

Analytical Partial Derivatives of the Q-Law Guidance Algorithm

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One method of producing low-thrust trajectories is the closed-loop Q-Law guidance algorithm. This work wraps Q-Law inside a nonlinear programming problem to search for locally optimal gain combinations and enforce nonlinear constraints on the initial state. Gradient-based trajectory optimization has been shown to benefit greatly when analytical partial derivatives are supplied to the optimizer. This paper presents the Q-Law optimization problem setup and derives the Q-Law thrust vector partial derivatives for the State Transition Matrix required by standard nonlinear optimization software.

Nomenclature

a	=	semi-major axis, km
e	=	eccentricity
i	=	inclination, deg
f_θ	=	Local frame circumferential perturbing acceleration
f_r	=	Local frame radial perturbing acceleration
f_h	=	Local frame normal perturbing acceleration
m	=	mass, kg
α	=	orbital elements
$\dot{\alpha}$	=	orbital element time derivatives
$\dot{\alpha}_{xx}$	=	orbital element max rate of change
P	=	periapsis constraint
Q	=	Q-Law Lyapunov function
$\dot{Q}$	=	Q-Law Lyapunov function time derivative
S_α	=	semi-major axis penalty
t	=	time, s
u_h	=	normal control component
u_θ	=	circumferential control component
u_r	=	radial control component
W_p	=	periapsis constraint switch
W_α	=	user-defined control gains
μ	=	central body gravitational parameter, km ³ /s ²
α	=	in-plane thrust angle, deg
β	=	out-of-plane thrust angle, deg
Ω	=	right ascension of ascending node, deg
ω	=	argument of periapsis, deg
θ	=	true anomaly, deg

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I. Introduction

Q-Law is a well-known Lyapunov guidance law for generating low-thrust spiral trajectories. The original Q-Law algorithm is presented in detail by Petropoulos[1, 2]. Typically, Q-Law is used within an evolutionary algorithm to stochastically vary the gains to achieve more optimal solutions. Although one Q-Law execution is very efficient, thousands of evaluations within an evolutionary algorithm will add up to considerable computation time. This work wraps Q-Law inside a nonlinear programming problem (NLP) to search for locally optimal gain combinations and enforce nonlinear constraints on the initial state.

NLP solvers can estimate the derivatives needed for each step through finite differences, but this approach is slow and can lead to derivative inaccuracy, resulting in the solver taking sub-optimal steps. To overcome this, analytical partial derivatives are highly sought after in spacecraft trajectory optimization. When available and correctly implemented, they can have drastic effects on the efficiency and accuracy of the solver [3, 4]. As a result, we derive the Q-Law thrust vector partial derivatives with respect to the gains and initial state. These derivatives are required to generate a State Transition Matrix (STM) for the NLP solver. To calculate some derivatives, Matlab's symbolic toolbox was used to generate expressions. Additionally, all derivatives were verified using complex-step.

II. Q-Law Thrust Vector Calculation

The Q-Law Lyapunov function is defined as

$$Q = (1 + W_P P) \sum_{\alpha} \mathbf{W}_{\alpha} \mathbf{S}_{\alpha} \left[\frac{d(\alpha, \alpha_T)}{\dot{\alpha}_{xx}} \right]^2 \quad (1)$$

with $\alpha = (a, e, i, \omega, \Omega)$.

In Equation 1, $\dot{\alpha}_{xx}$ is the max rate of change of each element over the local frame thrust angles α and β and true anomaly, θ . Also, $\mathbf{W}$ is the gain vector given to the orbital elements, and $d(\alpha, \alpha_T)$ is the difference between the values of the current and target orbital elements. The max rate of change of each element, $\dot{\alpha}_{xx}$, are defined as

$$\dot{a}_{xx} = 2f \sqrt{\frac{a^3(1+e)}{\mu(1-e)}} \quad (2)$$

$$\dot{e}_{xx} = \frac{2pf}{h} \quad (3)$$

$$\dot{i}_{xx} = \frac{pf}{h(\sqrt{1-e^2} \sin^2 \omega - e|\cos \omega|)} \quad (4)$$

$$\dot{\Omega}_{xx} = \frac{pf}{h \sin i (\sqrt{1-e^2} \cos^2 \omega - e|\sin \omega|)} \quad (5)$$

$$\dot{\omega}_{xx} = \frac{\dot{\omega}_{xxi} + b\dot{\omega}_{xxo}}{1+b} \quad (6)$$

$$(7)$$

where b is nominally 0.01 and

$$\dot{\omega}_{xxi} = \frac{f}{eh} \sqrt{p^2 \cos^2 \theta_{xx} + (p + r_{xx})^2 \sin^2 \theta_{xx}} \quad (8)$$

$$\dot{\omega}_{xxo} = \dot{\Omega}_{xx} |\cos i| \quad (9)$$

$$\cos \theta_{xx} = \left[\frac{1-e^2}{2e^3} + \sqrt{\frac{1}{4} \left(\frac{1-e^2}{e^3} \right)^2 + \frac{1}{27}} \right]^{1/3} - \left[-\frac{1-e^2}{2e^3} + \sqrt{\frac{1}{4} \left(\frac{1-e^2}{e^3} \right)^2 + \frac{1}{27}} \right]^{1/3} - \frac{1}{e} \quad (10)$$

$$r_{xx} = \frac{p}{1 + e \cos \theta_{xx}} \quad (11)$$

$$(12)$$

In Equation 1, S and P represent penalty functions that keep semi-major axis and periapsis from growing too large or too small. They are defined as

$$S = \begin{cases} \left[1 + \left(\frac{a-a_T}{ma_T} \right)^n \right]^{1/r} & \text{for } \alpha = a \\ 1 & \text{for } \alpha = e, i, \omega, \Omega \end{cases} \quad (13)$$

$$P = \exp \left[k \left(1 - \frac{r_p}{r_{pmin}} \right) \right] \quad (14)$$

with the nominal values of $m = 3$, $n = 4$, $r = 2$, and $k = 100$.

The proximity quotient Q is differentiated to minimize $\dot{Q}$, therefore driving Q to zero as quickly as possible to reach the desired orbit.

The derivative of Q can be calculated using the chain rule, as shown in Eq. (15),

$$\dot{Q} = \sum_{\alpha} \frac{\partial Q}{\partial \alpha} \dot{\alpha} \quad (15)$$

with

$$\dot{\alpha} = \begin{bmatrix} \frac{2a^2}{h} (e \sin \theta f_r + \frac{p}{r} f_{\theta}) \\ \frac{1}{h} (p \sin \theta f_r + [(p+r) \cos \theta + re] f_{\theta}) \\ \frac{r \cos(\theta+\omega)}{h} f_h \\ \frac{r \sin(\theta+\omega)}{h \sin i} f_h \\ \frac{1}{eh} [-p \cos \theta f_r + (p+r) \sin \theta f_{\theta}] - \frac{r \sin(\theta+\omega) \cos i}{h \sin i} f_h \end{bmatrix} \quad (16)$$

and can be rewritten as [5]

$$\dot{Q} = D_1 \cos \beta \cos \alpha + D_2 \cos \beta \sin \alpha + D_3 \sin \beta \quad (17)$$

The terms D_1 , D_2 and D_3 can be calculated by evaluating

$$D_1 = \sum_{\alpha} \frac{\partial Q}{\partial \alpha} \frac{\partial \dot{\alpha}}{\partial f_{\theta}} \quad (18)$$

$$D_2 = \sum_{\alpha} \frac{\partial Q}{\partial \alpha} \frac{\partial \dot{\alpha}}{\partial f_r} \quad (19)$$

$$D_3 = \sum_{\alpha} \frac{\partial Q}{\partial \alpha} \frac{\partial \dot{\alpha}}{\partial f_h} \quad (20)$$

To calculate the partials derivatives of Q , it is convenient to define the intermediate term $\mathbf{V}$.

$$V_{\alpha} = S_{\alpha} \left[\frac{d(\alpha, \alpha_T)}{\dot{\alpha}_{xx}} \right]^2 \quad (21)$$

$$\frac{\partial Q}{\partial \alpha} = W_P \frac{\partial P}{\partial \alpha} \mathbf{W}^T \mathbf{V} + (1 + W_P P) \mathbf{W}^T \frac{\partial \mathbf{V}}{\partial \alpha} \quad (22)$$

The optimal local frame thrust vector for reducing Q is

$$\mathbf{u} = \frac{-(D_2, D_1, D_3)}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (23)$$

III. NLP Problem Setup

For a particular Q-Law trajectory, the controls are the gains, W_{α} , and the spacecraft state is defined as

$$\mathbf{x} = [a, e, i, \omega, \Omega, \theta, m]^T \quad (24)$$

When Q-Law is wrapped in an NLP, the solver chooses the gains in order to optimize the cost function, J , as shown in Equation 25. Additionally, constraints like Equation 26 are enforced to ensure the final state after integration satisfies the desired terminal boundary condition. To effectively optimize the objective function and satisfy the constraints, the NLP decision vector, shown in Equation 27, includes the initial and final states as well as the gains. The final states are included to reduce sensitivity by allowing the optimizer to choose final states that obey the trajectory integration.

$$J = m_f \quad (25)$$

$$\mathbf{c} = \mathbf{x}_f - F(\mathbf{x}_0, W_{\alpha}) = 0 \quad (26)$$

$$\mathbf{X} = [a_0, e_0, i_0, \omega_0, \Omega_0, \theta_0, m_0, a_f, e_f, i_f, \omega_f, \Omega_f, \theta_f, m_f, W_a, W_e, W_i, W_{\omega}, W_{\Omega}]^T \quad (27)$$

For an NLP solver to enforce the constraint in Equation 26 and minimize the cost in Equation 25, the sensitivities of the final state to the initial state and gains are needed. These sensitivities make up the STM, Φ , which is defined as

$$\Phi = \begin{bmatrix} \frac{\partial \mathbf{x}_f}{\partial \mathbf{x}_0}, \frac{\partial \mathbf{x}_f}{\partial W_{\alpha}} \end{bmatrix} \quad (28)$$

and evolves according to Eqs 29 and 30.

$$\dot{\Phi} = A\Phi \quad (29)$$

$$A = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} \quad (30)$$

The STM is integrated alongside the trajectory to yield the final sensitivity matrix which is reported to the NLP solver. However, the state dynamics and A matrix depend on the Q-Law thrust vector, $\mathbf{u} = (u_r, u_{\theta}, u_h)$, which in turn depends on the current state and gains. In order to evaluate the A matrix at each time step during the integration, the partial derivatives of the thrust vector with respect to the current state, $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$, and gains, $\frac{\partial \mathbf{u}}{\partial W}$, are needed.

IV. Thrust Vector Partial Derivatives w.r.t. W_{α}

$$\frac{\partial^2 Q}{\partial \alpha \partial W_a} = \frac{\partial}{\partial W_a} \frac{\partial P}{\partial \alpha} \mathbf{W}^T \mathbf{V} + \frac{\partial}{\partial W_a} (1 + W_P P) \mathbf{W}^T \frac{\partial \mathbf{V}}{\partial \alpha} \quad (31)$$

$$\frac{\partial^2 Q}{\partial \alpha \partial W_a} = \frac{\partial P}{\partial \alpha} \frac{\partial \mathbf{W}^T}{\partial W_a} \mathbf{V} + (1 + W_P P) \frac{\partial \mathbf{W}^T}{\partial W_a} \frac{\partial \mathbf{V}}{\partial \alpha} \quad (32)$$

$$\frac{\partial^2 Q}{\partial \alpha \partial W_a} = \frac{\partial P}{\partial \alpha} [1, 0, 0, 0, 0] \mathbf{V} + (1 + W_P P) [1, 0, 0, 0, 0] \frac{\partial \mathbf{V}}{\partial \alpha} \quad (33)$$

$$\frac{\partial^2 Q}{\partial \alpha \partial W_a} = \frac{\partial P}{\partial \alpha} \mathbf{S}_a \left[\frac{d(a, a_T)}{\dot{a}_{xx}} \right]^2 + (1 + W_P P) \frac{\partial V_a}{\partial \alpha} \quad (34)$$

The partials of $\frac{\partial Q}{\partial \alpha}$ w.r.t. the other gains are derived in a similar fashion.

$$\frac{\partial^2 Q}{\partial \alpha \partial W_e} = \frac{\partial P}{\partial \alpha} \mathbf{S}_e \left[\frac{d(e, e_T)}{\dot{e}_{xx}} \right]^2 + (1 + W_P P) \frac{\partial V_e}{\partial \alpha} \quad (35)$$

$$\frac{\partial^2 Q}{\partial \alpha \partial W_i} = \frac{\partial P}{\partial \alpha} \mathbf{S}_i \left[\frac{d(i, i_T)}{\dot{i}_{xx}} \right]^2 + (1 + W_P P) \frac{\partial V_i}{\partial \alpha} \quad (36)$$

$$\frac{\partial^2 Q}{\partial \alpha \partial W_\omega} = \frac{\partial P}{\partial \alpha} \mathbf{S}_\omega \left[\frac{d(\omega, \omega_T)}{\dot{\omega}_{xx}} \right]^2 + (1 + W_P P) \frac{\partial V_\omega}{\partial \alpha} \quad (37)$$

$$\frac{\partial^2 Q}{\partial \alpha \partial W_\Omega} = \frac{\partial P}{\partial \alpha} \mathbf{S}_\Omega \left[\frac{d(\Omega, \Omega_T)}{\dot{\Omega}_{xx}} \right]^2 + (1 + W_P P) \frac{\partial V_\Omega}{\partial \alpha} \quad (38)$$

A. W_a

$$\frac{\partial D_1}{\partial W_a} = \frac{\partial^2 Q}{\partial \alpha \partial W_a} \frac{\partial \dot{\alpha}}{\partial F_\theta} \quad (39)$$

$$\frac{\partial D_2}{\partial W_a} = \frac{\partial^2 Q}{\partial \alpha \partial W_a} \frac{\partial \dot{\alpha}}{\partial F_r} \quad (40)$$

$$\frac{\partial D_3}{\partial W_a} = \frac{\partial^2 Q}{\partial \alpha \partial W_a} \frac{\partial \dot{\alpha}}{\partial F_h} \quad (41)$$

$$\frac{\partial u_r}{\partial W_a} = \frac{D_2(D_1 \frac{\partial D_1}{\partial W_a} + D_2 \frac{\partial D_2}{\partial W_a} + D_3 \frac{\partial D_3}{\partial W_a})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial W_a}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (42)$$

$$\frac{\partial u_\theta}{\partial W_a} = \frac{D_1(D_1 \frac{\partial D_1}{\partial W_a} + D_2 \frac{\partial D_2}{\partial W_a} + D_3 \frac{\partial D_3}{\partial W_a})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial W_a}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (43)$$

$$\frac{\partial u_h}{\partial W_a} = \frac{D_3(D_1 \frac{\partial D_1}{\partial W_a} + D_2 \frac{\partial D_2}{\partial W_a} + D_3 \frac{\partial D_3}{\partial W_a})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial W_a}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (44)$$

B. W_e

$$\frac{\partial D_1}{\partial W_e} = \frac{\partial^2 Q}{\partial \alpha \partial W_e} \frac{\partial \dot{\alpha}}{\partial F_\theta} \quad (45)$$

$$\frac{\partial D_2}{\partial W_e} = \frac{\partial^2 Q}{\partial \alpha \partial W_e} \frac{\partial \dot{\alpha}}{\partial F_r} \quad (46)$$

$$\frac{\partial D_3}{\partial W_e} = \frac{\partial^2 Q}{\partial \alpha \partial W_e} \frac{\partial \dot{\alpha}}{\partial F_h} \quad (47)$$

$$\frac{\partial u_r}{\partial W_e} = \frac{D_2(D_1 \frac{\partial D_1}{\partial W_e} + D_2 \frac{\partial D_2}{\partial W_e} + D_3 \frac{\partial D_3}{\partial W_e})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial W_e}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (48)$$

$$\frac{\partial u_\theta}{\partial W_e} = \frac{D_1(D_1 \frac{\partial D_1}{\partial W_e} + D_2 \frac{\partial D_2}{\partial W_e} + D_3 \frac{\partial D_3}{\partial W_e})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial W_e}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (49)$$

$$\frac{\partial u_h}{\partial W_e} = \frac{D_3(D_1 \frac{\partial D_1}{\partial W_e} + D_2 \frac{\partial D_2}{\partial W_e} + D_3 \frac{\partial D_3}{\partial W_e})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial W_e}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (50)$$

C. W_i

$$\frac{\partial D_1}{\partial W_i} = \frac{\partial^2 Q}{\partial \alpha \partial W_i} \frac{\partial \dot{\alpha}}{\partial F_\theta} \quad (51)$$

$$\frac{\partial D_2}{\partial W_i} = \frac{\partial^2 Q}{\partial \alpha \partial W_i} \frac{\partial \dot{\alpha}}{\partial F_r} \quad (52)$$

$$\frac{\partial D_3}{\partial W_i} = \frac{\partial^2 Q}{\partial \alpha \partial W_i} \frac{\partial \dot{\alpha}}{\partial F_h} \quad (53)$$

$$\frac{\partial u_r}{\partial W_i} = \frac{D_2(D_1 \frac{\partial D_1}{\partial W_i} + D_2 \frac{\partial D_2}{\partial W_i} + D_3 \frac{\partial D_3}{\partial W_i})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial W_i}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (54)$$

$$\frac{\partial u_\theta}{\partial W_i} = \frac{D_1(D_1 \frac{\partial D_1}{\partial W_i} + D_2 \frac{\partial D_2}{\partial W_i} + D_3 \frac{\partial D_3}{\partial W_i})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial W_i}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (55)$$

$$\frac{\partial u_h}{\partial W_i} = \frac{D_3(D_1 \frac{\partial D_1}{\partial W_i} + D_2 \frac{\partial D_2}{\partial W_i} + D_3 \frac{\partial D_3}{\partial W_i})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial W_i}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (56)$$

D. W_ω

$$\frac{\partial D_1}{\partial W_\omega} = \frac{\partial^2 Q}{\partial \alpha \partial W_\omega} \frac{\partial \dot{\alpha}}{\partial F_\theta} \quad (57)$$

$$\frac{\partial D_2}{\partial W_\omega} = \frac{\partial^2 Q}{\partial \alpha \partial W_\omega} \frac{\partial \dot{\alpha}}{\partial F_r} \quad (58)$$

$$\frac{\partial D_3}{\partial W_\omega} = \frac{\partial^2 Q}{\partial \alpha \partial W_\omega} \frac{\partial \dot{\alpha}}{\partial F_h} \quad (59)$$

$$\frac{\partial u_r}{\partial W_\omega} = \frac{D_2(D_1 \frac{\partial D_1}{\partial W_\omega} + D_2 \frac{\partial D_2}{\partial W_\omega} + D_3 \frac{\partial D_3}{\partial W_\omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial W_\omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (60)$$

$$\frac{\partial u_\theta}{\partial W_\omega} = \frac{D_1(D_1 \frac{\partial D_1}{\partial W_\omega} + D_2 \frac{\partial D_2}{\partial W_\omega} + D_3 \frac{\partial D_3}{\partial W_\omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial W_\omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (61)$$

$$\frac{\partial u_h}{\partial W_\omega} = \frac{D_3(D_1 \frac{\partial D_1}{\partial W_\omega} + D_2 \frac{\partial D_2}{\partial W_\omega} + D_3 \frac{\partial D_3}{\partial W_\omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial W_\omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (62)$$

E. W_Ω

$$\frac{\partial D_1}{\partial W_\Omega} = \frac{\partial^2 Q}{\partial \alpha \partial W_\Omega} \frac{\partial \dot{\alpha}}{\partial F_\theta} \quad (63)$$

$$\frac{\partial D_2}{\partial W_\Omega} = \frac{\partial^2 Q}{\partial \alpha \partial W_\Omega} \frac{\partial \dot{\alpha}}{\partial F_r} \quad (64)$$

$$\frac{\partial D_3}{\partial W_\Omega} = \frac{\partial^2 Q}{\partial \alpha \partial W_\Omega} \frac{\partial \dot{\alpha}}{\partial F_h} \quad (65)$$

$$\frac{\partial u_r}{\partial W_\Omega} = \frac{D_2(D_1 \frac{\partial D_1}{\partial W_\Omega} + D_2 \frac{\partial D_2}{\partial W_\Omega} + D_3 \frac{\partial D_3}{\partial W_\Omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial W_\Omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (66)$$

$$\frac{\partial u_\theta}{\partial W_\Omega} = \frac{D_1(D_1 \frac{\partial D_1}{\partial W_\Omega} + D_2 \frac{\partial D_2}{\partial W_\Omega} + D_3 \frac{\partial D_3}{\partial W_\Omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial W_\Omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (67)$$

$$\frac{\partial u_h}{\partial W_\Omega} = \frac{D_3(D_1 \frac{\partial D_1}{\partial W_\Omega} + D_2 \frac{\partial D_2}{\partial W_\Omega} + D_3 \frac{\partial D_3}{\partial W_\Omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial W_\Omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (68)$$

V. Thrust Vector Partial Derivatives w.r.t. $\mathbf{X}$

A. Semi-major Axis

$$\frac{\partial D_1}{\partial a} = \frac{\partial^2 Q}{\partial \alpha \partial a} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial a} \quad (69)$$

$$\frac{\partial D_2}{\partial a} = \frac{\partial^2 Q}{\partial \alpha \partial a} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial a} \quad (70)$$

$$\frac{\partial D_3}{\partial a} = \frac{\partial^2 Q}{\partial \alpha \partial a} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial a} \quad (71)$$

$$\frac{\partial u_r}{\partial a} = \frac{D_2(D_1 \frac{\partial D_1}{\partial a} + D_2 \frac{\partial D_2}{\partial a} + D_3 \frac{\partial D_3}{\partial a})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial a}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (72)$$

$$\frac{\partial u_\theta}{\partial a} = \frac{D_1(D_1 \frac{\partial D_1}{\partial a} + D_2 \frac{\partial D_2}{\partial a} + D_3 \frac{\partial D_3}{\partial a})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial a}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (73)$$

$$\frac{\partial u_h}{\partial a} = \frac{D_3(D_1 \frac{\partial D_1}{\partial a} + D_2 \frac{\partial D_2}{\partial a} + D_3 \frac{\partial D_3}{\partial a})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial a}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (74)$$

B. Eccentricity

$$\frac{\partial D_1}{\partial e} = \frac{\partial^2 Q}{\partial \alpha \partial e} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial e} \quad (75)$$

$$\frac{\partial D_2}{\partial e} = \frac{\partial^2 Q}{\partial \alpha \partial e} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial e} \quad (76)$$

$$\frac{\partial D_3}{\partial e} = \frac{\partial^2 Q}{\partial \alpha \partial e} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial e} \quad (77)$$

$$\frac{\partial u_r}{\partial e} = \frac{D_2(D_1 \frac{\partial D_1}{\partial e} + D_2 \frac{\partial D_2}{\partial e} + D_3 \frac{\partial D_3}{\partial e})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial e}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (78)$$

$$\frac{\partial u_\theta}{\partial e} = \frac{D_1(D_1 \frac{\partial D_1}{\partial e} + D_2 \frac{\partial D_2}{\partial e} + D_3 \frac{\partial D_3}{\partial e})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial e}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (79)$$

$$\frac{\partial u_h}{\partial e} = \frac{D_3(D_1 \frac{\partial D_1}{\partial e} + D_2 \frac{\partial D_2}{\partial e} + D_3 \frac{\partial D_3}{\partial e})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial e}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (80)$$

C. Inclination

$$\frac{\partial D_1}{\partial i} = \frac{\partial^2 Q}{\partial \alpha \partial i} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial i} \quad (81)$$

$$\frac{\partial D_2}{\partial i} = \frac{\partial^2 Q}{\partial \alpha \partial i} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial i} \quad (82)$$

$$\frac{\partial D_3}{\partial i} = \frac{\partial^2 Q}{\partial \alpha \partial i} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial i} \quad (83)$$

$$\frac{\partial u_r}{\partial i} = \frac{D_2(D_1 \frac{\partial D_1}{\partial i} + D_2 \frac{\partial D_2}{\partial i} + D_3 \frac{\partial D_3}{\partial i})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial i}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (84)$$

$$\frac{\partial u_\theta}{\partial i} = \frac{D_1(D_1 \frac{\partial D_1}{\partial i} + D_2 \frac{\partial D_2}{\partial i} + D_3 \frac{\partial D_3}{\partial i})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial i}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (85)$$

$$\frac{\partial u_h}{\partial i} = \frac{D_3(D_1 \frac{\partial D_1}{\partial i} + D_2 \frac{\partial D_2}{\partial i} + D_3 \frac{\partial D_3}{\partial i})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial i}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (86)$$

D. Argument of Periapsis

$$\frac{\partial D_1}{\partial \omega} = \frac{\partial^2 Q}{\partial \alpha \partial \omega} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial \omega} \quad (87)$$

$$\frac{\partial D_2}{\partial \omega} = \frac{\partial^2 Q}{\partial \alpha \partial \omega} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial \omega} \quad (88)$$

$$\frac{\partial D_3}{\partial \omega} = \frac{\partial^2 Q}{\partial \alpha \partial \omega} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial \omega} \quad (89)$$

$$\frac{\partial u_r}{\partial \omega} = \frac{D_2(D_1 \frac{\partial D_1}{\partial \omega} + D_2 \frac{\partial D_2}{\partial \omega} + D_3 \frac{\partial D_3}{\partial \omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial \omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (90)$$

$$\frac{\partial u_\theta}{\partial \omega} = \frac{D_1(D_1 \frac{\partial D_1}{\partial \omega} + D_2 \frac{\partial D_2}{\partial \omega} + D_3 \frac{\partial D_3}{\partial \omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial \omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (91)$$

$$\frac{\partial u_h}{\partial \omega} = \frac{D_3(D_1 \frac{\partial D_1}{\partial \omega} + D_2 \frac{\partial D_2}{\partial \omega} + D_3 \frac{\partial D_3}{\partial \omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial \omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (92)$$

E. Longitude of Ascending Node

$$\frac{\partial D_1}{\partial \Omega} = \frac{\partial^2 Q}{\partial \alpha \partial \Omega} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial \Omega} \quad (93)$$

$$\frac{\partial D_2}{\partial \Omega} = \frac{\partial^2 Q}{\partial \alpha \partial \Omega} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial \Omega} \quad (94)$$

$$\frac{\partial D_3}{\partial \Omega} = \frac{\partial^2 Q}{\partial \alpha \partial \Omega} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial \Omega} \quad (95)$$

$$\frac{\partial u_r}{\partial \Omega} = \frac{D_2(D_1 \frac{\partial D_1}{\partial \Omega} + D_2 \frac{\partial D_2}{\partial \Omega} + D_3 \frac{\partial D_3}{\partial \Omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial \Omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (96)$$

$$\frac{\partial u_\theta}{\partial \Omega} = \frac{D_1(D_1 \frac{\partial D_1}{\partial \Omega} + D_2 \frac{\partial D_2}{\partial \Omega} + D_3 \frac{\partial D_3}{\partial \Omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial \Omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (97)$$

$$\frac{\partial u_h}{\partial \Omega} = \frac{D_3(D_1 \frac{\partial D_1}{\partial \Omega} + D_2 \frac{\partial D_2}{\partial \Omega} + D_3 \frac{\partial D_3}{\partial \Omega})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial \Omega}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (98)$$

F. True Anomaly

$$\frac{\partial D_1}{\partial \theta} = \frac{\partial^2 Q}{\partial \alpha \partial \theta} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial \theta} \quad (99)$$

$$\frac{\partial D_2}{\partial \theta} = \frac{\partial^2 Q}{\partial \alpha \partial \theta} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial \theta} \quad (100)$$

$$\frac{\partial D_3}{\partial \theta} = \frac{\partial^2 Q}{\partial \alpha \partial \theta} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial \theta} \quad (101)$$

$$\frac{\partial u_r}{\partial \theta} = \frac{D_2(D_1 \frac{\partial D_1}{\partial \theta} + D_2 \frac{\partial D_2}{\partial \theta} + D_3 \frac{\partial D_3}{\partial \theta})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial \theta}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (102)$$

$$\frac{\partial u_\theta}{\partial \theta} = \frac{D_1(D_1 \frac{\partial D_1}{\partial \theta} + D_2 \frac{\partial D_2}{\partial \theta} + D_3 \frac{\partial D_3}{\partial \theta})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial \theta}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (103)$$

$$\frac{\partial u_h}{\partial \theta} = \frac{D_3(D_1 \frac{\partial D_1}{\partial \theta} + D_2 \frac{\partial D_2}{\partial \theta} + D_3 \frac{\partial D_3}{\partial \theta})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial \theta}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (104)$$

G. Mass

$$\frac{\partial D_1}{\partial m} = \frac{\partial^2 Q}{\partial \alpha \partial m} \frac{\partial \dot{\alpha}}{\partial F_\theta} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_\theta \partial m} \quad (105)$$

$$\frac{\partial D_2}{\partial m} = \frac{\partial^2 Q}{\partial \alpha \partial m} \frac{\partial \dot{\alpha}}{\partial F_r} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_r \partial m} \quad (106)$$

$$\frac{\partial D_3}{\partial m} = \frac{\partial^2 Q}{\partial \alpha \partial m} \frac{\partial \dot{\alpha}}{\partial F_h} + \frac{\partial Q}{\partial \alpha} \frac{\partial^2 \dot{\alpha}}{\partial F_h \partial m} \quad (107)$$

$$\frac{\partial u_r}{\partial m} = \frac{D_2(D_1 \frac{\partial D_1}{\partial m} + D_2 \frac{\partial D_2}{\partial m} + D_3 \frac{\partial D_3}{\partial m})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_2}{\partial m}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (108)$$

$$\frac{\partial u_\theta}{\partial m} = \frac{D_1(D_1 \frac{\partial D_1}{\partial m} + D_2 \frac{\partial D_2}{\partial m} + D_3 \frac{\partial D_3}{\partial m})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_1}{\partial m}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (109)$$

$$\frac{\partial u_h}{\partial m} = \frac{D_3(D_1 \frac{\partial D_1}{\partial m} + D_2 \frac{\partial D_2}{\partial m} + D_3 \frac{\partial D_3}{\partial m})}{(D_1^2 + D_2^2 + D_3^2)^{3/2}} - \frac{\frac{\partial D_3}{\partial m}}{\sqrt{D_1^2 + D_2^2 + D_3^2}} \quad (110)$$

VI. Conclusion and Future Work

In this work, partial derivatives of the Q-Law thrust vector calculation are derived for use in optimizing a Q-Law trajectory's input parameters using an NLP solver. The derivatives are taken with respect to the gains given to each orbital element and the initial spacecraft state. The current effort focuses on using these partials to generate a STM that will be integrated alongside the trajectory. This STM will be provided to a nonlinear optimizer to efficiently select the optimal gains and enforce potential nonlinear constraints on the initial spacecraft state.

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