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Generalized Process for Detecting Strong Convection Using an Airborne, Doppler, Weather Radar

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ABSTRACT

This document presents a generalized methodology for producing a measure of the convective wind field ahead of an aircraft using a modern, airborne, Doppler, weather radar. Building upon previous work that developed the methodology and corresponding algorithm, two new algorithms are presented. One is an extension of the original algorithm that eliminates two built-in assumptions, thereby expanding the applicability to a larger set of scenarios. The other algorithm is derived using an alternative approach which reduces complexity and eliminates intermediate steps while also increasing versatility. Relationships to the original algorithm are shown. Further, strengths and weaknesses of each algorithm are discussed and an examination of sensitivity to input parameters is provided.

1 Background

One limitation of all Doppler radar systems is that only radial velocity can be directly measured. Components of velocity orthogonal to the radar beam do not contribute to the Doppler measurement and must be estimated by other means. Airborne weather radar systems typically scan in a horizontal plane in front of the aircraft thereby producing a weather picture at the aircraft flight level along the projected flight path for a conventional Plan Position Indicator (PPI) display. This means that vertical winds (perpendicular to horizontal) cannot be estimated. However, these radar system have the capability to tilt the antenna in elevation and scan out of the horizontal plane. In this case, vertical winds will contribute to the radial velocity measurement during tilted scans. Often the applied tilt angles are small and only used to reduce ground clutter or produce a Range Height Indicator (RHI) display. However, in *Process for Detecting Strong Convection Using an Airborne, Doppler, Weather Radar*, Harrah et al. present an algorithm for estimating vertical velocity using sequences of tilted and non-tilted scans [1].

The algorithm (referred to as CONVEX) utilizes two or three scans to compute differences in radial velocity and estimate vertical velocity by accounting for differences in tilt angle between scans. The CONVEX algorithm developed in Reference 1 relies on at least one of the scans having a zero tilt angle. As will be shown, there are advantages to using scan sequences that include a non-tilted scan in which the vertical component of velocity is completely absent from the measurements. However, this also means that the algorithm must keep track of tilted versus non-tilted scans and cannot function when all scans are tilted. Additionally, when more than one scan in the sequence is tilted, the original CONVEX algorithm requires that they have the same tilt angle.

Upon further examination, these tilt sequence requirements can be eliminated and more generalized algorithms developed. The enhanced process is “generic” in the sense that any tilt angles may be used so long as sufficient changes in tilt are applied between scans. Analysis shows the significance of tilt changes to the robustness of the algorithms. Thus, a measure of sensitivity to tilt differences is made. It is also shown that the generalized CONVEX algorithm is mathematically equivalent to the original CONVEX algorithm when the original tilt sequences are used.

Two distinct mathematical approaches for generalizing the process are possible. The first, described in Section 2: Generalized Algorithm, follows the same derivation path used in Reference 1 to develop the CONVEX algorithm without prescribing tilt angles for any sweep. This results in an algorithm with similar 2-bar and 3-bar sequences. The second, described in Section 3: Generalized Matrix Algorithm, uses linear algebra to solve a system of linear equations. The resulting algorithm is equivalent for the 2-bar sequence and more direct for the 3-bar sequence.

2 Generalized Algorithm

To derive the CONVEX algorithm, a cylindrical coordinate system is defined in which the wind vector is represented by:

$$\vec{V} = V_\rho \hat{\rho} + V_\varphi \hat{\varphi} + V_V \hat{z}$$

and the radar look vector defined as:

$$\vec{R}(\theta) = \cos\theta\, \hat{\rho} + \sin\theta\, \hat{z}$$

where $\hat{\rho}$ is the radial unit vector in the horizontal plane and $\hat{z}$ is the vertical unit vector, and θ is the tilt (elevation) angle. The measured radial velocity is the dot product of these two vectors:

$$V_R(\theta) = \vec{V} \bullet \vec{R} = V_\rho \cos\theta + V_V \sin\theta$$

where V_ρ is the measurable component of the horizontal velocity and V_V is the vertical velocity.

Taking platform translation into account, V_ρ becomes:

$$V_\rho = V_H \cos(\alpha_T - \varphi)$$

where: V_H is the horizontal velocity, α_T is angle between wind vector and the longitudinal axis of the platform, and φ is the antenna azimuth angle. Because α_T does not change with platform translation[1] but φ does, V_ρ will change from one sweep to the next. The 2-bar sequence does not account for this change and considers V_ρ to be constant. This error is minimized when the change in φ is sufficiently small. Otherwise, the 3-bar sequence must be used to solve for α_T and V_H.

2.1 Two-Bar Sequence

As a shorthand notation, let $V_i = V_R(\theta_i)$ represent the radial velocity measurement for a given point in space taken during the i^th sweep. Starting with:

$$V_i = V_\rho \cos\theta_i + V_V \sin\theta_i$$

which contains two unknowns: V_ρ and V_V. If it is assumed that these remain constant from one sweep to

---

[1] α_T does change with platform rotation (heading changes), but this is easily accounted for as shown in Reference 1 and does not bear repeating.

the next, two sweeps provides a system of two linear equations.

$$V_{i-1} = V_\rho \cos\theta_{i-1} + V_V \sin\theta_{i-1}$$

Solving by substitution and utilizing the trigonometric identity [2]:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

yields:

$$V_V = \frac{V_i \cos\theta_{i-1} - V_{i-1}\cos\theta_i}{\sin(\theta_i - \theta_{i-1})}$$

Notice that if the ith sweep is tilted and the i-1 sweep is not ($\theta_{i-1} = 0^0$), the generalized 2-bar sequence equation reduces to the CONVEX algorithm 2-bar equation presented in Reference 1. Also notice that if the two sweeps have the same tilt angle ($\theta_i = \theta_{i-1}$), the denominator equals zero and it is impossible to solve for V_V.

Because V_ρ is not constant from one sweep to the next, the error in this vertical velocity estimate is proportional to the change in V_ρ. However, for measurements directly in front of the aircraft where the azimuth angle is close to zero ($\varphi \approx 0^0$), platform translation results in very little change to V_ρ between sweeps. Similarly, at the beginning of a new sweep too little time has elapsed since the previous sweep for much platform translation to occur resulting in negligible changes in azimuth angle ($\varphi_i \approx \varphi_{i-1}$) and consequently V_ρ. For these two special cases the 2-bar sequence provides a good vertical velocity estimate. As will be shown, these two special cases result in a singularity in the 3-bar sequence equations. The 2-bar and 3-bar sequences are therefore complementary and the algorithm uses both for maximum robustness.

2.2 Three-Bar Sequence

The 3-bar sequence does not assume V_ρ is constant and uses the system of three linear equations:

$$\begin{aligned} V_i &= V_H \cos(\alpha_T - \varphi_i)\cos\theta_i + V_V \sin\theta_i \\ V_{i-1} &= V_H \cos(\alpha_T - \varphi_{i-1})\cos\theta_{i-1} + V_V \sin\theta_{i-1} \\ V_{i-2} &= V_H \cos(\alpha_T - \varphi_{i-2})\cos\theta_{i-2} + V_V \sin\theta_{i-2} \end{aligned}$$

to solve for the three unknowns: V_H, α_T, and V_V.

Solving for V_V for the ith sweep yields:

$$V_V = \frac{V_i - V_H \cos(\alpha_T - \varphi_i)\cos\theta_i}{\sin\theta_i}$$

Substituting this into the other two equations and solving for V_H yields:

$$V_H = \frac{V_{i-1}\sin\theta_i - V_i \sin\theta_{i-1}}{\cos(\alpha_T - \varphi_{i-1})\cos\theta_{i-1}\sin\theta_i - \cos(\alpha_T - \varphi_i)\cos\theta_i \sin\theta_{i-1}}$$

and

$$V_H = \frac{V_{i-2}\sin\theta_i - V_i \sin\theta_{i-2}}{\cos(\alpha_T - \varphi_{i-2})\cos\theta_{i-2}\sin\theta_i - \cos(\alpha_T - \varphi_i)\cos\theta_i \sin\theta_{i-2}}$$

By setting these two equations equal to each other, the unknown quantity V_H is removed resulting in one equation with one unknown (α_T). Collecting like terms and utilizing the trigonometric identities [2]:

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\tan A = \frac{\sin A}{\cos A}$$

results in:

$$\tan \alpha_T = -\frac{C_0 \cos \varphi_i - C_1 \cos \varphi_{i-1} - C_2 \cos \varphi_{i-2}}{C_0 \sin \varphi_i - C_1 \sin \varphi_{i-1} - C_2 \sin \varphi_{i-2}}$$

where:

$$C_0 = (V_{i-2} \sin \theta_{i-1} - V_{i-1} \sin \theta_{i-2}) \cos \theta_i$$
$$C_1 = (V_{i-2} \sin \theta_i - V_i \sin \theta_{i-2}) \cos \theta_{i-1}$$
$$C_2 = (V_i \sin \theta_{i-1} - V_{i-1} \sin \theta_i) \cos \theta_{i-2}$$

which can be solved for α_T (the angle between the longitudinal axis of the platform and the horizontal wind vector).

Using α_T and either equation above, V_H may be computed depending on the scan sequence. The most sensible process would use the latest value since it should be from a shorter range and therefore a higher SNR than an older value. Notice that if the scan sequence includes two sweeps at zero tilt angle, then one of the equations will be unsolvable for V_H and the other must be used.

Using the values for α_T and V_H, the vertical velocity can be estimated using the original V_V expression above for any scan with non-zero tilt (i.e. $\theta \neq 0^0$).

2.2.1 *Comparison to the 0^0, θ, and 0^0 sequence*

The previous section describes the use of three scans to calculate the wind vector (V_H, α_T, V_V); however, these scans can have any elevation angle as long as at least one is tilted relative to the other two. If an alternating scan sequence is used where the elevation angle alternates between zero and non-zero for each sweep, then the solution is simplified. A three-sweep sequence will then have either elevation sequence 0^0, θ, 0^0 or elevation sequence θ, 0^0, θ where θ is a non-horizontal (tilted) scan[2]. For a comparison to the original CONVEX algorithm, let $\theta_i = \theta_{i-2} = 0^0$, and $\theta_{i-1} = \theta$. Then:

$C_0 = V_{i-2}\sin(\theta)$ $\quad$ $C_1 = 0$ $\quad$ $C_2 = V_i\sin(\theta)$

Since $\sin(\theta)$ appears in all terms in the numerator and denominator, it cancels out and:

$$\tan \alpha_T = -\frac{V_i \cos \varphi_{i-2} - V_{i-2} \cos \varphi_i}{V_i \sin \varphi_{i-2} - V_{i-2} \sin \varphi_i}$$

which is equivalent to the 0^0, θ, and 0^0 scan sequence solution in Reference 1.

2.2.2 *Comparison to the θ, 0^0, and θ sequence*

For the θ, 0^0, θ elevation sequence, the equations used to calculate the wind vector components differ slightly from those used for the other 3-scan sequence and from those used for the two-horizontal scan sequence developed in the previous sections. For a comparison to the CONVEX alternate scan sequence

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[2] θ has the same value for all non-horizontal scans to help simplify the expressions.

presented in Reference 1, let $\theta_{i-1} = 0^0$, and $\theta_i = \theta_{i-2} = \theta$. Then:

$C_0 = -V_{i-1}\sin(\theta)\cos(\theta)$ $C_1 = V_{i-2}\sin(\theta) - V_i\sin(\theta)$ $C_2 = -V_{i-1}\sin(\theta)\cos(\theta)$

Since $\sin(\theta)$ appears in all terms in the numerator and denomination, it cancels out and:

$$\tan\alpha_T = -\frac{(V_{i-2} - V_i)\cos\varphi_{i-1} - V_{i-1}\cos\theta\cos\varphi_{i-2} + V_{i-1}\cos\theta\cos\varphi_i}{(V_{i-2} - V_i)\sin\varphi_{i-1} - V_{i-1}\cos\theta\sin\varphi_{i-2} + V_{i-1}\cos\theta\sin\varphi_i}$$

Dividing the numerator and denominator by $-V_{i-1}\cos(\theta)$ results in:

$$\tan\alpha_T = -\frac{\frac{V_i - V_{i-2}}{V_{i-1}\cos\theta}\cos\varphi_{i-1} + \cos\varphi_{i-2} - \cos\varphi_i}{\frac{V_i - V_{i-2}}{V_{i-1}\cos\theta}\sin\varphi_{i-1} + \sin\varphi_{i-2} - \sin\varphi_i}$$

which is equivalent to the θ, 0^0, and θ scan sequence solution in Reference 1.

2.3 Assessment

Notice that if none of the sweeps are tilted ($\theta_i = \theta_{i-1} = \theta_{i-2} = 0^0$), then $C_0 = C_1 = C_2 = 0$ making it impossible to solve for α_T. Therefore, at least one sweep must have a non-zero tilt to solve for V_V as expected.

Likewise, if all of the sweeps have the same tilt angle ($\theta_i = \theta_{i-1} = \theta_{i-2}$), then:

$$V_H = \frac{V_{i-1} - V_i}{\cos(\alpha_T - \varphi_{i-1})\cos\theta - \cos(\alpha_T - \varphi_i)\cos\theta} = \frac{V_{i-2} - V_i}{\cos(\alpha_T - \varphi_{i-2})\cos\theta - \cos(\alpha_T - \varphi_i)\cos\theta}$$

providing insufficient information to solve for α_T. Therefore, at least one sweep must have a tilt angle different from the other two to solve for V_V.

Directly in front of the radar ($\varphi_i = \varphi_{i-1} = \varphi_{i-2} = 0^0$) the denominator of the $\tan(\alpha_T)$ equation equals zero making it impossible to solve for α_T. Therefore, it is not possible to compute V_V directly in front of the radar using the 3-bar sequence. In this case, the 2-bar sequence provides a viable solution as mentioned in Section 2.1.

For situations when the azimuth angle does not change significantly between sweeps ($\varphi_i \approx \varphi_{i-1} \approx \varphi_{i-2}$), such as nearly in front of the radar or when platform speed is sufficiently slow that little translation occurs over sweep intervals, then:

$$\tan\alpha_T \approx -\frac{\cos\varphi_i}{\sin\varphi_i}$$

which given the trigonometric identity [2]:

$$\tan A = \frac{\sin A}{\cos A}$$

it follows that:

$$\cos(\alpha_T - \varphi_i) \approx 0$$

making it impossible to compute V_V using the 3-bar sequence. In this case, the 2-bar sequence provides a viable solution as mentioned in Section 2.1.

Near the beginning of a new sweep, the two most recent measurements of a given point in space have occurred in quick succession. With little time for the platform to have moved between the measurements, $\varphi_i \approx \varphi_{i-1}$. Using the most recent sweep to compute V_H as prescribed, results in:

$$V_H = \frac{V_{i-1}\sin\theta_i - V_i\sin\theta_{i-1}}{\cos(\alpha_T - \varphi_i)\sin(\theta_i - \theta_{i-1})}$$

which cannot be solved unless the two most recent sweeps have different tilt angles.

2.3.1 Sign Ambiguity

Given the physical limitations of the radar antenna, it is safe to assume that azimuth angles are always acute ($-90^0 < \varphi < 90^0$). Because sin(-A) = -sin(A) and cos(-A) = cos(A) and the equation for tan(α_T) above contains only cosine terms in the numerator and sine terms in the denominator, negative azimuth angles ($\varphi < 0^0$) result in an ambiguity for the solution to α_T. A quadrant-aware inverse tangent function (e.g. atan2) will return an angle 180^0 away from the true angle. Consequently, the solution to the horizontal component of velocity (V_H) will have a negative value.

For straight platform translation, the azimuth angles for a point in space for all three sweeps will all have the same sign. Therefore the solution to α_T may be adjusted according to the sign of the known azimuth angles. However, if the platform changes direction, the azimuth angles for a given point in space can have different signs between sweeps even after accounting for the heading change.

The original CONVEX algorithm addressed this by using the absolute value ($|V_H|$) in the final computation for vertical velocity. Alternatively, Section 3 describes an approach which eliminates this ambiguity.

3 Generalized Matrix Algorithm

The algorithm derived in Section 2 was developed by solving a system of linear equations by substitution. As a result, it requires intermediate steps to solve for α_T and V_H which introduces sign ambiguities. Linear algebra provides an alternative means for solving such systems by constructing a matrix equation.

3.1 Two-Bar Matrix Sequence

Starting from the same system of two linear equations as in Section 2.1 and constructing a matrix equation results in:

$$\begin{bmatrix} V_i \\ V_{i-1} \end{bmatrix} = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ \cos\theta_{i-1} & \sin\theta_{i-1} \end{bmatrix} \begin{bmatrix} V_\rho \\ V_V \end{bmatrix}$$

Applying Cramer's Rule yields:

$$V_V = \frac{\begin{vmatrix} \cos\theta_i & V_i \\ \cos\theta_{i-1} & V_{i-1} \end{vmatrix}}{\begin{vmatrix} \cos\theta_i & \sin\theta_i \\ \cos\theta_{i-1} & \sin\theta_{i-1} \end{vmatrix}}$$

Completing the determinates and utilizing the trigonometric identities [2]:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\sin(-A) = -\sin(A)$$

yields:

$$V_V = \frac{V_i \cos\theta_{i-1} - V_{i-1}\cos\theta_i}{\sin(\theta_i - \theta_{i-1})}$$

which is equivalent to the 2-bar equation derived by substitution in Section 2.1.

3.2 Three-Bar Matrix Sequence

Starting from the same system of three linear equations as in Section 2.2 and using the trigonometric identity:

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

to expand the cosine expression yields:

$$V_i = V_H \cos\alpha_T \cos\varphi_i \cos\theta_i + V_H \sin\alpha_T \sin\varphi_i \cos\theta_i + V_V \sin\theta_i$$
$$V_{i-1} = V_H \cos\alpha_T \cos\varphi_{i-1} \cos\theta_{i-1} + V_H \sin\alpha_T \sin\varphi_{i-1} \cos\theta_{i-1} + V_V \sin\theta_{i-1}$$
$$V_{i-2} = V_H \cos\alpha_T \cos\varphi_{i-2} \cos\theta_{i-2} + V_H \sin\alpha_T \sin\varphi_{i-2} \cos\theta_i + V_V \sin\theta_{i-2}$$

Constructing a matrix equation results in:

$$\begin{bmatrix} V_i \\ V_{i-1} \\ V_{i-2} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} V_H \cos\alpha_T \\ V_H \sin\alpha_T \\ V_V \end{bmatrix}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} \cos\varphi_i \cos\theta_i & \sin\varphi_i \cos\theta_i & \sin\theta_i \\ \cos\varphi_{i-1} \cos\theta_{i-1} & \sin\varphi_{i-1} \cos\theta_{i-1} & \sin\theta_{i-1} \\ \cos\varphi_{i-2} \cos\theta_{i-2} & \sin\varphi_{i-2} \cos\theta_{i-2} & \sin\theta_{i-2} \end{bmatrix}$$

Applying Cramer’s rule yields:

$$V_V = \frac{\det(\boldsymbol{A_3})}{\det(\boldsymbol{A})}$$

where:

$$\boldsymbol{A_3} = \begin{bmatrix} \cos\varphi_i \cos\theta_i & \sin\varphi_i \cos\theta_i & V_i \\ \cos\varphi_{i-1} \cos\theta_{i-1} & \sin\varphi_{i-1} \cos\theta_{i-1} & V_{i-1} \\ \cos\varphi_{i-2} \cos\theta_{i-2} & \sin\varphi_{i-2} \cos\theta_{i-2} & V_{i-2} \end{bmatrix}$$

Completing the determinates, collecting like terms, and utilizing the trigonometric identities [2]:

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\sin(-A) = -\sin(A)$$

yields:

$$\det(\boldsymbol{A_3}) = V_i \cos\theta_{i-1} \cos\theta_{i-2} \sin(\varphi_{i-2} - \varphi_{i-1}) + V_{i-1} \cos\theta_i \cos\theta_{i-2} \sin(\varphi_i - \varphi_{i-2}) + V_{i-2} \cos\theta_i \cos\theta_{i-1} \sin(\varphi_{i-1} - \varphi_i)$$

and:

$$\det(\boldsymbol{A}) = \sin\theta_i \cos\theta_{i-1} \cos\theta_{i-2} \sin(\varphi_{i-2} - \varphi_{i-1}) + \cos\theta_i \sin\theta_{i-1} \cos\theta_{i-2} \sin(\varphi_i - \varphi_{i-2}) + \cos\theta_i \cos\theta_{i-1} \sin\theta_{i-2} \sin(\varphi_{i-1} - \varphi_i)$$

Thus vertical velocity may be estimated directly from the radial velocity measurements and known antenna angles with no intermediate steps required to estimate the speed and direction of the horizontal wind component (α_T and V_H).

For convenience, the solution may be rewritten as:

$$V_V = \frac{K_0 V_i + K_1 V_{i-1} + K_2 V_{i-2}}{K_0 \sin\theta_i + K_1 \sin\theta_{i-1} + K_2 \sin\theta_{i-2}}$$

where:

$$K_0 = \cos\theta_{i-1} \cos\theta_{i-2} \sin(\varphi_{i-2} - \varphi_{i-1})$$
$$K_1 = \cos\theta_i \cos\theta_{i-2} \sin(\varphi_i - \varphi_{i-2})$$
$$K_2 = \cos\theta_i \cos\theta_{i-1} \sin(\varphi_{i-1} - \varphi_i)$$

3.2.1 *Comparison to the 0^0, θ, and 0^0 sequence*

Unlike the 2-bar sequence, it is not immediately apparent that the linear algebra solution to the 3-bar sequence is equivalent to the solution found by substitution. However the special case alternating 0^0, θ, 0^0 elevation sequence can be evaluated for a comparison to the original CONVEX algorithm. Let $\theta_i = \theta_{i-2} = 0^0$, and $\theta_{i-1} = \theta$. Then:

$$\det(\boldsymbol{A_3}) = V_i \cos\theta \sin(\varphi_{i-2} - \varphi_{i-1}) + V_{i-1} \sin(\varphi_i - \varphi_{i-2}) + V_{i-2} \cos\theta \sin(\varphi_{i-1} - \varphi_i)$$

and

$$\det(\boldsymbol{A}) = \sin\theta \sin(\varphi_i - \varphi_{i-2})$$

Recalling: $V_i = V_H \cos(\alpha_T - \varphi_i) \cos\theta_i + V_V \sin\theta_i$

simplifying using the prescribed tilt angles:

$$V_i = V_H \cos(\alpha_T - \varphi_i)$$
$$V_{i-2} = V_H \cos(\alpha_T - \varphi_{i-2})$$

and substituting back into det(**A**$_3$):

$$\det(\boldsymbol{A_3}) = V_H \cos(\alpha_T - \varphi_i) \cos\theta \sin(\varphi_{i-2} - \varphi_{i-1}) + V_{i-1} \sin(\varphi_i - \varphi_{i-2}) + V_H \cos(\alpha_T - \varphi_{i-2}) \cos\theta \sin(\varphi_{i-1} - \varphi_i)$$

Expanding the sine and cosine expressions, combining like terms, and applying trigonometric identities results in:

$$\det(\boldsymbol{A_3}) = V_{i-1} \sin(\varphi_i - \varphi_{i-2}) - V_H \cos(\alpha_T - \varphi_{i-1}) \cos\theta \sin(\varphi_i - \varphi_{i-2})$$

Finally:

$$V_V = \frac{\det(\boldsymbol{A_3})}{\det(\boldsymbol{A})} = \frac{V_{i-1} - V_H \cos(\alpha_T - \varphi_{i-1}) \cos\theta}{\sin\theta}$$

which is equivalent to the 0^0, θ, and 0^0 scan sequence solution in Reference 1.

3.3 Assessment

Notice that if none of the sweeps are tilted ($\theta_i = \theta_{i-1} = \theta_{i-2} = 0^0$), then det(**A**) = 0 and the matrix **A** is singular. Therefore, at least one sweep must have a non-zero tilt to solve for V_V as expected.

Likewise, if all of the sweeps have the same tilt angle ($\theta_i = \theta_{i-1} = \theta_{i-2}$), then det(**A**) ≈ 0 for all realistic tilt and scan angles[3] and the matrix **A** is approximately singular. Therefore, at least one sweep must have a tilt angle different from the other two to solve for V_V.

Directly in front of the radar ($\varphi_i = \varphi_{i-1} = \varphi_{i-2} = 0^0$) then det(**A**) = 0 and the matrix **A** is singular. Therefore, it is not possible to compute V_V directly in front of the radar using the 3-bar sequence. In this case, the 2-bar sequence provides a viable solution as mentioned in Section 2.1.

For situations when the azimuth angle does not change significantly between sweeps ($\varphi_i \approx \varphi_{i-1} \approx \varphi_{i-2}$), such as nearly in front of the radar or when platform speed is sufficiently slow that little translation occurs over sweep intervals, then det(**A**) ≈ 0 and the matrix **A** is singular. Therefore, the radar platform must be moving to solve for V_V using the 3-bar sequence as expected. As the speed of the platform approaches zero, the angles φ become closer together and consequently det(**A**) approaches zero. In this case, the 2-bar sequence provides a viable solution as mentioned in Section 2.1.

Near the beginning of a new sweep, the two most recent measurements of a given point in space have occurred in quick succession. With little time for the platform to have moved between the measurements, $\varphi_i \approx \varphi_{i-1}$. As a result:

$$V_V \approx \frac{V_i \cos\theta_{i-1} - V_{i-1} \cos\theta_i}{\sin(\theta_i - \theta_{i-1})}$$

Notice that the computation is completely independent of the third (oldest) sweep and equivalent to the computation for the 2-Bar process. Also, if the two most recent sweeps have the same tilt, $\theta_i = \theta_{i-1}$ causing the denominator to become zero, and making it impossible to solve for V_V. Likewise, near the end of a sweep, the two previous measurements of a given point in space happened in quick succession. With $\varphi_{i-1} \approx \varphi_{i-2}$ the result is:

$$V_V \approx \frac{V_{i-1} \cos\theta_{i-2} - V_{i-2} \cos\theta_{i-1}}{\sin(\theta_{i-1} - \theta_{i-2})}$$

Which is completely independent of the current measurement. Again, if the previous two sweeps had the same tilt, $\theta_{i-1} = \theta_{i-2}$ causing the denominator to become zero, and making it impossible to solve for V_V. Therefore, changing the tilt angle between every sweep is necessary for a solution near the azimuth angle extents.

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[3] See Section 3.3.1 for a more thourough analysis.

3.3.1 Sensitivity Analysis

All forms of the vertical velocity calculation contain the radial velocity measurements in the numerator and trigonometric functions in the denominator providing a means to gauge the sensitivity of the result to uncertainties in the inputs. Errors in the vertical velocity estimate are proportional to radial velocity measurement uncertainty and dominated by the sine of the change in tilt:

$$\varepsilon V_V \propto \frac{\sigma_V}{\sin(\Delta\theta)}$$

where: εV_V represents the error in the vertical velocity estimate, σ_V is the uncertainty in the radial velocity measurements, and $\Delta\theta$ is the change in tilt angle between sweeps. This is readily apparent from the 2-bar solution or any of the special cases resulting in a simplified 3-bar solution. For all reasonable changes in tilt angle (i.e. $\Delta\theta \leq 30^0$), the denominator is never larger than one-half, meaning errors in the vertical velocity estimate are always at least twice the uncertainty in the radial velocity measurements. As the change in tilt angle is reduced, the errors in the result increase exponentially. For tilt angle changes on the order of a beamwidth (e.g. $\Delta\theta \approx 5^0$), the resulting errors are ten times the measurement uncertainty.

For the generalized 3-bar solution, this analysis is complicated by the fact that the denominator is a function of six independent variables: $\det(\mathbf{A}) = f(\theta_i,\theta_{i-1},\theta_{i-2},\varphi_i,\varphi_{i-1},\varphi_{i-2})$. It is convenient to reexamine the special case where all of the sweeps have the same tilt angle ($\theta_i = \theta_{i-1} = \theta_{i-2} = \theta$) to gain an understanding of the limits. In this case:

$$det(\boldsymbol{A}) = f(\theta)g(\varphi)$$

where:

$$f(\theta) = \cos^2\theta \sin\theta$$

$$g(\varphi) = \sin(\varphi_{i-2} - \varphi_{i-1}) + \sin(\varphi_i - \varphi_{i-2}) + \sin(\varphi_{i-1} - \varphi_i)$$

The maximum of $f(\theta)$ is found by taking the first derivative and using the trigonometric identity:

$$\sin^2 A + \cos^2 A = 1$$

to solve for the critical points:

$$\frac{df}{d\theta} = (3\cos^2\theta - 2)\cos\theta = 0$$

resulting in:

$$\cos\theta = \pm\sqrt{\frac{2}{3}}$$

it follows that the maximum[4] value is:

$$f(\theta) = \pm\frac{2\sqrt{3}}{9} at\ \theta \cong \pm 35.26°$$

as illustrated in Figure 1.

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[4] The minimum value of $f(\theta) = 0$ at $\theta = 0^0, 90^0, \ldots$

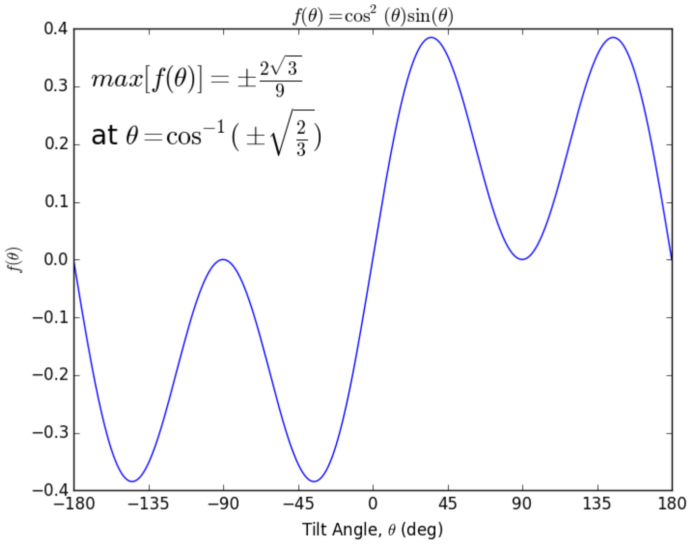

Figure 1: Sensitivity of det(A) to Tilt Angle When All Sweeps Have the Same Tilt

Likewise, the maximum value of g(φ) is found by rewriting it as:

$$g(\alpha, \beta) = \sin\alpha - \sin(\alpha + \beta) + \sin\beta$$

where:

$$\alpha = \varphi_{i-2} - \varphi_{i-1}$$
$$\beta = \varphi_{i-1} - \varphi_i$$

and taking the partial derivatives with respect to α and β to solve for the critical points:

$$\frac{\partial g}{\partial \alpha} = \cos\alpha - \cos(\alpha + \beta) = 0$$
$$\frac{\partial g}{\partial \beta} = \cos\beta - \cos(\alpha + \beta) = 0$$

resulting in the maximum[5] value:

$$g(\varphi) = \pm\frac{3\sqrt{3}}{2}\ at\ \alpha = \beta = \pm 120°$$

as illustrated in Figure 2.

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[5] The minimum value of g(α,β) = 0 at (α = 0^0, β = any), (α = any, β = 0^0), and α = -β.

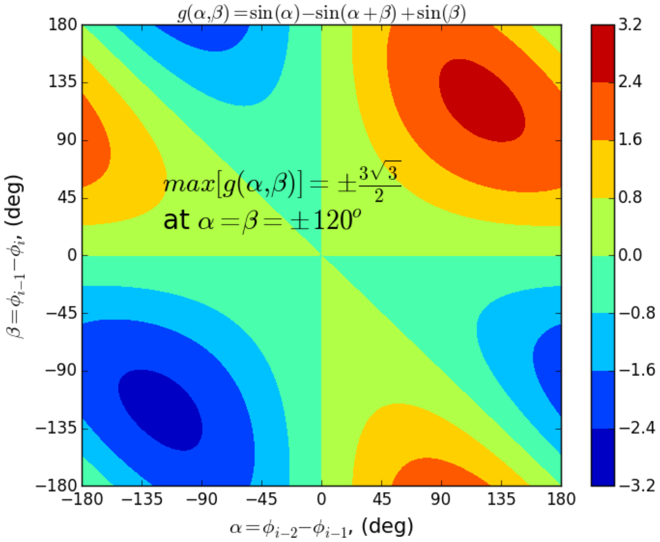

Figure 2: Sensitivity of det(A) to Azimuth Angle When All Sweeps Have the Same Tilt

Finally, in this case, the maximum value of det(**A**) is found:

$$\det(\boldsymbol{A}) = \left(\pm\frac{2\sqrt{3}}{9}\right)\left(\pm\frac{3\sqrt{3}}{2}\right) = \pm 1$$

Since the denominator is never greater than one, the error in the vertical velocity estimate will always be greater than the uncertainty in the radial velocity measurements. Further, for all realistic applications of a forward looking airborne weather radar, α + β cannot exceed the antenna azimuth angle limit (i.e. $\alpha + \beta \leq 90^0$ and often $\alpha + \beta \leq 60^0$). Therefore, for most practical uses the maximum is:

$$\det(\boldsymbol{A}) = \left(\frac{2\sqrt{3}-3}{9}\right) \cong 0.05 \text{ at } \theta \cong 35.26° \; and \; \alpha = \beta = 30°$$

which would multiply velocity measurement uncertainty by a factor of twenty; confirming the assertion in Section 3.3 that det(**A**) ≈ 0 when all of the sweeps have the same tilt angle ($\theta_i = \theta_{i-1} = \theta_{i-2}$)[6].

Since the maximum of g(φ) occurs when α = β, revisiting det(**A**) for this best case scenario yields:

$$\det(\boldsymbol{A}) = \sin\theta_i \cos\theta_{i-1} \cos\theta_{i-2} \sin(\alpha) + \cos\theta_i \sin\theta_{i-1} \cos\theta_{i-2} \sin(-2\alpha) \\ + \cos\theta_i \cos\theta_{i-1} \sin\theta_{i-2} \sin(\alpha)$$

The critical points to maximize det(**A**) define the optimum tilt angle sequence, and are found by taking the partial derivatives:

$$\frac{\partial det(\boldsymbol{A})}{\partial\theta_i} = 0 \qquad \frac{\partial det(\boldsymbol{A})}{\partial\theta_{i-1}} = 0 \qquad \frac{\partial det(\boldsymbol{A})}{\partial\theta_{i-2}} = 0$$

which results in elevation sequences of 90^0, 0^0, 90^0 and 0^0, 90^0, 0^0. While a 90^0 tilt angle is unrealistic, the sequencing suggests that the optimum tilt sequence alternates between opposite limits between each sweep. This is consistent with the conclusions for the 2-bar sequence and the 3-bar sequence near the azimuth angle

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[6] Especially since the tilt angle cannot exceed the antenna elevation limit (i.e. often $|\theta| \leq 15^0$)

extents, where changing tilt between every sweep is necessary and larger changes minimize the resulting errors. Combinatorial analysis of the algorithm constrained by realistic antenna elevation angle limits shows the same trend.

3.3.2 Combinatorial Analysis

While closed form mathematical analysis is useful for discovering the theoretical limits of an algorithm, operational constraints imposed on the system in this case mean that it never functions at the maxima. With the goal of identifying the optimum tilt angle sequence (i.e. one that maximizes det(**A**)) given physical antenna angle limits, a combinatorial approach is used [3]. The antenna elevation angle was limited to fifteen degrees ($|\theta| \leq 15^0$) and the azimuth angle was limited to sixty degrees ($|\varphi| \leq 60^0$). The best case scenario for azimuth angles was assumed (i.e $\alpha = \beta = 30^0$ such that $\alpha + \beta \leq 60^0$) to focus on the contribution due to tilt sequencing.

The analysis produces a 3-D matrix of values for det(**A**) with each of the three elevation angles as one dimension. The maximum value in the matrix and corresponding elevation angles is easily found but produces little insight. More is gained by an assessment of changing results at various locations along one dimension. With the first elevation angle fixed, 2-D contour representations of the values relative to the second and third elevation angle are produced.

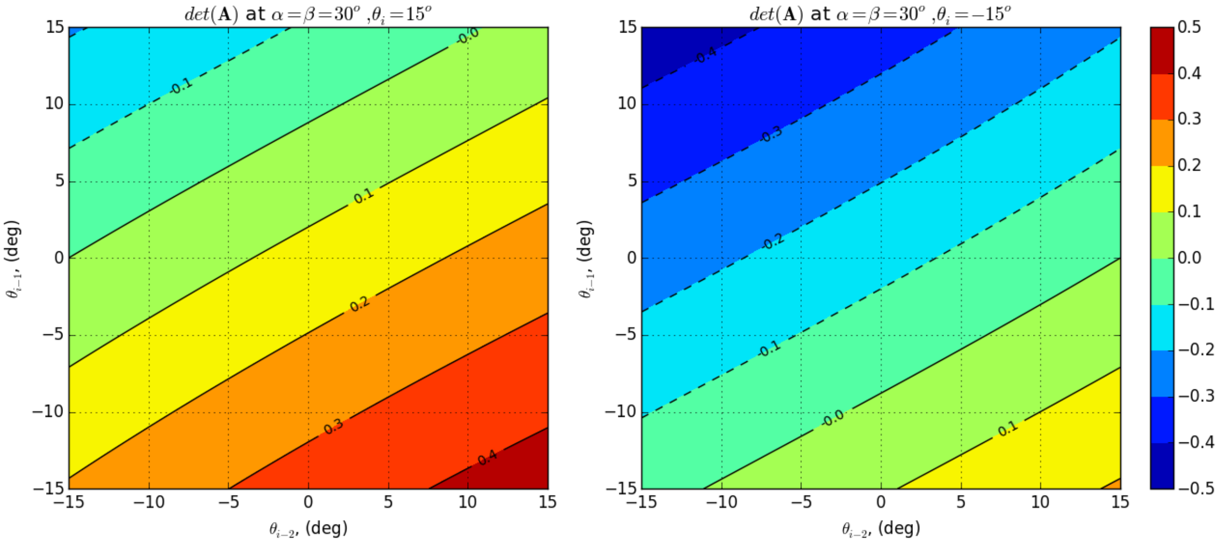

Figure 3: Sensitivity of det(A) to Tilt Angle Sequences (with Best Case Azimuth Angles)

Clearly, Figure 3 shows that alternating between the upper and lower elevation angle limits between every sweep maximizes det(**A**), consequently minimizing the error in the vertical velocity estimate. Therefore the best tilt sequences are 15^0, -15^0, 15^0 or -15^0, 15^0, -15^0 which result in det(**A**) ≈ ±0.45. Likewise, if operational considerations require that zero tilt sweeps are included in the sequence (e.g. for situational awareness at flight level), then the best tilt sequences alternate between zero and either the upper or lower limit (i.e. 15^0, 0^0, 15^0 or -15^0, 0^0, -15^0) which result in det(**A**) = ±0.25. This means that in the best case scenario with realistic radar operating constraints, the error in the vertical velocity estimate is two to four times greater than the uncertainty in the radial velocity measurements.

Somewhat counter intuitively, including a zero tilt sweep between two others at opposite elevation angle limits (i.e. 15^0, 0^0, -15^0 or -15^0, 0^0, 15^0) produces the worst result. This can be seen by revisiting the expression and setting the middle sweep elevation angle to zero ($\theta_{i-1} = 0^0$):

$$\det(\boldsymbol{A}) = \sin(\theta_i + \theta_{i-2})\sin(\alpha)$$

which is zero when $\theta_i = -\theta_{i-1}$ and $\alpha = \beta$ producing a singularity and making it impossible to solve for vertical velocity. This turns out not to be true when the zero-tilt sweep is not between sweeps at opposite elevation angle limits. If the zero-tilt sweep is after the two at opposite elevation angle limits (i.e. 15^0, -15^0, -0^0 or -15^0, 15^0, 0^0), then the result is somewhat better than alternating between zero tilt and one or the other elevation angle limit with det(A) ≈ ±0.34. Likewise, as illustrated in Figure 4, beginning the tilt sequence with a zero-tilt sweep followed by two sweeps at opposite elevation angle limits (i.e. 0^0, -15^0, 15^0 or 0^0, 15^0, -15^0) produces the same result. However, repeating the tilt sequence pattern will produce the singularity case every third sweep unless the pattern is modified. For example, the tilt sequence 0^0, -15^0, 15^0, 0^0, -15^0, 15^0… contains the singularity while the tilt sequence 0^0, -15^0, 15^0, 0^0, 15^0, -15^0… does not.

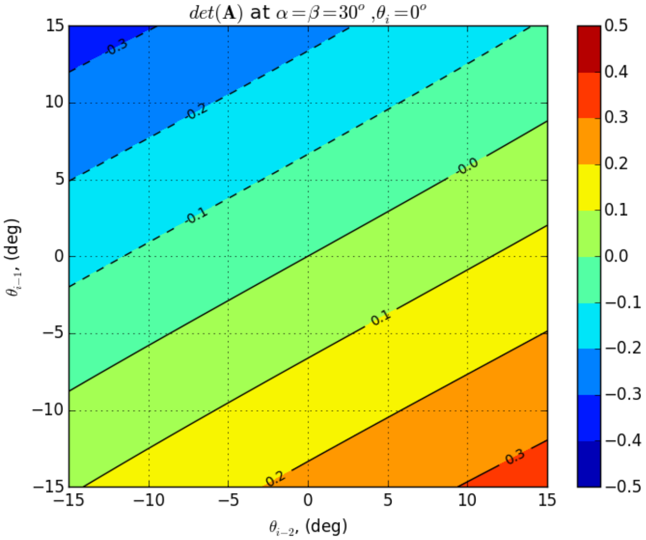

Figure 4: Sensitivity of det(A) to Tilt Sequences Including a Zero Tilt Sweep (with Best Case Azimuth Angles)

Also shown in Figure 4 are the corresponding sets for the tilt sequences alternate between zero and either the upper or lower limit (i.e. 0^0, 15^0, 0^0 and 0^0, -15^0, 0^0). While the results are similar, they are not the same as in the other cases. With two zero-tilt sweeps, det(A) ≈ ±0.22 rather than det(A) = ±0.25 when two sweeps are tilted. This can be seen by revisiting the expression and setting the first and third sweep elevation angles to zero ($\theta_i = \theta_{i-2} = 0^0$):

$$\det(\boldsymbol{A}) = \sin(\theta_{i-1})\sin(-2\alpha)$$

which is slightly less for the elevation and azimuth angles assumed for the analysis.

4 Conclusions and Future Work

A generalized methodology for producing a measure of the convective wind field ahead of an aircraft using a modern, airborne, Doppler, weather radar was built upon previous work that developed the original methodology and corresponding algorithm. The algorithm utilizes two or three scans to compute differences in radial velocity and estimate vertical velocity by accounting for differences in tilt angle between scans. To simplify the derivation, the original algorithm relies on a scan sequence with tilt angle alternating between zero and a fixed non-zero tilt.

Two new algorithms were developed to eliminate the tilt sequence requirements so that any tilt angles may be used so long as sufficient changes in tilt are applied between scans. This first is an extension of the original algorithm derived by solving a system of linear equations by substitution. The second is derived by applying Cramer's rule to solve a matrix equation. The matrix approach reduces complexity in the final algorithm and eliminates intermediate steps which can exhibit ambiguities. Relationships to the original algorithm were illustrated along with strengths and weaknesses of each algorithm. An examination of sensitivity to input parameters was provided to show the significance of tilt changes to the robustness of the algorithms and a measure of sensitivity to tilt differences.

The 3-bar sequence solution encounters singularities under certain conditions, however errors in the 2-bar solution are minimal under the same conditions. A robust algorithm must then apply the appropriate 2- or 3-bar solution depending on the scenario. Additionally, changing the tilt angle between every sweep is required for the 2-bar solution and extends the applicability of the 3-bar solution.

An implementation of this generalized algorithm is no more difficult than for the original algorithm, and may be accomplished by software changes in modern weather radar systems without any hardware changes required. Because the generalized algorithm is shown to be mathematically equivalent to the original CONVEX algorithm, it is expected that previous ground test results [1] are applicable. However, ground testing (or reprocessing of previously recorded ground testing data) is warranted. More importantly though, flight testing where the actual vertical winds are sampled would provide more conclusive validation of the algorithm.

5 References

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[2] Spiegel, Murray R., Schaum's Outline Series Mathematical Handbook of Formulas and Tables, McGraw-Hill, New York, 1997.

[3] Hoffman, Joe D. and Steven Frankel, Numerical Methods for Engineers and Scientists, Second Edition, CRC Press, May 31, 2001.