

WMLES of Boundary-Layer Transition

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Abstract

An analysis of the potential for the accurate prediction of laminar-turbulent transition by Wall-Modeled Large-Eddy Simulation (WMLES) indicates that satisfactory prediction of transition by WMLES is possible, provided a) a sufficiently accurate wall model for the initial stage of transition is employed, b) a satisfactory transition is made between this initial-stage wall model and the wall model for the downstream fully-developed turbulence, and c) limitations on temporal and spatial resolution, and on the thickness of the wall-modeled region, imposed by the need to accurately resolve the disturbance waves and the transition region are observed. This is illustrated with WMLES computations of subharmonic transition in a flat-plate boundary layer that yield reasonable agreement with DNS results. However, the wall-modeled region thickness, and thus the cost savings, is less than is typical for WMLES of fully developed turbulence.

I. Introduction

Wall-modeled large-eddy simulation (WMLES) of the full laminar-turbulent transition process requires that the wall model be effective during the development of the disturbance modes, through the breakdown of discrete disturbances and the development of broader spectrum disturbances and finally the evolution into fully developed turbulence. Wall-resolved large-eddy simulation (WRLES) has been successful¹ when a dynamic subgrid model is employed, but whether a wall-modeled computation is possible remains to be determined. Related work, employing DNS to study transition, includes the study of H-type, K-type and bypass transition by Wu and Moin,² the study of final-stage transition mechanisms in H-type and K-type transitions by Sayadi et al.³ and the associated reduced-order modeling Sayadi et al.,⁴ and the work connecting various forms of Parabolized Stability Equation (PSE) analysis, WMLES, WRLES and DNS by Lozano-Duran et al.⁵

In the present paper, a wall model is developed for the initial, pretransition, region of the flow, characterized by the linear and nonlinear growth of distinct disturbance waves. This wall model is valid in the bottom five or ten percent of the boundary layer, below the critical-layer of the disturbances, and thus promises significant computational cost savings due to a reduced number of grid points and, most importantly, due to an increased time step resulting from the elimination of the smallest grid cells.

To model the wall behavior during transition itself, a simple approach is followed of blending the new pretransition wall model with an equilibrium wall model for fully developed turbulence. This punts on important questions about the nature of the transition region and how to model it, but, at a minimum, this approach provides a starting point from which these questions may be addressed. The degree of blending of the two constituent wall models at each wall location may be determined in a variety of ways; the local position and local mean wall shear stress are employed in this work. The subgrid-scale model for the transition region is blended in a similar way.

The classic experiment by Kachanov and Levchenko⁶ of subharmonic transition in a flat-plate boundary layer provides a good test case for these ideas. Transition is due to the resonance of three-dimensional subharmonic secondary waves with a two-dimensional Tollmein-Schlichting primary wave. When the amplitude growth from the resonance gets large enough, further nonlinear mechanisms come into play and transition to

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turbulence takes place. In the following, the effectiveness of the proposed pretransition and transition wall models in capturing the essential dynamics of these stages is evaluated.

The test case and models are detailed in the following section. Section III is concerned with numerics and Section IV presents DNS computations validating the numerics against previous computations for the pretransition and transition regions. The pretransition wall model is employed in WMLES of the pretransition region in Section V and the blended transition model is tested in Section VI with several blending strategies. The paper concludes with a summary and assessment of the results.

II. Problem Description and Models

The H-type subharmonic mode of laminar-turbulent transition has been chosen as the test case for the present investigation. Subharmonic disturbance modes in resonance as a precursor for laminar-turbulent transition were identified by Kachanov and Levchenko⁶ in 1984, followed soon by Herbert's stability analysis⁷ further clarifying the nature of the modes involved and their interaction. A highly accurate DNS of the disturbance-wave interaction has been presented by Joslin, Streett and Chang.⁸

II.A. Resonant Triad and Inflow Prescription

The subharmonic resonant triad consists of a two-dimensional Tollmein-Schlichting primary wave,

$$u_i = a_{20}\phi_{20,i}(y)E + \text{complex conjugate}, \quad (1)$$

and two three-dimensional secondary waves,

$$u_i = a_{11} (\phi_{11,i}^+(y)e^{i\beta z} + \phi_{11,i}^-(y)e^{-i\beta z}) E^{\frac{1}{2}} + \text{complex conjugate}, \quad (2)$$

where $E = e^{i(\alpha x - \omega t)}$ and α , β and ω are the streamwise and spanwise wavenumbers and the frequency (numerical values are specified below). The amplitude a_{nm} is that of the wave with frequency $n\omega/2$ and spanwise wavenumber $m\beta$ and the eigenfunction $\phi_{nm,i}^{\{\pm\}}$ is that for the i th velocity component u_i of the primary wave or $\pm$ secondary wave, with frequency and spanwise wavenumber n and m as for the amplitudes. These waves form a resonant triad: the quadratically nonlinear Navier-Stokes equations generate products of the two secondary waves that force the primary wave and products of the primary wave with each of the secondary waves that force the other secondary wave. This means the effect of the nonlinear interaction is felt at the order of the disturbance amplitude, rather than the disturbance amplitude squared, as would be the case for an arbitrary collection of disturbance waves. The resonant triad interaction is thus capable of initiating transition with much smaller initial disturbance amplitudes.

These expressions, along with the Blasius boundary-layer solution, provide the inflow condition for the computations to follow.

II.B. Wall Modeling

As introduced by Cabot and Moin,⁹ the WMLES wall model is a thin-layer approximation to the Reynolds-averaged Navier-Stokes equations employing a mixing-length model for the eddy viscosity. In the initial stage of transition, the eddy viscosity is zero and, in the simplest approximation, the thin-layer equations reduce to one-dimensional, parallel-flow, equations for the near-wall flow. This is the analog, in the present context, of the equilibrium wall model for fully developed turbulence. While it clearly lacks the physics to capture the dynamics of the critical layer and requires modification to provide sufficient accuracy, the basic concept will be seen to provide satisfactory results if the wall-modeled region is entirely below the critical layer. In the present subsection, this pretransition model, valid for the initial stage of transition, is described, along with its blending with the equilibrium model for fully developed turbulence to permit computations of the full transition process.

II.B.1. Pretransition Wall Model

The principal contribution of the present work is the introduction of a pretransition wall model and its incorporation with a traditional wall model for fully developed turbulence to make a comprehensive wall

model for the full transition process. We begin by clarifying what must be true of a pretransition wall model in order that it in fact provides an improvement in computational cost over simply resolving the wall region.

The combination of base flow and disturbances that comprise the flow prior to transition are solutions of the pure Navier-Stokes equations, with no turbulence modeling. In the most general case of wall modeling, nonequilibrium wall modeling, the Navier-Stokes equations are solved with a subgrid-scale model in the LES region, away from the wall, and the Navier-Stokes equations are solved with a RANS model (the RANS equations) near the wall. In the pretransition region, where there is no turbulence, the subgrid-scale and RANS models must be turned off and the pure Navier-Stokes equations are solved in both the near-wall layer and the LES region. This is effectively DNS; consequently DNS-like resolution is required and there is no computational cost savings.

Computational cost savings is possible through wall modeling only if the wall model involves a cost-saving approximation to the pure Navier-Stokes equations. The most obvious way to do this is with an equilibrium wall model,⁹ a thin-layer approximation that permits simplification of the full Navier-Stokes and thus computational savings.

The wall model as a thin-layer approximation is emphasized here because it provides a systematic means of improving the accuracy of the model beyond the simple, first-order, parallel-flow approximation. That the computation of the resonant triad, in particular the primary wave, requires a high degree of accuracy was observed by Herbert⁷ in his original discussion of subharmonic resonance and is evident in any DNS of this phenomenon (e.g., Ref. 8).

It will be helpful in understanding the nature of the problem to be solved to make the thin-layer nature of the equilibrium modeling assumption explicit. Thus, the incompressible Navier-Stokes equations are considered in a domain between a wall at $y = 0$ and an external flow zone considered to start at $y = h$, $h \ll Re^{-\frac{1}{2}} \ll 1$:

$$\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t - \left(\left(\nu_t + \frac{1}{Re} \right) u_y \right)_y + p_x &= -(uu_x + vu_y + wu_z) + \nabla_\perp \cdot \left(\left(\nu_t + \frac{1}{Re} \right) \nabla_\perp u \right) \\
w_t - \left(\left(\nu_t + \frac{1}{Re} \right) w_y \right)_y + p_z &= -(uw_x + vw_y + ww_z) + \nabla_\perp \cdot \left(\left(\nu_t + \frac{1}{Re} \right) \nabla_\perp w \right) \\
p_y &= -(v_t + uv_x + vv_y + vw_z) + \nabla \cdot \left(\left(\nu_t + \frac{1}{Re} \right) \nabla v \right)
\end{aligned} \tag{3}$$

where t is time, x , y and z are the streamwise, wall-normal and spanwise spatial coordinates, respectively, and u , v and w are the corresponding velocities. The pressure is p , Re is the Reynolds number and ν_t is the eddy viscosity. The three-dimensional gradient operator is ∇ and $\nabla_\perp$ is the gradient operator in the two dimensions x and z . No-slip boundary conditions are imposed at the wall, $y = 0$, and the solution is matched to the LES solution at $y = h$.

Equation 3 is written with the higher-order terms on the right to emphasize the thin-layer aspects of the equilibrium model and how that model may be refined without losing its computational efficiency. In particular, the convective terms are higher order than the time-derivative and viscous terms because, first, the wall layer is thinner than the boundary-layer thickness $Re^{-\frac{1}{2}}$ (these terms all balance when $h = Re^{-\frac{1}{2}}$, giving the traditional boundary-layer equations), and, second, the wall layer is below the critical layer, where the disturbance wave's phase velocity and the base flow velocity are comparable and so the streamwise convective term is comparable with the leading-order terms.

The essential feature of the thin-layer approximation that allows an equilibrium wall model to yield computational savings is that the leading-order operators of the momentum equations (the left-hand sides in Equation 3) involve only derivatives with respect to the wall-normal direction, y , and time. Thus, the computation of the wall model reduces to a set of one-dimensional computations (one at each wall grid point), which is much cheaper than a fully coupled three-dimensional solution in the near-wall region.

For example, in the case of the traditional equilibrium wall model for fully developed (statistically steady) turbulence,⁹ the right-hand sides of Equation 3 are considered to be negligible, as are the time derivatives, and one has, for, say, the streamwise momentum equation,

$$\left(\left(\nu_t + \frac{1}{Re} \right) u_y \right)_y = 0. \tag{4}$$

If a mixing-length model is employed, the eddy viscosity is linear in y and this equation may be integrated to yield what is essentially the familiar log-layer expression for the mean velocity in a turbulent wall-bounded shear flow.

A similar approach to a pretransition wall model involves keeping the time derivatives, since the magnitude of the time derivative relative to the viscous term for the disturbances is $\omega h^2 Re = O(1)$. The higher-order terms on the right-hand side of Equation 3 also need to be included for sufficient accuracy; the solution involving these terms can be obtained either perturbatively or iteratively without compromising the one-dimension-plus-time structure that is responsible for the computational cost savings. Results from this approach of directly integrating Equation 3 will be presented in a forthcoming publication.

In the present work, however, an algebraic wall model is developed as an alternative to this ordinary differential equation wall model. The preceding analysis, in establishing the conditions under which the equilibrium differential-equation model can be expected to be valid ($h \ll Re^{-\frac{1}{2}} \ll 1$, $y = h$ below the critical layer), also prescribes the same conditions for this corresponding equilibrium algebraic wall model. Just as with the equilibrium differential-equation model, the algebraic model is computed at each wall location independently, preserving the equilibrium-model cost savings. Instead of solving differential equations at each wall location, however, a specific functional form $u_i = f(y, \alpha_0, \dots, \alpha_m)$ with parameters α_j is fit to the instantaneous LES velocity field at each wall location, with the fitting of the parameters analogous to applying the boundary conditions at $y = h$ for a differential-equation model. The resulting wall-layer profile is then differentiated to provide the shear stress: $\tau|_{y=h} = \mu f_y(h, \alpha_0, \dots, \alpha_m)$. This is just like using an algebraic wall model based on a log-law mean velocity profile for fully developed turbulence: the LES velocity value at the exchange point $y = h$ is used to fit the wall-layer mean velocity profile $u_i|_{y=h} = U_{\log \text{ law}}(h, u_\tau)$, from which the wall shear stress may be extracted via the friction velocity u_τ .

In the case of the pretransition wall model, the differential Equation 3 is too complex to yield an explicit solution like the log-law mean velocity profile, so an n th-order polynomial is employed. Thus, $f(y, \alpha_0, \dots, \alpha_m) = \sum_{j=0}^m \alpha_j y^j$, and the polynomial coefficients are determined by imposing the no-slip boundary condition at the wall and the condition that the polynomial fit, in the least-square sense, the m points closest to the wall of the instantaneous LES velocity profile. The derivative of the polynomial then provides the shear stress.

This approach has a number of potential advantages. Implementation is straightforward and the computational cost is competitive with, and possibly an improvement on, that of the corresponding differential-equation equilibrium wall model. Unlike a fully turbulent solution, the solutions for the disturbances in the pretransition region are known to be fairly smooth in the neighborhood of the wall, increasing the likelihood that a low- to moderate-order polynomial could effectively approximate them.

In keeping with wall model problem as specified by Equation 3 and its boundary conditions, the velocity derivative (or shear stress) passed back to the LES solution as a boundary condition is evaluated at $y = h$, rather than $y = 0$. The use of the wall shear stress (velocity derivative evaluated at $y = 0$) is customary in WMLES of fully developed turbulence, where a fundamental aspect of the wall-bounded turbulent shear flow profile is that the shear stress is constant in the near-wall and log layers. Thus, the shear stress is the same at $y = 0$ and $y = h$ and applying a given shear stress at $y = 0$ yields the desired shear stress at $y = h$, too. In the case of pretransition disturbances, on the other hand, there are amplitude and phase differences between the values at $y = 0$ and $y = h$ and the correct value of the shear stress is imposed directly at $y = h$.

The process of applying the wall boundary condition to the LES computation via the wall model is illustrated in Figure 1. At the beginning of each time step, polynomial profiles for the wall region are constructed for the streamwise and spanwise velocity components using a least-squares fit to the wall value ($j = 0$, where the no-slip condition is applied) and the values from the LES computation at the first m points in the LES grid. The normal velocity is computed via the continuity equation. These velocity profiles are used to determine the normal derivatives of the velocity at $y = h$, which are then employed as boundary conditions for the LES computation.

The time-splitting algorithm used in this work requires wall modeling of the normal intermediate velocity that results from the first step in the algorithm. Because the intermediate velocity field does not satisfy the continuity equation, a polynomial is constructed for the normal intermediate velocity, as described above for the spanwise and streamwise velocities, and the required normal derivative is determined from that polynomial.

The results of computations using this approach exhibit a continuous variation as m is changed, so a means for determining the best m is needed. Naturally, the ultimate consideration is whether the disturbance

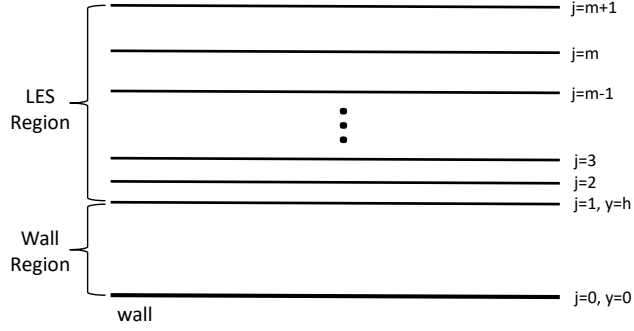


Figure 1. Diagram showing context for construction of wall-model polynomials.

waves are propagated accurately across the computational domain, with correct growth rates, wave speeds, etc., but a more local criterion is necessary if the wall model is to be a practical tool for WMLES, particularly if complex geometries are involved. It will be seen in the results to follow that the residuals in reasonably small domains adjacent to the wall show a clear minimum at a specific value of m , providing at least a semilocal criterion. The contributions of each of the m points to the least-squares computation are equally weighted in the present work; unequal weighting could be employed if results from any given m are not close enough. The same value of m is used everywhere in the present computations, but m could vary from point to point in more complex flow situations.

II.C. Blending Pretransitional and Fully Turbulent Wall and Subgrid-Scale Models

To perform a wall-modeled computation of the complete transition process, the wall model must, at a minimum, switch from the pretransition model to the fully turbulent model in the neighborhood of the rapid rise in wall shear stress. In the present work, this switch takes the form of a linear blending based on a parameter λ , which may be controlled by some sensor function such as streamwise location, wall shear stress, etc. Thus, for example, the normal derivative at $y = h$ of the streamwise velocity provided to the LES solution is

$$u_y = (1 - \lambda)u_{y\text{pre}} + \lambda u_{y\text{turb}}, \quad (5)$$

where $u_{y\text{pre}}$ is the velocity derivative from the present pretransition model and $u_{y\text{turb}}$ is the derivative from an algebraic fully turbulent equilibrium wall model. It is important that this latter model includes the viscous sublayer, as well as the log layer, in the velocity profile, because it will need to be valid for y^+ as low as five.

The subgrid-scale model is handled similarly: the blending parameter is employed to switch from the pretransition subgrid-scale model $\nu_t = 0$ (which is to say, no model) to the fully turbulent model $\nu_t = \nu_{\text{smag}}$,

$$\nu_t = \lambda \nu_{\text{smag}}. \quad (6)$$

These expressions represent the bare minimum in content that complete wall models and subgrid-scale models for the transition process must have, but they will be seen to be reasonably effective in the results presented below. However, by constructing the complete models out of only the pretransition and fully turbulent models, the assumption is made that there is no new physics and no new modeling required for the transition process itself and this assumption is likely false.

III. Numerics

The DNS and WMLES computations presented in this paper were performed with a mixed finite-difference/spectral code previously employed in hybrid LES-RANS computations.^{10–14} The streamwise and normal directions are discretized on a structured grid with fourth-order finite differences, and the spanwise

direction is discretized spectrally. Splitting is employed for time advancement; in the present computations, a Crank-Nicholson/explicit-Runge-Kutta scheme was used for the viscous/convective terms and then the pressure correction is computed from a Poisson equation.

The computational domain is sized following Ref. 5, with inflow boundary at $x = 1$ (where $Re = 180,000$), outflow boundary at $x = 5$, plate surface at $y = 0$ and upper boundary at $y = 0.3535$. All lengths are nondimensionalized by the distance between the plate leading edge and the inflow. The span is the spanwise wavelength of the disturbances introduced at the inflow boundary. For the DNS, the uniform streamwise spacing is $\Delta x^+ = 11.3$ and the number of spanwise modes is 96, corresponding to a uniform grid size of $\Delta z^+ = 5.6$. The first grid point from the wall is at $y^+ = 0.11$. The computations presented here employ various subsets of the full computational domain and the WMLES computations are performed on grids that are coarsened and have near-wall points removed. Further details are provided in the discussion below.

Inflow conditions are specified using the Blasius boundary layer solution and eigenfunction solutions for the disturbances.¹⁵ The frequency and wavenumbers of the disturbances (defined above) are $\omega = 22.311$, $\alpha = 60.695$ and $\beta = 58.447$. The amplitude of the primary wave is 0.0048 and that of the secondary wave is 1.45×10^{-5} . The Blasius boundary layer solution is also specified at the upper boundary, following Ref. 5. No-slip conditions are specified at the plate surface and the standard convective boundary condition is used at the outflow boundary.¹⁶

IV. Direct Numerical Simulation

We begin with some validation computations, involving partial-domain computations that will be convenient in the analysis to follow.

IV.A. Initial-Stage DNS

In the initial stage of the computation, the flow is characterized by the Blasius boundary-layer solution perturbed by the relatively small number of plane-wave disturbances that make up the subharmonic resonance triad and their low-order harmonics. These are the waves employed in the successful PSE computations of this stage of transition.^{5,15}

DNS for this initial stage of transition employing the code described in the previous section was performed on a truncated section of the computational domain (Figure 2). The region from $x = 1$ to $x = 2.5$ is gridded uniformly as described in the previous section; aggressive grid stretching and a sponge layer following Ref. 8 are employed beyond $x = 2.5$ to reduce reflections from the outflow boundary. Four spanwise modes are employed.

The streamwise evolution of mode amplitudes in Figure 2 shows good agreement with the computations of Ref. 5. The computed mode shapes for the primary and secondary waves at $x = 1.9$ are compared with those computed in Ref. 8 in Figure 3.

IV.B. Final-Stage DNS

In the final stage of transition, the flow dynamics are characterized by spectrum broadening, an end to small numbers of easily identifiable modes and an evolution into fully developed turbulence. It is convenient for validation purposes, and it will be even more convenient in the WMLES experimentation to follow, to consider this final stage on its own, without computing the full run up to transition in the original computational domain. Thus, the final-stage computations (see Figure 2) will have inflow conditions specified at $x = 2$; these inflow conditions are based on the initial-stage computation discussed above. No grid stretching or sponge layer is employed at the outflow boundary, as the flow is fully turbulent there, and the convective outflow boundary condition¹⁶ is sufficient. The plots of skin friction and mode amplitudes vs. streamwise position in Figure 4 show the present computation agrees well with that of Ref. 5, once some initial transients are past.

Further comparisons may be made by considering mean-velocity and Reynolds stresses after the flow has reached fully developed turbulence. At the streamwise location $x = 3.8$, for example, the momentum-thickness Reynolds number is about one thousand. The streamwise mean velocity profile there is shown in Figure 5, along with that of the computation of Ref. 5 and that of the DNS of Ref. 17. Reynolds-stress profiles from these computations are shown in Figure 6.

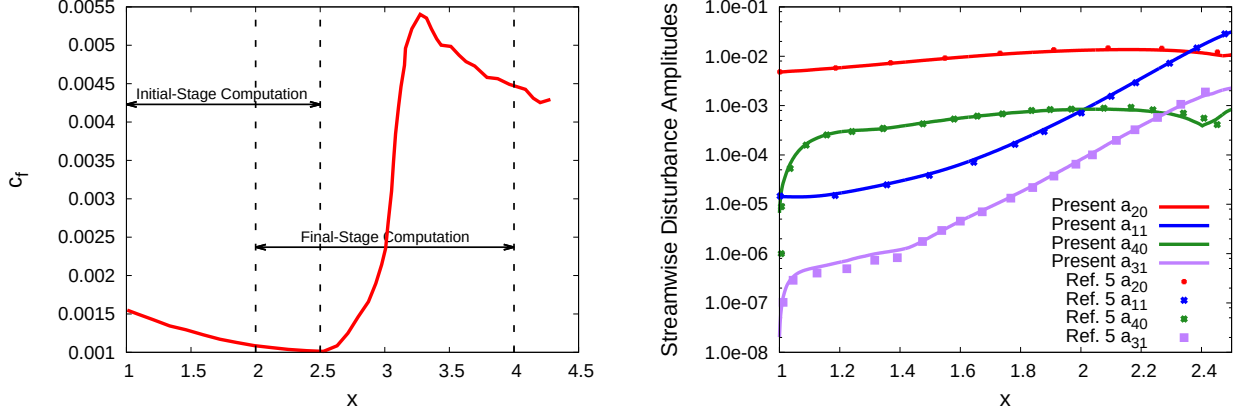


Figure 2. Left: Definition of initial- and final-stage computational domains. Right: Streamwise growth of disturbance-mode amplitudes. Present DNS results compared with those of Ref. 5.

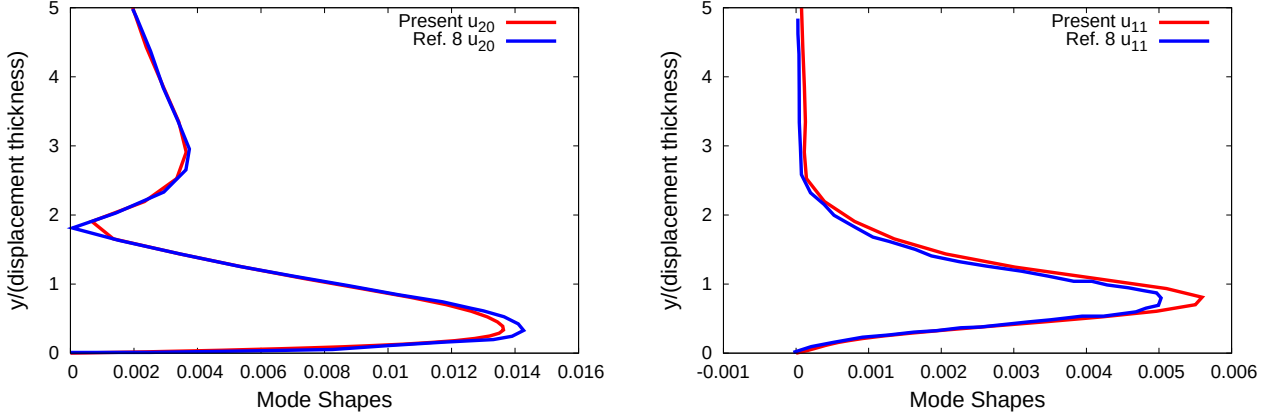


Figure 3. Primary-wave (left) and secondary-wave (right) mode functions (red curves) compared with those of Joslin et al.⁸ (blue curves).

V. WMLES of Initial-Stage Transition

To illustrate the present wall-modeling approach, it is applied first to the range $x = 1$ to $x = 1.4$ with only the primary wave included. As has been noted ever since the original analysis of subharmonic resonance by Herbert,⁷ it is the computation of the primary waves that requires extremely high accuracy. Fourth-order polynomials are employed and the case is run with m ranging from 6 to 9. The streamwise evolution of the primary-wave amplitude is shown in Figure 7 and the computations for the various m values are seen to bracket the result from the full-resolution DNS. Examination of the time-averaged norm of the residual from a flow region encompassing half a wavelength in the streamwise direction and most of the boundary layer in the vertical direction (Figure 7) suggests $m = 7$ yields the lowest residual values. The corresponding computation is seen in Figure 7 to give results closest to the DNS, so this provides a semilocal means for determining m . Clearly the more local the method of determining m is, the better for accuracy and ease of application to complex geometries; further investigation may well improve on this. While the differences in the residuals are quite small, it must be remembered the primary-wave amplitude is less than one percent of the base flow and so its contribution to the residual is comparable.

Results employing the pretransition model for the full initial stage are shown in Figure 8. In these

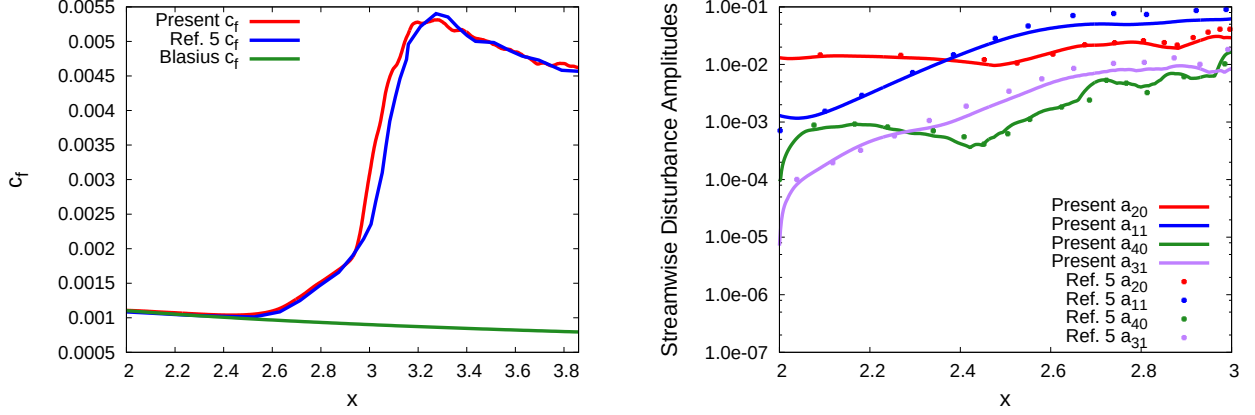


Figure 4. Left: Skin friction from final-stage DNS compared with that of Ref. 5 and Blasius solution. Right: Streamwise evolution of mode amplitudes from final-stage DNS compared with that of Ref. 5.

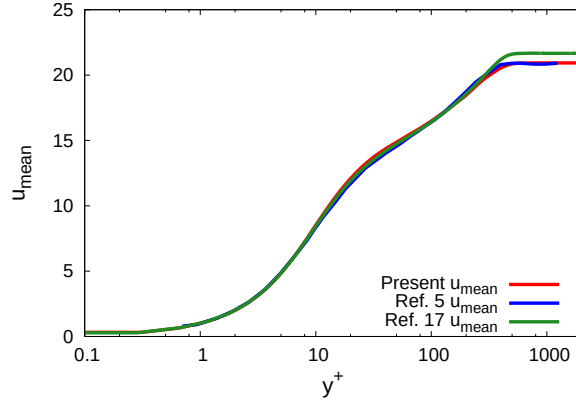


Figure 5. Mean streamwise velocity profile from final-stage DNS at $Re_\theta \approx 1000$ compared with those of Lozano-Duran et al.⁵ and Jimenez et al.¹⁷

computations, the first grid points from the wall are at $0.05\delta_0$, $0.075\delta_0$ and $0.1\delta_0$, where $\delta_0 = \delta_{99}$ at the inflow boundary. The normal spacing is otherwise similar to the DNS grid, as is the streamwise spacing. As in the initial-stage DNS, four spanwise modes are employed. While there is some loss of accuracy in the disturbance-amplitude predictions as the size of the wall-modeled region is increased, the results are overall fairly good.

VI. WMLES of Final-Stage Transition

A WMLES of the full transition process will now be made using the blended wall model presented above. In these computations, the blending parameter λ is a function of position: $\lambda = (x - x_0)/(x_1 - x_0)$, where $x_0 = 2.5$ and $x_1 = 3.0$. The need to specify x_0 and x_1 in advance is awkward and will be addressed in the next section, but the position of the start of transition is clearly indicated by the c_f results from the pretransition model (e.g., Figure 8, upper left), independent of user intervention, and so this issue could be managed, if necessary. Sixteen spanwise modes are employed in this computation and the time step is twice that of the DNS. The streamwise grid spacing, and the normal distribution of points above the first grid point, follow those of the DNS. As for the initial-stage transition wall model in the previous section, the first grid point away from the wall is placed at $0.05\delta_0$, or $y^+ = 5$ at $Re_\theta = 1000$ ($x = 3.8$).

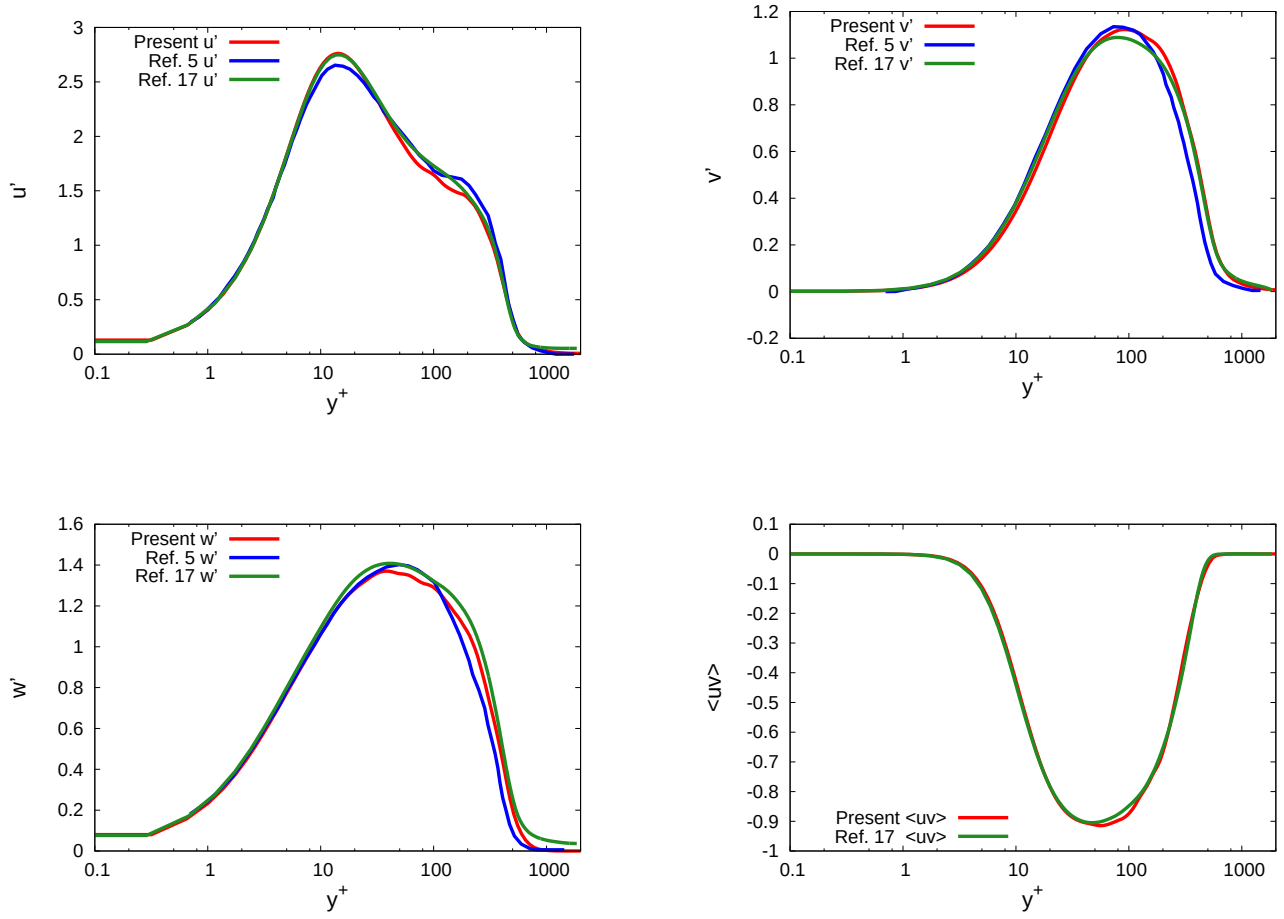


Figure 6. Streamwise (upper left), normal (upper right) and spanwise (lower left) rms velocity profiles and Reynolds shear stress profile (lower right) from final-stage DNS at $Re_\theta \approx 1000$ (red curves) compared with those of Lozano-Duran et al.⁵ (blue curves) and Jimenez et al.¹⁷ (green curves).

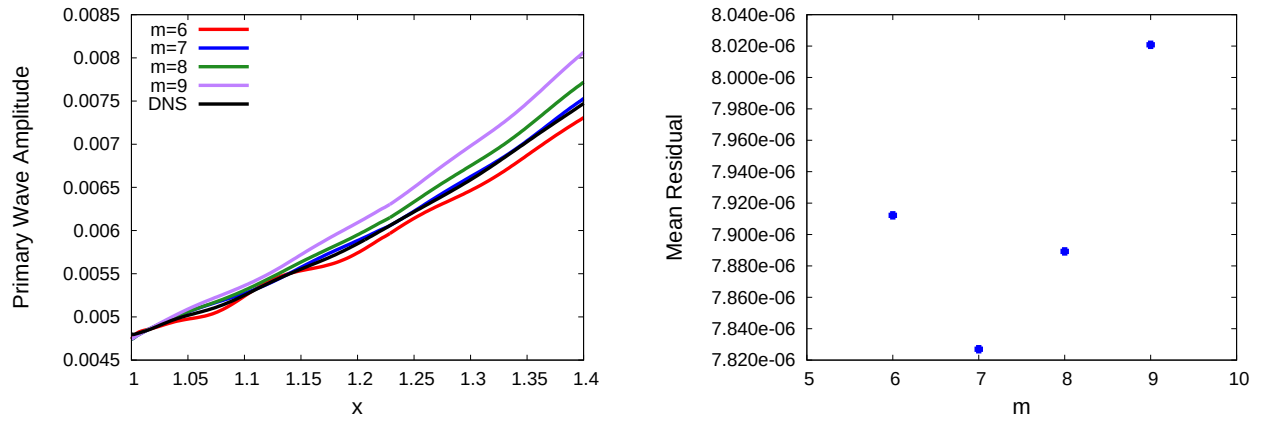


Figure 7. Left: Streamwise growth of primary-wave amplitudes for various choices of m . Right: Variation of mean residual with m .

Results from this computation for skin friction and the evolution of the disturbance amplitudes are shown in Figure 9. Agreement with the DNS is good overall, though the rise in c_f is a little early and a little ragged.

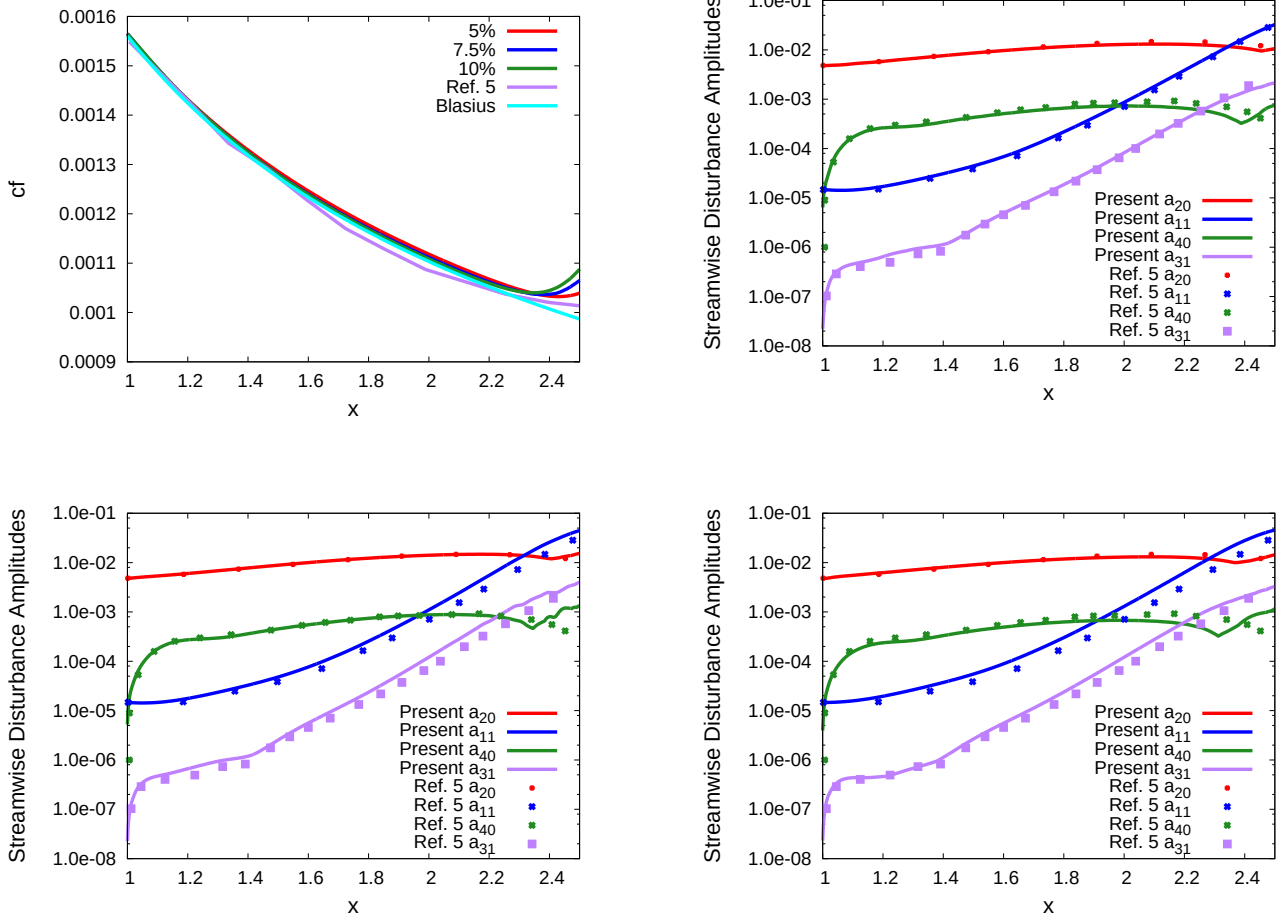


Figure 8. Pretransition WMLES results for various wall-layer thicknesses. Upper left: wall shear stress. Upper right: streamwise disturbance amplitudes, $h = 5\%$. Lower left: streamwise disturbance amplitudes, $h = 7.5\%$. Lower right: streamwise disturbance amplitudes, $h = 10\%$.

Mean velocity and Reynolds stress profiles at $Re_\theta = 1000$ are shown in Figures 10 and 11, where agreement is also seen to be good overall, though there is some deterioration in the accuracy of the Reynolds-stress predictions.

It is of interest to note that the fairly good results obtained here depend on employing the full turbulence wall model (both eddy and molecular viscosities) alluded to above, in spite of the fact that the exchange point, $y^+ = 5$, is, at most, at the edge of the viscous sublayer. Use of only the linear profile of the viscous sublayer yields a peak c_f of only about 0.004 (Figure 12), suggesting the instantaneous velocity reaches buffer-zone amplitudes sufficiently often to cause the difference between a linear sublayer profile and the full sublayer-plus-log-layer profile to affect the results materially.

It is also of interest to note that these WMLES results were obtained without the explicit subgrid-scale model, so that damping is due entirely to numerical dissipation. When the explicit subgrid-scale model is employed (Equation 6), the c_f overshoot is underpredicted (Figure 12), as expected from LES in which a dynamic subgrid-scale model is not used.¹ The reasons for the present computation giving a fairly good prediction of the overshoot while employing no explicit subgrid-scale model, static or dynamic, are currently under investigation for potential development as a subgrid model. For the purposes of the present work, we will continue to take advantage of this fact and not include the subgrid-scale model.

VI.A. Variations

Having established a baseline capability to model rather than resolve the near-wall region in the pretransition zone and blend that model with a traditional wall model for fully turbulent flow, we consider some variations

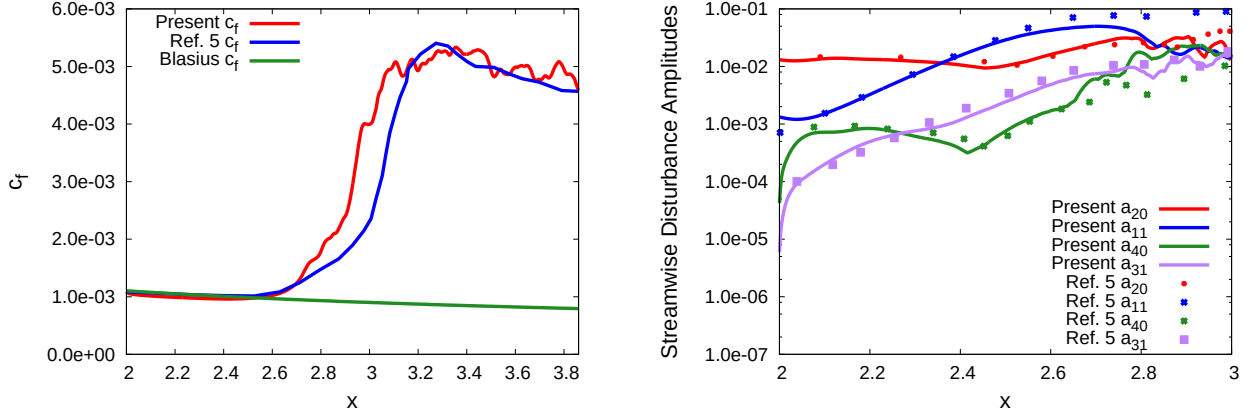


Figure 9. Left: Final-stage WMLES results for skin friction compared with DNS results of Ref. 5. Right: Streamwise growth of disturbance-mode amplitudes. Final-stage WMLES results compared with DNS results of Ref. 5.

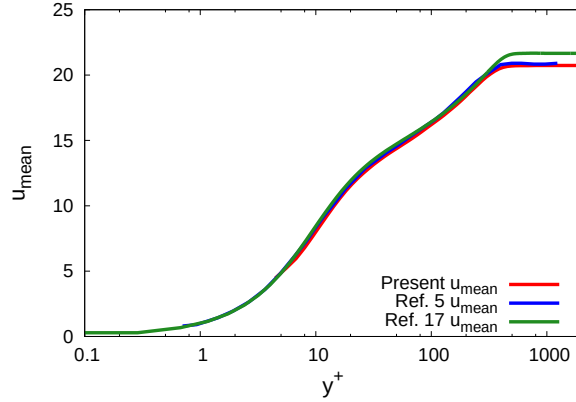


Figure 10. Mean streamwise velocity profile from final-stage WMLES at $Re_\theta \approx 1000$ compared with DNS of Lozano-Duran et al.⁵ and Jimenez et al.¹⁷

on the manner in which these computations are carried out.

VI.A.1. Temporal and Spatial Resolution

As a significant portion of the computational savings of WMLES is due to the increased time step permitted through modeling rather than resolving the near-wall region, we consider next the results of a computation with twice the time step of that of the previous subsection. The skin friction, shown in Figure 13, does depart somewhat from the baseline case, but the fully turbulent mean velocity and streamwise rms velocity profiles shown in Figure 14 agree well with it.

Decreasing the streamwise spatial resolution is less successful. The results in Figures 15 and 16 show the skin friction fails to slope downward after the transitional overshoot, the mean velocity is correspondingly reduced, and the streamwise fluctuations are overpredicted.

VI.A.2. Alternative Sensor: Skin-Friction-Blended Wall Model

The chief virtue of combining the pretransition and fully turbulent wall models through position blending is the robustness of the resulting computation: with the position of the handover from pretransition model to the full turbulence model fixed from the start, the potential instabilities a handover based on local flow

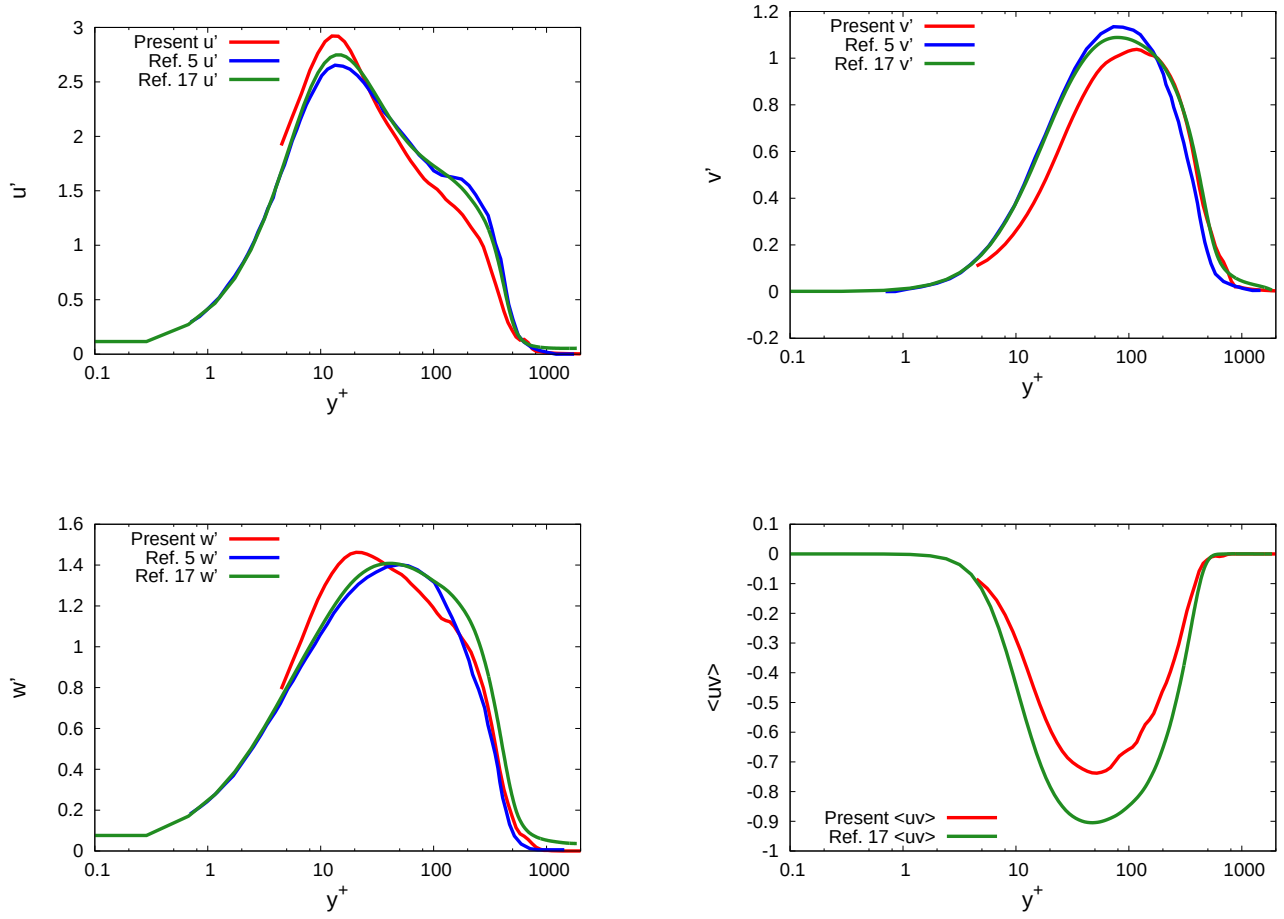


Figure 11. Streamwise (upper left), normal (upper right) and spanwise (lower left) rms velocity profiles and Reynolds shear stress profile (lower right) from final-stage WMLES at $Re_\theta \approx 1000$ (red curves) compared with those of Lozano-Duran et al.⁵ (blue curves) and Jimenez et al.¹⁷ (green curves).

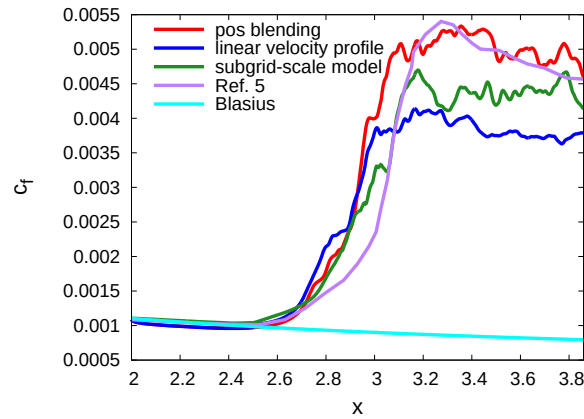


Figure 12. Wall shear stress for baseline WMLES (red curve), WMLES with viscous sublayer only in wall model (blue curve) and WMLES with subgrid-scale model (green curve) compared with DNS of Lozano-Duran et al.⁵ (purple curve) and Blasius solution (light blue curve).

quantities is prey to are avoided. Specifying the approximate position in advance is feasible (the upturn in wall stress is clearly indicated by the pretransition computation, for example), but does entail the additional

work of some computation to determine it. The present subsection is devoted to consideration of an alternative, a transition sensor based on the mean wall shear stress.

The transition from the pretransition model to the fully-developed turbulence model will be supposed to take place between two values of wall shear stress, τ_{w1} and τ_{w2} . Thus, at any point with local wall shear stress τ_w , the blending parameter is $\lambda = (\tau_w - \tau_{w1})/(\tau_{w2} - \tau_{w1})$, limited to values between 0 and 1.

Choosing $\tau_{w1} = 0.0018$ and $\tau_{w2} = 0.0028$ yields the results shown in Figures 17 and 18 for the 5% and 7.5% grids. The computation with the 5% grid gives results similar to that employing position blending, except that the wall shear stress is elevated in the pretransition region. The elevated wall shear stress is present in the 7.5% grid computation, as well, but here the transition is significantly delayed as well. The mean velocity profile is raised slightly and the magnitude of the streamwise rms velocity profile peak is noticeably reduced.

These results show it is possible to capture the transition without specifying its position a priori, but there are problems with the solutions and they get worse when the size of the wall region is increased. In an attempt to address these problems, a modified blending function is employed, in which a cubic spline is used to make the variation of the blending parameter smoother: we define $\xi = (\tau_w - \tau_{w1})/(\tau_{w2} - \tau_{w1})$, with the value of ξ limited to values between 0 and 1, then $\lambda = \xi^2(3 - 2\xi)$. In particular, the onset of the blending parameter is smoother, with both λ and its first derivative with respect to streamwise position being zero as the change $\lambda = 0$ to $\lambda = 1$ begins. Computations were performed using this modified blending on the 5% grid, and, as we see in Figure 19, it does achieve the desired result of eliminating the problem with elevated wall shear stress in the pretransition region. However, the skin-friction overshoot is reduced significantly and there is a curious shift of the peak in the streamwise rms velocity profile toward the wall (Figure 20). On the other hand, the results exhibit an admirable consistency as τ_{w1} and τ_{w2} are varied over a fairly wide range.

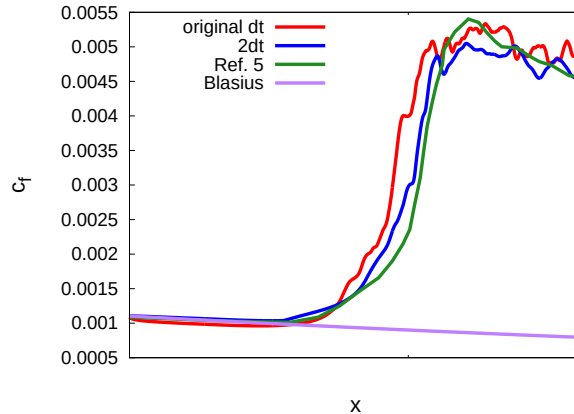


Figure 13. Baseline WMLES (red curve) compared with simulation at twice time step (blue curve).

VII. Conclusion

There are two new ingredients that must be added to traditional WMLES of fully developed turbulent shear layers to get a complete WMLES of a transitional flow. The first is a wall model for the pretransition region that permits the grid to be coarsened near the wall so that the time step may be increased for the entire computation. The second is wall and subgrid-scale models for the transition itself. In the present work, a good start has been made on both these ingredients. The pretransition model proposed here is relatively easy to implement, effective and should be extendible to more complex flows. The blending of pretransition and fully turbulent wall models to make a complete transition model incorporates no new physics of the transition region, but in spite of this yields fair results and improved blending approaches promise to do even better.

As an initial assessment of the potential for WMLES for transition, the results presented here are fairly encouraging. If care is taken to respect the limitations imposed by the different dynamics in the initial and

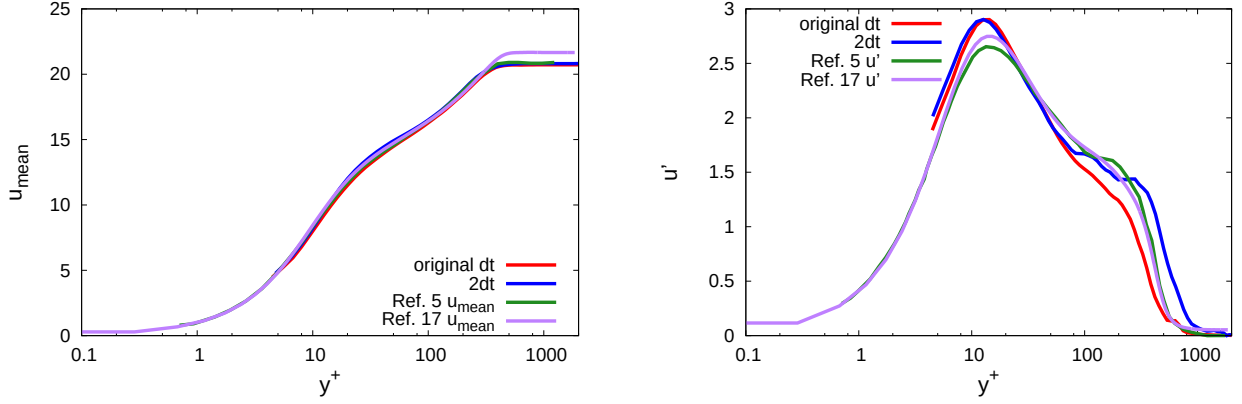


Figure 14. Profiles in fully turbulent region, baseline WMLES (red curves) and WMLES with twice time step (blue curves). Left: Mean velocity. Right: Streamwise rms velocity.

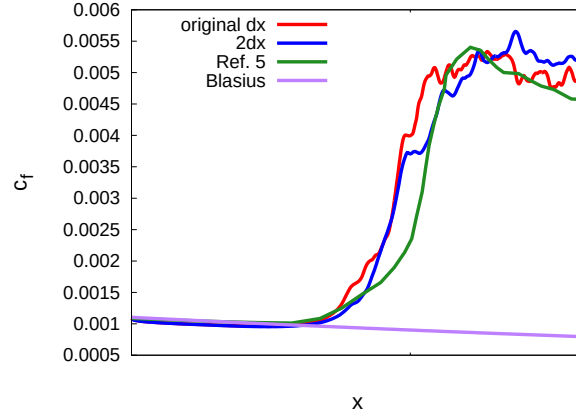


Figure 15. Baseline WMLES (red curve) compared with WMLES with half the streamwise resolution (blue curve).

final stages of transition versus the dynamics of fully developed turbulence, significant computational savings are possible. The results with the 5% grid ($h = 0.05\delta_0$) represent a factor of 20 improvement in computational cost versus the DNS. Improvements in blending the pretransition and fully turbulent wall models to permit h to be increased to 10% or more of δ_0 would make this cost savings even greater. Significant lengthening of the time step, however, or increasing the streamwise spacing beyond that of the DNS, appears to noticeably degrade the prediction of the primary-wave disturbance amplitude. Thus, even the present relatively modest wall-modeled region thickness may have at least partially shifted the chief obstacle to an efficient computation from the resolution of near-wall flow features (grid spacing in the wall-normal direction) to resolution of the primary wave (grid spacing in the streamwise direction and time step). The flow dynamics at transition, in the neighborhood of the rise in c_f , could well impose further constraints.

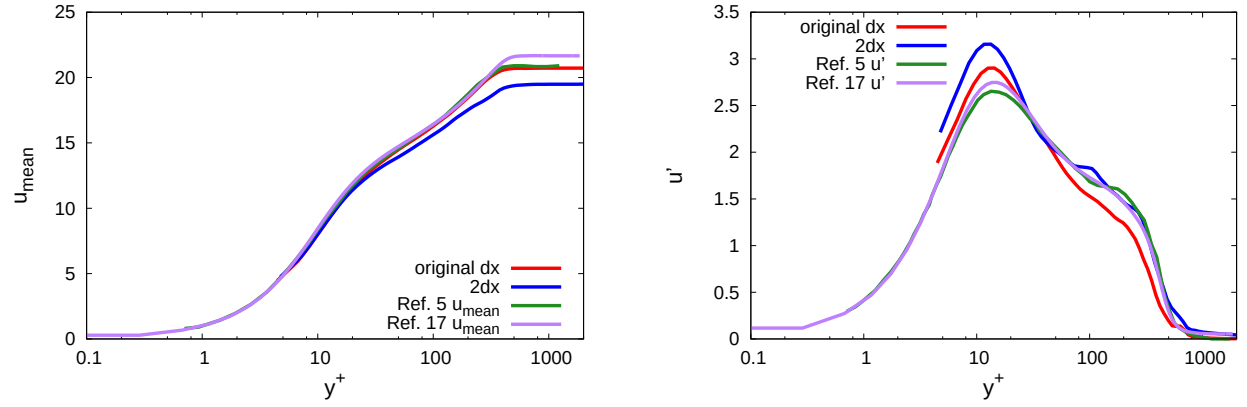


Figure 16. Profiles in fully turbulent region, baseline WMLES (red curves) and WMLES with half the streamwise resolution (blue curves). Left: Mean velocity. Right: Streamwise rms velocity.

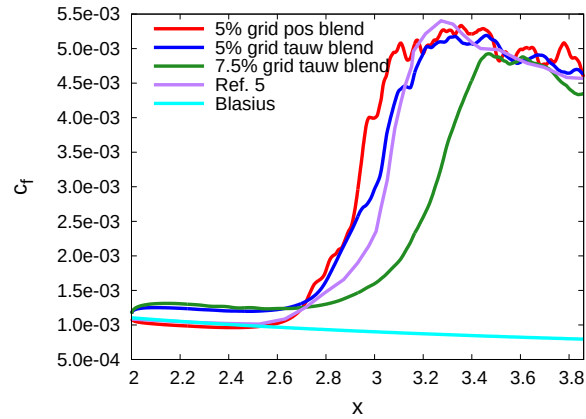


Figure 17. Baseline position-blended WMLES (red curve) compared with WMLES with wall-stress blending (blue curve) and WMLES with wall-stress blending and 7.5% grid (green curve). DNS (purple curve) and Blasius (light blue curve) results are also shown.

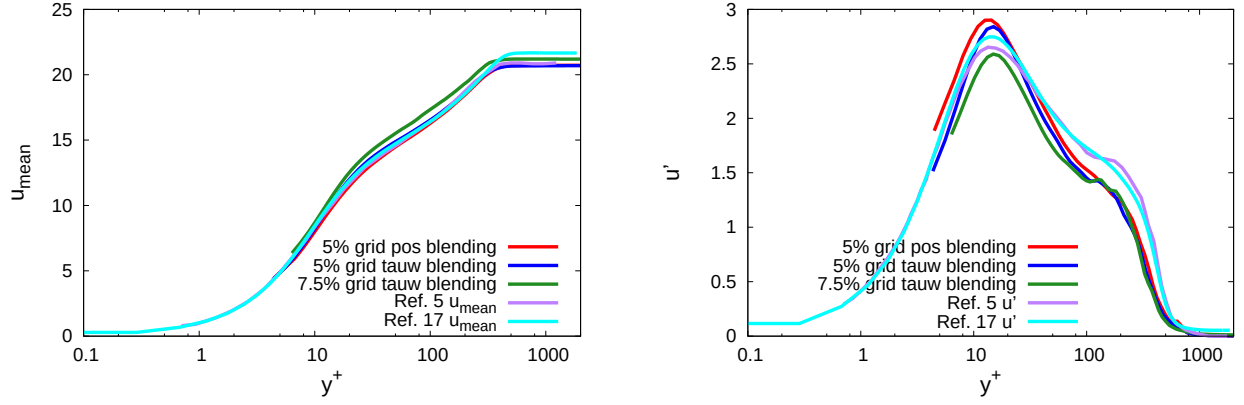


Figure 18. Baseline position-blended WMLES (red curve) compared with WMLES with wall-stress blending (blue curve) and WMLES with wall-stress blending and 7.5% grid (green curve) and DNS (purple curve). Left: Mean velocity. Right: Streamwise rms velocity.

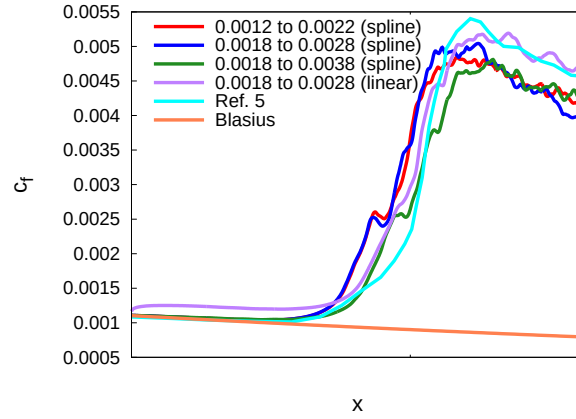


Figure 19. Wall-stress blending with spline blending function: $\tau_w = 0.0012 - 0.0022$, red curve; $\tau_w = 0.0018 - 0.0028$, blue curve; $\tau_w = 0.0018 - 0.0038$, green curve; $\tau_w = 0.0018 - 0.0028$ with linear blending, purple curve; DNS, light-blue curve; Blasius, orange curve.

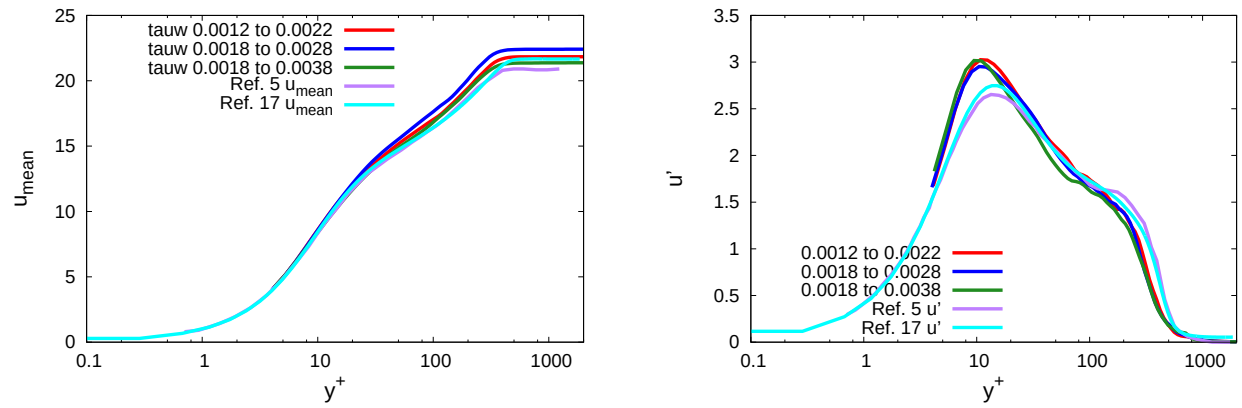


Figure 20. Wall-stress blending with spline blending function. $\tau_w = 0.0012 - 0.0022$, red curve; $\tau_w = 0.0018 - 0.0028$, blue curve; $\tau_w = 0.0018 - 0.0038$, green curve. Left: Mean velocity. Right: Streamwise rms velocity.

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