

Formalizing Ordinary Differential Equations in PVS

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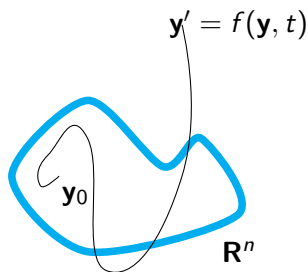
The Full Scope of the Project

The focus of this project is the development of an ordinary differential equation (ODE) Library in the Prototype Verification System (PVS) Including:

- Semi-algebraic sets.
- Existence and uniqueness of solutions to ODE's.
- Analytic functions
- How a solution to a polynomial ODE would enter or leave a semi-algebraic set.

Applications

The Purpose of ODE theory is to develop enough machinery for Differential Dynamic Logic to be implemented in PVS. This will allow formal reasoning of hybrid programs which can describe safety-critical systems. Ex: Drones maintaining well-clear in crowded airspace (or urban air mobility).



Ordinary Differential Equations

Definition 1 (nth order ODE)

Let $D \subseteq \mathbb{R}^{n+1}$ be a nonempty, open, connected subset of $\mathbb{R}^{n+1}$ and let $F \subseteq C(D, \mathbb{R})$. An n th order ordinary differential equation is of the form

$$\frac{d^{(n)}y}{dt^{(n)}} = F\left(t, y, \frac{dy}{dt}, \dots, \frac{d^{(n-1)}y}{dt^{(n-1)}}\right)$$

An n th order system of differential equations has the form

$$\begin{pmatrix} \tilde{y}_1^{(n)} \\ \tilde{y}_2^{(n)} \\ \dots \\ \tilde{y}_p^{(n)} \end{pmatrix} = \begin{bmatrix} f_1\left(t, \tilde{y}, \frac{d\tilde{y}}{dt}, \dots, \frac{d^{(n-1)}\tilde{y}}{dt^{(n-1)}}\right) \\ f_2\left(t, \tilde{y}, \frac{d\tilde{y}}{dt}, \dots, \frac{d^{(n-1)}\tilde{y}}{dt^{(n-1)}}\right) \\ \dots \\ f_p\left(t, \tilde{y}, \frac{d\tilde{y}}{dt}, \dots, \frac{d^{(n-1)}\tilde{y}}{dt^{(n-1)}}\right) \end{bmatrix}$$

where $\tilde{y} = [y_1 \ y_2 \ \dots \ y_p]^T$ and $[\cdot]^T$ is the transpose.

Properties of ODE's proven in PVS

These are a few of the properties of higher order ODE's that are now proven in PVS:

- Solutions to nth order ODE's on a closed interval, all of its domain.
- Solutions to ODE's with initial values.
- Solutions to ODE's are unique for initial value problems.

Theorem 2 (Picard Lindelöf theorem)

If $\exists L > 0$ s.t. the vector field $f(t, \tilde{y})$ satisfies

$$|f(t, \tilde{y}_1) - f(t, \tilde{y}_2)| \leq L|\tilde{y}_1 - \tilde{y}_2|$$

in a domain $\mathbb{R}$ and f is continuous at t , then there is a unique solution $\tilde{y}(t)$ of the differential system $\frac{d\tilde{y}}{dt} = f(t, \tilde{y})$, that satisfies a given initial condition $\tilde{y}(t_0) = c$ in $\mathbb{R}$.

Reduction of Order in PVS

Consider the IVP:

$$y^{(n)} = F(t, y, y^{(1)}, \dots, y^{(n-1)})$$

with IC's: $y^{(i-1)}(\tau) = \xi_i$ where $i = 1, 2, \dots, n$ on the open interval $I=(a,b)$.

We define $x_1, x_2, \dots, x_n$ as:

$$\begin{aligned} x_1 &= y \\ x_2 &= y^{(1)} \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned} \tag{1}$$

Reduction of Order in PVS

Now differentiating w.r.t t :

$$\begin{aligned}x_1' &= y^{(1)} \\x_2' &= y^{(2)} \\&\vdots \\x_n' &= y^{(n)}\end{aligned}\tag{2}$$

rewriting yields first order ODE:

$$\begin{aligned}x_1' &= x_2 \\x_2' &= x_3 \\&\vdots \\x_n' &= F(t, x_1, x_2, \dots, x_n)\end{aligned}\tag{3}$$

Reduction of Order in PVS

As you would want and expect, the solution to an n th order differential equation and the solution to the corresponding system of first order differential equations are equivalent.

```
od_red_same_sol: LEMMA FORALL(n,k:posnat, f:od_fk(n,k))
  (y:{ yy:r2v_n(n) | derivable_k?(k,yy) }):
    od_order_k_sol?(n,k,f)(y)
      IMPLIES
    od_sol?(n*k,red(n,k,f))(y)
```

Definition of a Polynomial ODE

Definition 3 (Polynomial Differential Equation)

Let $D \subseteq \mathbb{R}^{n+1}$ be a nonempty, open, connected subset of $\mathbb{R}^{n+1}$ and let $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a polynomial in $\mathbb{R}^n$. Then,

$$y' = P(t, y)$$

is the polynomial differential equation with the initial condition $y(0)$

Definition of Analytic and Smooth Functions

Definition 4 (Analytic Function)

A function is analytic at t_0 if there exists an open neighborhood around t_0 such that the Taylor series expansion $T(t)$ of a function $f(t)$ at any point t_0 in its domain:

$$T(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!} (t - t_0)^n$$

for each t in the open neighborhood.

Definition 5 (Smooth Function)

A smooth function is a function such that $f^{(n)}(t)$ exists $\forall n \in \mathbb{N}$ and $t \in d$. The collection of these functions is called C^∞ .

Definition of Analytic and Smooth Functions

NOTE: You are **not** guaranteed that a solution to a polynomial ODE is Lipschitz.

An example is $x' = x^2$, a non-linear separable polynomial ODE with solution $\frac{1}{c_1 - t}$.

Notice that certain values of c_1 found from the initial value, the interval of existence of the unique solution is arbitrarily small.

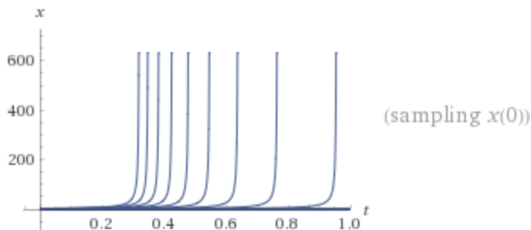


Figure: Sample Solution Family [1]

Extending Analytic Functions and Properties of Linearity

Definition 6 (Analytic on $\mathbb{R}^n \rightarrow \mathbb{R}$)

A map $f : U \rightarrow \mathbb{R}$ where $U \subset \mathbb{R}^n$ is analytic if every component of f is analytic.

Definition 7 (Linearity of Analytic Functions)

Let $f(t)$ and $g(t)$ be analytic functions on a domain D and let $c \in \mathbb{R}$. Then:

$(f + c * g)(t)$ is analytic.

Defining High Order Analytic Functions in PVS

The following steps are taken to define Analytic on $\mathbb{R}^n \rightarrow \mathbb{R}$

- Pre-allocate the memory bins
- Make vectors of length n such that k elements are 1 and the rest are 0
- Then exploit bijections from multisets to these “choose” vectors. Where a multiset with n choices of size k gets mapped to choose vectors corresponding to $\binom{n+k-1}{k}$

Extending Analytic Functions

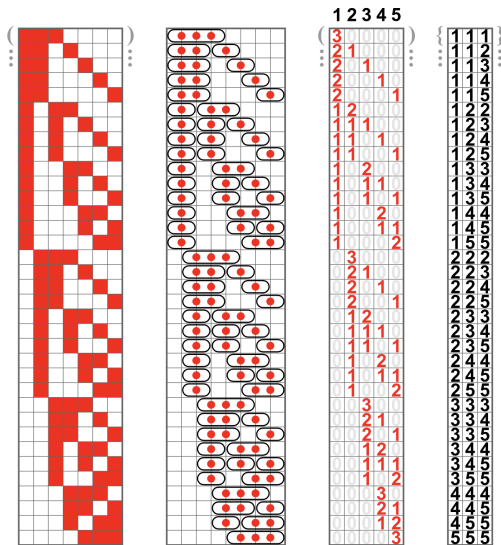


Figure: Counting Multisets [2]

Future work

- Reduction of order has equivalence of solutions
- Existence and uniqueness for higher order ODEs
- Existence and uniqueness for Locally Lipschitz condition
- Existence and uniqueness for Polynomial ODEs
- Solution of poly ODE is analytic
- Peano's Existence Theorem

Takeaways

- Semi-Algebraic sets.
- Always question your domain transformations.
- The real importance of computer science in the future of advanced mathematics and automation.

Thank You! Any Questions?

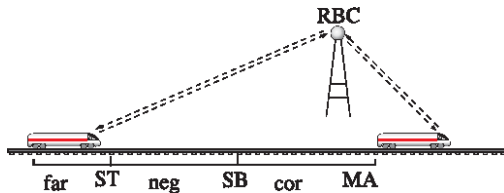


Figure: Application of DDL for Automated Train Network [3]

1. <https://www.wolframalpha.com/input/?i=x>
2. <https://en.wikipedia.org/wiki/Multiset>
3. <http://symbolaris.com/logic/dL.html>
4. Title= Ordinary Differential Equations
Author= Morris Tenenbaum Harry Pollard
Year= 1963
ISBN= 9738-0-486-64940-5
Publisher= Dover Publications