Explicit Discontinuous Galerkin Methods for Conservation Laws

Korea Society for Computational Fluids Engineering

KSCFE 25th Conference

October 18-21, 2020, Jeju, Korea

H. T. Huynh

NASA Glenn Research Center, Cleveland, Ohio

Supported by NASA Transformational Tools and Technologies Project

Computational Fluid Dynamics (CFD) on Boeing 787

Computational Fluid Dynamics Today

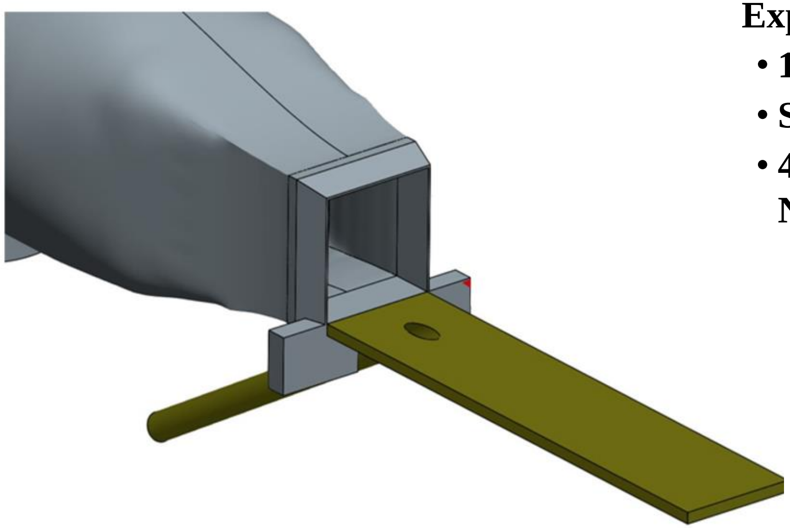
"In spite of considerable successes, reliable use of CFD has remained confined to a small but important region of the operating design space due to the inability of current methods to reliably predict turbulent-separated flows", NASA CFD Vision 2030 Study, 2014.

Numerical Methods

- Current popular CFD methods are second-order accurate.
- The most promising methods to tackle turbulent compressible flows are high-order (third and higher) methods such as Discontinuous Galerkin (DG), Spectral Element, or Flux Reconstruction (FR), …

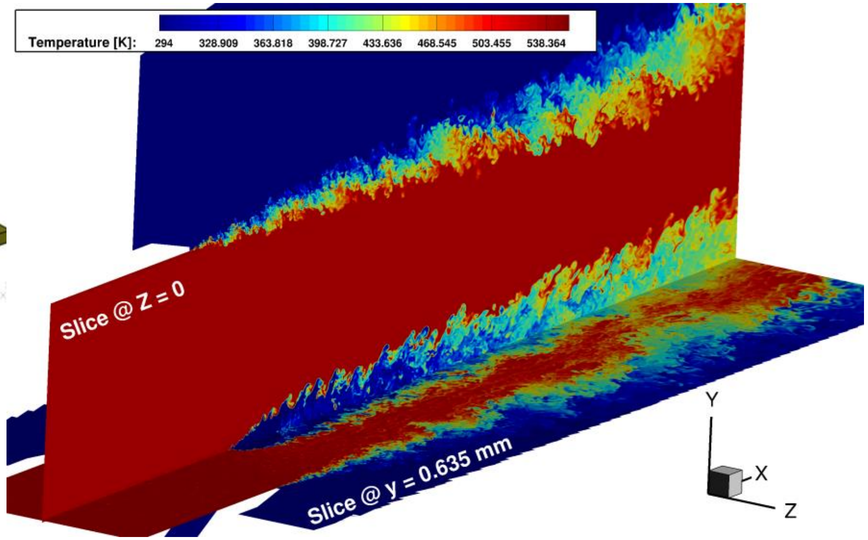
On-Going Numerical Simulations for Turbulent Heat Flux Experiment Using Glenn Flux Reconstruction (GFR) Code

- **Large Eddy Simulation results for Turbulent Heat Flux Experiment (by Dr. Seth Spiegel)**
  - **13 million high-order grid cells**
  - **Solution containing 1.6 billion Degrees of Freedom**
  - **40,000 CPU cores of Electra supercomputer housed at NASA Ames Research Center**



THX 3: Single Cooling Hole

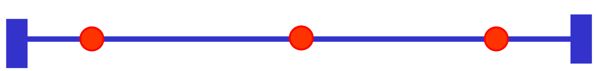
Fundamental aerodynamic & thermodynamic study



Instantaneous temperature contours

Motivation for Current Work

- Time discretization with such high-order spatial methods results in
  - a small time step size for an explicit scheme (but can be efficiently implemented on super computers and GPUs)



Piecewise Parabolic

Piecewise Cubic

  - a very large system of equations for an implicit scheme (inefficient on super computers and GPUs)
- Efforts to improve time discretization has had limited success
- Time stepping is a pacing item in High-order CFD methods.

Explicit Time Stepping

The explicit methods discussed here are of **predictor-corrector** type:

- **Predictor** via Taylor series expansion (or Cauchy–Kowalevski CK procedure) involves **no interaction** of data among cells
- **Corrector** on a space-time domain is where **interaction** takes place

The two predictor-corrector methods studied are:

- Space-Time Expansion **STE-DG** by Lörcher, Gassner, Munz, 2007…
  - CFL condition considerably smaller than 1 in 1D
- **Moment** scheme by Huynh ICCFD 2004, …
  - CFL condition 1 in 1D for all orders.

Simple idea can lead to significant improvement.

Outline

- Van Leer’s scheme III for advection (1977): exact evolution and projection
- Time discretization of **predictor-corrector** type.
- Space-Time Expansion **STE-DG** method by Lörcher, Gassner, and Munz (JSC-2007, …)
- **Moment** scheme by Huynh (ICCFD3-2004, AIAA-2013)
- Preliminary numerical results
- Discuss 2D extensions
- Conclusions and discussion

Advection Equation

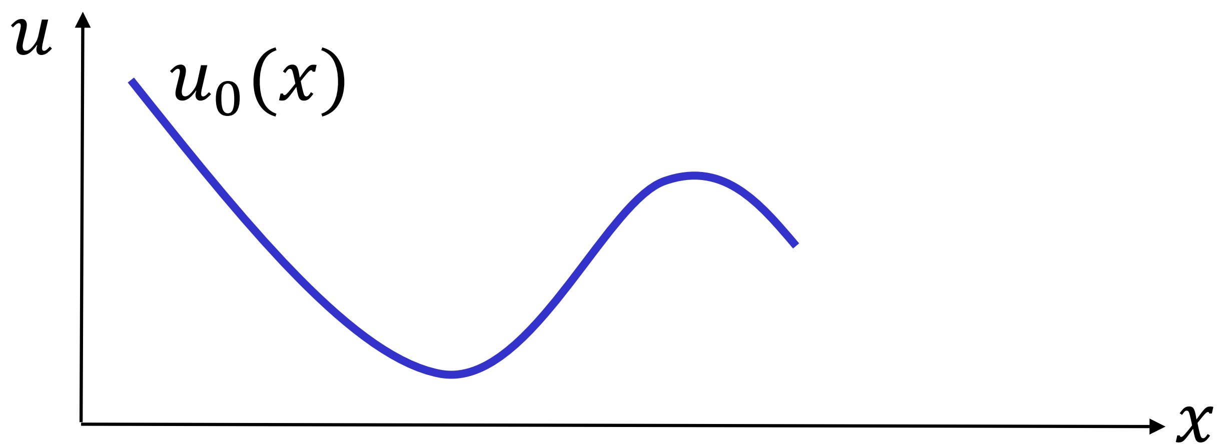
$$u_t + a\,u_x = 0.$$

Initial condition

$$u(x,0) = u_0(x).$$

Solution

$$u(x,t) = u_0(x - at).$$



Advection Equation

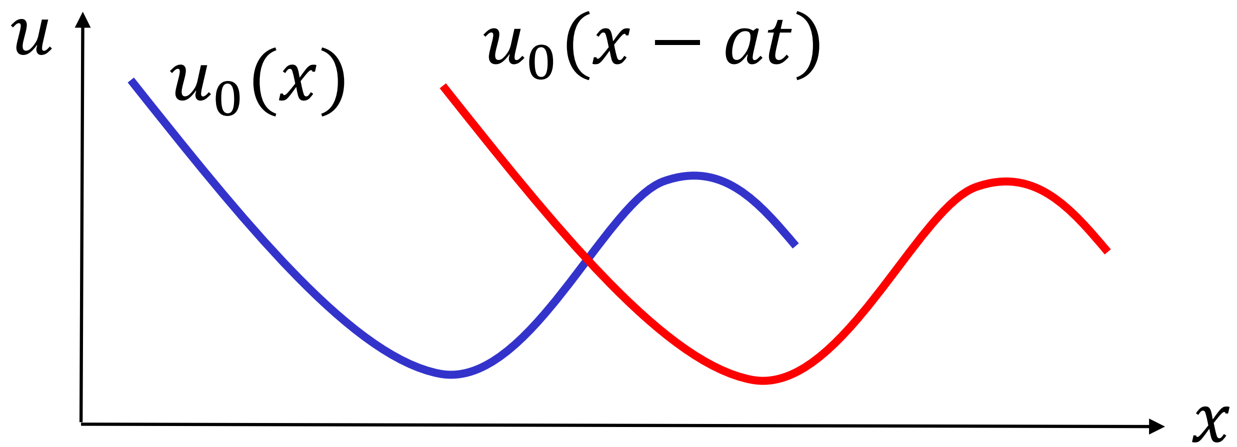
$$u_t + a\,u_x = 0.$$

Initial condition

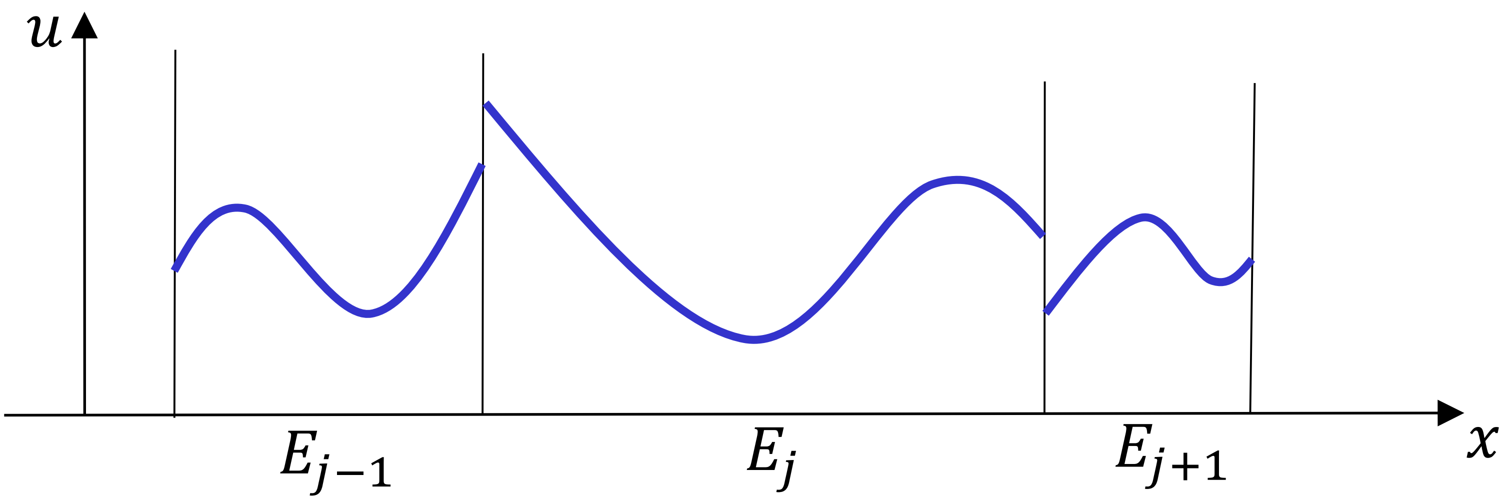
$$u(x,0) = u_0(x).$$

Solution

$$u(x,t) = u_0(x - at).$$



Space Discretization



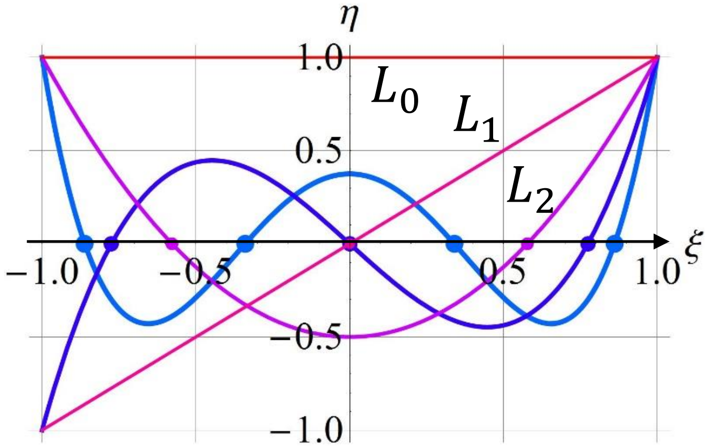
Approximate the solution in each cell E_j by a polynomial of degree p

$$u_j(x,t) = \sum_{k=0}^{p} u_{j,k}(t)\phi_k(x)\,.$$

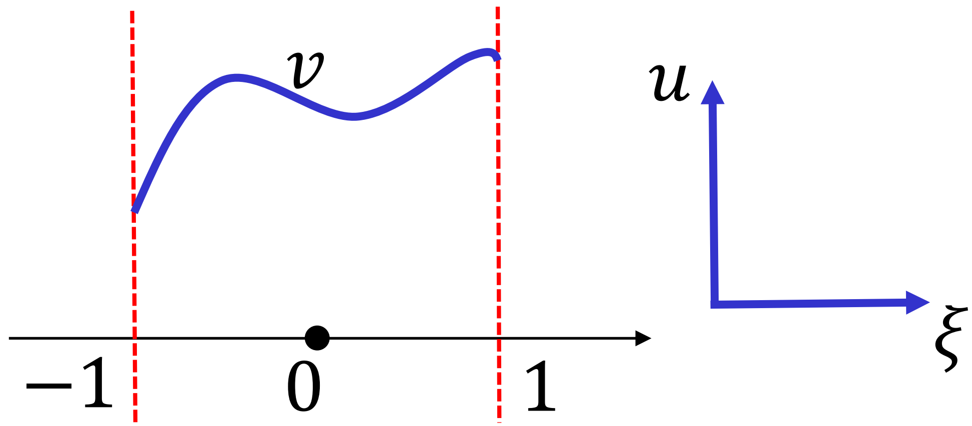
Dot product of two functions on element (cell) $E = E_j$:

$$(v,w) = \int_E v(x)w(x)dx\,.$$

Projection Using Legendre Polynomials



Legendre Polynomials



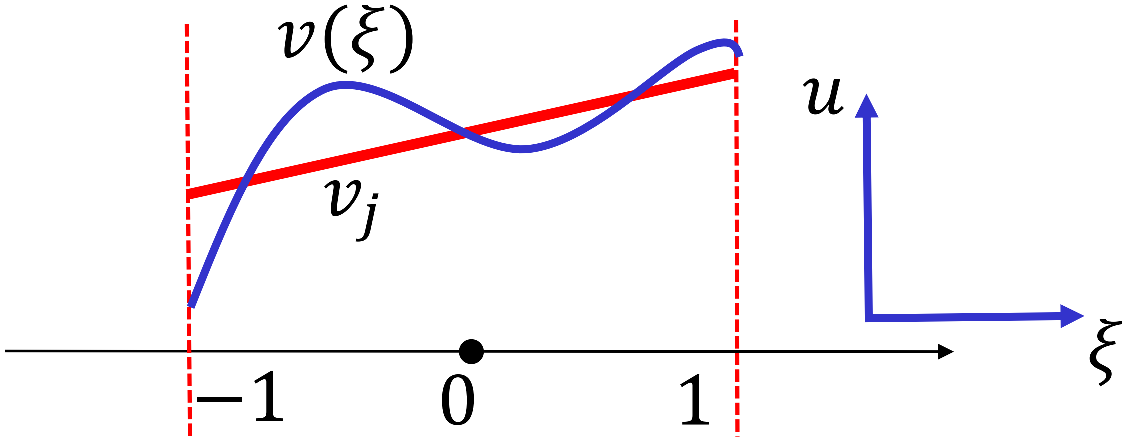
Let L_k be the Legendre polynomial of degree k. In the local coordinate ξ, the projection of any function v onto the space of polynomials of deg $\leq p$ is

$$v_j(\xi) = \sum_0^p v_{j,k} L_k$$

where, in the reference frame, $v_{j,k}$ is given by a dot product,

$$v_{j,k} = \frac{\int_{-1}^{1} v(\xi)\, L_k(\xi)\, d\xi}{\|L_k\|^2}$$

Projection, Linear Case



$$v_j(\xi) = v_{j,0} + v_{j,1}\xi$$

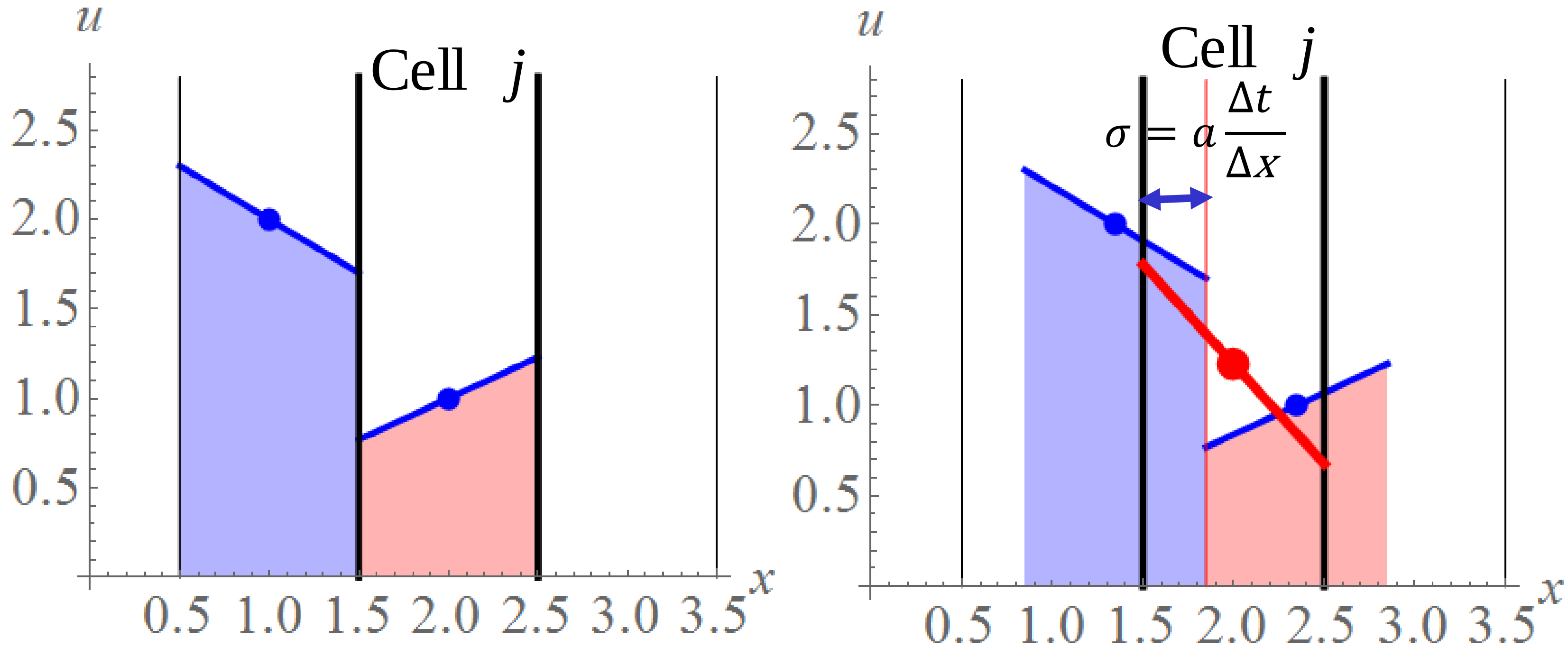
where,

$$v_{j,0} = \frac{1}{2}\int_{-1}^{1} v(\xi)\, d\xi$$

and, since $L_1(\xi) = \xi$ and $\|L_1\|^2 = \frac{2}{3}$

$$v_{j,1} = \frac{3}{2}\int_{-1}^{1} v(\xi)\, \xi\, d\xi$$

Van Leer's Scheme III (1977): Shift and Project

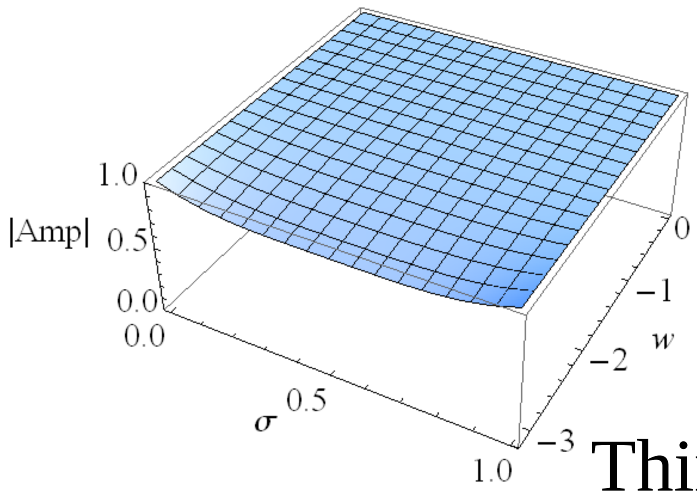


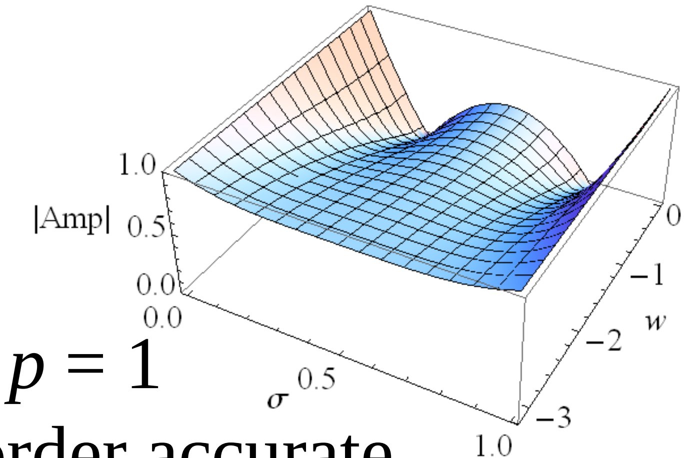
(a) Data (b) Linear solution

$$u_{j,0}^{n+1} = u_{j,0} + \sigma(u_{j-1,0} + u_{j-1,1} - u_{j,0} - u_{j,1}) + \sigma^2(-u_{j-1,1} + u_{j,1})$$

$$u_{j,1}^{n+1} = u_{j,1} + 3\sigma(-u_{j-1,0} - u_{j-1,1} + u_{j,0} - u_{j,1}) + 3\sigma^2(u_{j-1,0} + 2u_{j-1,1} - u_{j,0}) + \boxed{2\sigma^3}(-u_{j-1,1} + u_{j,1})$$

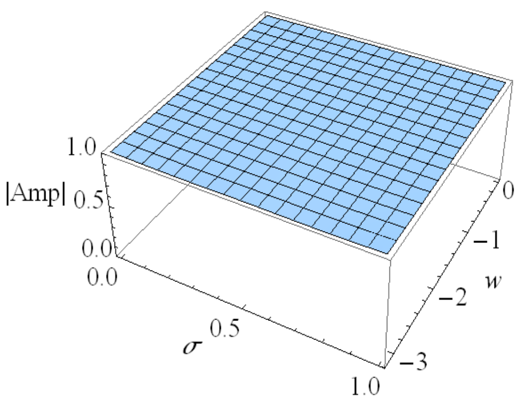
Fourier Analysis: Plots of Absolute Values of Eigenvalues

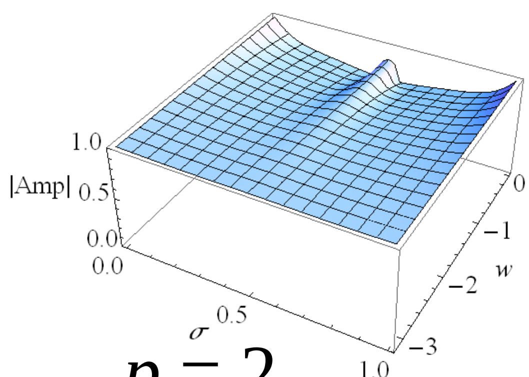


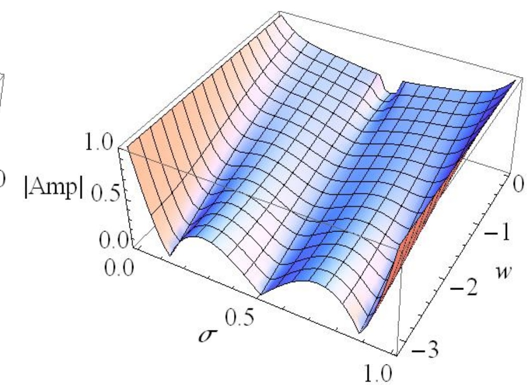


$p = 1$

Third-order accurate







$p = 2$

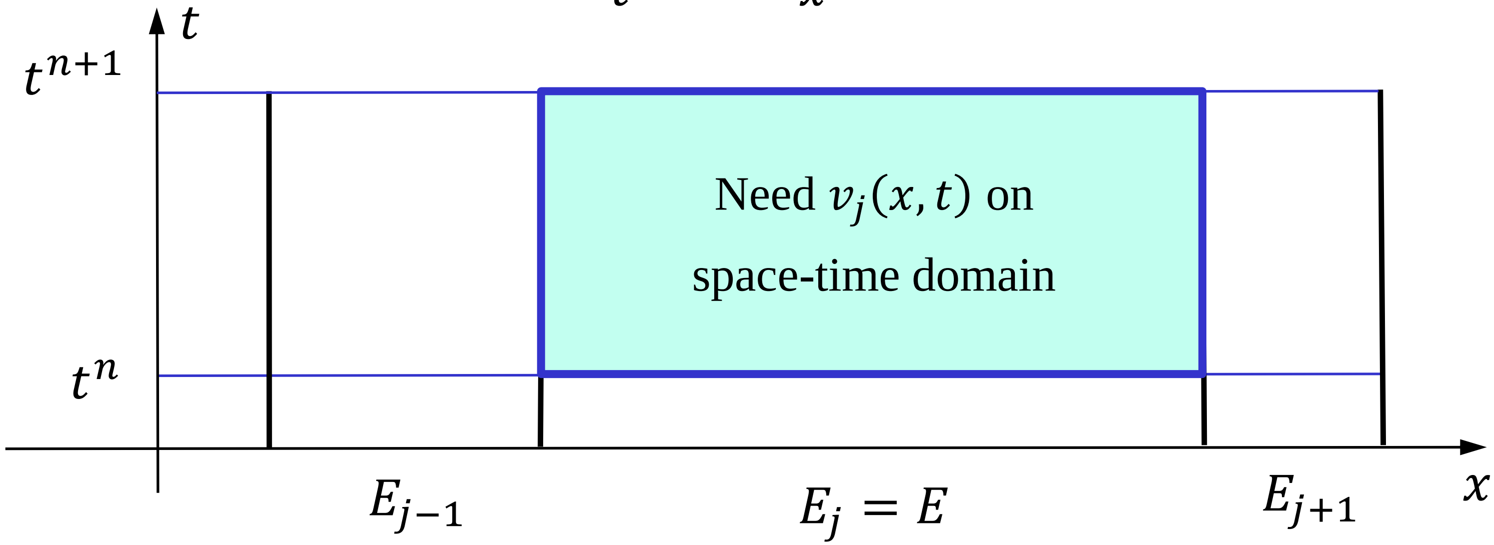
Fifth-order accurate

Difficulty

- Concerning the extension of scheme III (1977), after nearly 28 years, Van Leer wrote in 2005 (Van Leer and Nomura AIAA): “When trying to extend these schemes beyond advection, viz., to a nonlinear hyperbolic system like the Euler equations, the first author ran into insuperable difficulties because the exact shift operator no longer applies, ...”
- This difficulty can be overcome by using a space-time domain. The result is the Moment scheme (Huynh, ICCFD 2004).

Extension to Systems Using Space-Time Domain

$$u_t + a\,u_x = 0.$$



First step (predictor): space-time polynomial expansions.

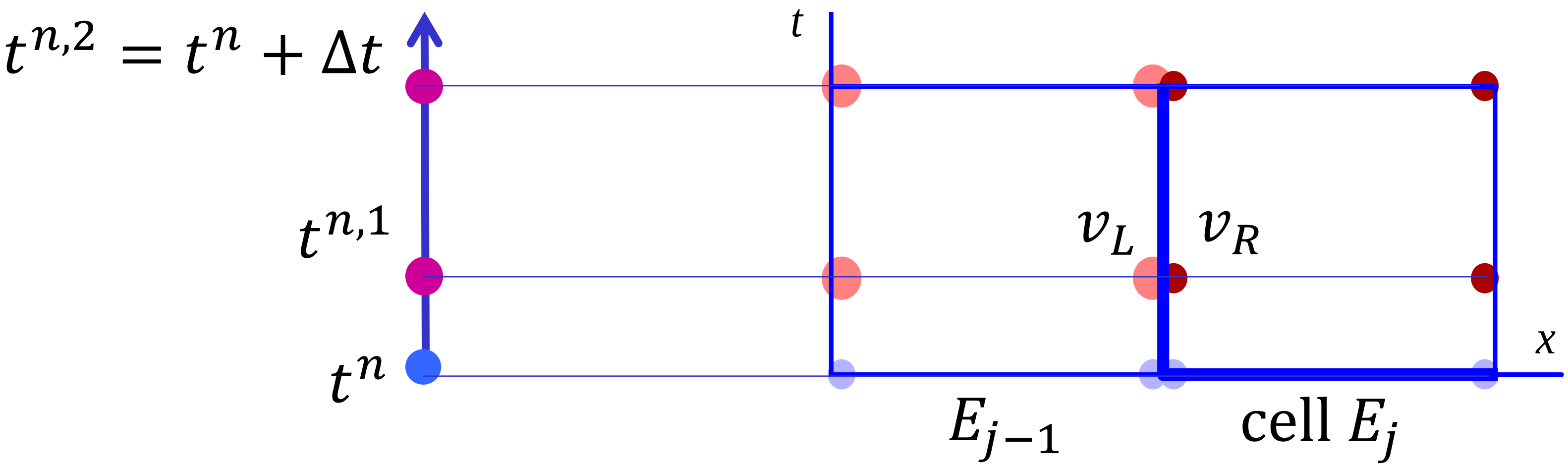
For the piecewise linear case, with $(u_x)_j$ known,

$$(u_t)_j = -a(u_x)_j.$$

The space-time expansion **(no data interaction among cells)** is given by

$$v_j(x,t) = u_j + (u_x)_j(x - x_j) + (u_t)_j(t - t^n).$$

Data Interaction via Upwind Flux



At each interface $j - 1/2$, for each Radau time level, e.g., $\frac{1}{3}\Delta t$, using the space-time polynomials, set

$$v_L = v_{j-1}\left(1,\ \tfrac{1}{3}\Delta t\right), \quad v_R = v_j\left(-1,\ \tfrac{1}{3}\Delta t\right),$$

and

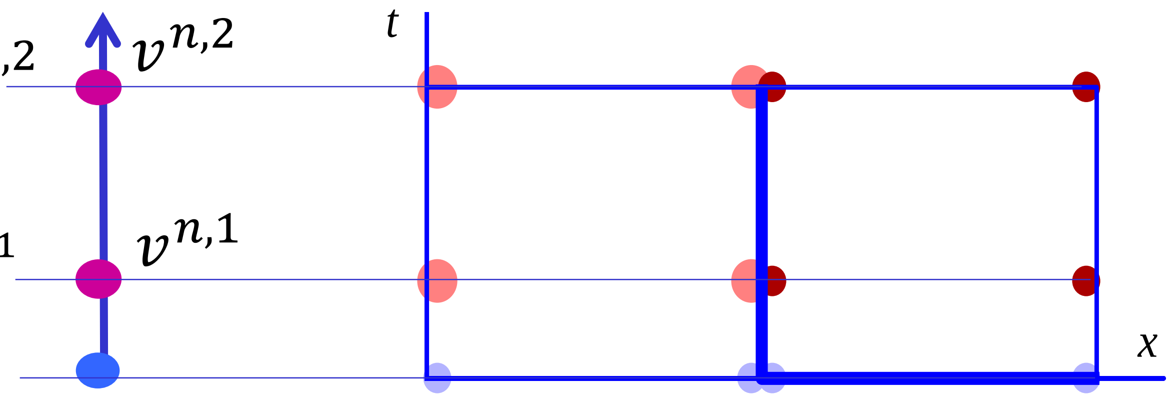
$$f^{\text{com}} = f_{\text{upwind}}\left(v_L, v_R\right).$$

Time Integration via Radau Quadrature

$$t^{n+1} = t^n + \Delta t = t^{n,2}$$

$$t^n + \frac{1}{3}\Delta t = t^{n,1}$$

$$t^n$$



Denote the data at purple dots by

$$v^{n,1} \;\text{ at }\; t^{n,1} = t^n + \frac{1}{3}\Delta t \qquad \text{and} \qquad v^{n,2} \;\text{ at }\; t^{n,2} = t^n + \Delta t$$

We need quadratures for time integration from t^n to $t^{n,k}$

$$\int_{t^n}^{t^n+\frac{1}{3}\Delta t} v(\xi)\, d\xi = \Delta t \left(\frac{5}{12} v^{n,1} - \frac{1}{12} v^{n,2}\right)$$

$$\int_{t^n}^{t^n+\Delta t} v(\xi)\, d\xi = \Delta t \left(\frac{3}{4} v^{n,1} + \frac{1}{4} v^{n,2}\right)$$

Integrate On Space-Time Domain

$$u_t + f_x = 0.$$

Dot product with Legendre polynomial $L_k = \phi$ on cell E (integrate in x):

$$(u, \phi)_t \; + \; (f_x, \phi) = 0.$$

Apply integration by parts to (f_x, ϕ)

$$(u, \phi)_t \; + \; [f^{\text{com}} \phi]_{\partial E} - (f, \phi_x) = 0.$$

Integrate in time on $[t^n, t^*]$,

$$\int_E u(x, t^*)\, \phi(x)\, dx \; - \; \int_E u(x, t^n)\, \phi(x)\, dx$$

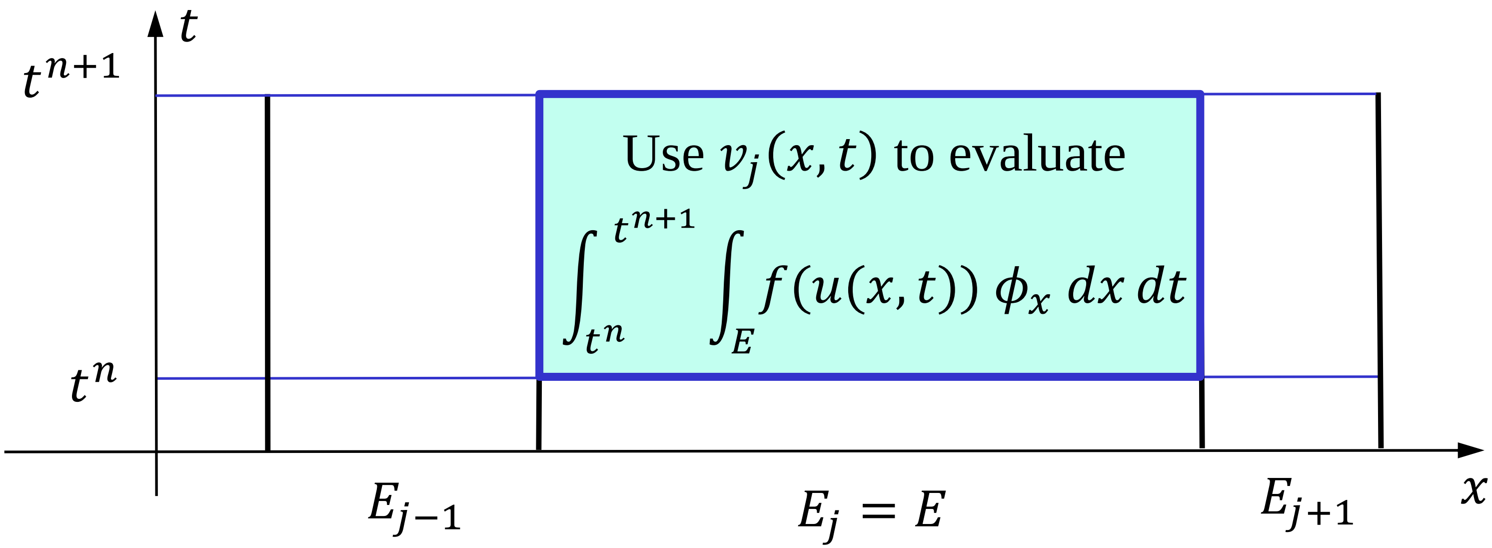
Surface Integral: Trivial

$$\boxed{+\int_{t^n}^{t^*} f^{\text{com}}\big(x_{j+1}, t\,\big)\, \phi\big(x_{j+1}\big)\, dt - \int_{t^n}^{t^*} f^{\text{com}}\big(x_{j-1}, t\,\big)\, \phi\big(x_{j-1}\big)\, dt}$$

$$\boxed{+\int_{t^n}^{t^*} \int_E f\, \phi_x\, dx\, dt \;\; = \;\; 0.}$$

Volume Integral VI

STE-DG Scheme (Lörcher, Gassner, Munz)



For advection, $f = u$. The cell average solution is trivial and as before,

$$u_{j,0}^{n+1} = u_{j,0} + \sigma(u_{j-1,0} + u_{j-1,1} - u_{j,0} - u_{j,1}) + \sigma^2(-u_{j-1,1} + u_{j,1})$$

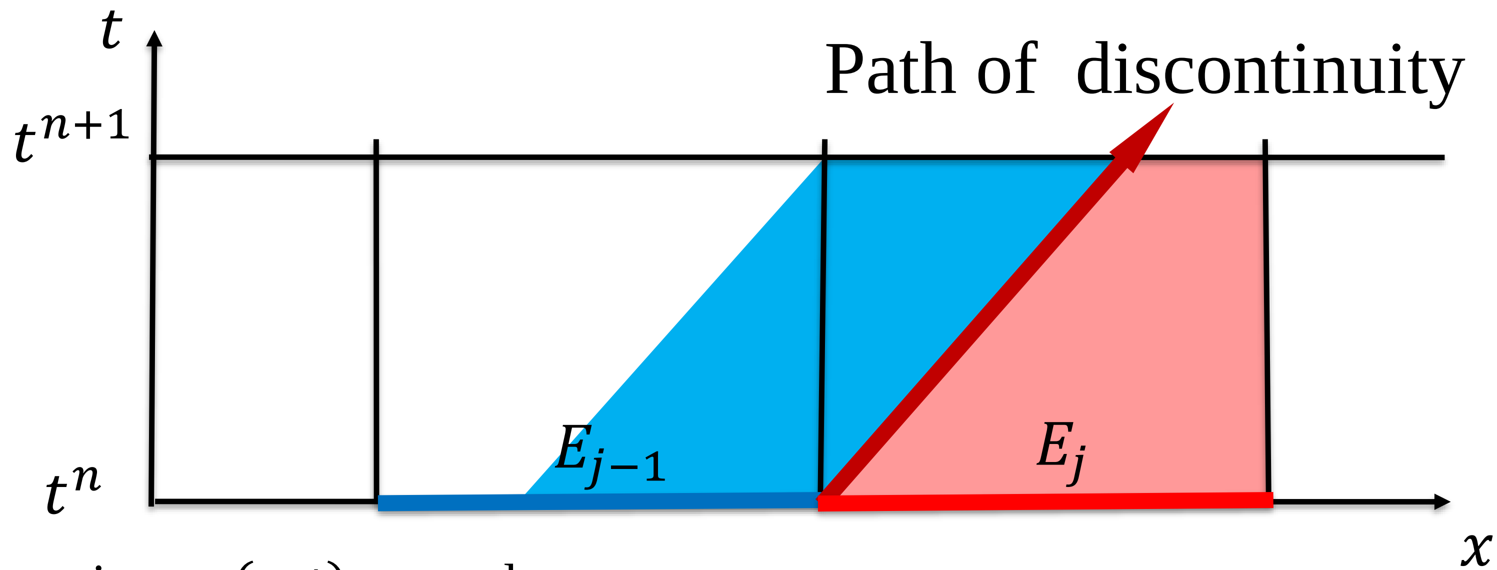
By using $v_j(x,t)$ to evaluate the volume integral,

$$u_{j,1}^{n+1} = u_{j,1} + 3\sigma(-u_{j-1,0} - u_{j-1,1} + u_{j,0} - u_{j,1}) + \boxed{3\sigma^2}(u_{j-1,1} - u_{j,1})$$

STE-DG Scheme

For arbitrary p, the STE-DG method (Lörcher, Gassner, and Munz, JSC 2007…) is accurate to order $p+1$ and has a restrictive CFL condition.

Explicit Discontinuous Galerkin (DG) Schemes



- By using $v_j(x,t)$ to evaluate

$$\mathrm{VI} = \int_{t^n}^{t^{n+1}} \int_E f(v_j(x,t))\,(\phi_k)_x\,dx\,dt\,,$$

  the influence of the data from E_{j-1} is ignored. The resulting STE-DG method (Lörcher et al.) is accurate to order p and has a restrictive CFL condition.

- In order to account for this influence, we use the Legendre polynomials $\phi = L_k$. What is **important is that $L_k{}'(\xi)$ is of 1 degree lower than the degree of L_k.**

Moment Scheme: On Space-Time Domain

For the volume integral

$$\int_{t^n}^{t^*} \int_E u(\xi, t)\, \phi_\xi \, d\xi \, dt\,,$$

first, set $\phi = L_0 = 1$. Then $\phi_\xi = 0$, and thus, VI $= 0$.

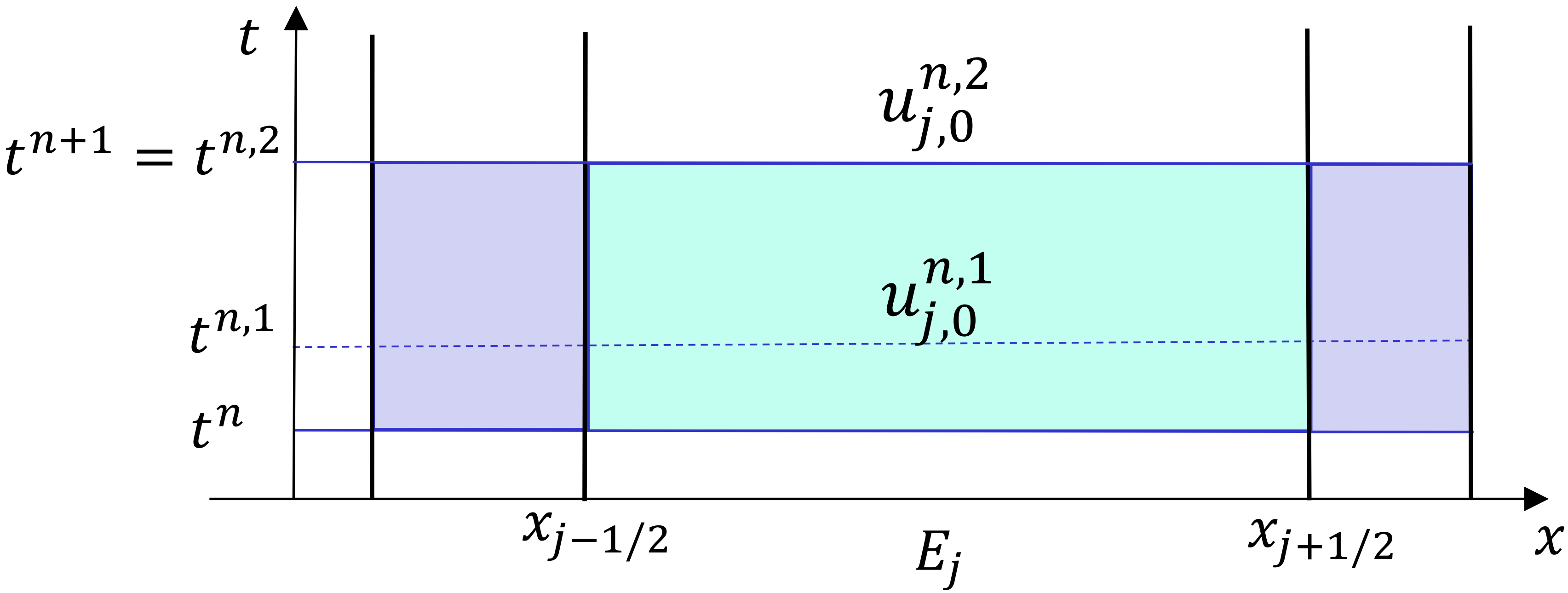
Update the cell average solutions

$$u_{j,0}^{n,1} \quad \text{at} \quad t^{n,1} = t^n + \frac{1}{3}\Delta t$$

and

$$u_{j,0}^{n,2} \text{ at } t^{n,2} = t^n + \Delta t.$$

Moment Scheme: First, get cell average solutions for all stages



The cell average solutions $u_{j,0}^{n,1}$ and $u_{j,0}^{n,2}$ **have influence** from neighboring data

Moment Scheme: Slope Update

In the local frame, set $\phi = L_1 = \xi$, then $L_1{}'(\xi) = 1$, and

$$\text{VI} = \int_{t^n}^{t^{n+1}} \int_E u(x,t)\,dx\,dt$$

$$= \Delta t \left(\frac{3}{4} \int_E u(x, t^{n,1})\,dx + \frac{1}{4} \int_E u(x, t^{n,2})\,dx \right)$$

$$= \Delta t \left(\frac{3}{4} u_{j,0}^{n,1} + \frac{1}{4} u_{j,0}^{n,2} \right)$$

Get slope update

$$u_{j,1}^{n,2} \;\text{ at }\; t^{n,2} = t^n + \Delta t.$$

Moment Scheme: General Case

$$\int_E u(x,t^*)\, L_k(x)\, dx = \int_E u(x,t^n)\, L_k(x)\, dx \; +$$

$$\int_{t^n}^{t^*} f^{\mathrm{com}}\big(x_{j-1},t\,\big)\, L_k\big(x_{j-1}\big)\, dt \; - \int_{t^n}^{t^*} f^{\mathrm{com}}\big(x_{j+1},t\,\big)\, L_k\big(x_{j+1}\big)\, dt$$

$$+ \; \int_{t^n}^{t^*} \int_E f(u(x,t))\, (L_k)_x\, dx\, dt\,.$$

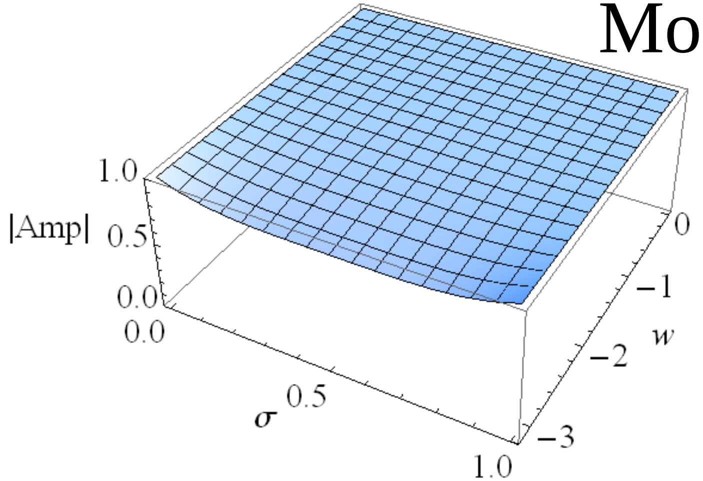
Calculate for $k = 0, 1, 2, \ldots,$ successively

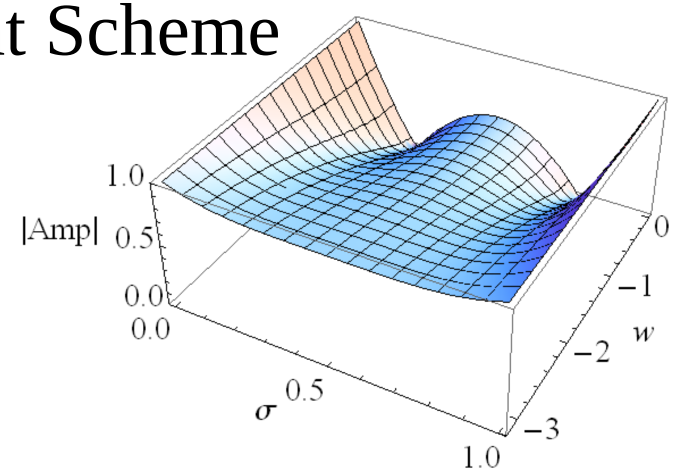
Moment Scheme (1D)

- For 1D convection with constant speed, the moment scheme boils down to shifting and projecting and yields a result identical that of Van Leer's scheme III.
- The moment scheme extends to systems of equations with ease.
- For arbitrary order p, the method is accurate to order $2p + 1$, and has a CFL condition of 1.
- For a vanishingly small time step, the moment scheme reduces to RK-DG.

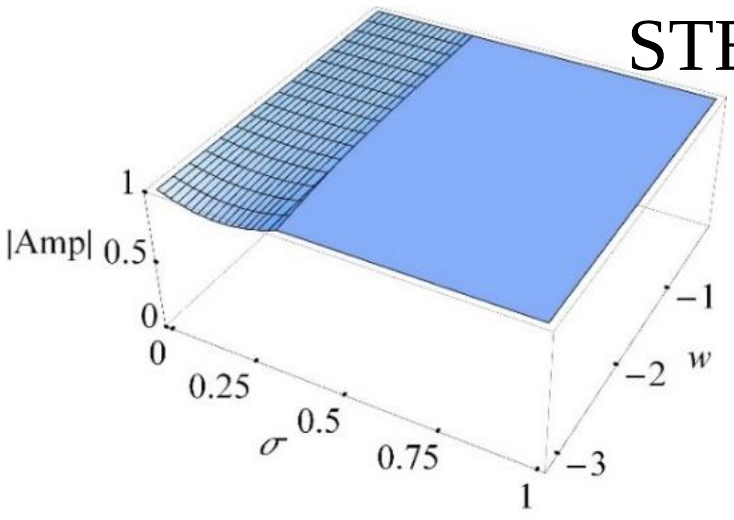
Von Neumann Analysis: Absolute Values of the Two Eigenvalues, p = 1

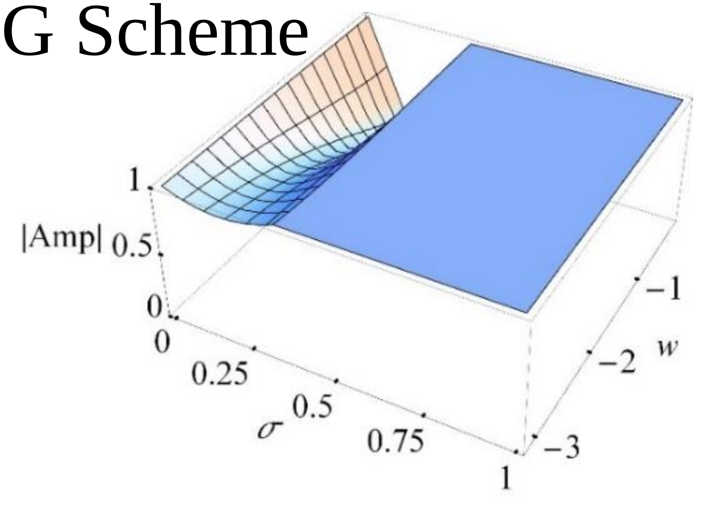
Moment Scheme





STE-DG Scheme





CFL Conditions

Degree p	0	1	2	3	4
RKDG (RK same order as DG)	1	0.33	0.20	0.14	0.11
STE-DG Scheme	1	0.33	0.17	0.10	0.069
Moment Scheme	1	1	1	1	1

Propagating a Sound Wave with Moment Scheme

- Propagating a sound wave by Moment Scheme using 1D Euler equations:

$$\boldsymbol{U}_t + \boldsymbol{F}_x = 0,$$

$$\boldsymbol{U} = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix}, \qquad \boldsymbol{F} = \begin{bmatrix} m \\ (m^2/\rho) + p \\ (e+p)m/\rho \end{bmatrix}.$$

- The initial data is

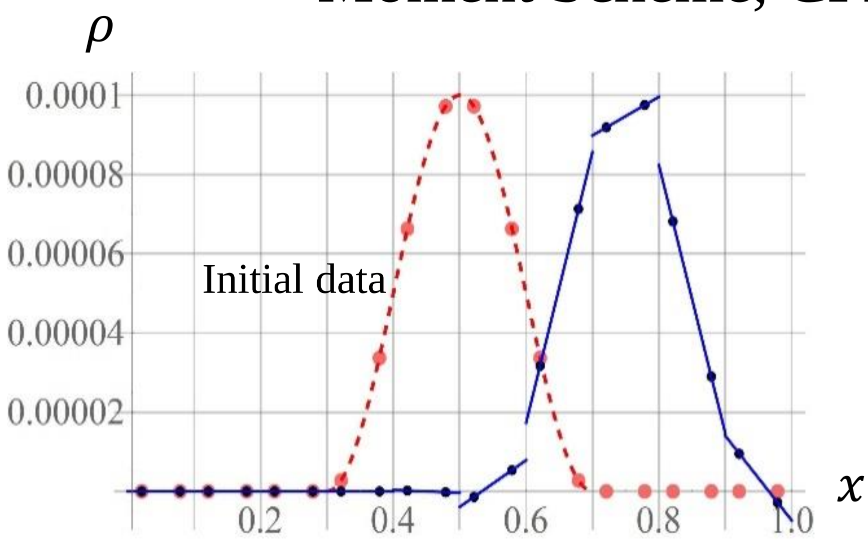
$$(\rho_0, u_0, p_0) = (1, 0, 1/\gamma).$$

- The domain $[0, 1]$ is divided into 10 cells with periodic BC. Let the fluctuation $\Delta\rho$ be

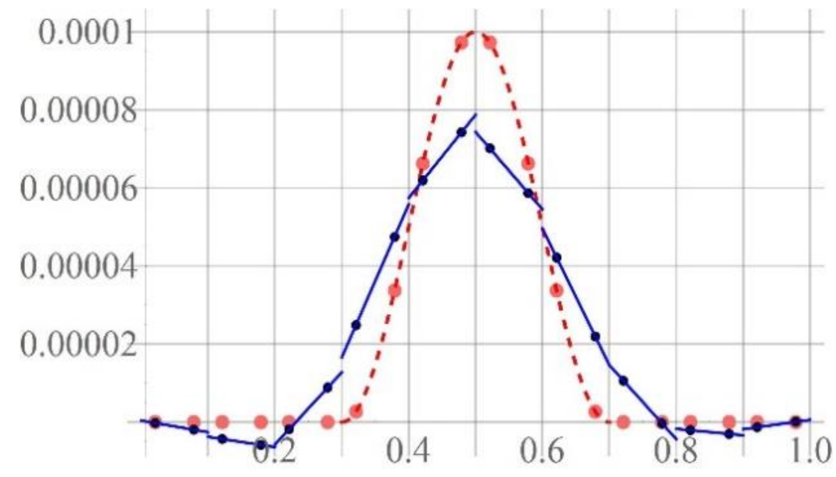
$$\Delta\rho = \begin{cases} 0 & \text{if } \ x < 0.3 \ \text{ or } \ x > 0.7, \\ 10^{-4}\sin^2(\pi(x-0.3)/0.4) & \text{if } \quad 0.3 \le x \le 0.7 \end{cases}$$

$$(\Delta\rho, \Delta u, \Delta p) = \Delta\rho(1, a_0/\rho_0, a_0^2) = \Delta\rho(1, 1, 1).$$

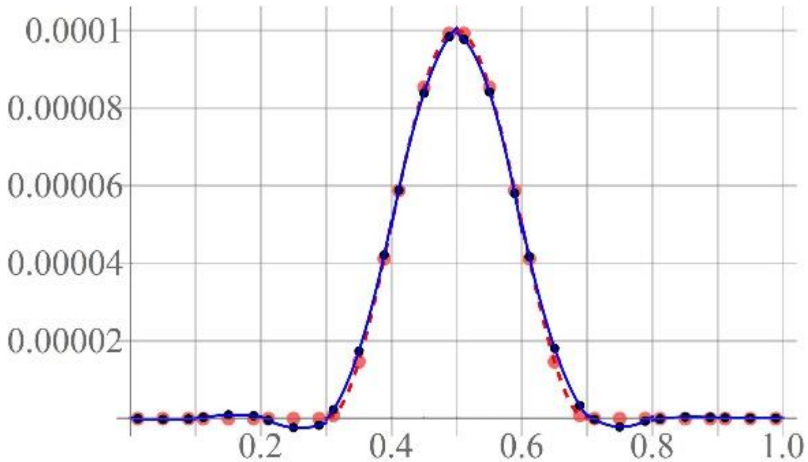
Moment Scheme, CFL = $10/12 = 0.8333$



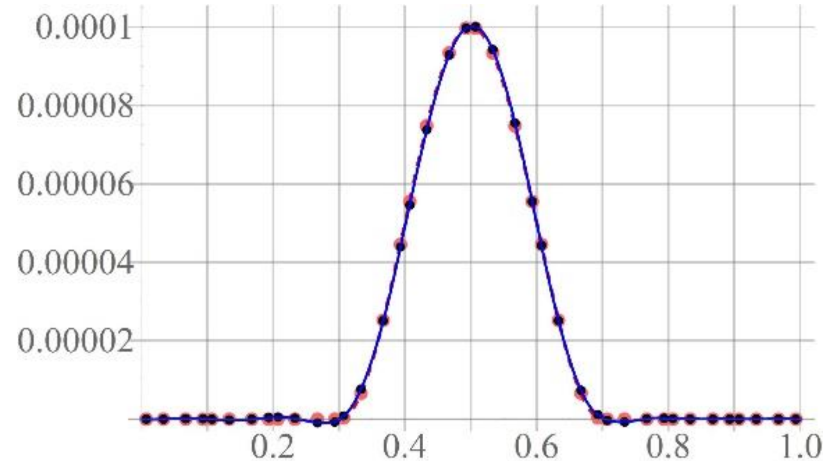
(a) Linear Solution, 3 time steps



(b) Linear Solution,120 time steps

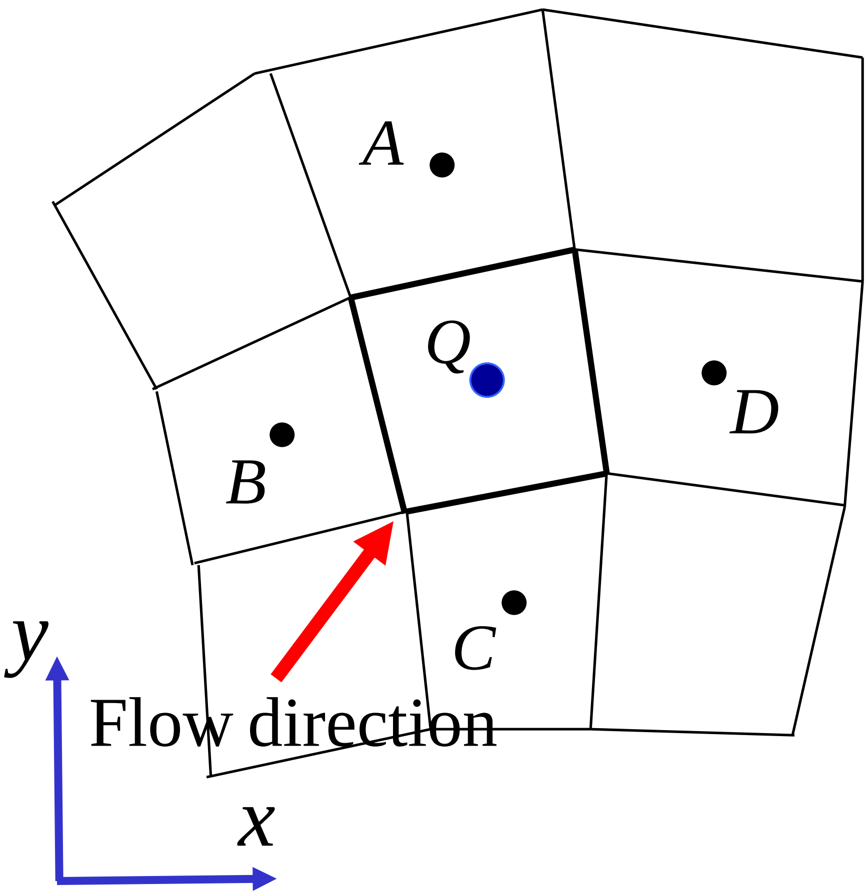


(c) Quadratic Solution, 120 time steps



(d) Cubic Solution, 120 time steps

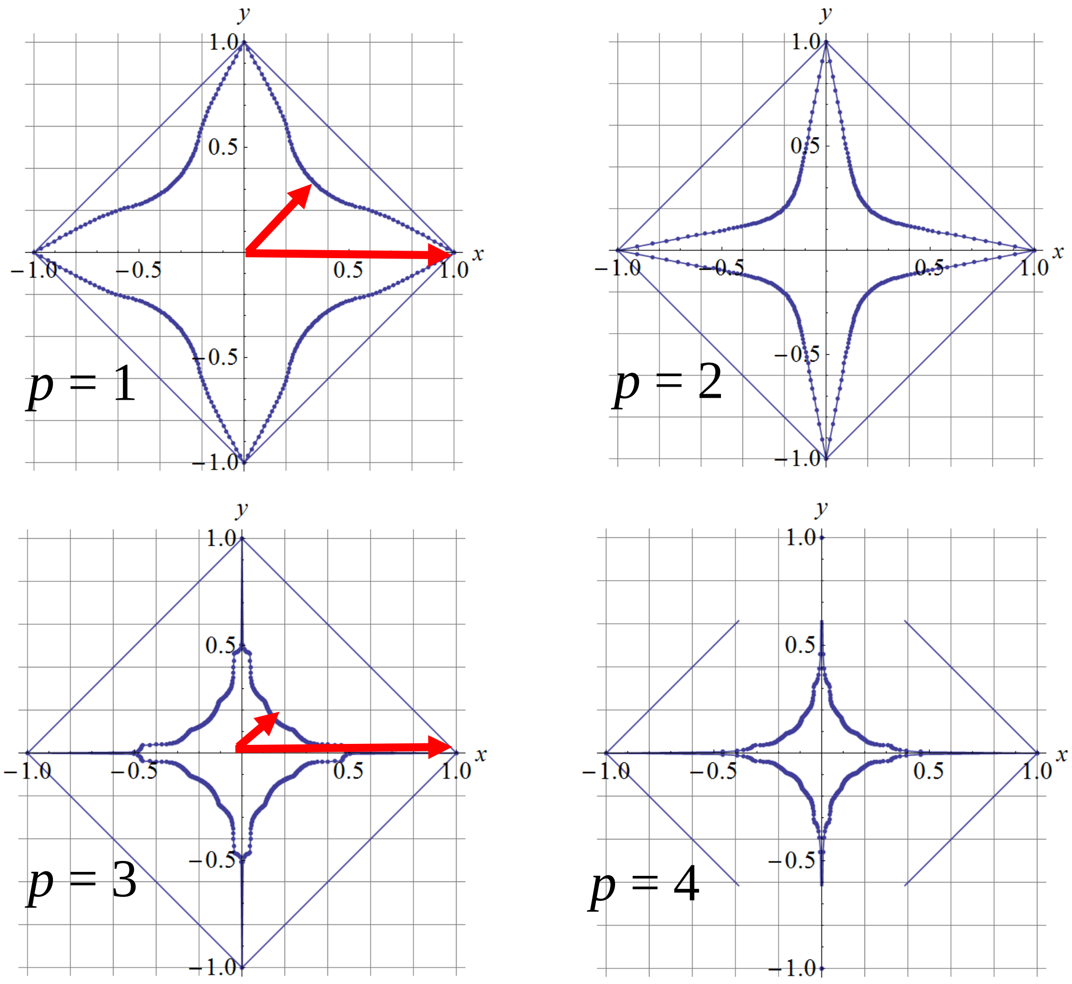
Moment Scheme in 2 Spatial Dimension



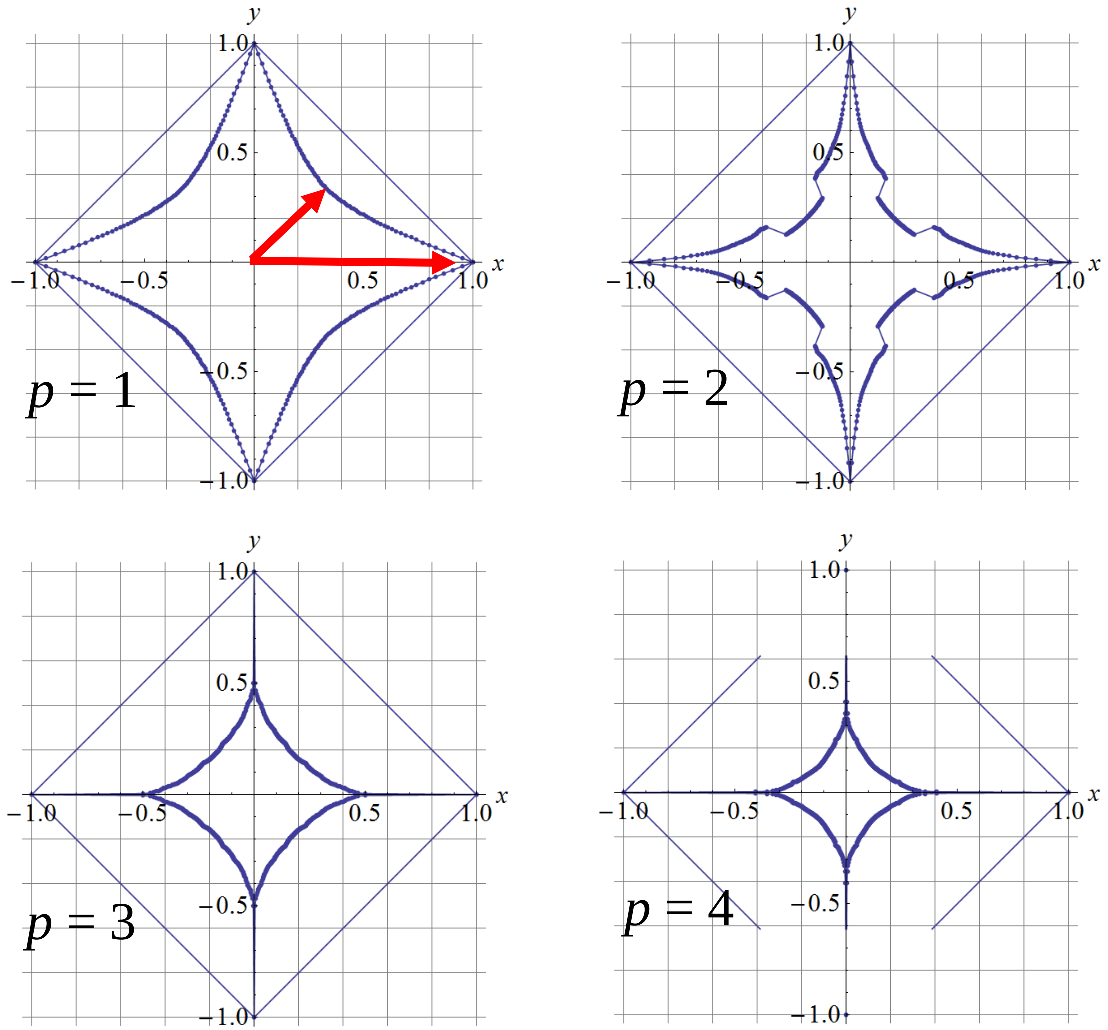
For a quad mesh, the solution for each cell involves the data of the four immediate neighbors, but does NOT involve corner neighbors.

This property manifests itself in the form of a reduction in CFL condition for flows in diagonal directions.

2D Fourier Analysis for Moment Scheme: Stability Regions (Not Tensor Product)



2D Fourier Analysis for Moment Scheme: Stability Regions, Tensor Products



Need New Ideas

Can

- implicit time stepping or
- *p* multigrid

help reduce stiffness along diagonal directions?

Conclusions

- Explicit time method STE-DG (Lörcher, Gassner, and Munz) and Moment Scheme (Huynh) were discussed.
- The moment scheme takes into account the physics of advection. It has a CFL condition of 1 in 1D for all orders, but a restrictive CFL condition along diagonal directions in multiple dimensions.
- Can the moment scheme be employed in some iterative manner in conjunction with the implicit space-time DG or p multigrid to improve stability?
- Further research on time-stepping methods is needed.

Thank you.