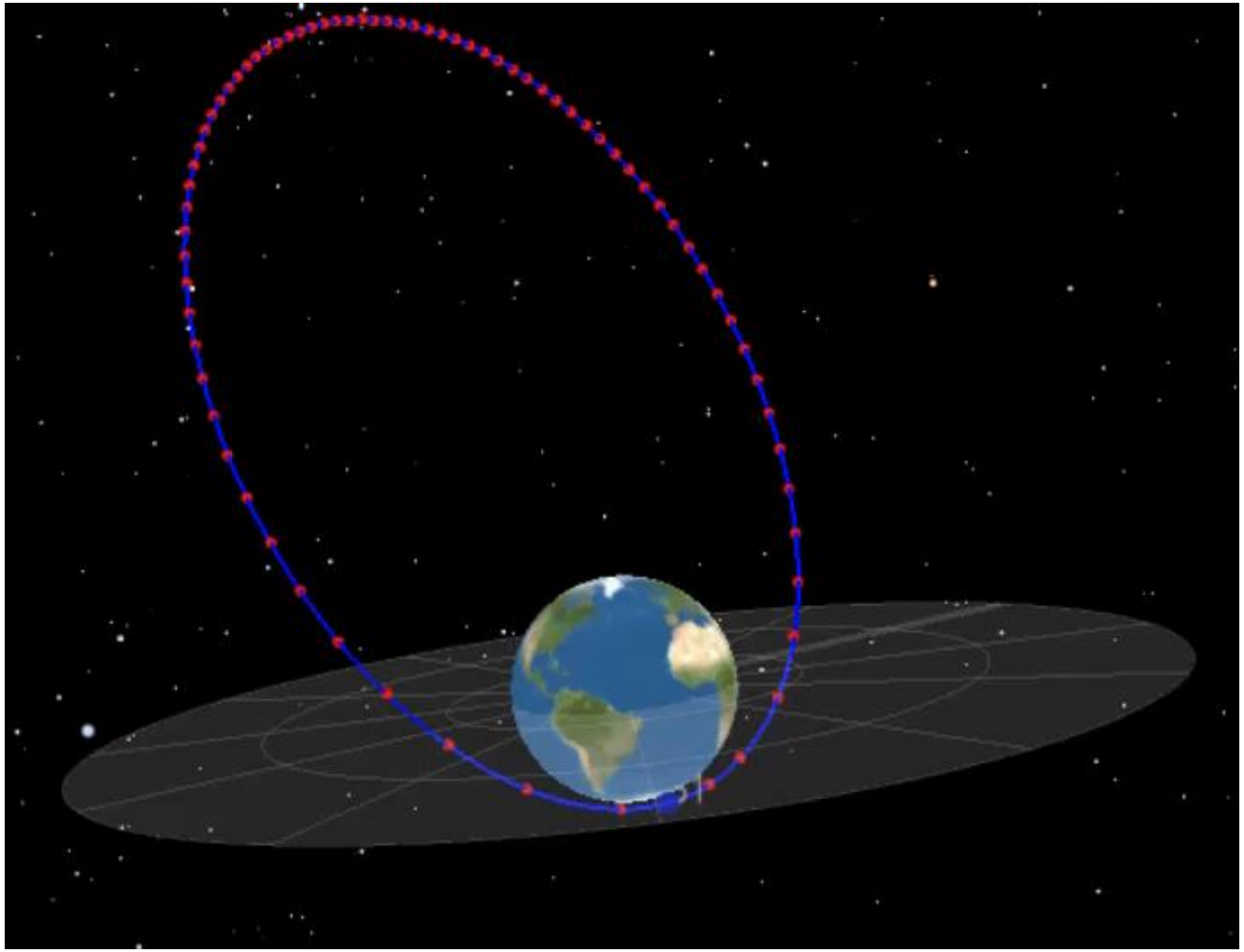
$$(4 - 5\sin^2 i) = 0$$

This is true when the inclination is $i = 63.4°$

Example: Molniya Orbit

The spacecraft spends much of its orbit at high altitude, at high latitude, moving slowly -- appearing nearly stationary when near apoapsis;

Orbit is designed so that apoapsis stays “locked” into the same position over time.



Molniya Type Orbit (Time points in red are 10 minutes apart)

The Effect of Orbit Perturbations on the Thermal Environment

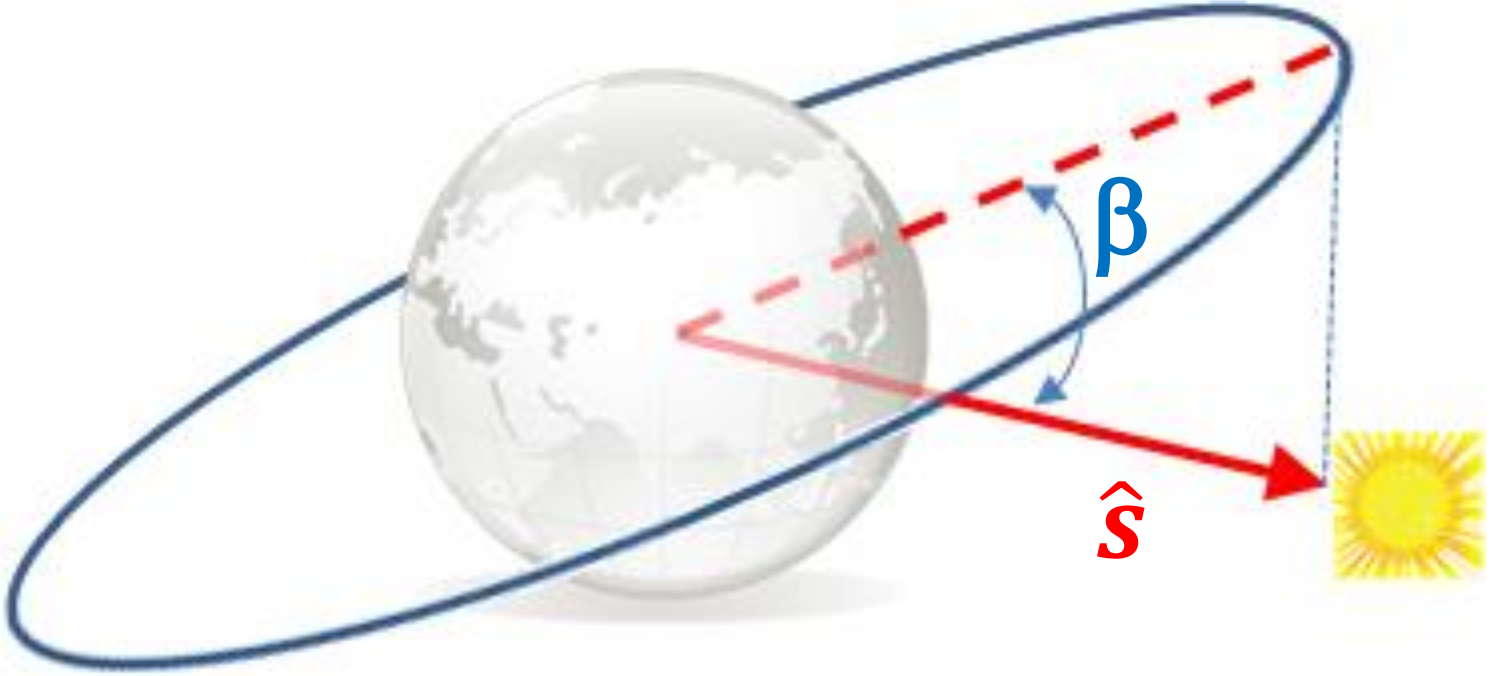
The precession of the ascending node changes the angle at which sunlight falls onto the orbit plane – this angle is referred to at the β angle. As β changes, an orbiting spacecraft will experience a variety of thermal environments.

β angle is one parameter that affects how much environmental heating a spacecraft surface experiences.

β also affects how much time a spacecraft spends in eclipse.

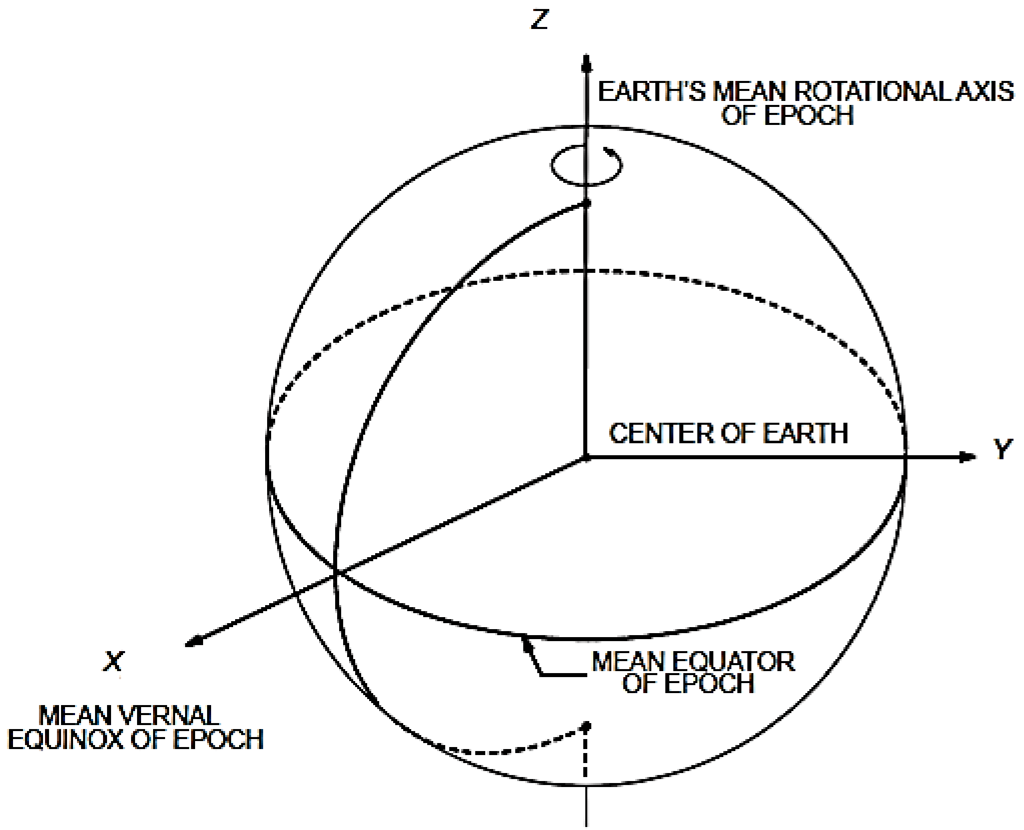
The Beta Angle

The beta angle, β is defined as the angle between the solar vector, $\hat{s}$ and its projection onto the orbit plane.



The Celestial Inertial Coordinate System

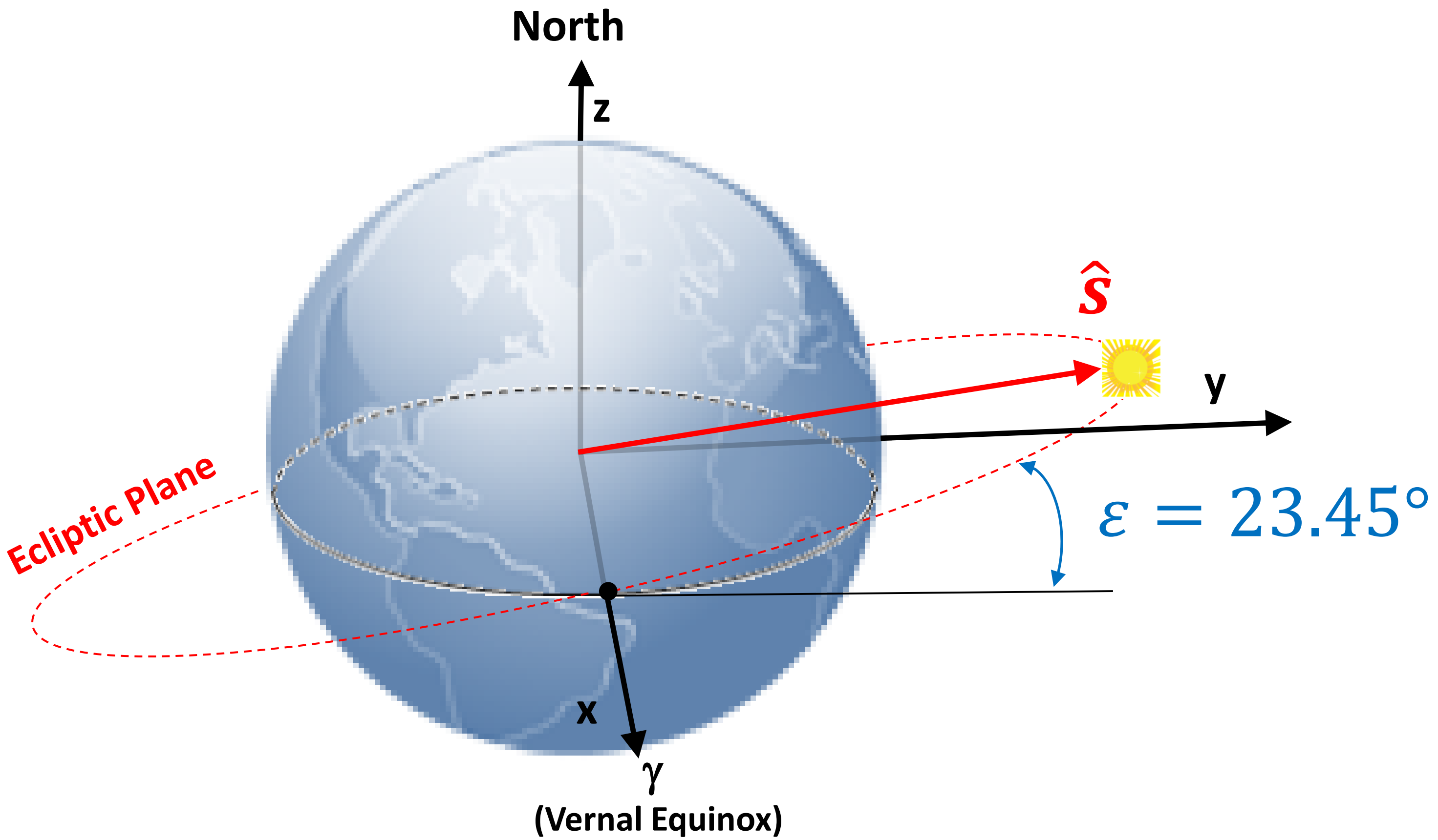
In the celestial inertial coordinate system shown at the right, the $X_{J2000} - Y_{J2000}$ plane is the mean Earth's equator of epoch, the X_{J2000} axis is directed toward the mean vernal equinox of epoch, the Z_{J2000} axis is directed along Earth's mean rotational axis of epoch and is positive north, and the Y_{J2000} axis completes the right handed system.



Adapted from: Space Station Reference Coordinate Systems, SSP 30219, Revision J, May 1, 2008 (found at: https://pims.grc.nasa.gov/plots/user/tibor/SSP%2030219J%20ISS%20Coord%20Systems.pdf)

The Beta Angle

We define the solar vector, $\hat{s}$ as a unit vector in the celestial inertial coordinate system that points toward the sun.

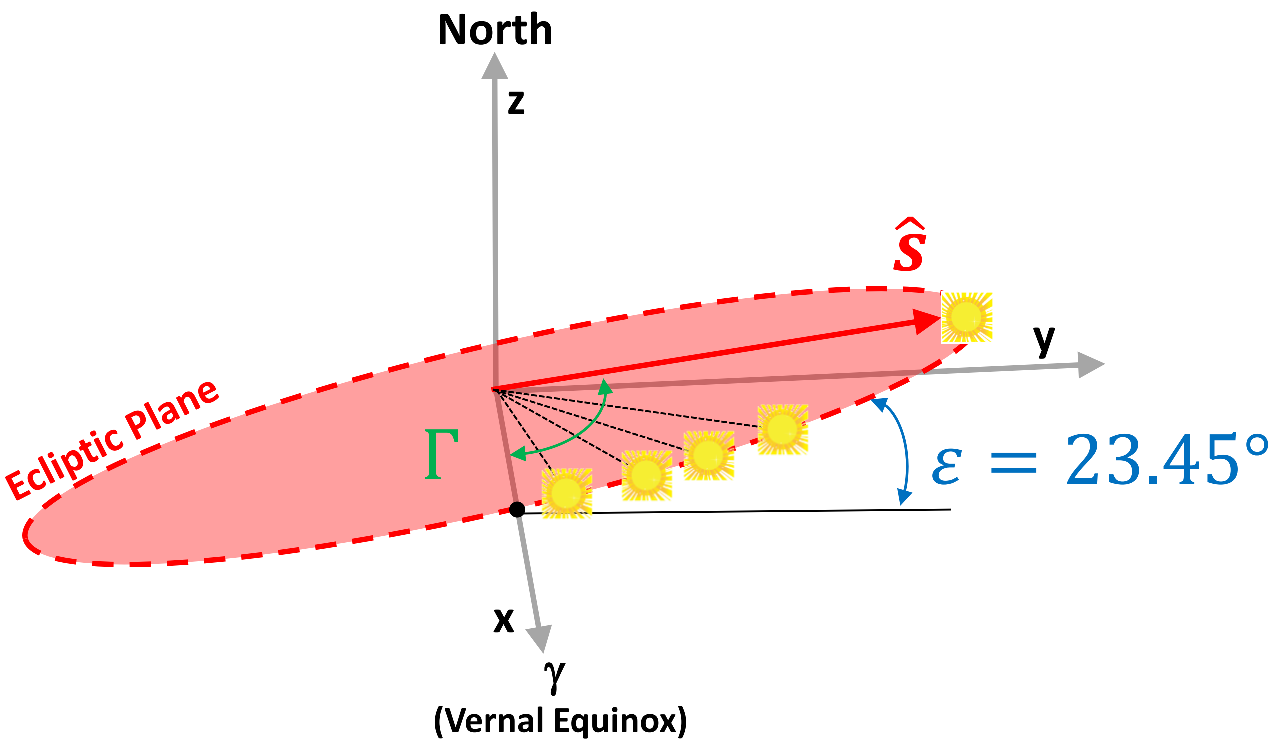


The Beta Angle

The apparent motion of the sun is constrained to the Ecliptic Plane and is governed by two parameters: Γ and ε.

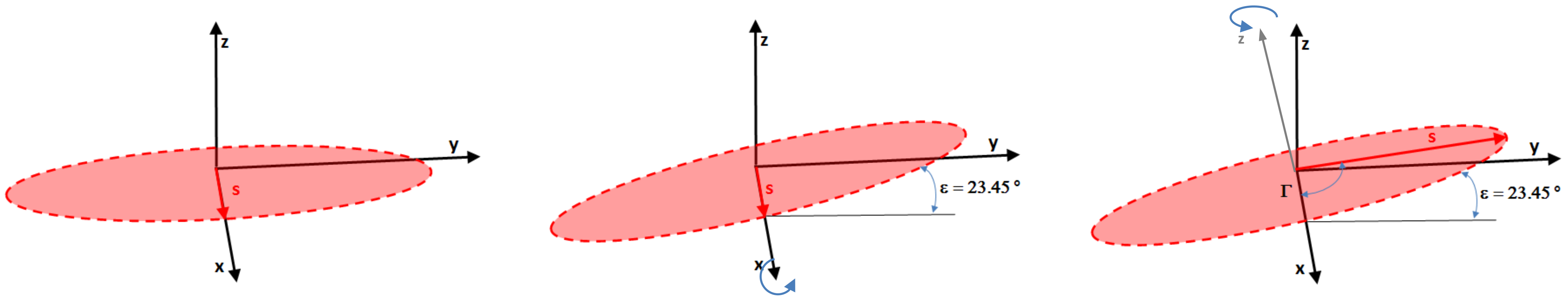
Γ is the **Ecliptic True Solar Longitude** and changes with date. $\Gamma = 0°$ when the sun it at the Vernal Equinox.

ε is the **Obliquity of the Ecliptic** and, for Earth, is presently $23.45°$



The Solar Vector

We can form the solar vector via two Euler angle transformations: first a rotation of the unit vector of ε about the x-axis and then a rotation of Γ about the *new* z-axis.



Unit Vector, No Rotation

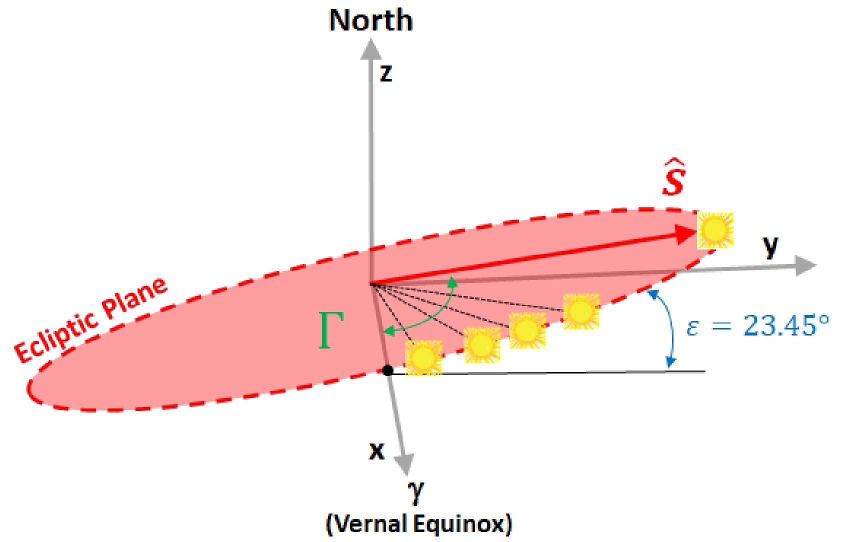
First Rotation, ε about x-axis

Second Rotation, Γ about new z-axis

The Solar Vector

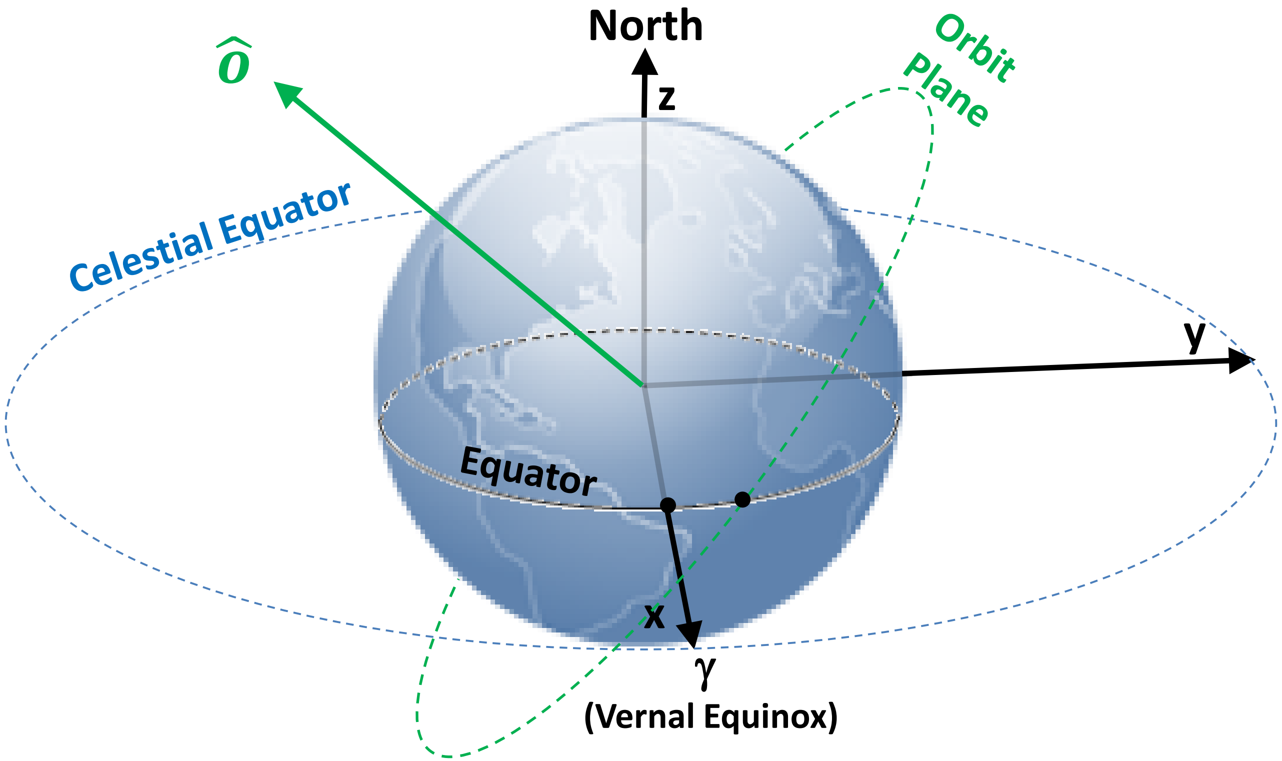
Mathematically, the transformation is expressed as:

$$\hat{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \begin{bmatrix} \cos\Gamma & -\sin\Gamma & 0 \\ \sin\Gamma & \cos\Gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \cos\Gamma \\ \sin\Gamma\cos\varepsilon \\ \sin\Gamma\sin\varepsilon \end{Bmatrix}$$



The Orbit Normal Vector

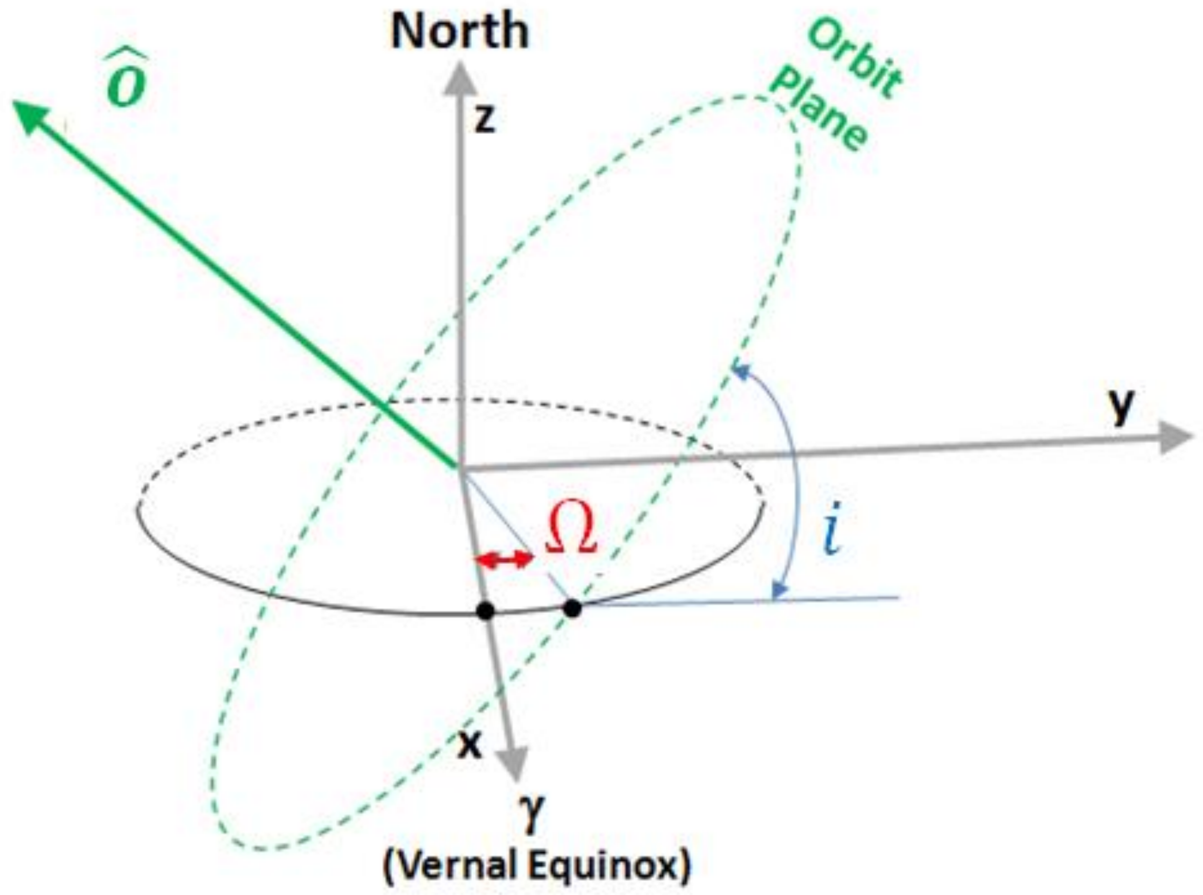
In the same celestial inertial coordinate system, we define the vector, $\hat{o}$, as a unit vector pointing normal to the orbit plane.



The Orbit Normal Vector

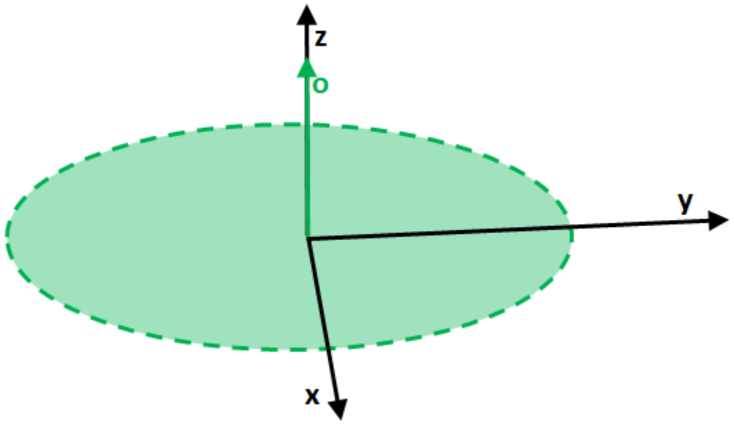
i is the **Orbit Inclination** -- a measure of angular tilt from the equatorial plane;

Ω is the **Right Ascension of the Ascending Node** -- a measure of angle between the x-axis at the point where the orbit cross the equatorial plane going from south to north.

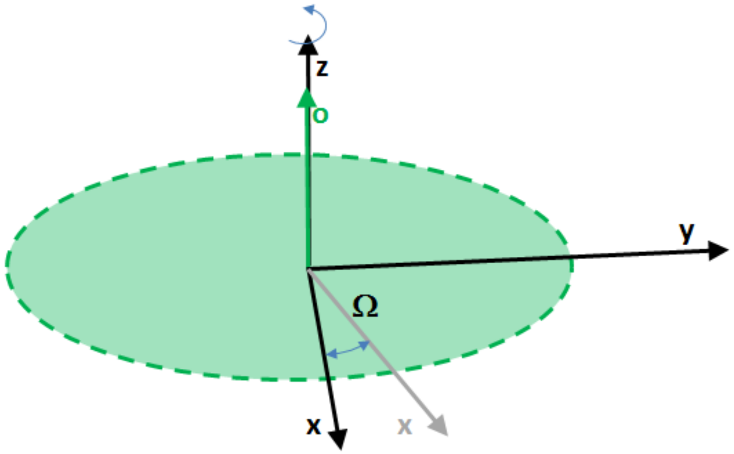


The Orbit Normal Vector

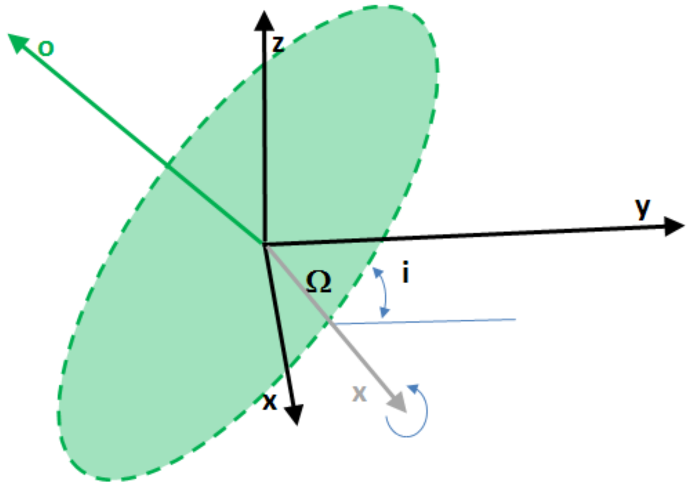
We can form the orbit normal vector via two Euler angle transformations: first a rotation of the unit vector of Ω about the z-axis and then a rotation of i about the *new* x-axis.



Unit Vector,
No Rotation



First Rotation,
Ω about z-axis

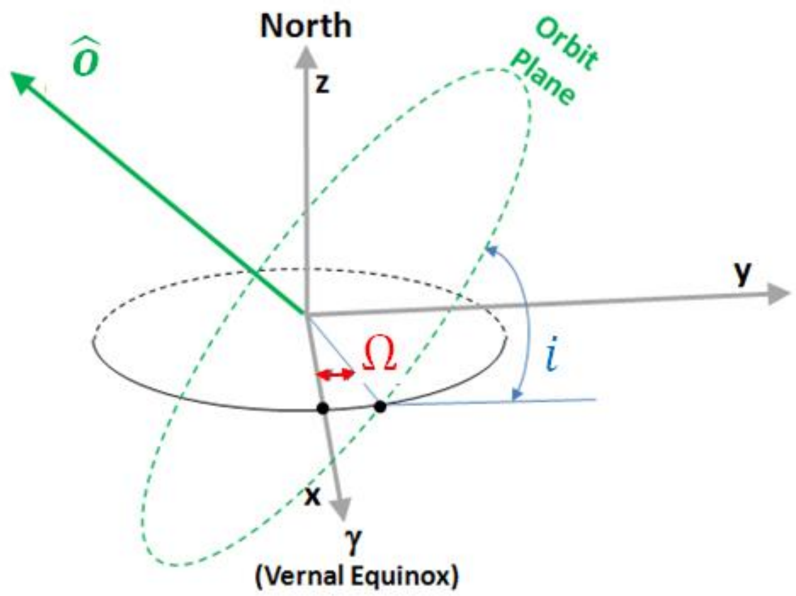


Second Rotation,
i about new x-axis

The Orbit Normal Vector

Mathematically, the transformation is expressed as:

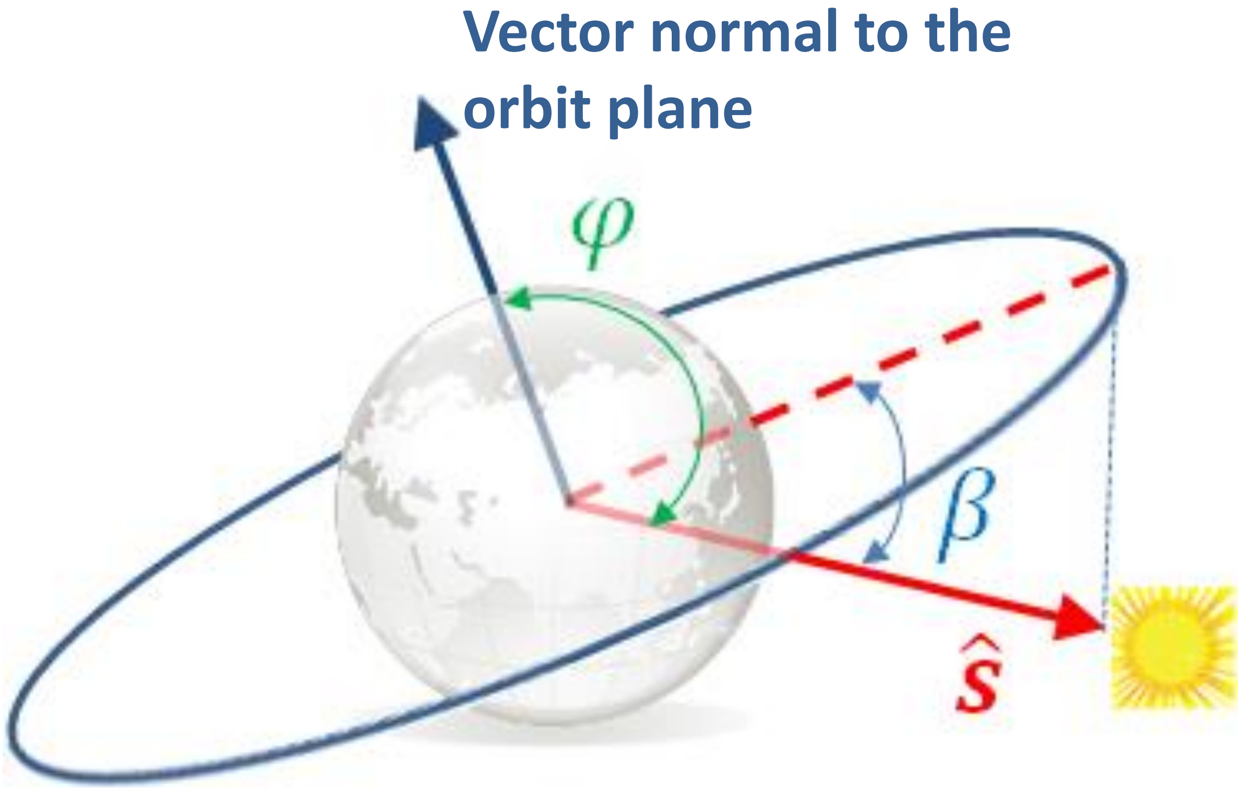
$$\hat{o} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{Bmatrix}$$



Calculating the Beta Angle

To most calculate the angle between a vector, $\hat{s}$ and a plane, it is necessary to determine the angle between the vector and a vector *normal* to the plane, denoted here by φ;

The angle between the vector of interest and the orbit plane, then, is $\beta = \varphi - \frac{\pi}{2}$ radians.



Calculating the Beta Angle

The beta angle, β then, is given by:

$$\cos\varphi = \hat{\boldsymbol{o}} \cdot \hat{\boldsymbol{s}} = \begin{Bmatrix} \sin\Omega \sin i \\ -\cos\Omega \sin i \\ \cos i \end{Bmatrix}^T \begin{Bmatrix} \cos\Gamma \\ \sin\Gamma \cos\varepsilon \\ \sin\Gamma \sin\varepsilon \end{Bmatrix}$$

$$\cos\varphi = \cos\Gamma \sin\Omega \sin i - \sin\Gamma \cos\varepsilon \cos\Omega \sin i + \sin\Gamma \sin\varepsilon \cos i$$

But, since $\beta = \varphi - \frac{\pi}{2}$ radians:

$$\boxed{\beta = \sin^{-1}(\cos\Gamma \sin\Omega \sin i - \sin\Gamma \cos\varepsilon \cos\Omega \sin i + \sin\Gamma \sin\varepsilon \cos i)}$$

Calculating the Beta Angle

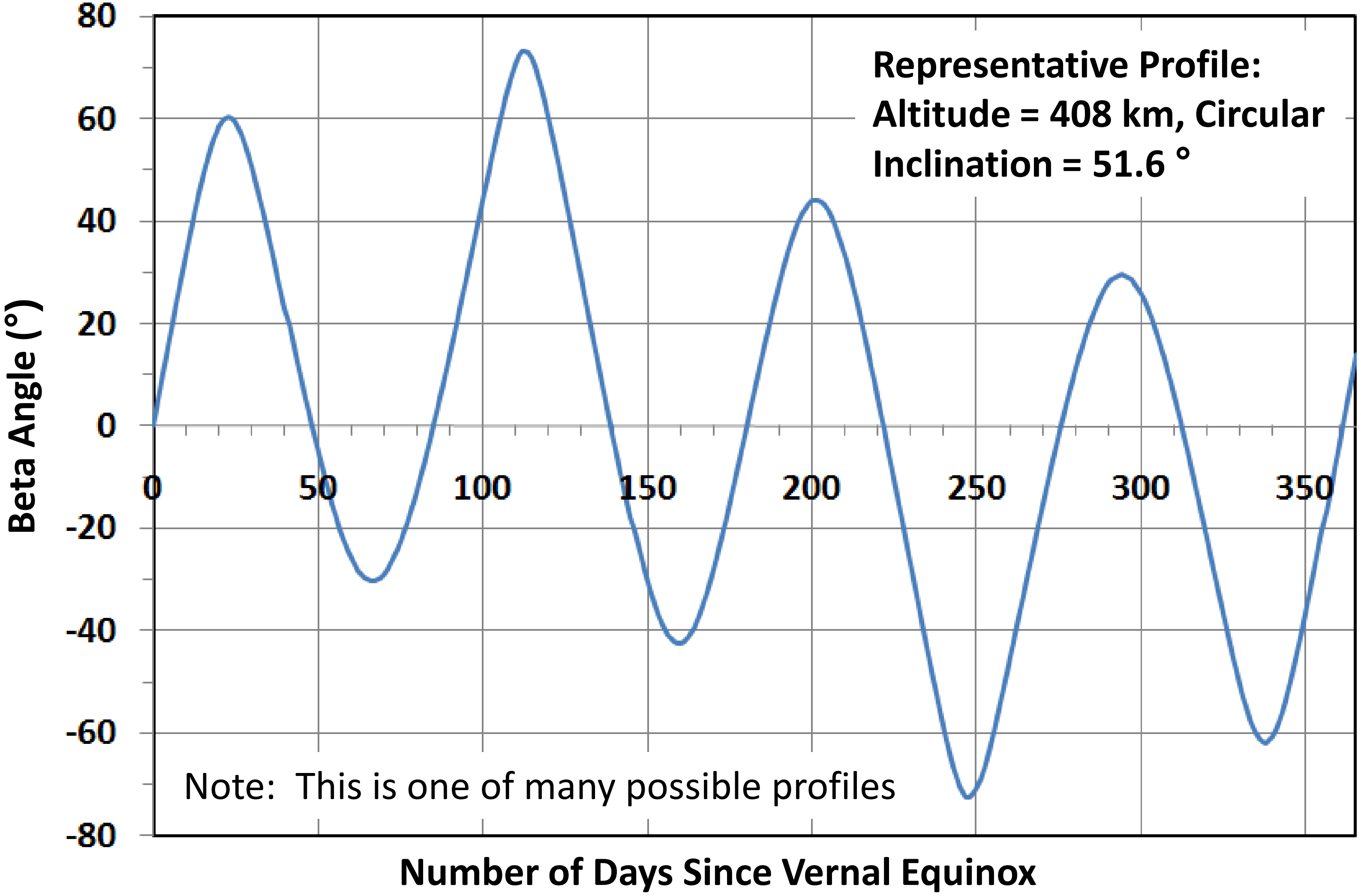
We see that β is limited by:

$$\beta = \pm(\varepsilon + |i|)$$

over the range of $-90° \leq \beta \leq +90°$ $\left(-\frac{\pi}{2} \leq \beta \leq +\frac{\pi}{2}\right)$

Beta angles where the sun is north of the orbit plane are considered positive and beta angles where the sun is south of the orbit are considered negative.

Variation of the Beta Angle Due to Seasonal Variation and Orbit Precession



Consequences of Beta Angle Variation

As β changes, there are two consequences of interest to thermal engineers:

1) The time spent in eclipse (i.e., planet shadow) varies;

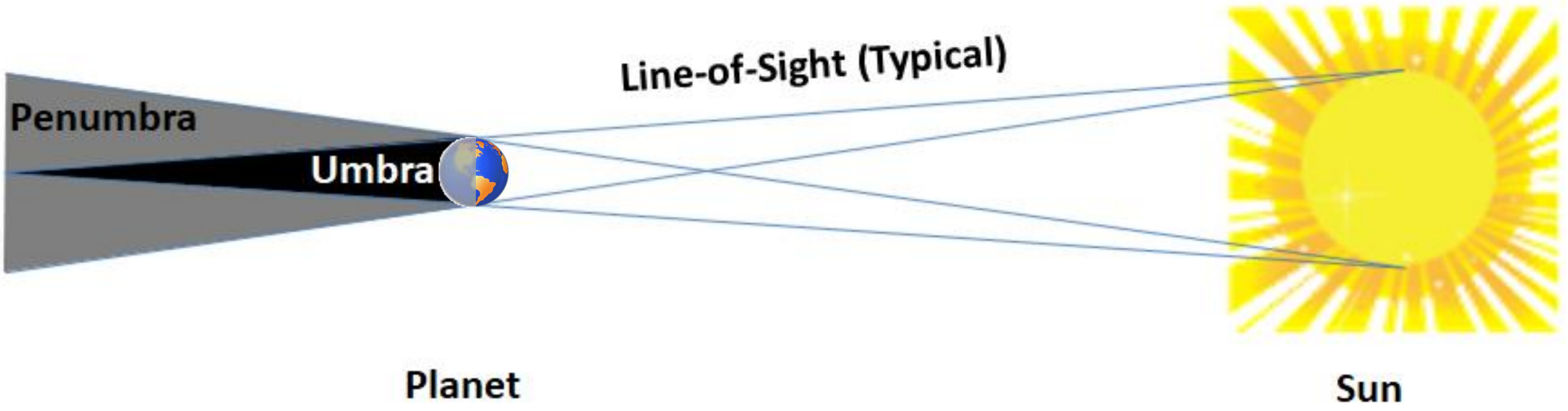
2) The intensity and direction of heating incident on spacecraft surfaces changes;

Let's explore each of these effects.

Eclipse: Umbra and Penumbra

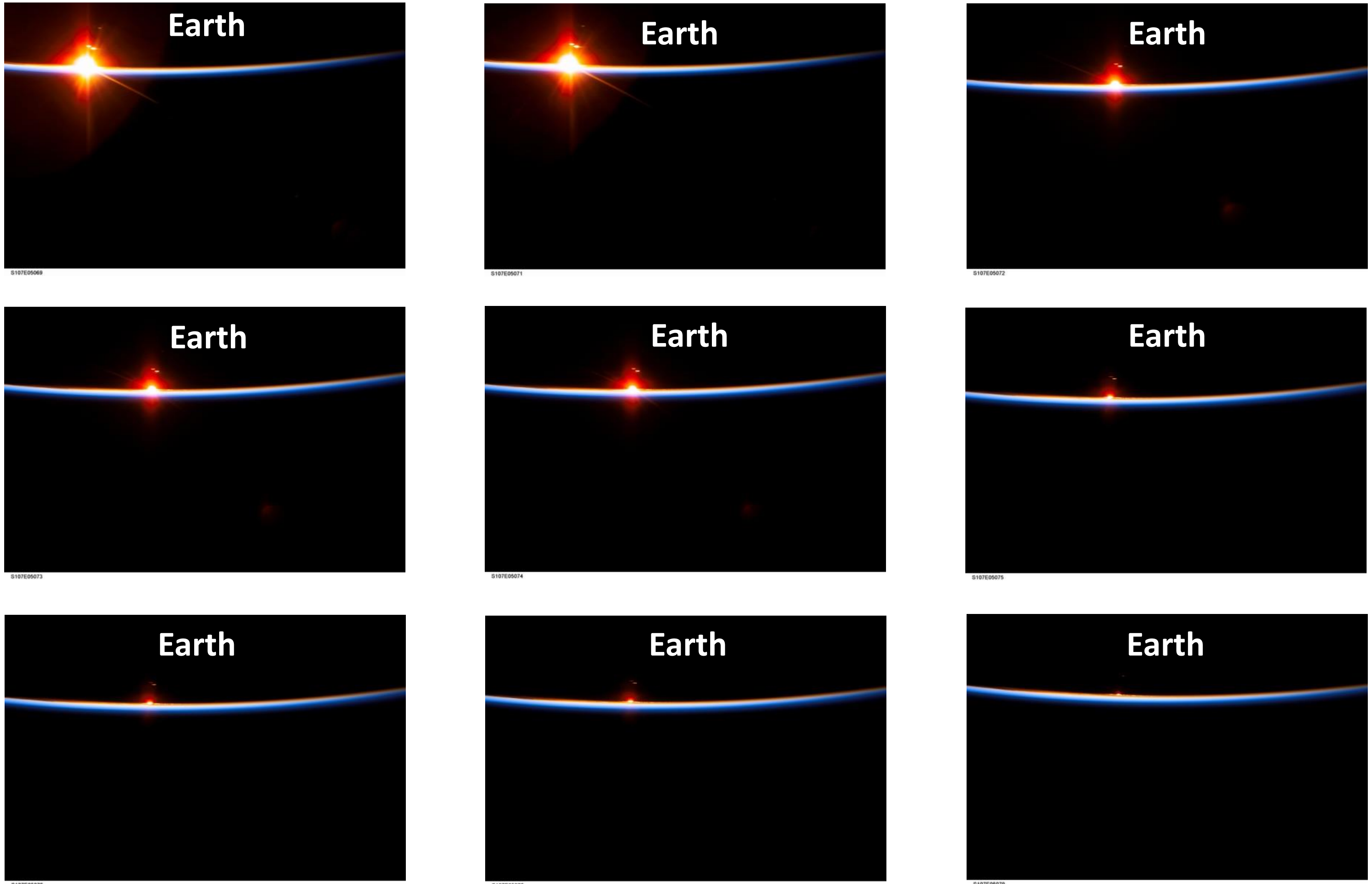
Umbral region - sunlight is completely obscured;

Penumbral region - sunlight is partially obscured.



Note: Diagram not to scale

Orbital Sunset: From Penumbra to Umbra



NASA Photos

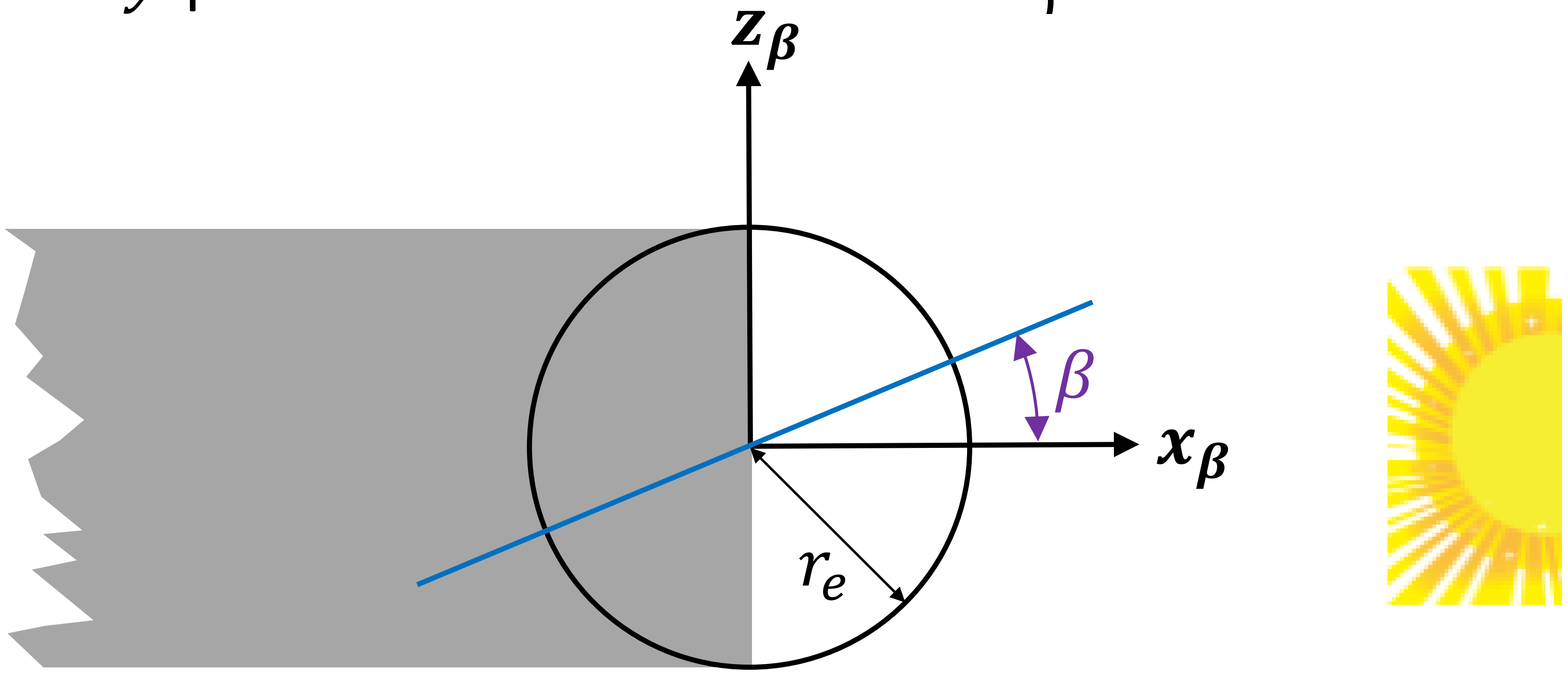
Eclipse: Umbra and Penumbra

If time in penumbra is minimal (i.e., *if* it can be neglected), analysis may be simplified using a cylindrical shadow assumption.

Cylindrical Shadow Approximation

Geometry for Eclipse Calculation (Low, Circular Orbit Only)

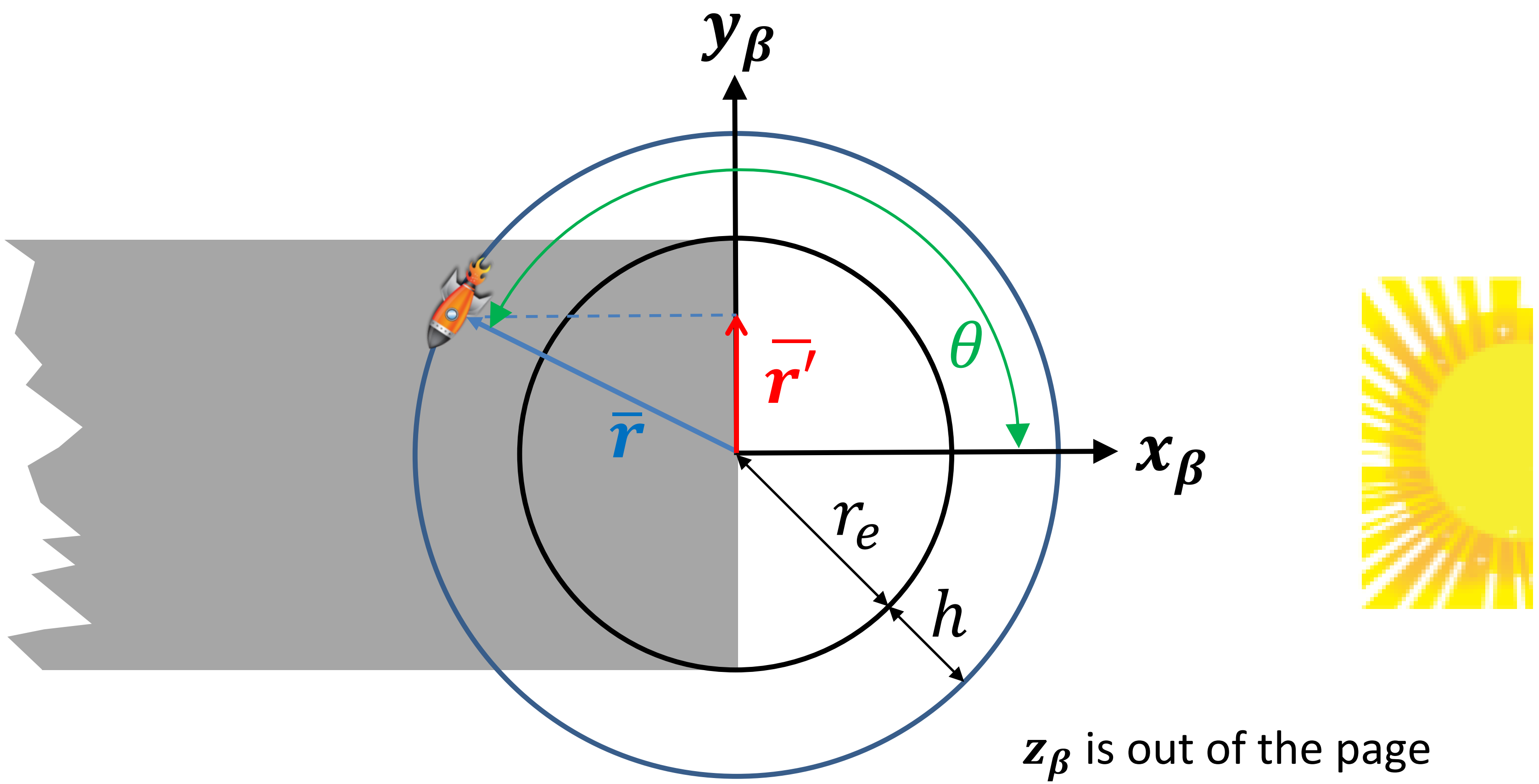
We create a new coordinate system (subscripted with β) where the sun is always in the xy-plane and the orbit is inclined β:



y_β is into the page

Geometry for Eclipse Calculation (Low, Circular Orbit Only)

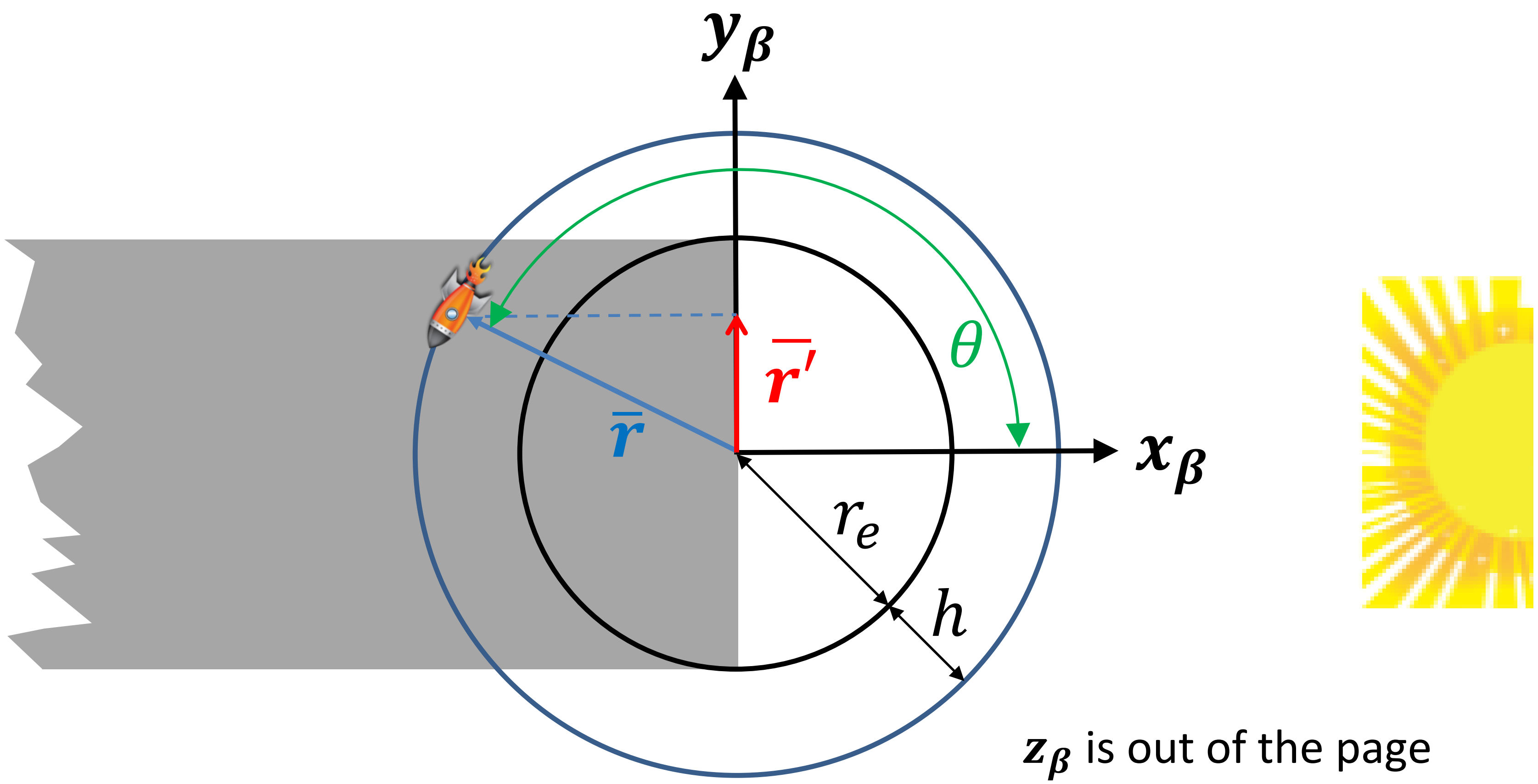
Looking down onto the orbit plane gives us this geometry (when β = 0°).



z_β is out of the page

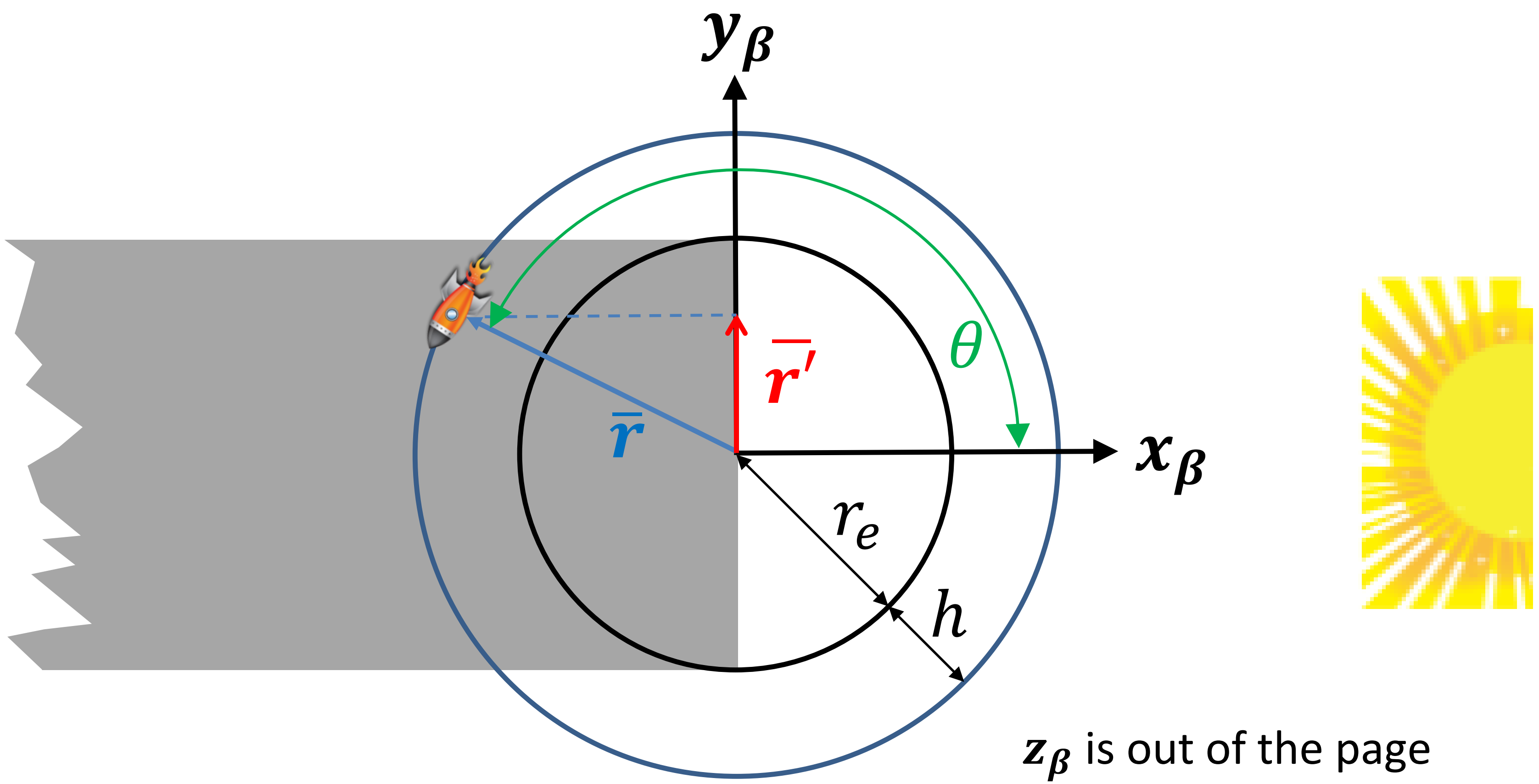
Geometry for Eclipse Calculation
(Low, Circular Orbit Only)

We seek an expression for $\bar{\boldsymbol{r}}'$ which is a projection of $\bar{\boldsymbol{r}}$ onto the $y_\beta z_\beta$ -plane.



Geometry for Eclipse Calculation
(Low, Circular Orbit Only)

When $|\bar{r}'| < r_e$, the spacecraft is in the umbral shadow.



z_β is out of the page

Calculating Umbral Eclipse Entry (Low, Circular Orbit Only)

The spacecraft position vector, $\bar{\boldsymbol{r}}$, can be expressed as a function of altitude above planet, h, planet radius, r_e, angle from orbit noon, θ, and beta angle, β:

$$\vec{r} = (r_e + h)\left[\cos\theta\cos\beta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + \cos\theta\sin\beta\,\hat{k}\right]$$

The projection of this vector onto the $y_\beta z_\beta$ -plane is given by:

$$\vec{r}\,' = (r_e + h)\left[\sin\theta\,\hat{\jmath} + \cos\theta\sin\beta\,\hat{k}\right]$$

Calculating Umbral Eclipse Entry
(Low, Circular Orbit Only)

And the magnitude is given by:

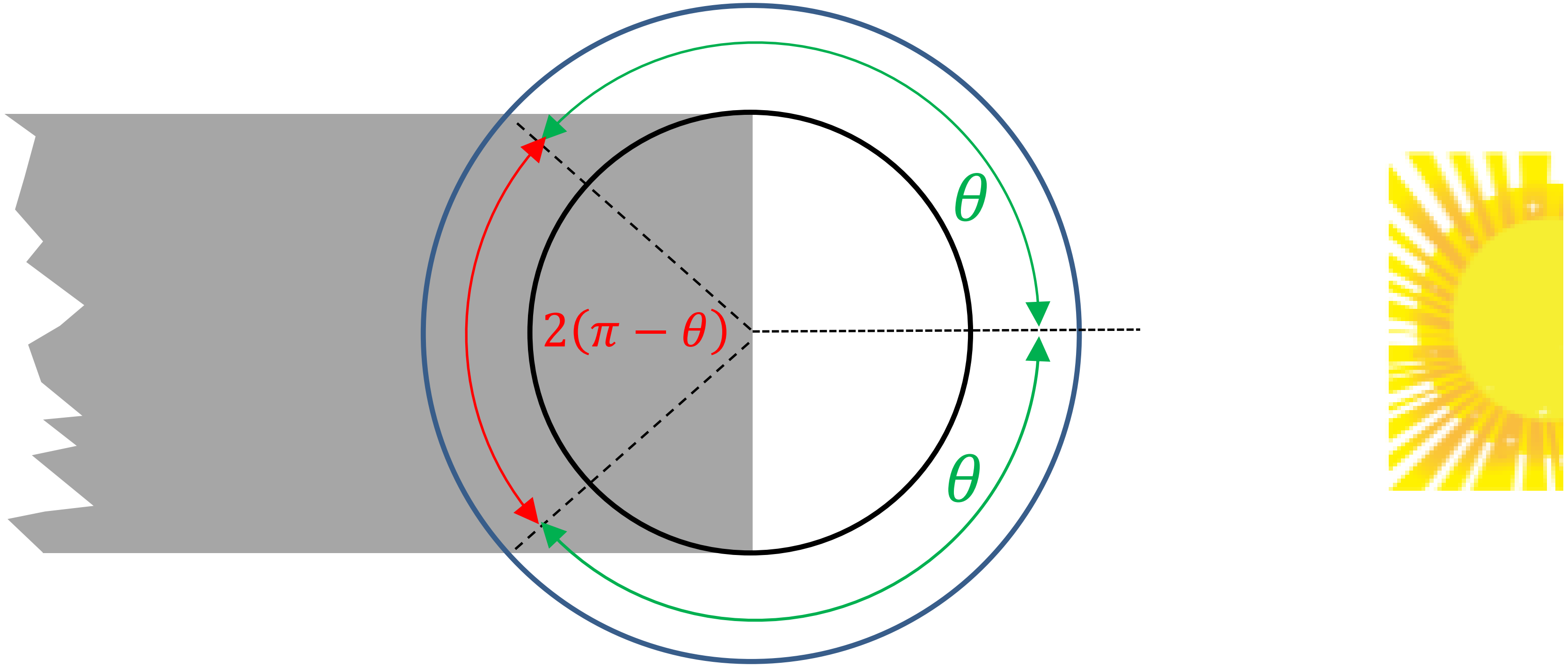
$$|\vec{r}'| = (r_e + h)\sqrt{\sin^2\theta + \cos^2\theta\sin^2\beta}$$

The onset of shadowing occurs when $|\bar{\boldsymbol{r}}'| < r_e$:

$$\sin\theta = \sqrt{\frac{1}{\cos^2\beta}\left[\left(\frac{r_e}{r_e + h}\right)^2 - \sin^2\beta\right]}$$

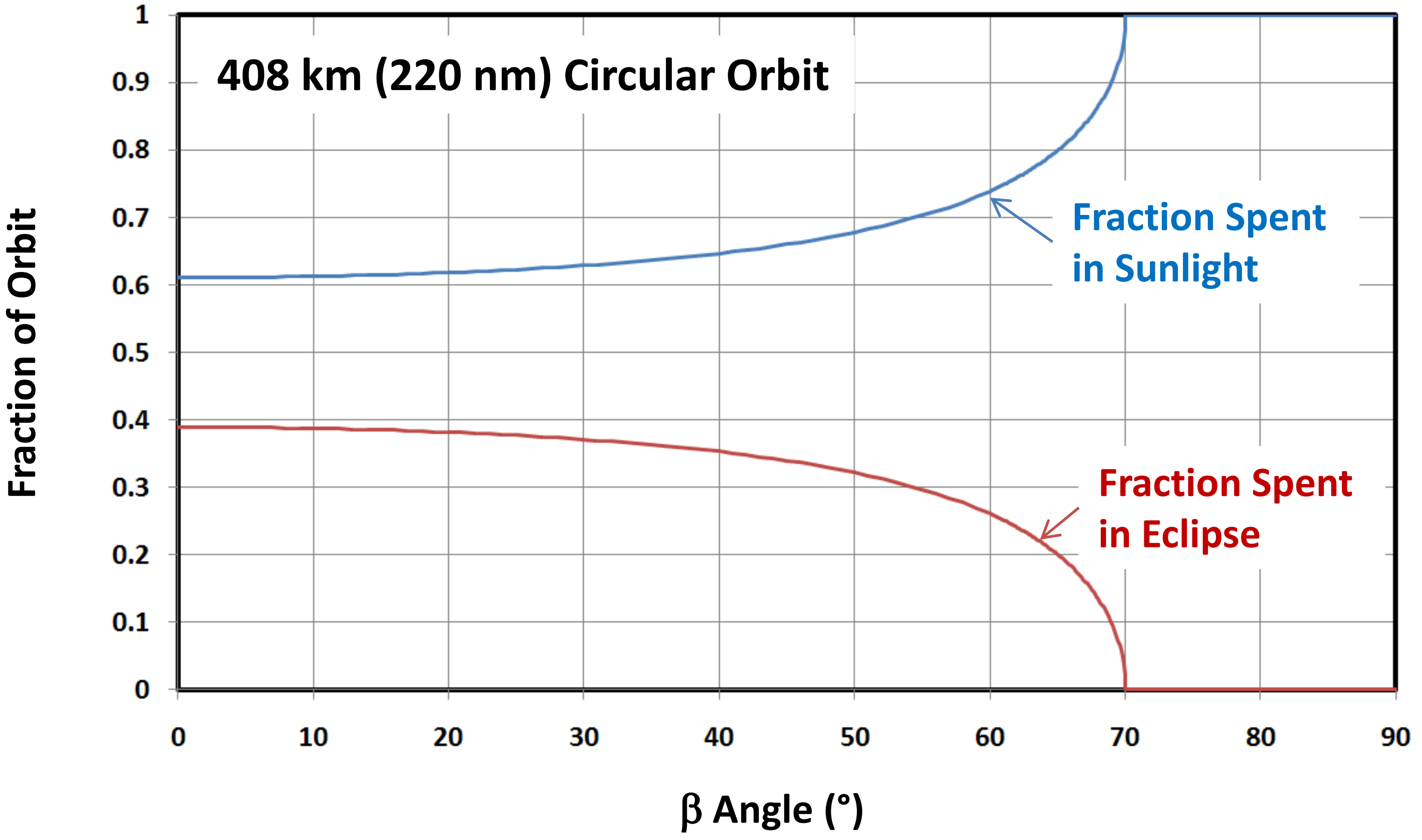
Calculating Umbral Eclipse Entry/Exit (Low, Circular Orbit Only)

Now that the θ of eclipse onset is known, it is a simple matter to determine the entire eclipse period for a circular orbit by noting that the total angle shadowed is $2(\pi - \theta)$:



Fraction of Orbit Spent in Sunlight/Eclipse

The fraction of orbit spent in sunlight and eclipse for a circular orbit is clearly related to β:



Example: Eclipse Season for a Geostationary Orbit

Geostationary orbits, as we saw earlier, have an orbit inclination, $i = 0°$ with respect to the equator.

Since the orbit inclination is zero, the limits of β are:

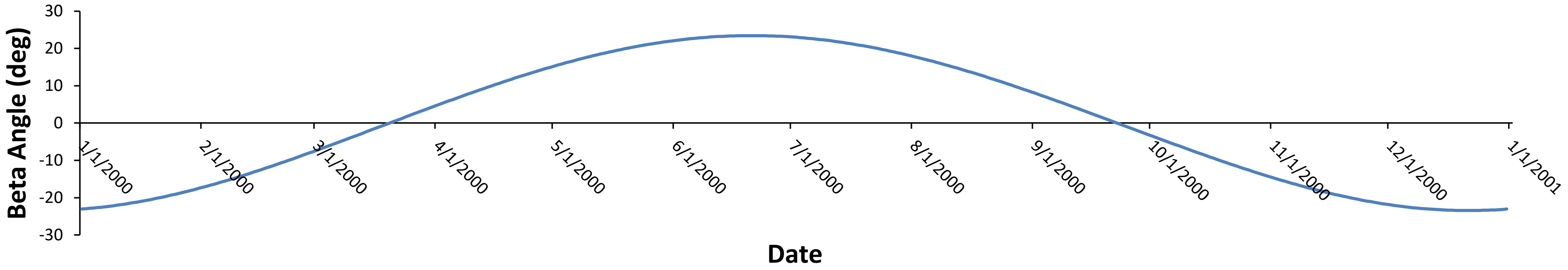
$$\beta = \pm(\varepsilon + |i|) = \pm(23.45° + 0°) = \pm 23.45°$$

Therefore, we expect to see only a seasonal variation in β and, hence, spacecraft eclipse.

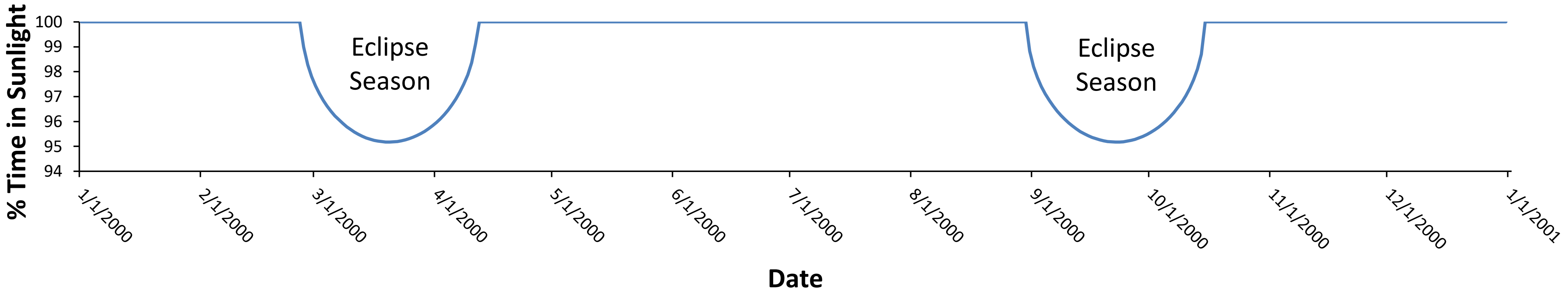
This gives rise to "eclipse seasons" for geostationary spacecraft.

Example: Eclipse Season for a Geostationary Orbit

β Angle versus Time for a Spacecraft in Geostationary Orbit

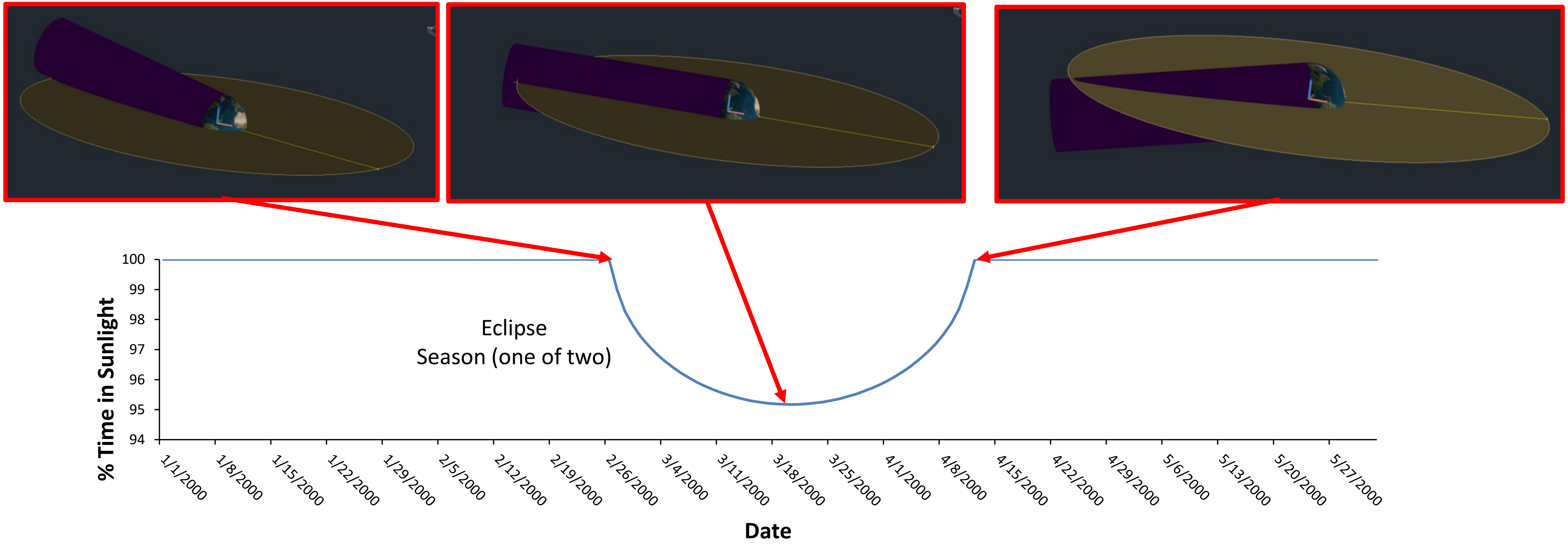


Percent of Orbit Period Spent in Sunlight



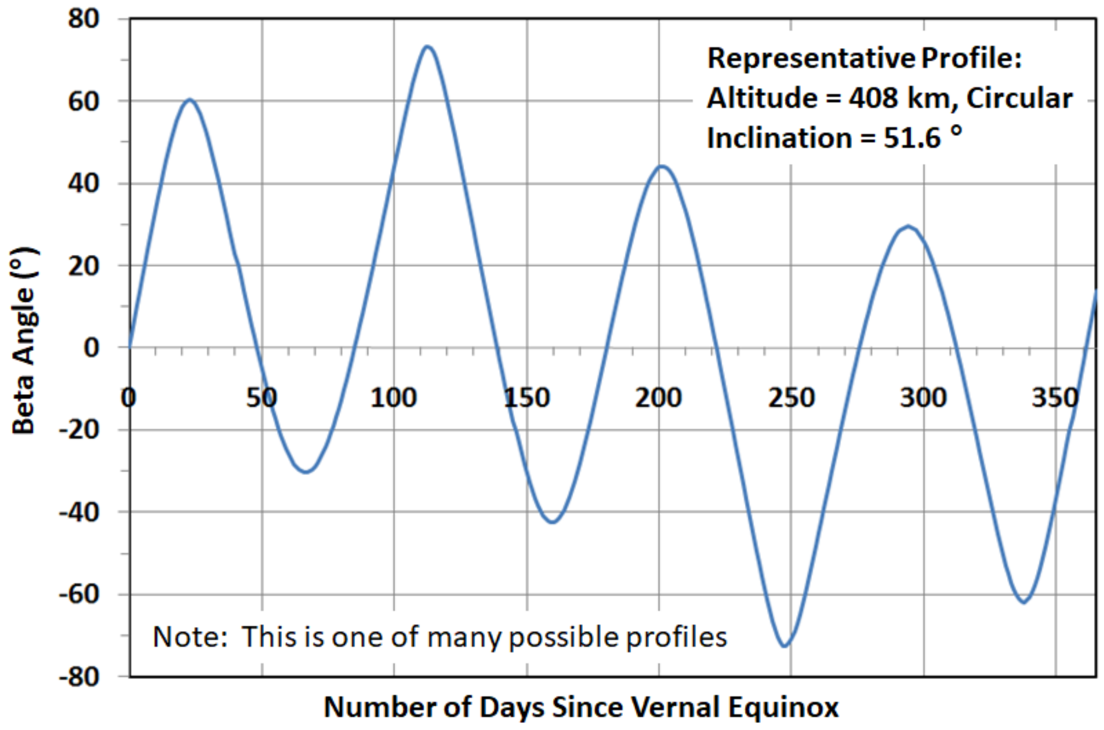
Note: Altitude adjusted slightly so as to produce an orbit period of 1 day.

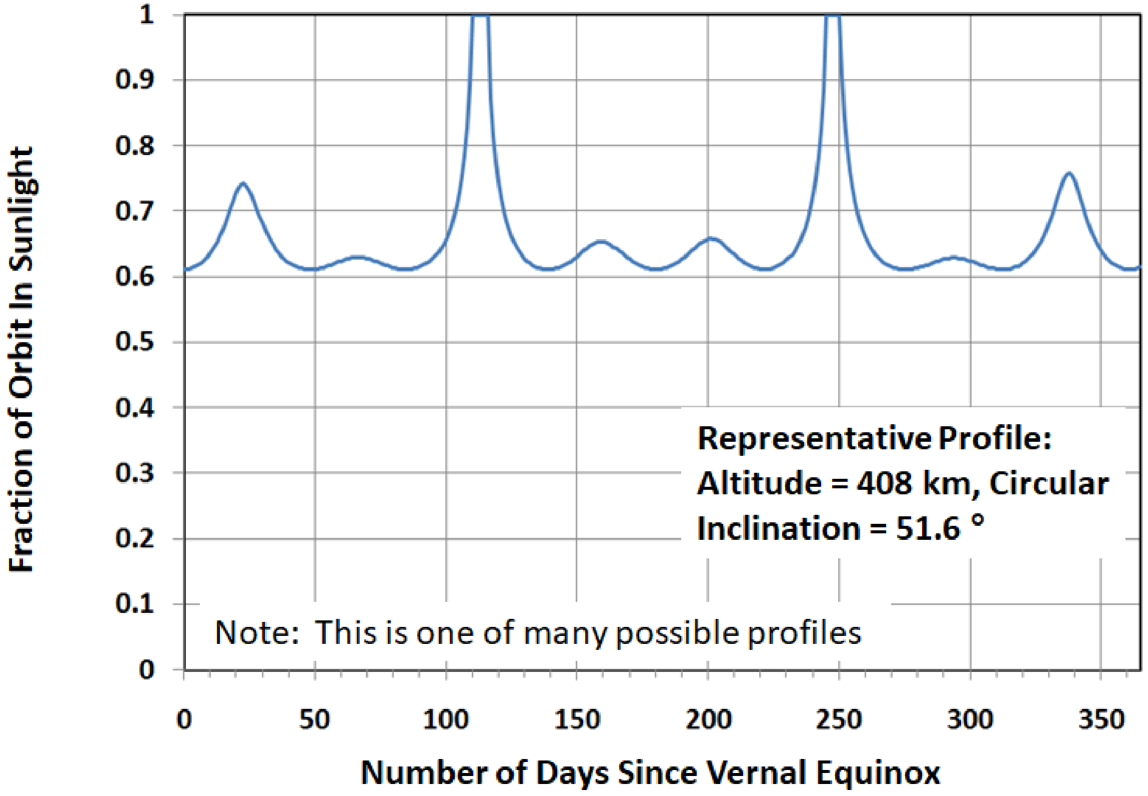
Example: Eclipse Season for a Geostationary Orbit



Note: Altitude adjusted slightly so as to produce an orbit period of 1 day.
Screen shots from: Thermal Desktop® by Cullimore and Ring Technologies, Inc. for visualization purposes only

Example: ISS Orbit





Example: Sun Synchronous Orbit

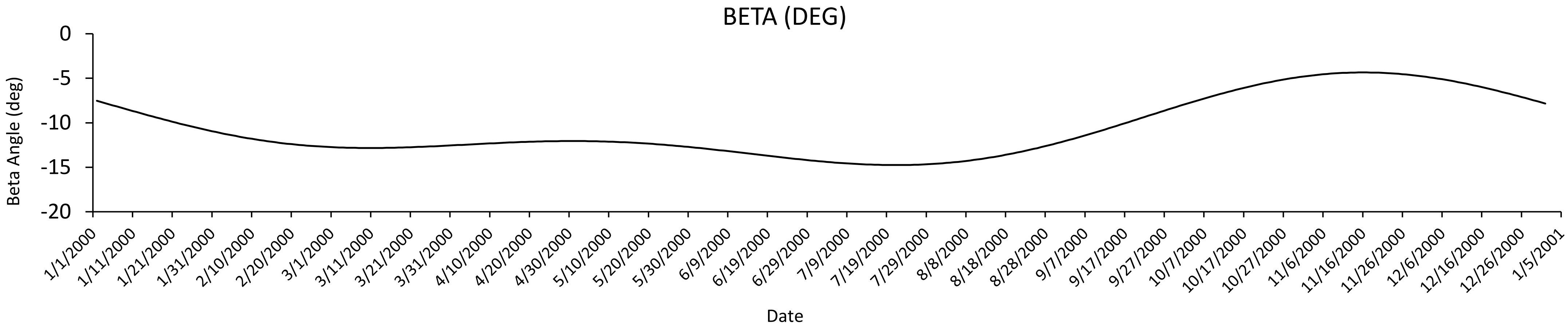
Sun synchronous orbits are designed such that the orbit ascending node moves in the *same direction* and at the *same average rate as the sun's motion* about the ecliptic plane.

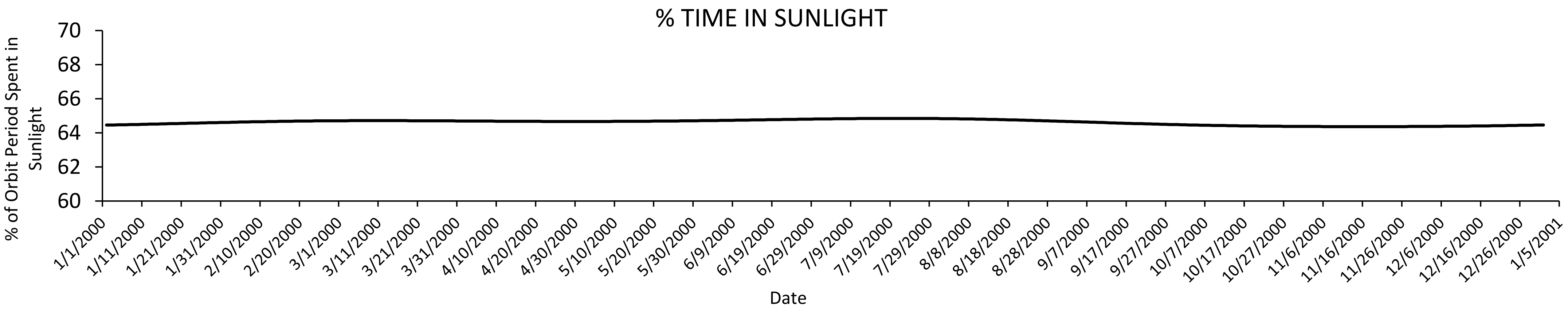
This can be accomplished by selecting the right combination of altitude, h and inclination, i. But note that in all cases, i must be $> 90°$.

For our example:

$$h = 705\ km$$
$$i = 98.2°$$

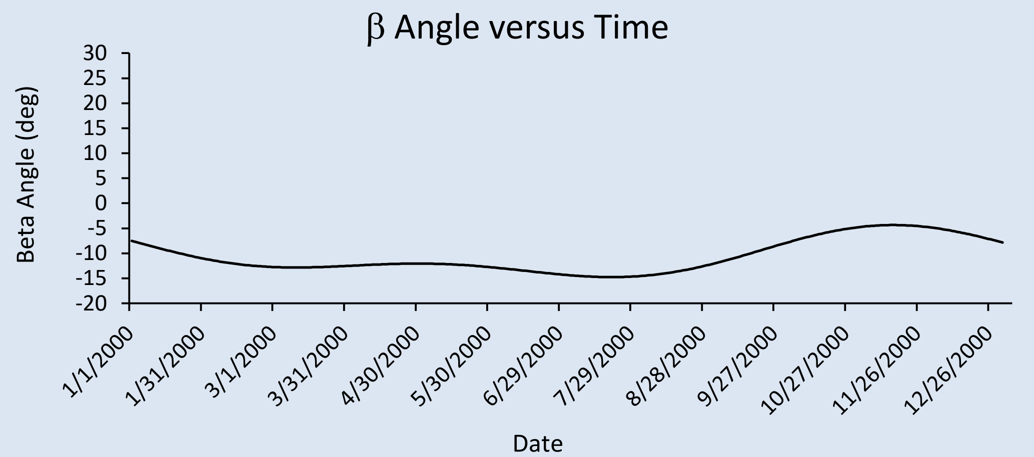
Example: Sun Synchronous Orbit

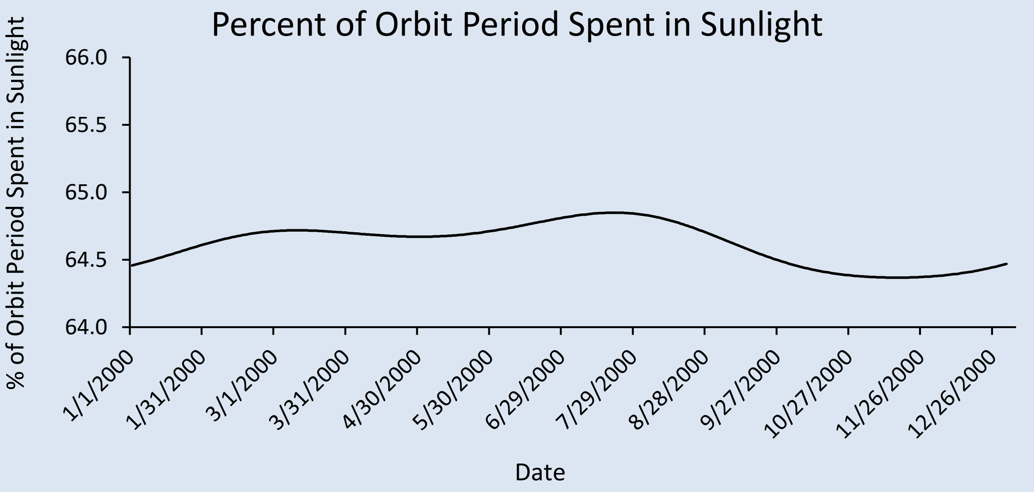




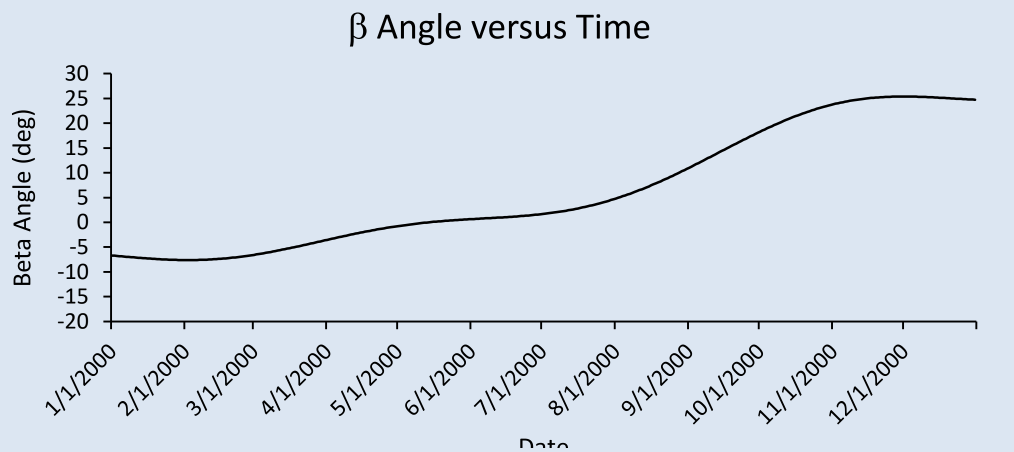
Example: What if the Orbit Isn't Quite Sun Synchronous

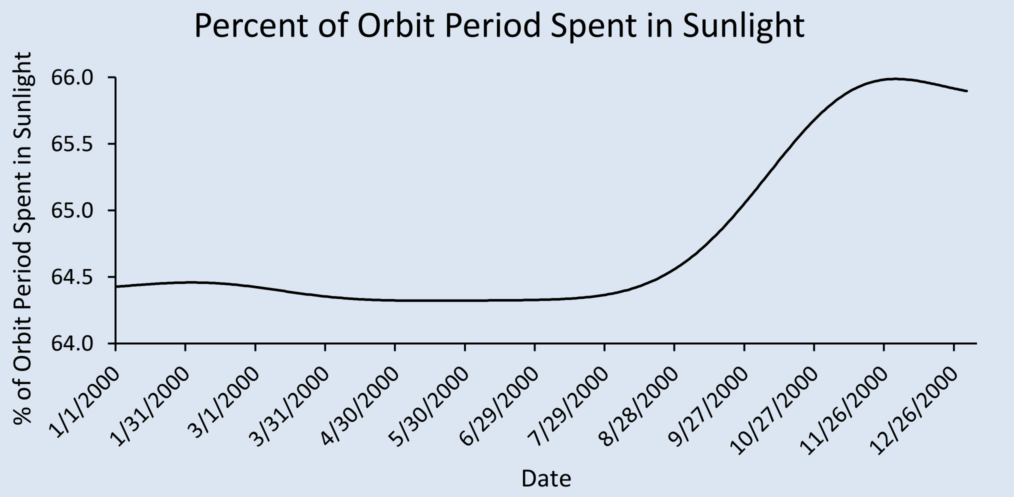
$i = 98.2°, h = 705\ km, \Omega = 270°$





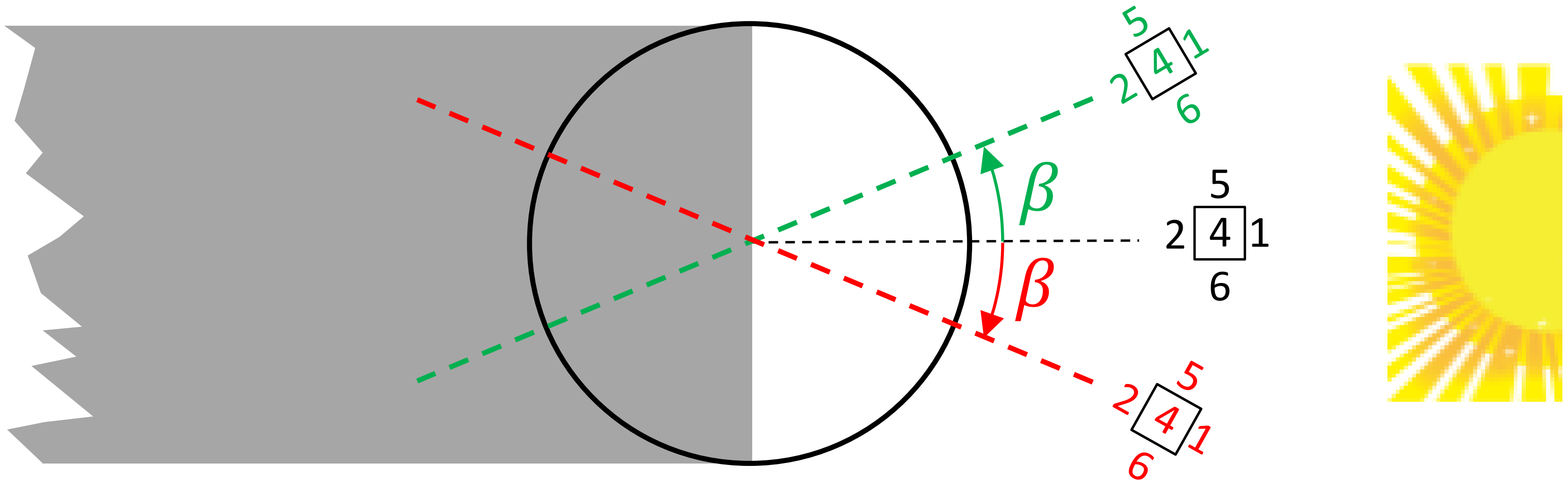
$i = 99.0°, h = 705\ km, \Omega = 270°$





Effect of Beta Angle on Flux Incident on Spacecraft Surfaces

Note: Side 3 not shown



Note: Side 1 is “Zenith” facing. Side 2 is “Nadir” facing.

Part 3 – Wrap-Up

In Part 3, we considered the effect of orbit perturbations arising from Earth's oblateness. This led to our understanding of the precession of the orbit ascending node and the precession of the orbit periapsis. These perturbations can be exploited to create useful orbits such as the sun synchronous and the Molniya orbits.

The effect of the perturbations on the beta angle and the consequences for spacecraft eclipse were discussed.

Part 4 – Advanced Orbit Concepts

Part 4 -- Contents

In this lesson, we'll briefly discuss a number of advanced orbit concepts including:

- Transfer Orbit
- Orbit Plane Change
- Aerobraking Orbit
- Gravity Assists
- The Restricted Three-Body Problem
- Halo Orbits
- Artemis I
- Gateway (Near Rectilinear Halo Orbit)

Transfer Orbit

Transfer orbits are used to raise a spacecraft orbit after launch.

They are also used for interplanetary trajectories.

Orbits are changed by changing the energy of the orbit.

It is useful to consider the minimum energy required to attain the desired orbit transfer.

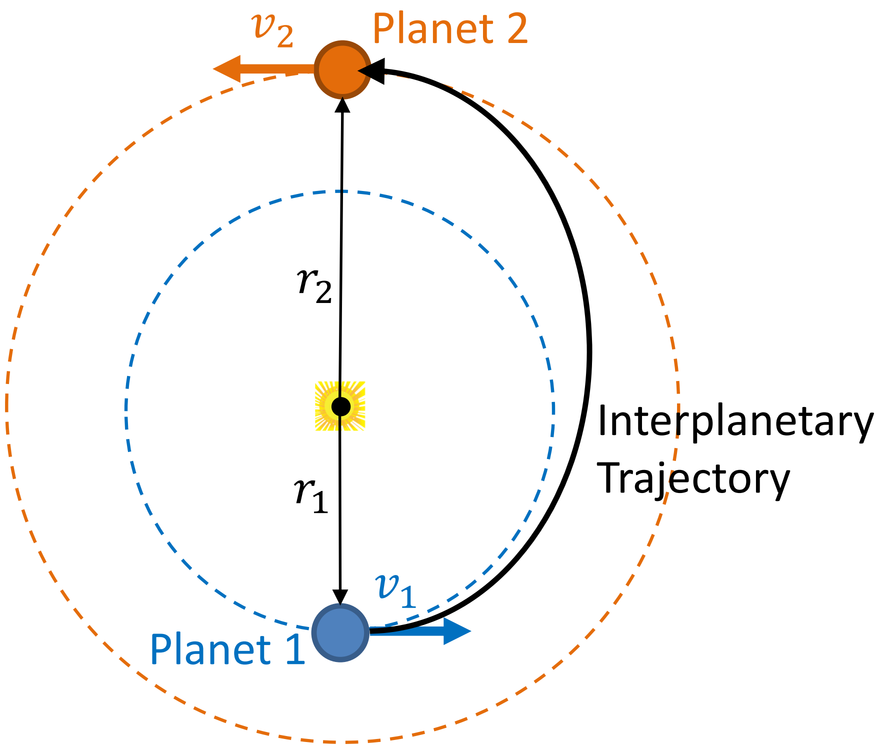
This minimum energy trajectory is referred to as a Hohmann transfer.

Transfer Orbit

We desire an interplanetary trajectory to take us from Planet 1 (shown at departure) to Planet 2 (shown at arrival), in co-planar, circular orbits;

The lowest energy transfer orbit occurs when the speed change, Δv is the lowest (and also takes the longest);

For the circular orbits, Planet 1 is traveling at velocity, v_1 and Planet 2 is traveling at velocity, v_2.



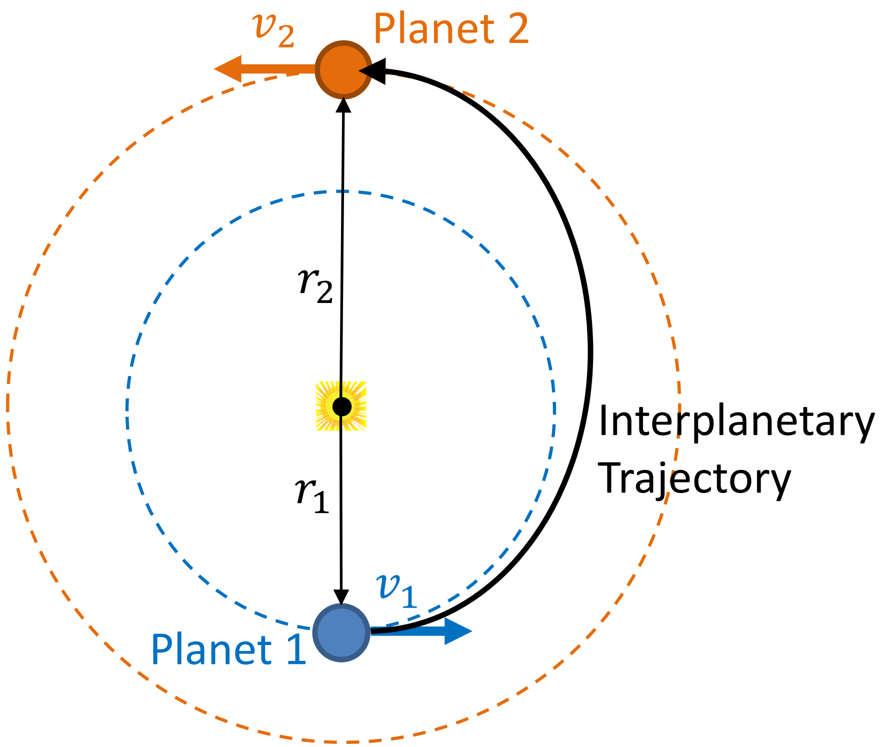
From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Transfer Orbit

For a successful transfer, we need to ensure the extent of our new orbit reaches from one planet to the other;

We have constructed half an ellipse – the semimajor axis of the transfer ellipse, a_t is…

$$a_t = \frac{r_1 + r_2}{2}$$



From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Transfer Orbit

Recall, the total specific energy for an orbit is given by:

$$E = -\frac{\mu}{2a}$$

And note that the total specific energy of the orbit is the sum of the specific kinetic energy and specific gravitational potential energy:

$$E = \frac{v^2}{2} - \frac{\mu}{r}$$

From: https://en.wikipedia.org/wiki/Hohmann_transfer_orbit

Transfer Orbit

Equating the two expressions for total specific energy…

$$-\frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

We can rearrange the equation to form an expression for the velocity, v…

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Adapted from: https://en.wikipedia.org/wiki/Hohmann_transfer_orbit

Transfer Orbit

At Planet 1, we are already traveling at the circular orbit velocity, v_1 …

$$v_1 = \sqrt{\mu\left(\frac{2}{r_1} - \frac{1}{r_1}\right)} = \sqrt{\frac{\mu}{r_1}}$$

But we need to be traveling at the transfer speed, v_t at Planet 1 to be on the elliptical trajectory to Planet 2…

$$v_t = \sqrt{\mu\left(\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right)} = \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right)}$$

From: https://en.wikipedia.org/wiki/Hohmann_transfer_orbit

Transfer Orbit

So, the change in velocity, Δv required to establish the transfer orbit from Planet 1 to Planet 2 is…

$$\Delta v = v_t - v_1 = \sqrt{2\mu\left(\frac{1}{r_1} - \frac{1}{r_1+r_2}\right)} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)$$

The time, T_t it takes to travel from Planet 1 to Planet 2 is one half of the entire elliptical orbital period…

$$T_t = \left(\frac{1}{2}\right) 2\pi\sqrt{\frac{a_t^3}{\mu}} = \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}}$$

Adapted from: https://en.wikipedia.org/wiki/Hohmann_transfer_orbit

Plane Change

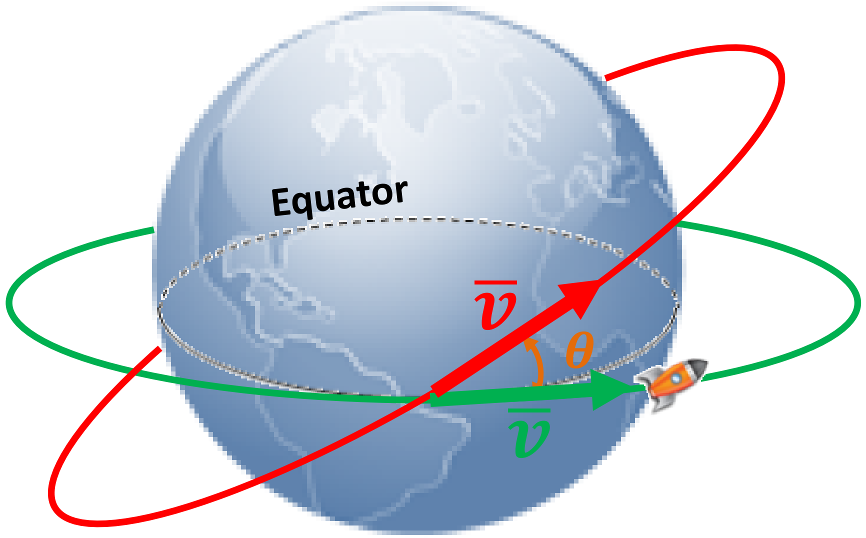
A plane change is used to change the inclination of a spacecraft's orbit.

Consider the diagram at the right – the spacecraft is currently in the circular green orbit and the associated plane traveling at velocity , $\bar{v}$.

We wish to change the spacecraft orbit to the circular red orbit and its associated plane.

The difference in inclination between the orbit planes is an angle, θ.

Note here, we are assuming $v = |\bar{v}| = |\bar{v}|$



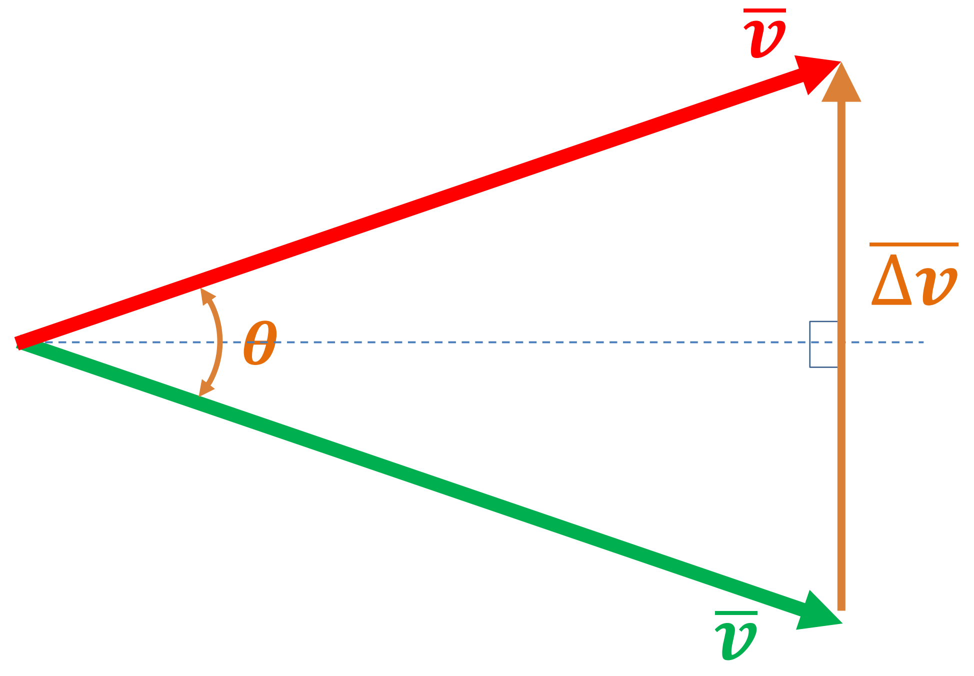
From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

Plane Change

Let's examine the velocity vectors more closely.

We see to change the trajectory from the green velocity vector to the red velocity vector, a change in velocity by $\overline{\boldsymbol{\Delta v}}$ is required.

$$\Delta v = |\overline{\boldsymbol{\Delta v}}| = 2v\sin\left(\frac{\theta}{2}\right)$$



From: Bate, R. R., Mueller, D. D., and White, J. E., Fundamentals of Astrodynamics, Dover Publications, New York, 1971.

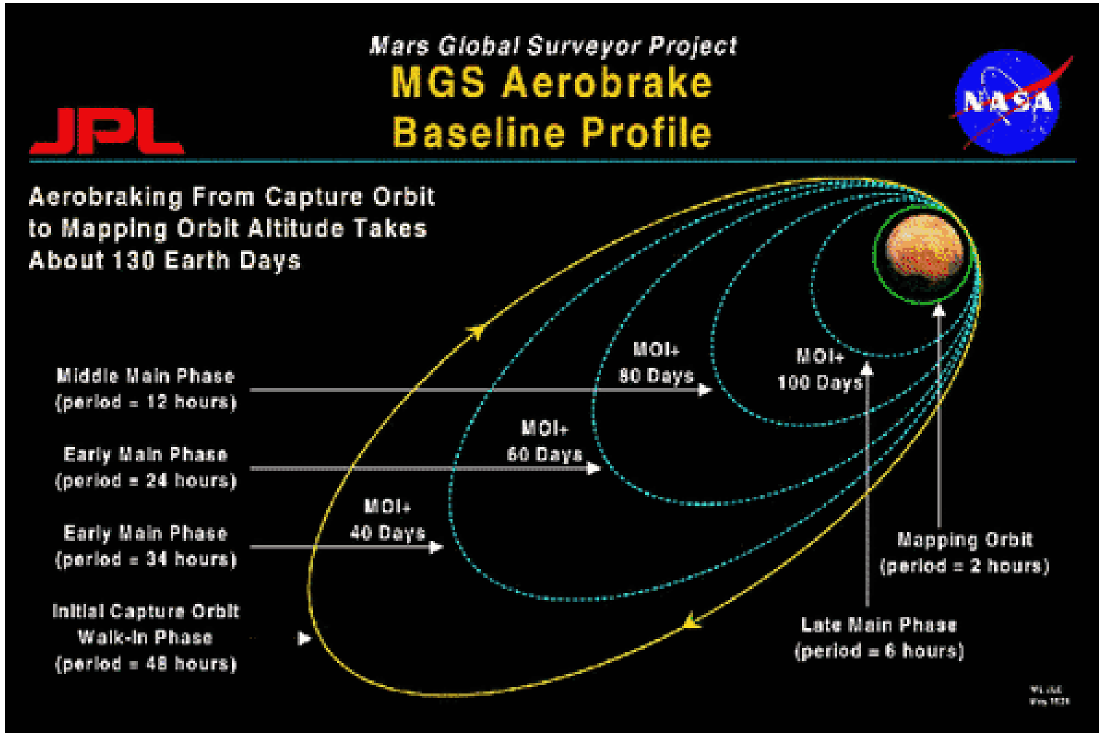
Aerobraking Orbit

Aerobraking is a technique used to reduce the amount of fuel required to slow down a spacecraft.

This was used for Mars Global Surveyor (MGS) spacecraft as it approached Mars.

The MGS spacecraft used the drag of the Martian atmosphere on its solar panels to slow down as an alternative to using thrusters.

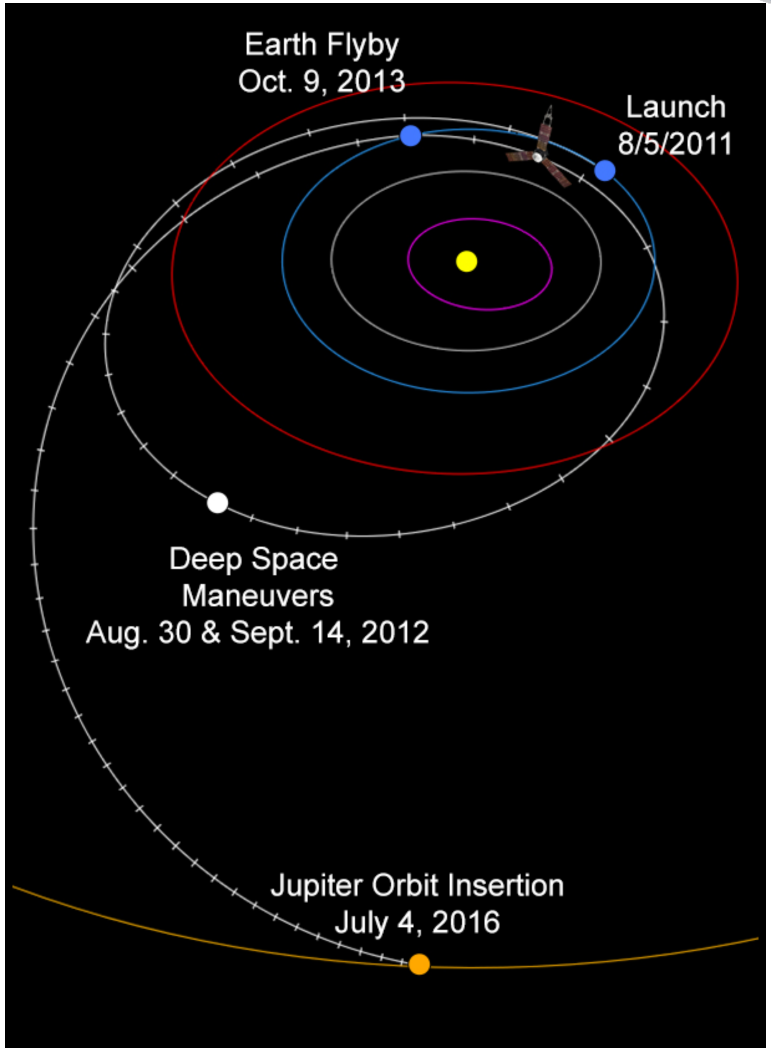
The duration of the aerobraking phase is directly related to how fast Mars' relatively thin atmosphere reduces the spacecraft's velocity.



Excerpt from: https://mgs-mager.gsfc.nasa.gov/overview/aerobraking.html

Gravity Assist Orbit

Spacecraft orbits may be redirected using gravity assist maneuvers where a close fly-by of a planet is be used to change the direction of and orbit and add energy to it.



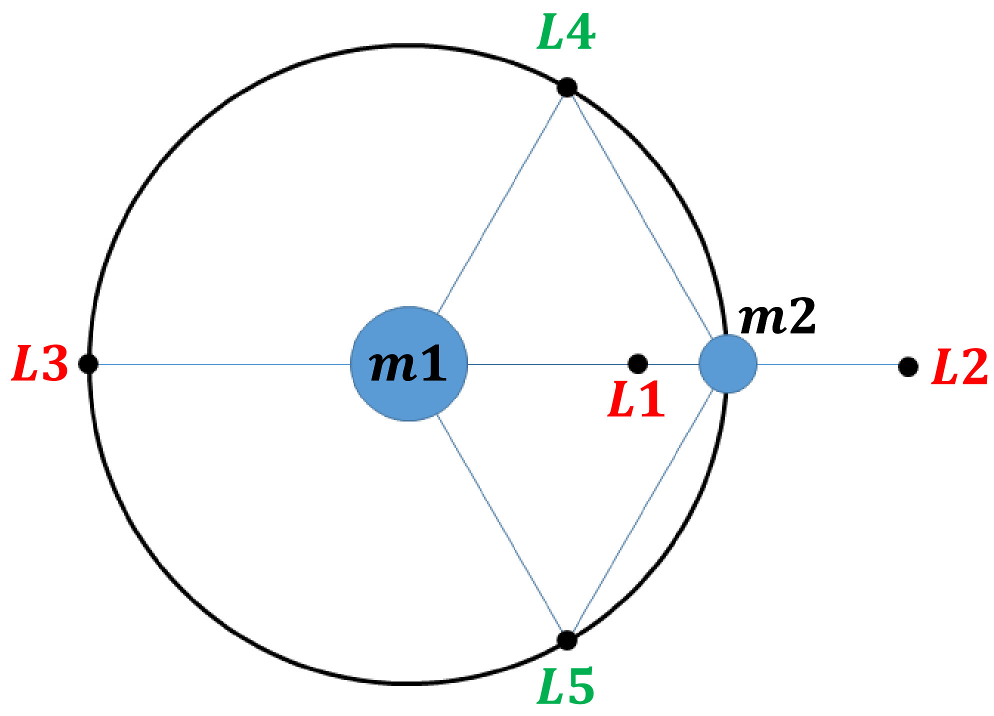
Juno Spacecraft Trajectory

Image Credit: NASA/JPL/Caltech
https://www.jpl.nasa.gov/images/juno/earthflyby/JunoCruiseTraj_20130815.jpg

The Restricted Three Body Problem

When two large masses, $m1$ and $m2$, are orbiting one another, regions in space can serve as gravitational nodes where spacecraft or other celestial bodies can collect – these are called Lagrange points;

$L1$, $L2$ and $L3$ and unstable.
$L4$ and $L5$ are stable.

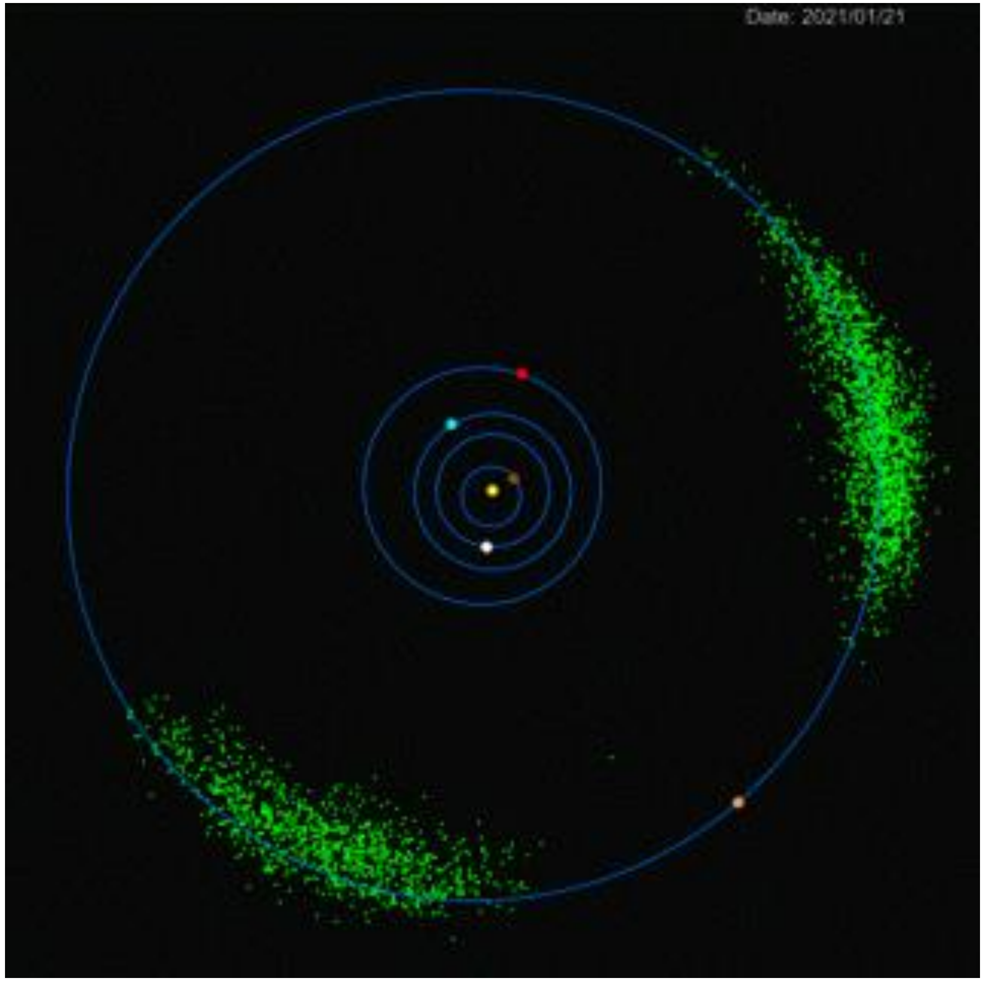


Dynamics of the Three-Body System

Three Body Orbits

In 1772, using three-body assumptions, Joseph-Louis Lagrange believed asteroids might be trapped near the L4 and L5 points because they are stable;

The first confirmed observation of a Jupiter Trojan was made by Max Wolf in 1906.



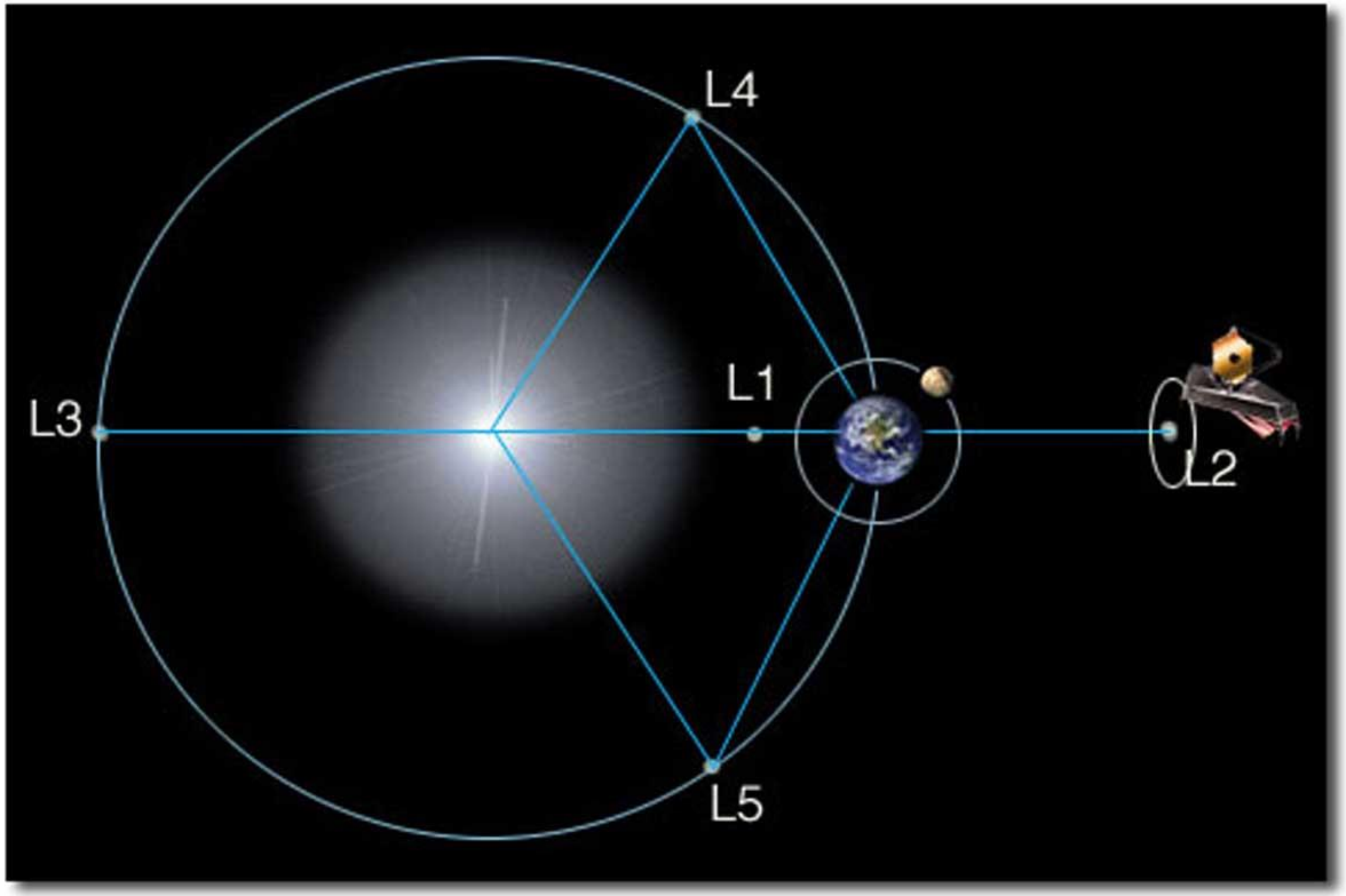
Jupiter's Trojan Asteroids

Animation Credit: CAS/Petr Scheirich. Used with permission.
Source: https://en.wikipedia.org/wiki/Jupiter_trojan

Three Body Orbits

Lagrange points are also used for spacecraft;

The James Webb Space Telescope (JWST) will be located at the Sun-Earth L2 point.

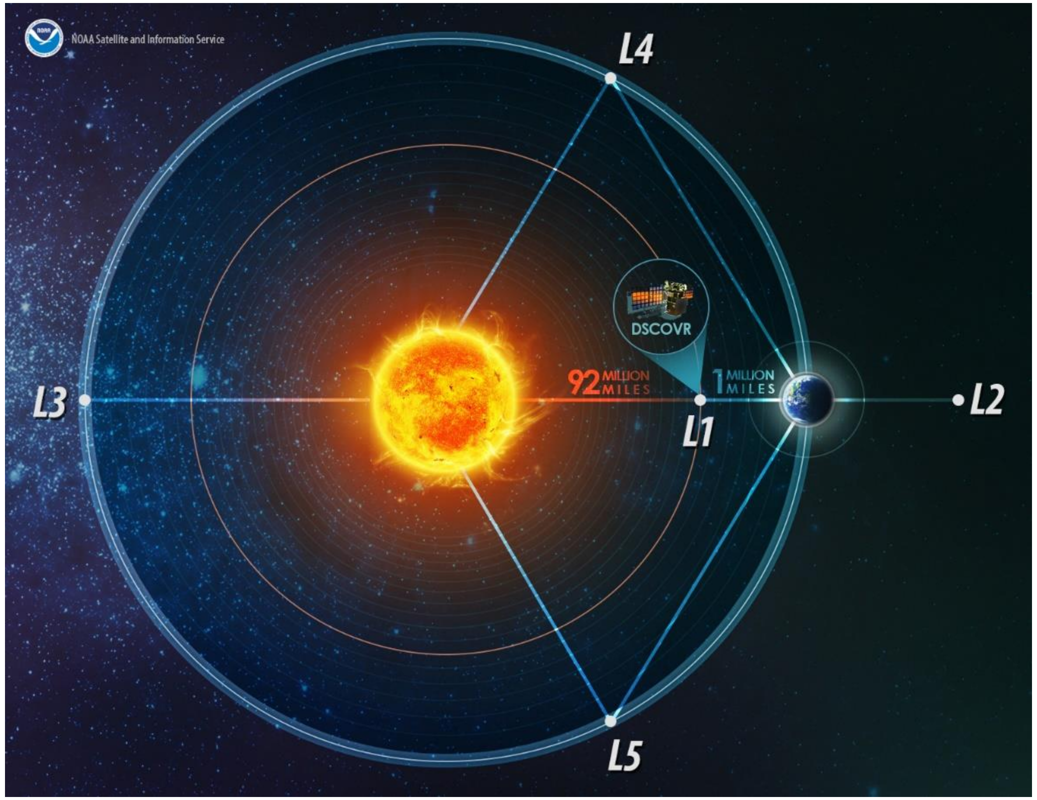


JWST Orbit at the Earth-Moon L2 Point

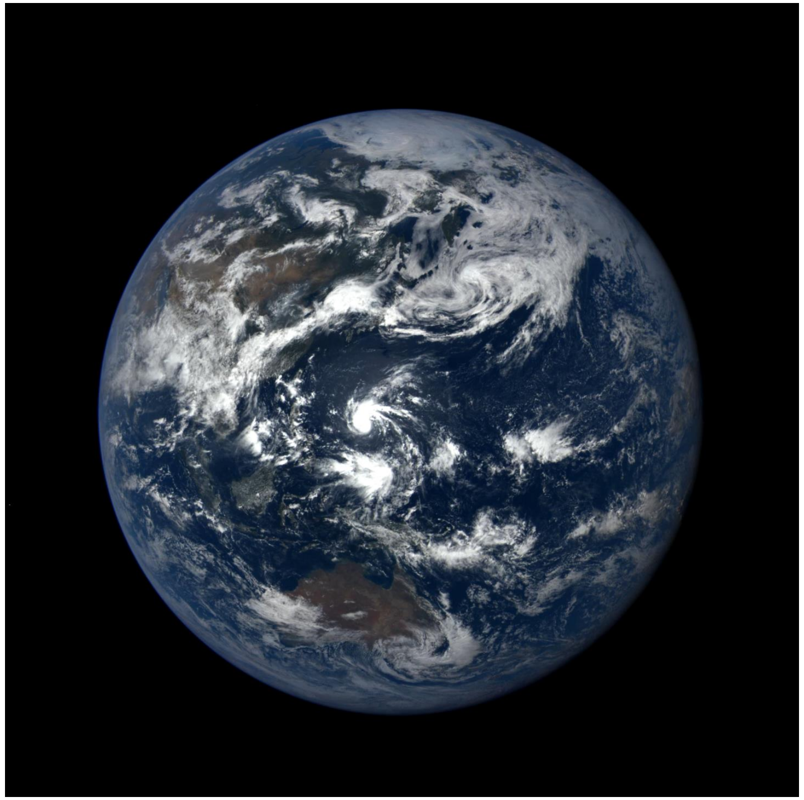
Image Credit: NASA
https://www.nasa.gov/images/content/463480main_lagrange_point_lg_1.jpg

Three Body Orbits

The DSCOVR spacecraft is located at the Sun-Earth L1 point.



DSCOVR Spacecraft at the Sun-Earth L1 Point

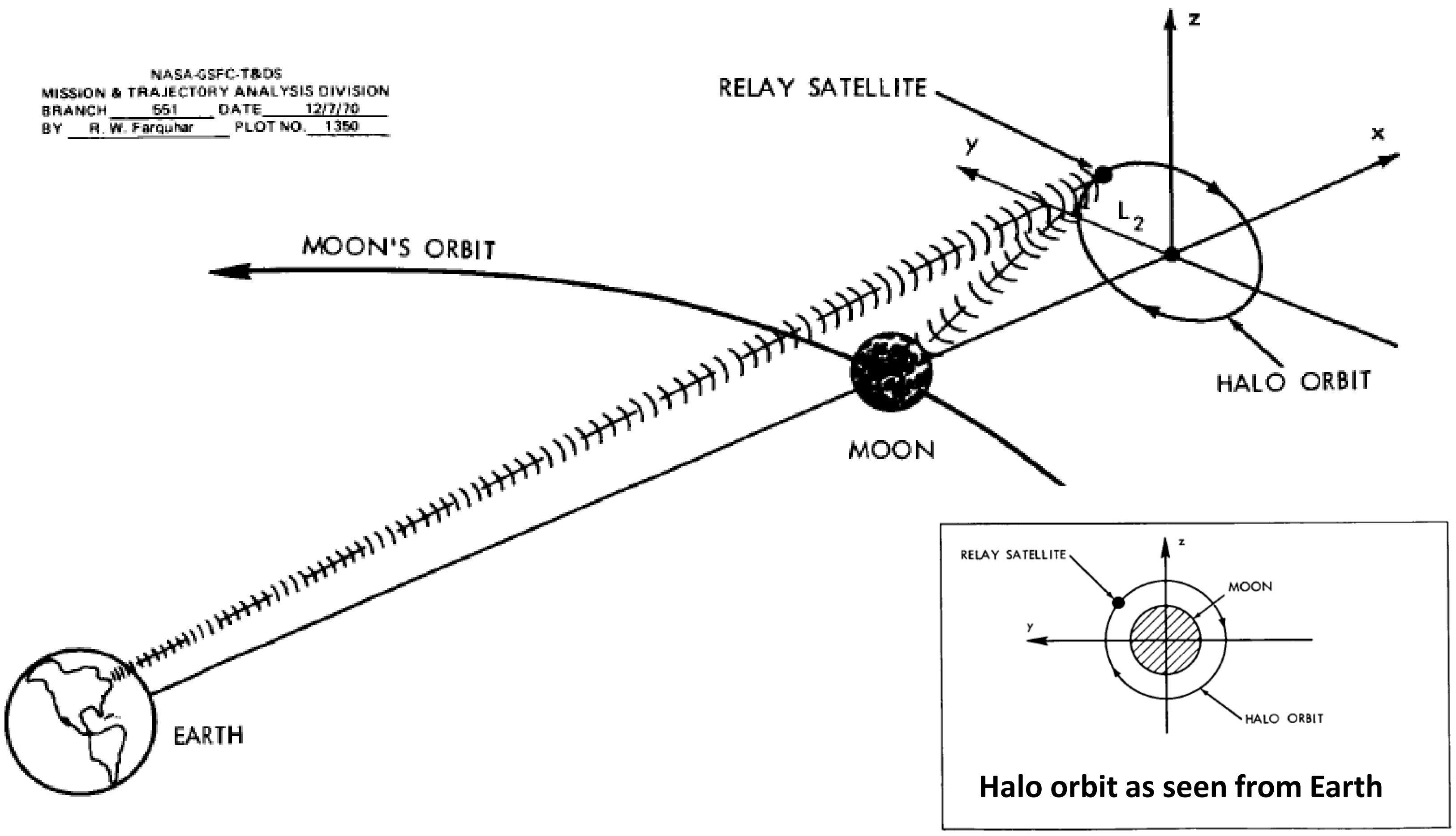


The Earth-Moon System as Seen from the DSCOVR Spacecraft

Image Credit: NOAA
Video Credit: DSCOVR: EPIC Team

Halo Orbits

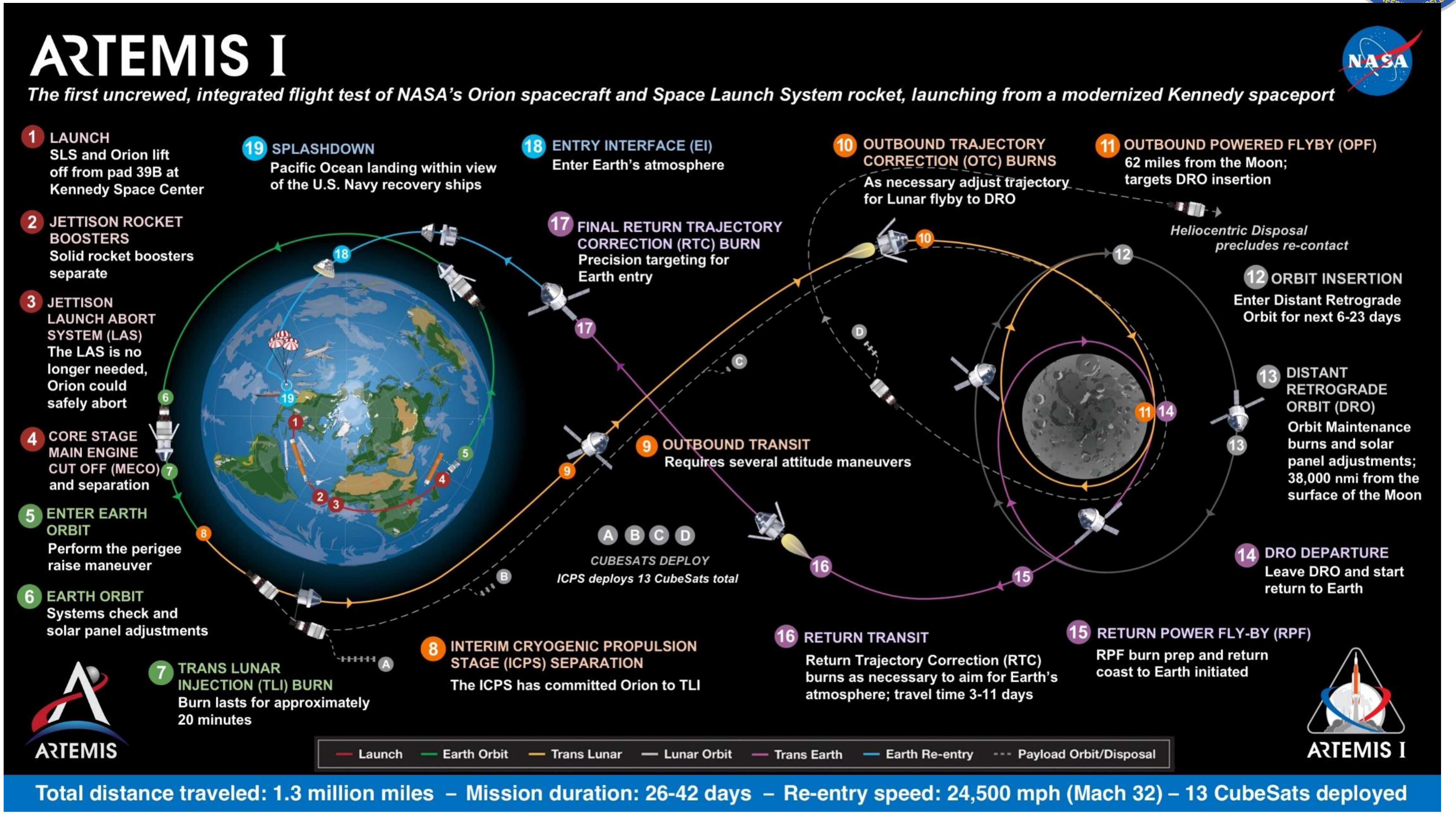
A halo orbit can be established about the Earth-Moon L2 point to serve as a communication link between the lunar far side and Earth.



Lunar Far-Side Communications with a Halo Satellite

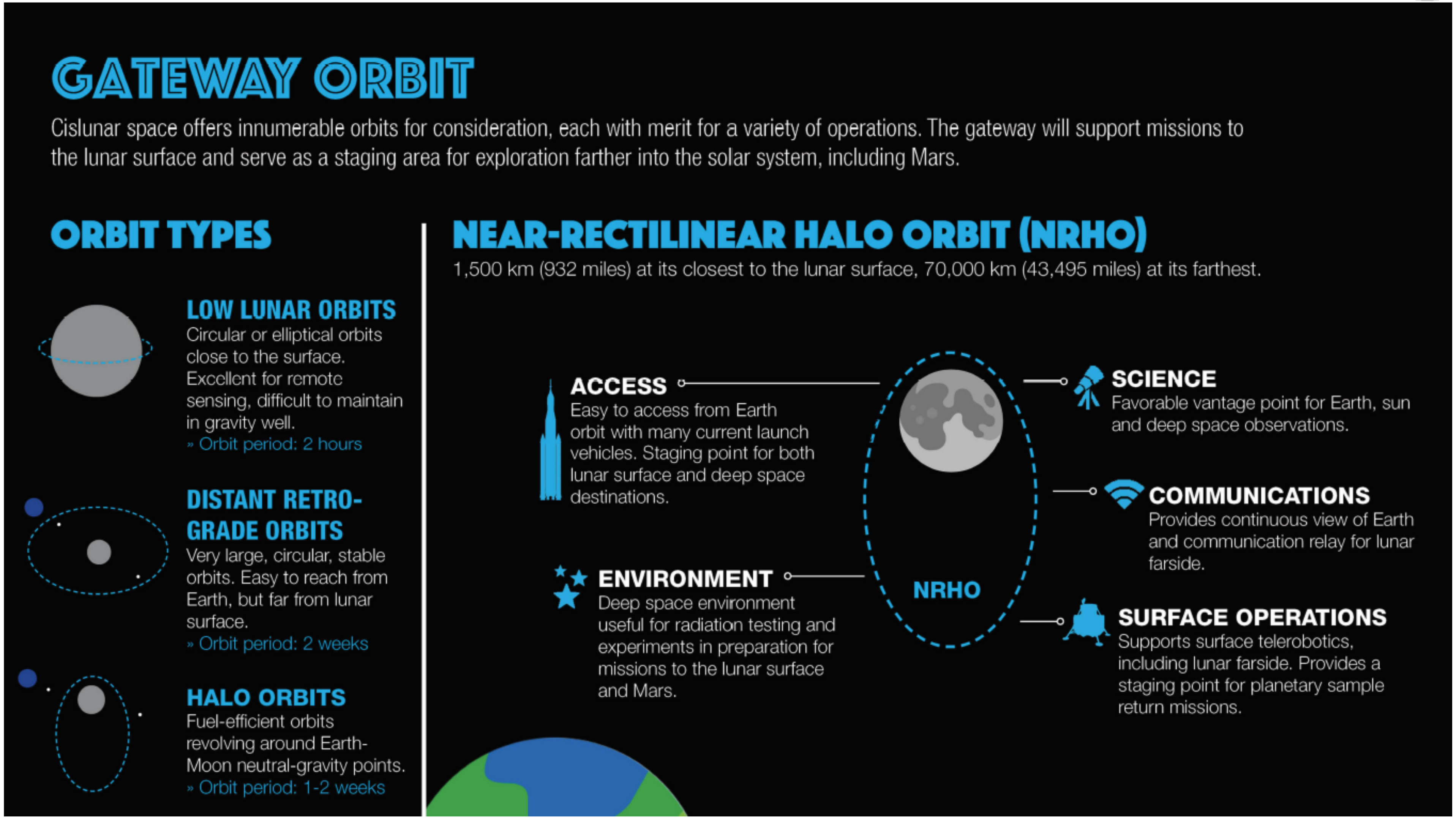
Graphics from: Farquhar, R. W., The Utilization of Halo Orbits in Advanced Lunar Operations, X-551-70-449, NASA Goddard Space Flight Center, December 1970

Artemis I



https://www.nasa.gov/image-feature/artemis-i-map

Gateway



GATEWAY ORBIT
Cislunar space offers innumerable orbits for consideration, each with merit for a variety of operations. The gateway will support missions to the lunar surface and serve as a staging area for exploration farther into the solar system, including Mars.
ORBIT TYPES
LOW LUNAR ORBITS
Circular or elliptical orbits close to the surface. Excellent for remote sensing, difficult to maintain in gravity well.
» Orbit period: 2 hours
DISTANT RETRO-GRADE ORBITS
Very large, circular, stable orbits. Easy to reach from Earth, but far from lunar surface.
» Orbit period: 2 weeks
HALO ORBITS
Fuel-efficient orbits revolving around Earth-Moon neutral-gravity points.
» Orbit period: 1-2 weeks
NEAR-RECTILINEAR HALO ORBIT (NRHO)
1,500 km (932 miles) at its closest to the lunar surface, 70,000 km (43,495 miles) at its farthest.
ACCESS
Easy to access from Earth orbit with many current launch vehicles. Staging point for both lunar surface and deep space destinations.
ENVIRONMENT
Deep space environment useful for radiation testing and experiments in preparation for missions to the lunar surface and Mars.
NRHO
SCIENCE
Favorable vantage point for Earth, sun and deep space observations.
COMMUNICATIONS
Provides continuous view of Earth and communication relay for lunar farside.
SURFACE OPERATIONS
Supports surface telerobotics, including lunar farside. Provides a staging point for planetary sample return missions.

Part 4 – Wrap-Up

In Part 4, we considered some advanced orbit concepts.

Transfer orbits are useful for changing from one orbit to another such as in interplanetary missions.

An orbit plane change can be used to change the inclination of an orbit.

Aerobraking orbits can be used to lower an orbit by using passes through an atmosphere to remove energy from the orbit.

Gravity assists can be employed in interplanetary missions to impart additional energy to a spacecraft.

Part 4 – Wrap-Up (Continued)

Study of the Restricted Three-Body Problem explains the existence of asteroids at Lagrange points. Solutions to the Three Body Problem are also useful for the placement of spacecraft such as the James Webb Space Telescope.

Halo Orbits are useful for continuous communications with the lunar far side.

Upcoming missions such as Artemis I require more complex orbit solutions to accommodate mission requirements as it travels between the Earth and Moon.

The Gateway outpost is planned to use a Near Rectilinear Halo Orbit to allow easy access, provide the desired environment, meet communications requirements, serve as a science platform, and support surface operations.

Part 5 -- Spacecraft Attitudes

Part 5 -- Contents

In this fifth and final part of the lesson, we'll focus on spacecraft attitudes. We'll discuss, both, the Local Vertical – Local Horizontal and Celestial Inertial reference frames and provide an attitude transformation strategy.

These transformations orient spacecraft surfaces with respect to the solar, albedo and planetary heating sources. We'll spend some time showing how to calculate the view factor to these sources.

Finally, we'll tie it all together with an illustrative example.

Spacecraft Attitudes

Spacecraft attitude, in concert with the orbit is important to thermal engineers as these must be known to determine the on-orbit thermal environment required to determine spacecraft thermal response.

As we have seen, the orbit is used to determine the distance from the sun and, hence, the magnitude of the solar flux. This, in turn, affects albedo and planetary heating components. The evolution of the orbit over time affects periods of spacecraft eclipse and the orbit beta angle.

The attitude is required to determine where on the spacecraft the environment is applied.

Reference Frames

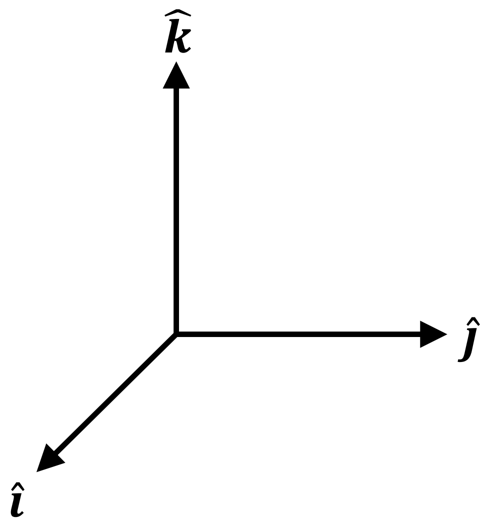
A reference frame can be thought of as a basis or starting point for a subsequent series of rotations.

All axes of the coordinate system to be subsequently transformed are aligned with the principal axes of the reference frame coordinate system.

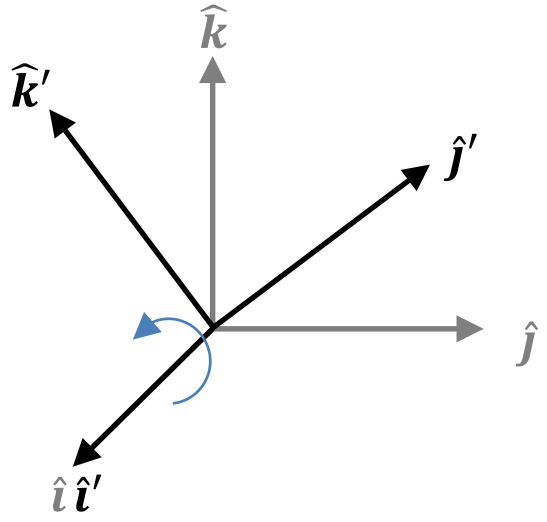
In other words, no rotations have yet taken place.

Reference Frames

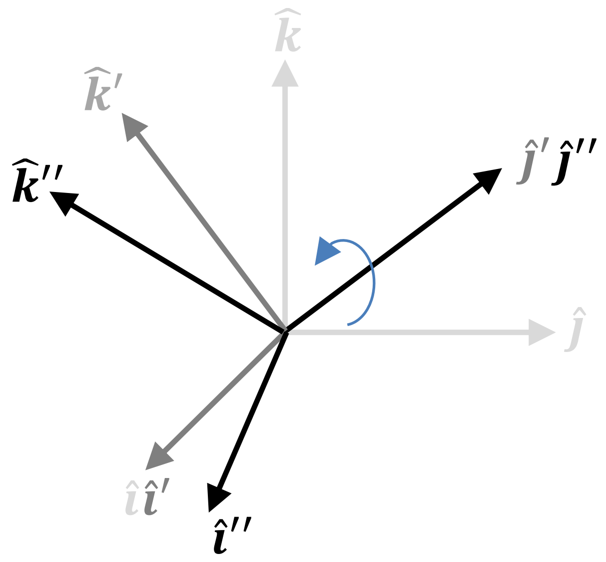
Consider the Euler rotation sequence shown below – the rotations must be referenced to some starting point which we will call the reference frame.



Reference Frame



After One Rotation (x −axis)

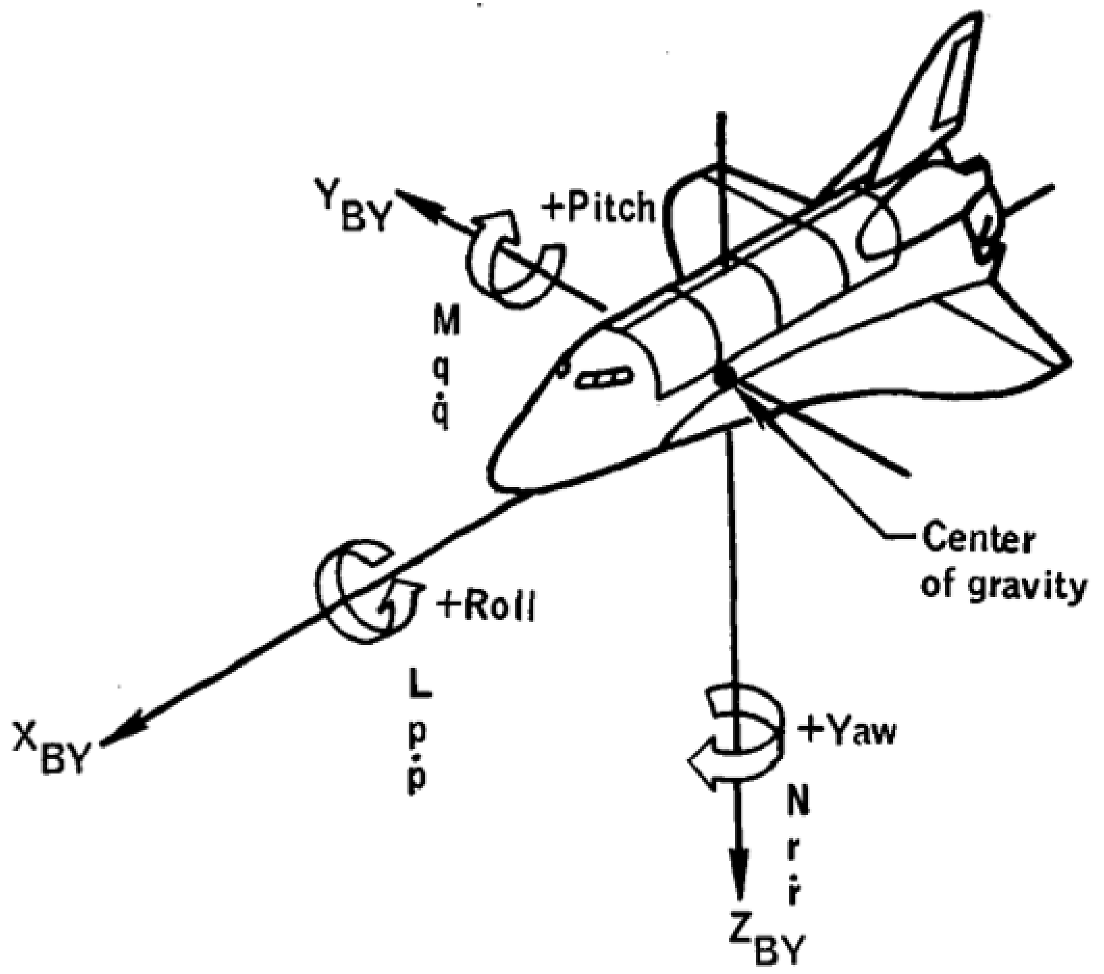


After Two Rotations (x −axis, then y −axis)

Vehicle Body Axes

But if we want to transform a spacecraft within a reference frame, we must establish a meaningful coordinate system on the spacecraft.

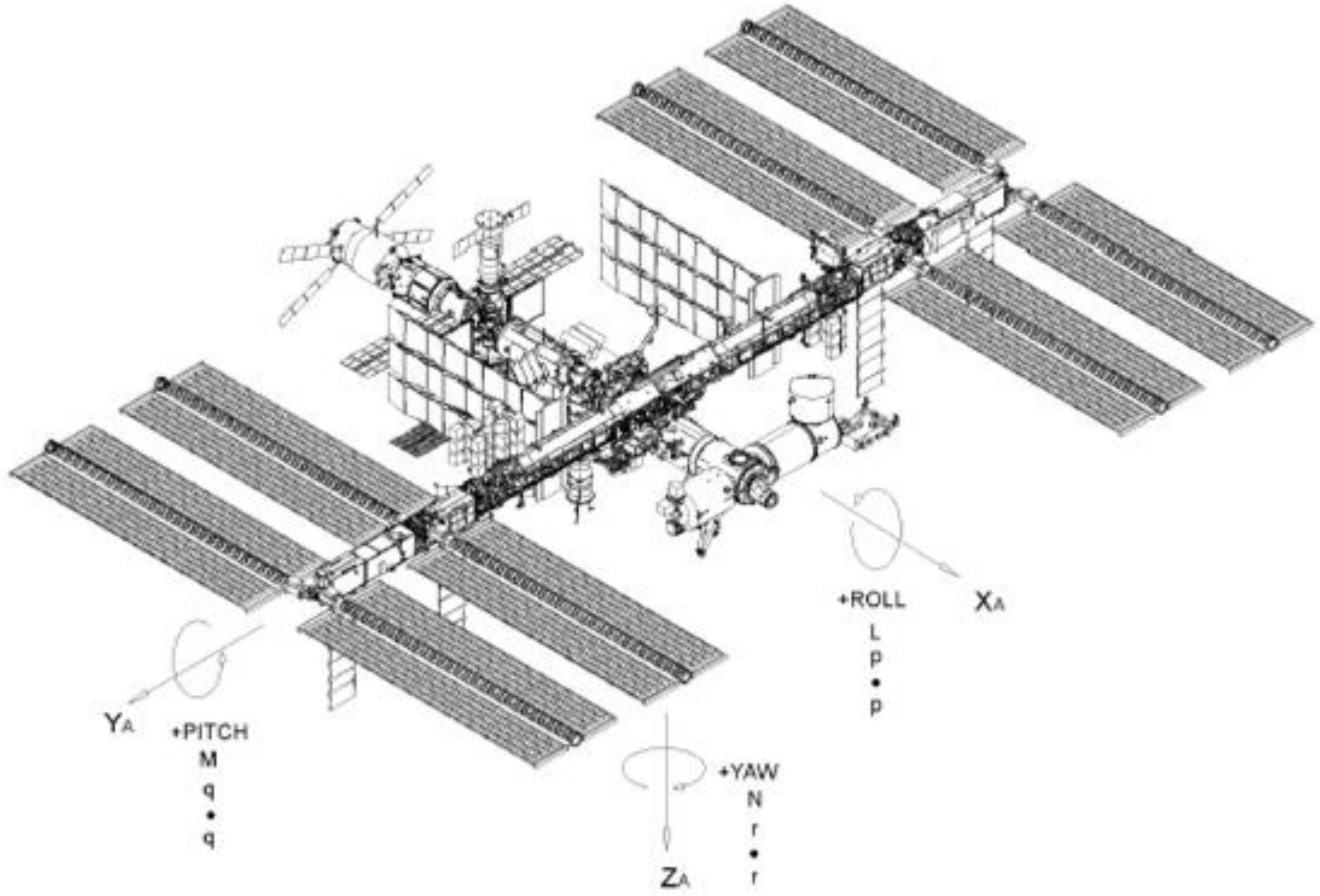
As an example, consider the body axes designated for the Space Shuttle Orbiter.



From: Davis, L. D., Coordinate Systems for the Space Shuttle Program, NASA Technical Memorandum, JSC-09084, NASA TM X-58153, October 1974

Vehicle Body Axes

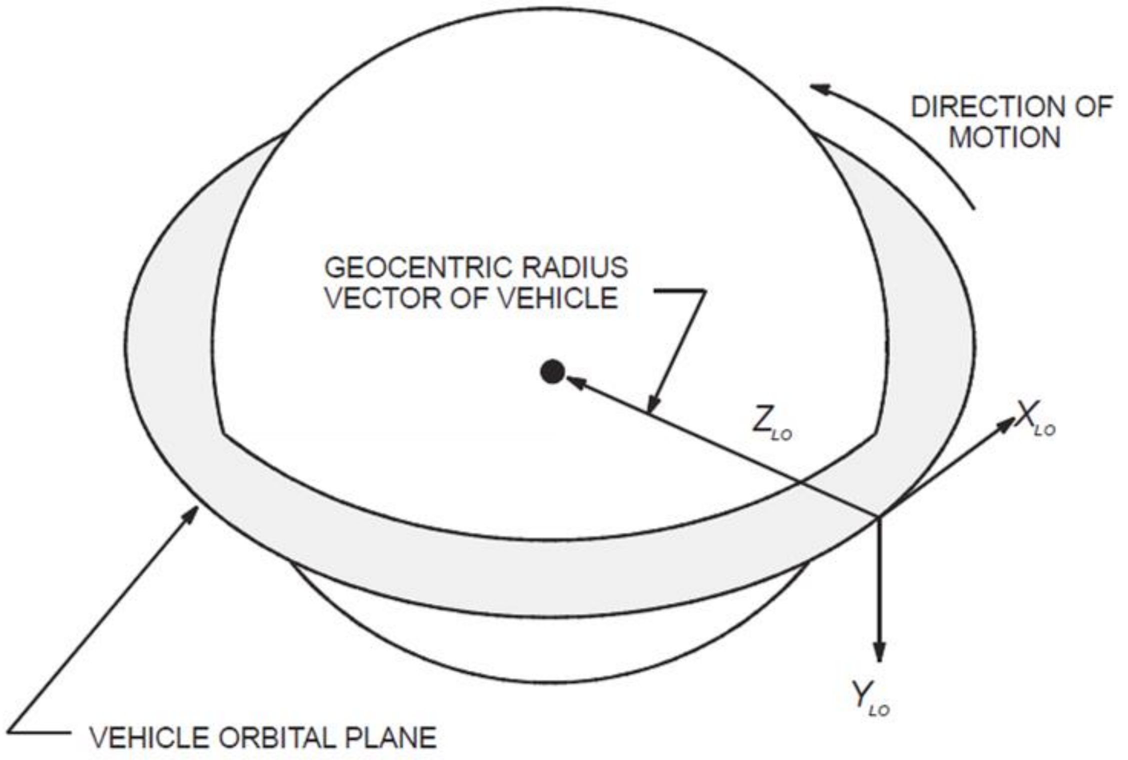
As another example, here is the coordinate system definition for the International Space Station.



From: Space Station Reference Coordinate Systems, SSP 30219, Revision J, May 1, 2008 (found at: https://pims.grc.nasa.gov/plots/user/tibor/SSP%2030219J%20ISS%20Coord%20Systems.pdf)

Local Vertical-Local Horizontal (LVLH)

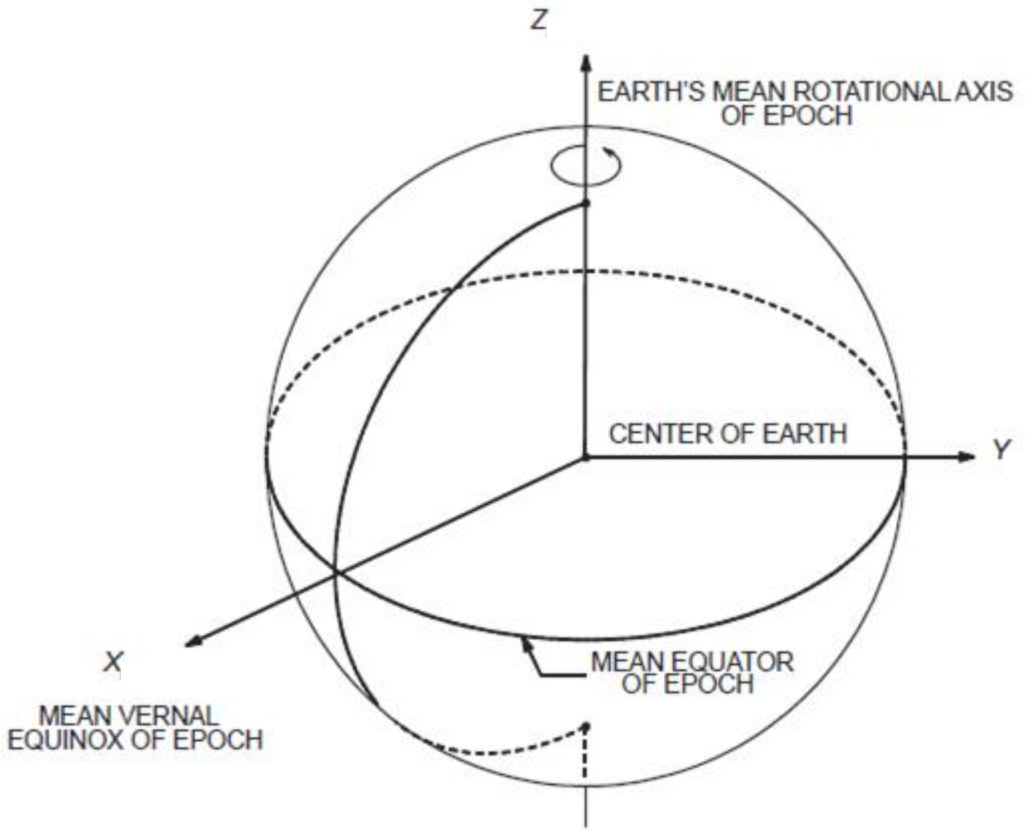
In the local vertical-local horizontal frame shown at the right, the $X_{LO} - Z_{LO}$ plane is the instantaneous orbit plane at the time of interest, the Y_{LO} axis is normal to the orbit plane, Z_{LO} points toward the center of the planet, and the X_{LO} axis completes the right handed system and is positive in the direction of motion.



From: Space Station Reference Coordinate Systems, SSP 30219, Revision J, May 1, 2008 (found at: https://pims.grc.nasa.gov/plots/user/tibor/SSP%2030219J%20ISS%20Coord%20Systems.pdf)

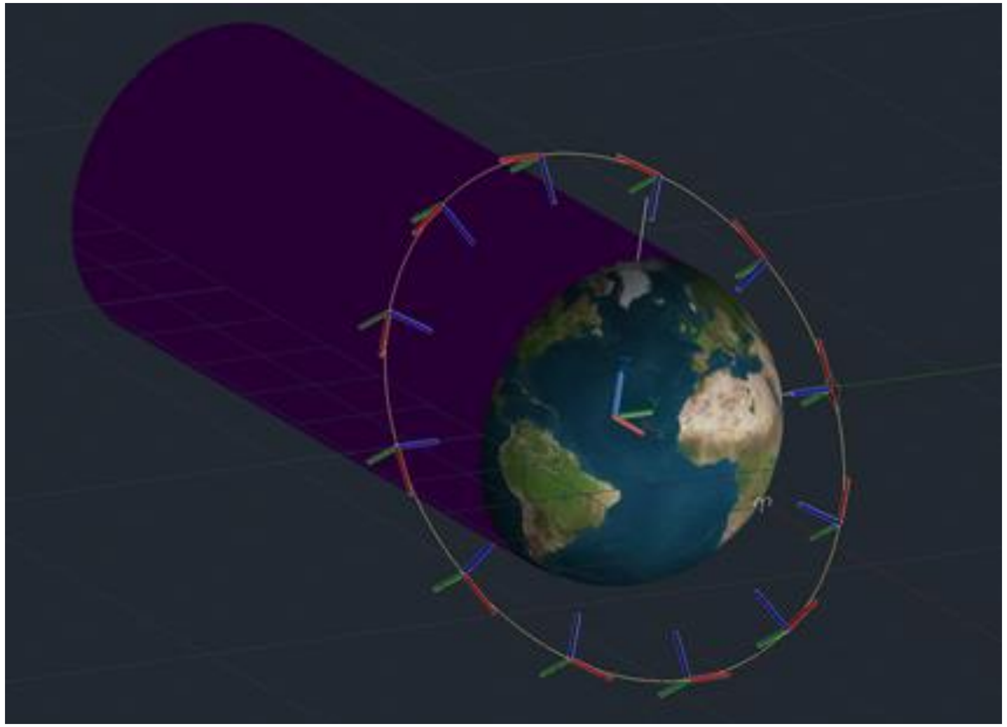
Celestial Inertial (CI)

In the celestial inertial coordinate system shown at the right, the $X_{J2000} - Y_{J2000}$ plane is the mean Earth's equator of epoch, the X_{J2000} axis is directed toward the mean vernal equinox of epoch, the Z_{J2000} axis is directed along Earth's mean rotational axis of epoch and is positive north, and the Y_{J2000} axis completes the right handed system.

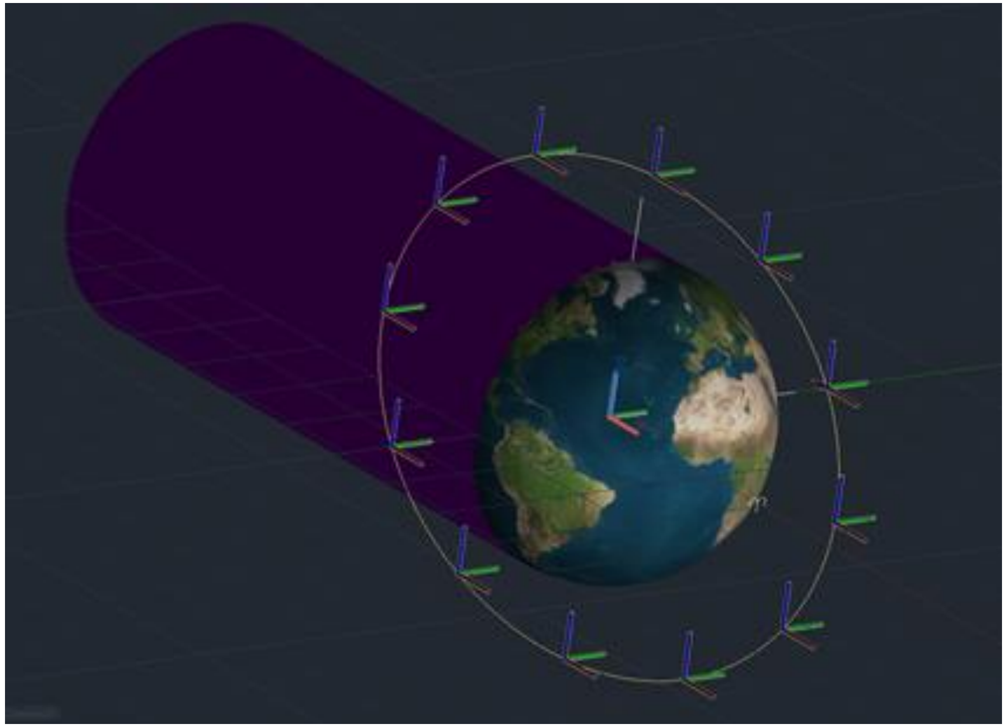


Adapted from: Space Station Reference Coordinate Systems, SSP 30219, Revision J, May 1, 2008 (found at: https://pims.grc.nasa.gov/plots/user/tibor/SSP%2030219J%20ISS%20Coord%20Systems.pdf)

Comparing LVLH and CI Reference Frames (No Rotations)



LVLH Coordinate System Progression Throughout Orbit



CI Coordinate System Progression Throughout Orbit

In both images:

$+x$ axis

$+y$ axis

$+z$ axis

Images created using: Thermal Desktop® by Cullimore and Ring Technologies, Inc.

Attitude Transformation Strategy

Our ultimate strategy is to transform surface normals (representing spacecraft surfaces of interest) into the same coordinate system in which unit vectors describing the location of the sun and planet are expressed;

Once all vectors are transformed, angles between vectors of interest may be calculated and view factors to the sun and planet may be readily determined;

It is most convenient to transform all surface normal vectors into the celestial inertial system.

Transforming Attitudes in CI

If a spacecraft is flying in a celestial inertial reference frame, then unit vectors representing surface normals are transformed as follows, assuming a pitch, yaw, roll sequence executed in the specified order:

$$[Transformed\ Vectors] = [P][Y][R][Untransformed\ Vectors]$$

where…
$[P]$ is a y-axis transformation matrix
$[Y]$ is a z-axis transformation matrix
$[R]$ is an x-axis transformation matrix

Transforming LVLH into CI

For a spacecraft flying in the local vertical-local horizontal frame, then unit vectors representing surface normals are transformed as follows, assuming a pitch, yaw, roll sequence executed in the specified order:

$$[Transformed\ Vectors] = [\Omega][i][\omega][\nu][REF][P][Y][R][Untransformed\ Vectors]$$

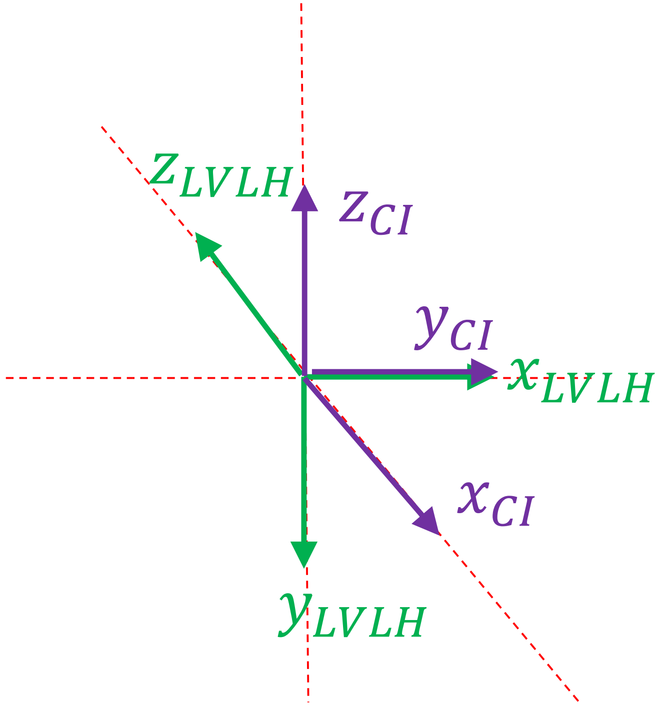
where…

$[\Omega]$ is a z-axis transformation matrix for orbit right ascension
$[i]$ is an x-axis transformation matrix for orbit inclination
$[\omega]$ is a z-axis transformation matrix for argument of periapsis
$[\nu]$ is a z-axis transformation matrix for true anomaly
$[REF]$ is the reference change matrix
$[P]$ is a y-axis transformation matrix
$[Y]$ is a z-axis transformation matrix
$[R]$ is an x-axis transformation matrix

Transforming LVLH into CI

The reference change matrix $[REF]$ is used to flip the LVLH reference coordinate system into the CI coordinate system.

$$[REF] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$



Aside: View Factor to Planet as a Function of Orbit and Attitude

To determine the view factor from a planar spacecraft surface to the planet, we will consider the following geometry.

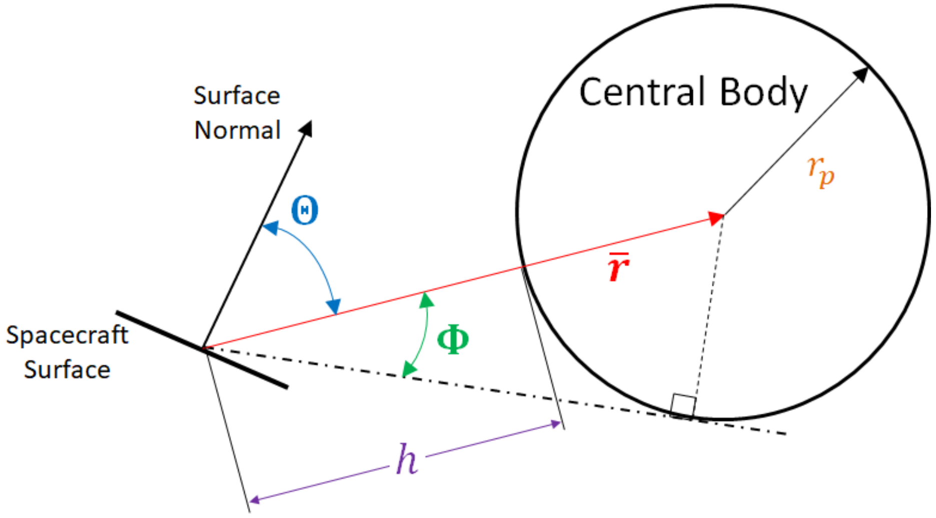
$\bar{r}$ is the vector from the spacecraft surface to the center of the central body – its magnitude is $r_p + h$

Θ is the angle between the surface normal and the vector to the center of the central body

Φ is the angle half angle subtended by the central body as seen from the planar surface

r_p is the central body radius

h is the altitude above the planet



Adapted from: http://www.thermalradiation.net/sectionb/B-43.html

Aside: View Factor to Planet as a Function of Orbit and Attitude

The view factor (VF_{planet}) from the plate to the central body (i.e., planet) is:

$$H = \frac{r_p + h}{r_p} \qquad \text{and} \qquad \Phi = \sin^{-1}(1/H)$$

For $\frac{\pi}{2} - \Phi \leq \Theta \leq \frac{\pi}{2} + \Phi$:

$$VF_{planet} = \frac{1}{2} - \frac{1}{\pi}\sin^{-1}\left[\frac{(H^2-1)^{1/2}}{H\sin\Theta}\right] + \frac{1}{\pi H^2}\left\{\cos\Theta\cos^{-1}\left[-(H^2-1)^{1/2}\cot\Theta\right] - [H^2-1]^{1/2}[1 - H^2 cos^2\Theta]^{1/2}\right\}$$

For $\Theta \leq \frac{\pi}{2} - \Phi$:

$$VF_{planet} = \frac{\cos\Theta}{H^2}$$

Adapted from: http://www.thermalradiation.net/sectionb/B-43.html
Note that the parameter definition for h, and subsequently, H presented here is different than that in the reference.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

Using h and r_p, , we can determine Φ.

To determine Θ, we will also need the orientation of the spacecraft surface with respect to the planet and to do that, we will need to perform attitude transformations to determine the direction of the surface normal of interest.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

A cubical spacecraft circular Earth orbit is oriented in a $P = -45°, Y = 0°, R = 0°$ Euler angle sequence in the local vertical/local horizontal reference frame.

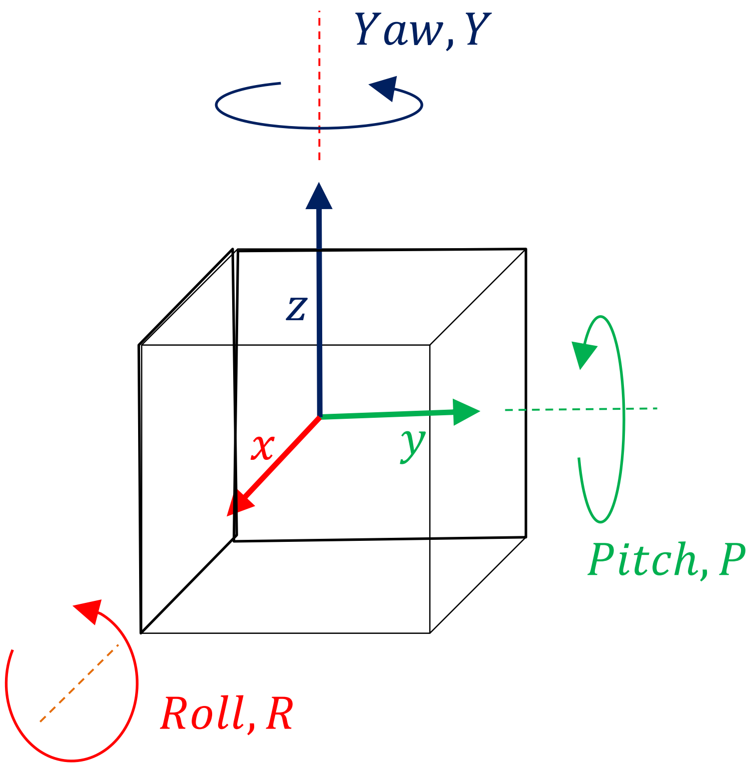
For the specified orbit and environment parameters, determine the solar, albedo and planetary heating flux on the surface pointing 45 toward the nadir from the velocity vector (i.e., ram) direction as a function of time.

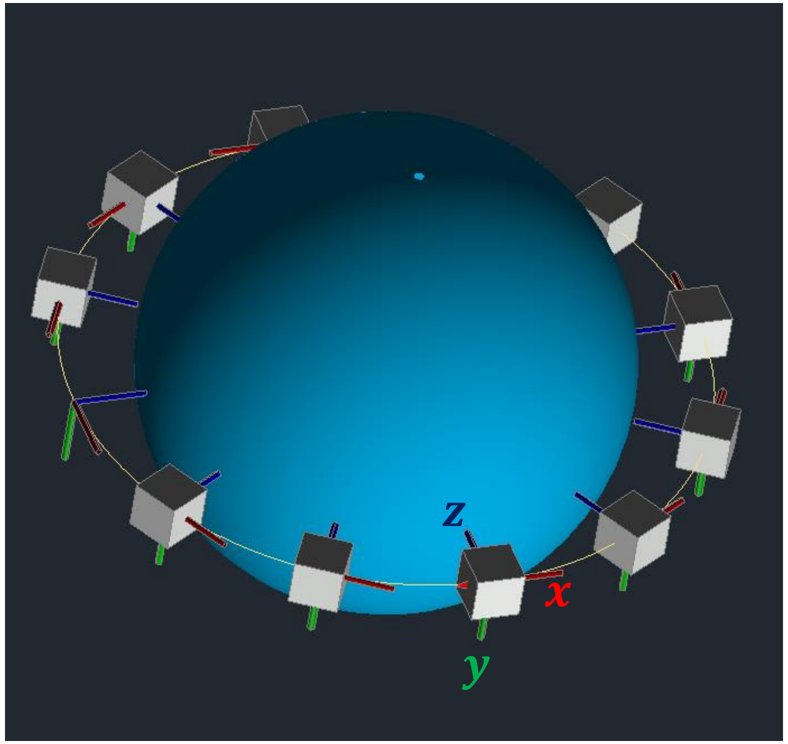
Also determine the beta angle profile over time assuming only the J_2 oblateness perturbation and calculate the percent of time spent in eclipse.

Parameter	Value
h	$500\ km$
e	0.0
i	$28.5°$
Ω	$270°$
ω	$Undefined$
Date/Time	$March\ 20, 2020\ \ 03{:}49\ UTC$
$\dot{q}_{solar}$	$1371\ \ W/m^2$
a	0.3
$\dot{q}_{OLR}$	$237\ \ W/m^2$

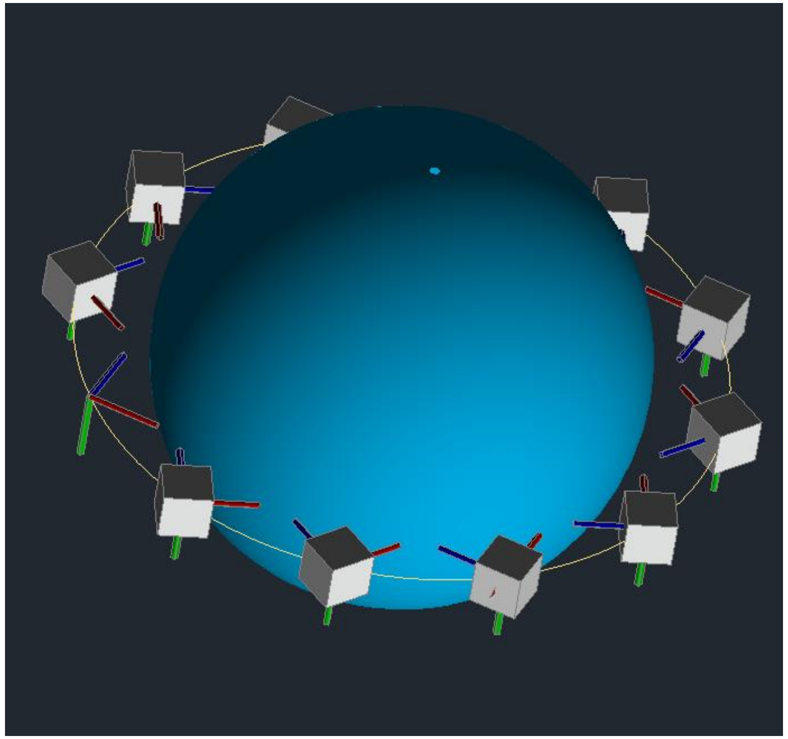
Date/Time info for 2020 vernal equinox from: https://www.timeanddate.com/calendar/march-equinox.html

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude





Box in LVLH Reference Position



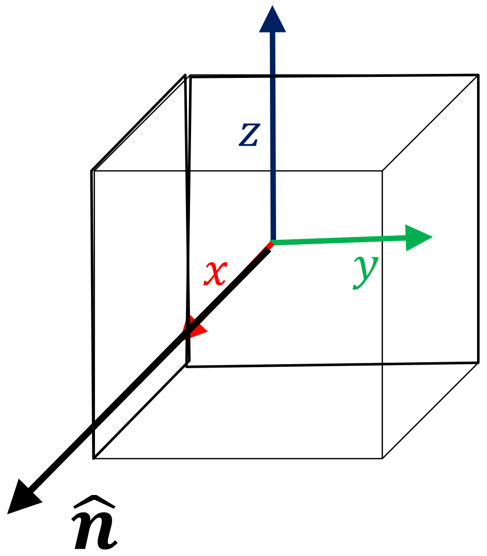
Box Pitched -45 deg

Images created using: Thermal Desktop® by Cullimore and Ring Technologies, Inc.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

We form the unit vector for the surface normal facing in the $+x$ direction in the *spacecraft body coordinate system*.

$$\{\boldsymbol{n}\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$



We will be calculating heating for this surface once it is tilted 45 degrees toward nadir.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

Next, we form the Euler angle sequence to transform the $+x$ facing unit vector through the prescribed pitch, yaw and roll formation. Executing the rotation sequence in this order requires first, a y −axis rotation, then a z-axis rotation, and finally, an x −axis rotation.

$$[P][Y][R] = \underbrace{\begin{bmatrix} \cos P & 0 & \sin P \\ 0 & 1 & 0 \\ -\sin P & 0 & \cos P \end{bmatrix}}_{\boldsymbol{y-axis}} \underbrace{\begin{bmatrix} \cos Y & -\sin Y & 0 \\ \sin Y & \cos Y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{z-axis}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos R & -\sin R \\ 0 & \sin R & \cos R \end{bmatrix}}_{\boldsymbol{x-axis}} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

We must also form the Euler angle sequence to position the spacecraft within the reference frame. Remember, we ultimately aim to express everything in the celestial inertial (CI) coordinate system so these transformations transform from LVLH to CI.

$$[\Omega][i][\omega][\nu] = \underbrace{\begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{z-axis}} \underbrace{\begin{bmatrix} \cos i & 0 & \sin i \\ 0 & 1 & 0 \\ -\sin i & 0 & \cos i \end{bmatrix}}_{\boldsymbol{y-axis}} \underbrace{\begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{z-axis}} \underbrace{\begin{bmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{z-axis}}$$

Note: For circular orbits, the argument of periapsis is undefined so a value of $\omega = 0°$ is used and the corresponding matrix $[\boldsymbol{\omega}]$ becomes the identity matrix.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

Remember, to complete the transformation from LVLH to CI coordinates, the reference change matrix must be applied.

$$[Transformed\ Vectors] = [\Omega][i][\omega][\nu]\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}[P][Y][R][Untransformed\ Vectors]$$

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

To calculate the angle between the transformed surface normal and the center of the Earth, we see that a unit vector that points from the center of the Earth to the spacecraft location is given by

$$\{\hat{\boldsymbol{r}}\} = [\Omega][i][\omega][\nu]\begin{Bmatrix}1\\0\\0\end{Bmatrix}$$

But we need a vector that points from the spacecraft to the Earth which is given by:

$$\{-\hat{\boldsymbol{r}}\} = [\Omega][i][\omega][\nu]\begin{Bmatrix}-1\\0\\0\end{Bmatrix}$$

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

To calculate the angle between a transformed surface normal and the center of the Earth (COE):

$$\cos(Angle\ to\ COE) = [\Omega][i][\omega][\nu]\begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} \cdot \{Transformed\ Surface\ Normal\}$$

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

To calculate the angle between the transformed surface normal and the sun, we can use our previously derived expression for the solar vector

$$\{\hat{\boldsymbol{s}}\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \begin{bmatrix} \cos\Gamma & -\sin\Gamma & 0 \\ \sin\Gamma & \cos\Gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \cos\Gamma \\ \sin\Gamma\cos\varepsilon \\ \sin\Gamma\sin\varepsilon \end{Bmatrix}$$

The angle between the solar vector and a transformed surface normal is:

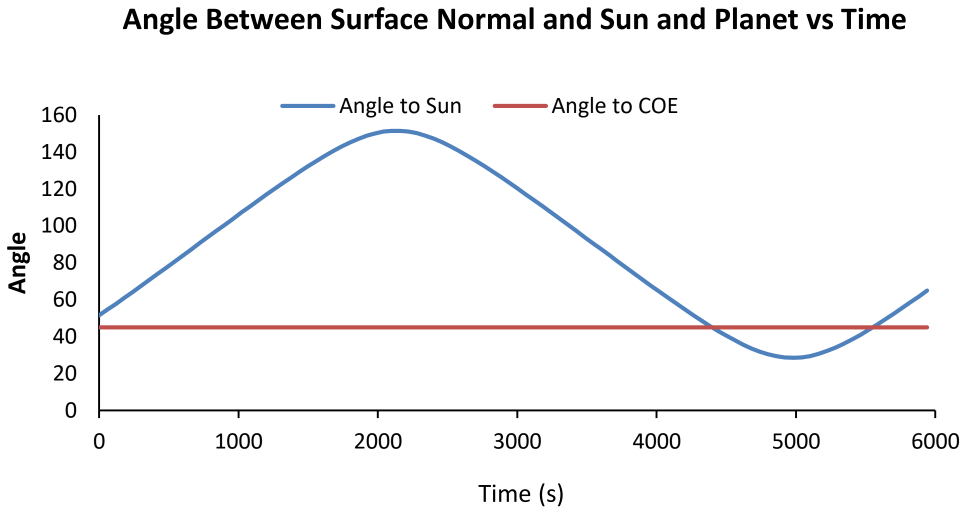
$$\cos(Angle\ to\ Sun) = \begin{Bmatrix} \cos\Gamma \\ \sin\Gamma\cos\varepsilon \\ \sin\Gamma\sin\varepsilon \end{Bmatrix} \cdot \{Transformed\ Surface\ Normal\}$$

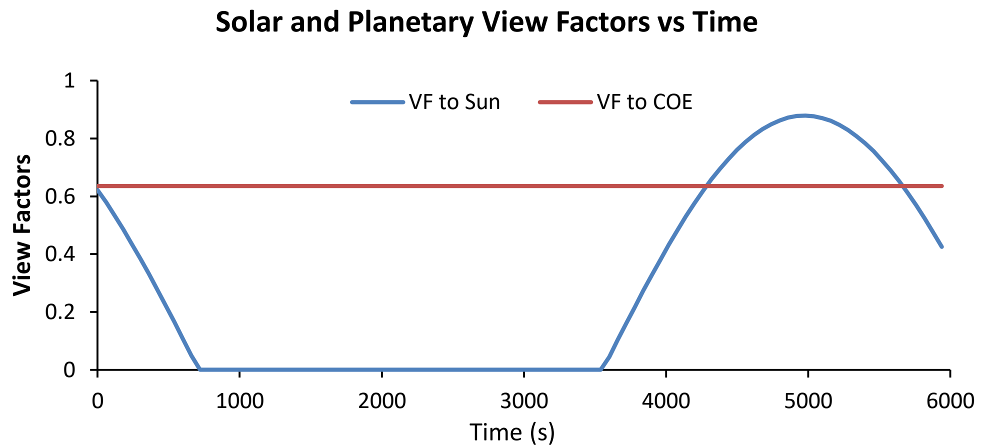
For a flat surface, the view factor to the sun, $VF_{solar} = \cos(Angle\ to\ Sun)$ when the $Angle\ to\ the\ Sun\ <\ 90°$.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

We calculate the angle from the unit normal to the center of the Earth (COE) and to the sun.

From this, we can calculate the view factor to the planet (if altitude is known) as well as the sun.





Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

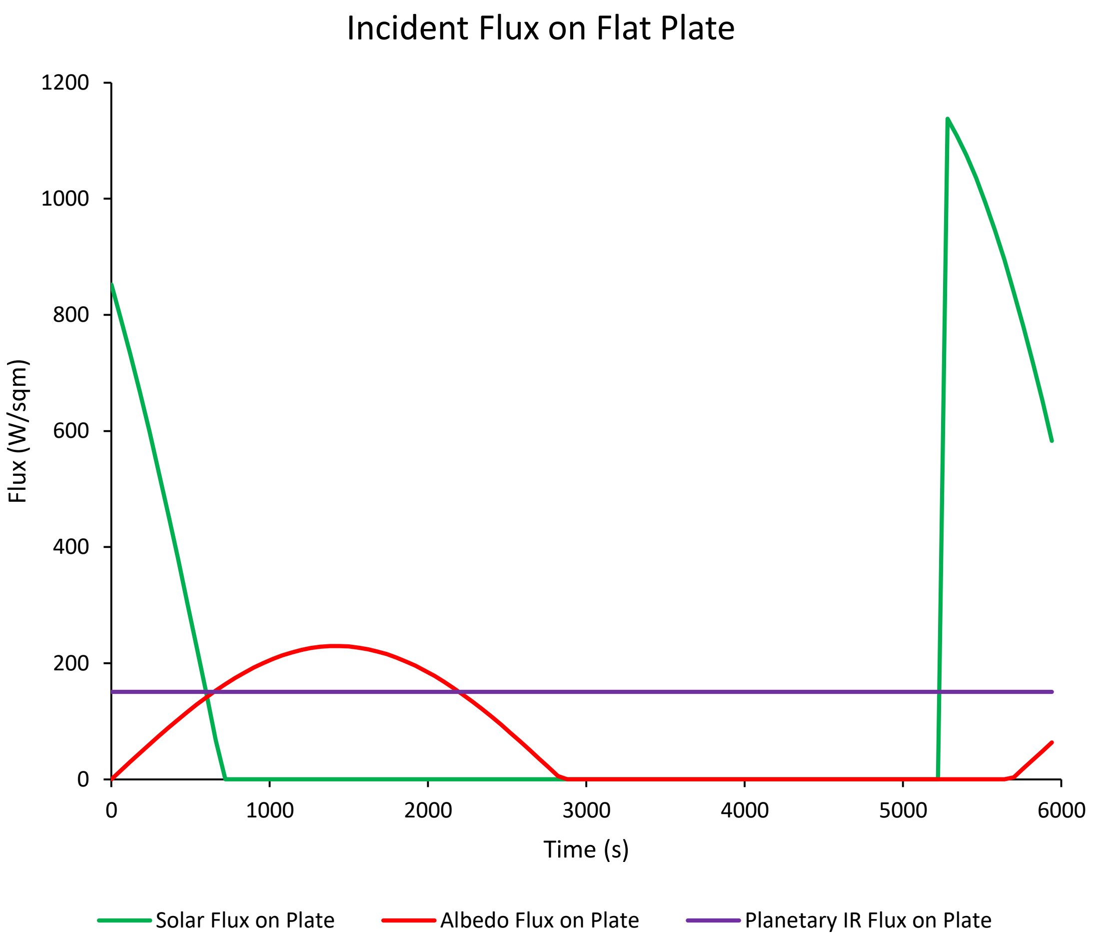
Once the view factors are known, the solar, albedo and planetary (OLR) fluxes incident on the plate are calculated using:

$$\dot{q}_{solar,node}(t) = \dot{q}_{solar} VF_{solar}(t)$$
$$\dot{q}_{albedo,node}(t) = a\dot{q}_{solar} VF_{planet}(t)\cos\big(\theta(t)\big)$$
$$\dot{q}_{OLR,node}(t) = \dot{q}_{OLR} VF_{planet}(t)$$

where…

$\theta(t) = \cos^{-1}\big(\widehat{\boldsymbol{r}(\boldsymbol{t})} \cdot \widehat{\boldsymbol{s}(\boldsymbol{t})}\big)$ is the angle between the solar vector and the vector from the center of the Earth to the spacecraft and applies only when $\cos(\theta) > 0$

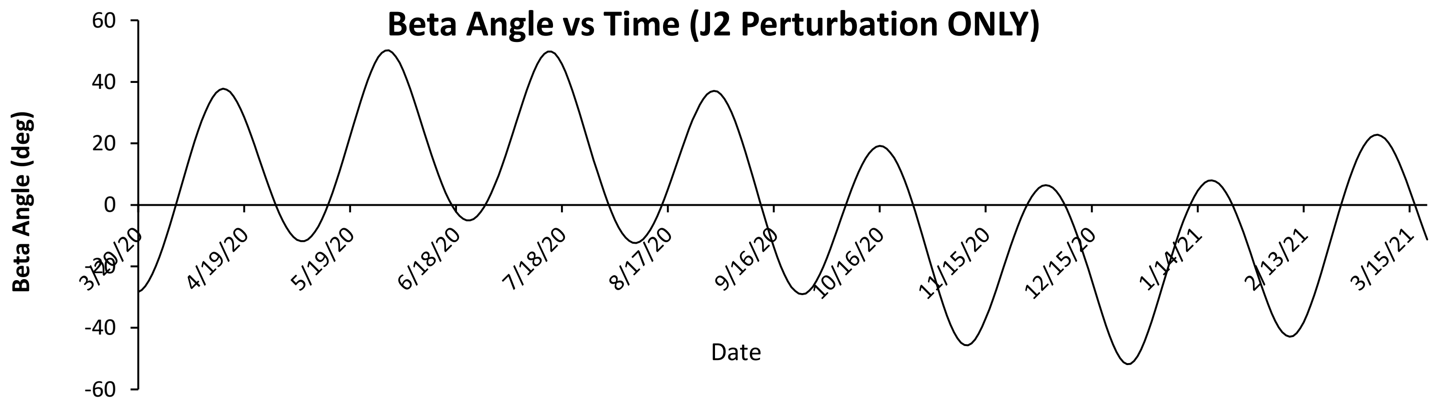
The plot also includes calculation of eclipse entry and exit.

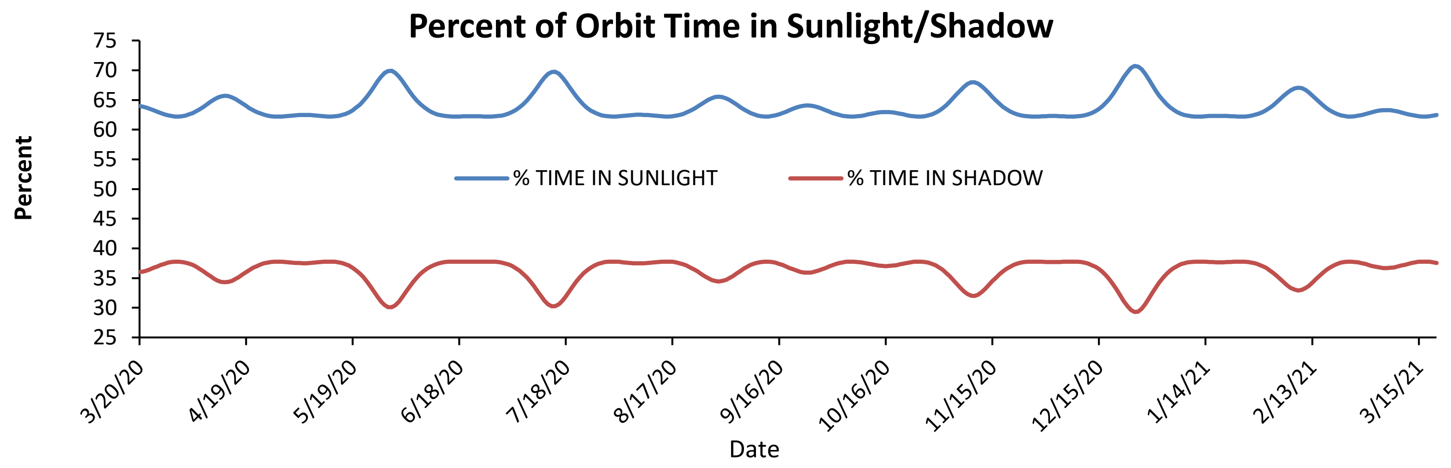


Note: The albedo model is highly simplified and is used for illustrative purposes only.

Example: Heating to Spacecraft Surfaces as a Function of Orbit and Attitude

We can calculate the progression of the angle throughout the year* as well as the time spent in sunlight/shadow.





*Note: This example is for illustrative purposes only as it considers perturbation from the J_2 term only. Other perturbations would likely change this profile.

Part 5 – Wrap-Up

In this fifth and final part of the lesson, we focused on spacecraft attitudes.

We discussed, both, the Local Vertical – Local Horizontal and Celestial Inertial reference frames and demonstrated an attitude transformation strategy.

These transformations were used to orient spacecraft surfaces with respect to the solar, albedo and planetary heating sources. View factors to the sun and planet were calculated and used to calculate heating to a spacecraft surface.

Conclusion

To fully understand on orbit thermal environments, knowledge of orbital mechanics and spacecraft attitudes is required.

An introduction to orbital mechanics with focus on the two-body problem has been presented.

Numerous examples demonstrating the effect of orbital parameters and progression on parameters of interest to thermal engineers has been demonstrated.

Spacecraft attitudes and reference frames were introduced and their effect on thermal environments experienced by orbiting spacecraft was examined.

Acknowledgements

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