

Analytical Investigation of Propellant Slosh Stability Boundary on a Space Vehicle

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The effect of propellant sloshing upon the stability of a liquid-propelled space vehicle has been studied extensively. For a typical space vehicle with thrust vector control and a single slosh tank, the Bauer slosh danger zone is located between the vehicle center of mass and the center of percussion while ignoring effects like aerodynamics and axial acceleration. If the slosh mass is located between these two locations, baffles may be required to provide additional damping for stability. Although this criteria is widely used in the launch vehicle flight controls community, its application is limited to the aforementioned configuration and assumptions. In this work, the methodology behind the classic single-tank criteria is extended to other tank configurations and includes previously ignored effects. First, the variation of the classic danger zone for a tandem tank configuration where the tanks are located along the vehicle centerline or symmetrically offset from the centerline is investigated. Second, a vehicle with a single centerline tank and pair of outboard reaction jets, rather than thrust vectoring, for attitude control is studied. Finally, the impacts of aerodynamics and vehicle axial acceleration on the classic single-tank criteria are analyzed.

I. Introduction

Liquid propellant makes up a significant portion, as much as 90 percent during liftoff, of a launch vehicle gross weight. The effect of fuel sloshing on vehicle dynamics and control must be considered in the design of the flight control system. This is especially true for slosh tanks with large diameters in which the frequencies of liquid sloshing motion approach the rigid body control frequency. Typically, in the formulation of the linearized equations of motion for a launch vehicle,¹⁻³ slosh dynamics is modeled as either a spring-mass or a pendulum. In both cases, the displacement of the slosh mass is assumed to be small. Several key slosh parameters that affect vehicle stability are: 1) amount of the slosh mass or portion of the total liquid mass that is assumed to behave as an equivalent pendulum, 2) frequency of the slosh mode, which is dependent on the length and mass of the equivalent pendulum representation and vehicle axial acceleration, and 3) location of the slosh mass along the length of the vehicle. Reference [4] demonstrated that for a launch vehicle with a single centerline slosh tank the danger zone for slosh instability occurs when the slosh mass location is between the vehicle center of mass and center of percussion (also referred to as the instantaneous center of rotation⁷). Reference [2] shows that the center of percussion for a rocket can be expressed as:

$$L_Q = \frac{I_{zz}}{L_C M} \quad (1)$$

Here L_Q is defined as the distance to the center of percussion on the vehicle forward from the center of mass. I_{zz} is the transverse moment of inertia of the rocket with respect to the center of mass, L_C is the distance from the center of mass to the thrust application point, and M is the total mass of the vehicle. Other references in literature including [8] and [9] appear to be only partially correct when stating that the slosh mass is stable if the pendulum hinge point is below the center of mass.

Once inside the danger zone, the slosh mass has unfavorable phasing with respect to rigid body control or is phase unstable. In other words, the control effector is indirectly adding energy to the slosh mode when trying to actively control the rigid body attitude. Figure 1 highlights the difference between a phase stable

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vs. unstable slosh mode on a root locus plot in the absence of damping (zoomed in on the slosh pole-zero pair located on the imaginary axis). In the stable scenario, the natural frequency, Ω , of the slosh pole is greater than that of the zero ($\Omega_p^2 > \Omega_z^2$), and the angle of departure is into the left half of the s-plane. Hence the closed-loop slosh pole is stable for all values of feedback gains. For the unstable scenario, the slosh pole is below the zero, the angle of departure is into the right half of the s-plane and the mode is unstable. As a result, additional damping is required (e.g. baffles) to ensure the closed-loop slosh pole remains stable. Figure 2 illustrates the difference on a Nichols chart, where M_p refers to the slosh mass. The frequency response associated with a stable slosh mode naturally projects away from the critical point $(-1, 0)$; whereas, for an unstable slosh mode the frequency response protrudes towards the critical point.

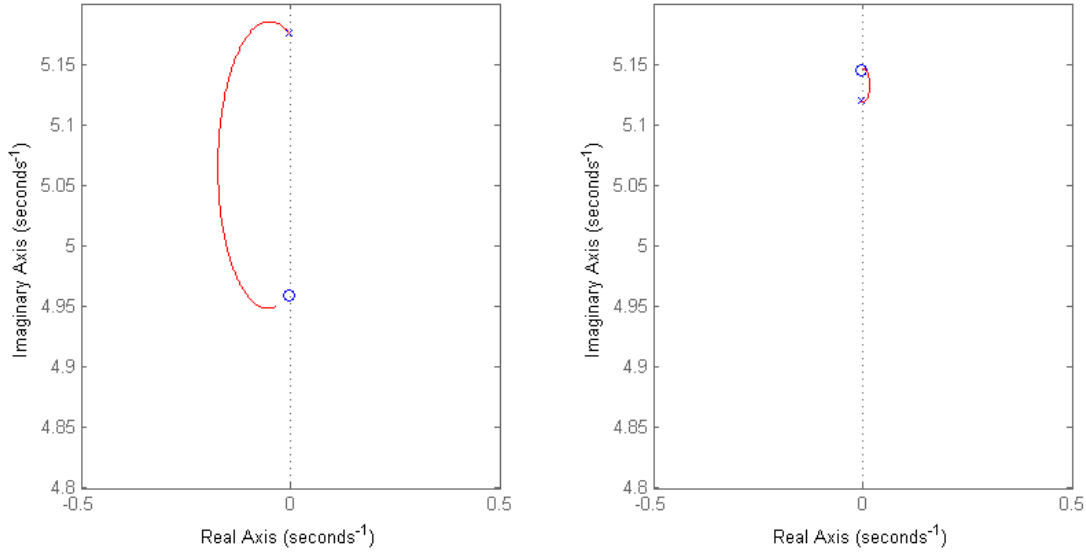


Figure 1. a) Phase Stable Slosh Mode, b) Phase Unstable Slosh Mode

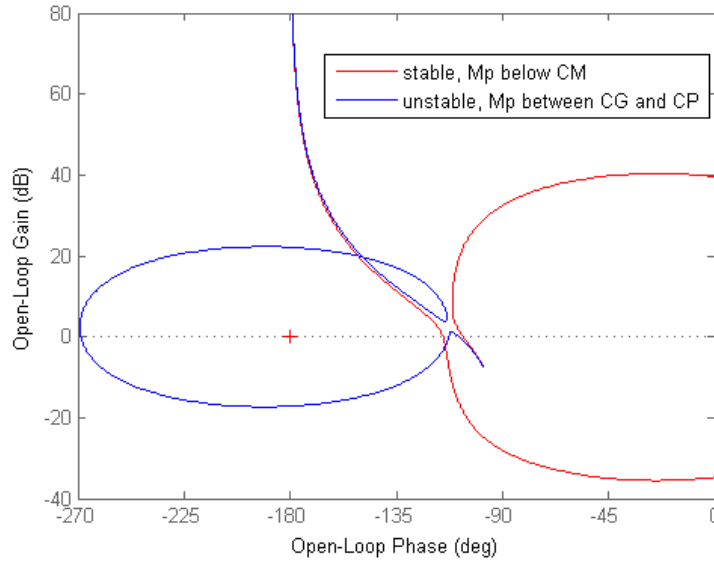


Figure 2. Nichols Chart Illustration: Phase Stable vs. Unstable Slosh Mode

The remainder of the paper is organized as follows. Sections II and III re-derives the center of percussion for a rocket and the classic single tank danger zone criteria from first principles. Sections IV and V examine

the slosh danger zone for the tandem tank configuration where the tanks are located either along the vehicle centerline or symmetrically offset from the centerline. Section VI derives the single tank danger zone criteria for a vehicle with pairs of outboard reaction jets rather than thrust vectoring, for attitude control. Section VII explores the impacts of aerodynamics and axial acceleration on the classic single tank criteria. Conclusions are presented in Sec. VIII.

II. Rocket Center of Percussion

In this section, the center of percussion, denoted by point Q in Fig. 3, is derived for a simplified planar model of a space vehicle with thrust vector control (TVC). The center of mass is denoted by point G , whereas L_C and L_Q represent the distances from G to the thrust application point and the center of percussion, respectively. I_{zz} is the transverse moment of inertia of the rocket with respect to G and M is the mass of the vehicle. Δ represents the deflection of the thrust vector and T_c is the engine thrust force. $\hat{\mathbf{b}}$ denotes the body frame and $\hat{\mathbf{n}}$ denotes the inertial frame.

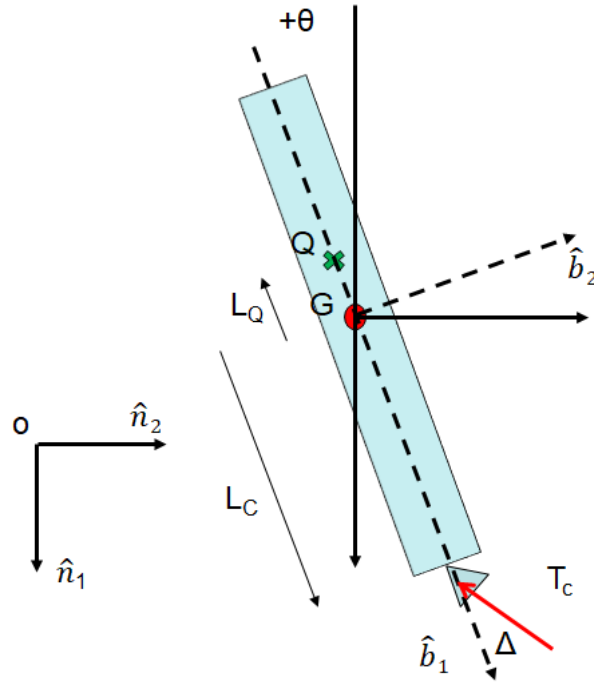


Figure 3. Simplified Rocket

The pitch equation about $\hat{\mathbf{b}}_3$ with respect to the center of mass, G (for positive, small Δ angle), is as follows:

$$I_{zz}\ddot{\theta} = -T_C L_C \sin \Delta \approx -T_C L_C \Delta \quad (2)$$

Resolving the acceleration of point Q with respect to o in the body frame gives:

$$a_{Q/o} = (\ddot{X} \cos \theta + \ddot{Y} \sin \theta - \dot{\theta}^2 L_Q) \hat{\mathbf{b}}_1 + (-\ddot{X} \sin \theta + \ddot{Y} \cos \theta - \ddot{\theta} L_Q) \hat{\mathbf{b}}_2 \quad (3)$$

Acceleration of point Q along $\hat{\mathbf{b}}_2$ is approximately $-\frac{T_C \sin \Delta}{M}$:

$$M(-\ddot{X} \sin \theta + \ddot{Y} \cos \theta - \ddot{\theta} L_Q) \approx -T_C \sin \Delta \approx -T_C \Delta \quad (4)$$

Assuming $\ddot{Y} \approx 0$ at point Q , θ is small, solving Eq. 2 for $\ddot{\theta}$, and substituting into Eq. 4, L_Q can then be expressed as:

$$L_Q = -\frac{I_{zz}}{M L_C} \quad (5)$$

This expression is consistent with Eq. 1. For a uniform rod, $I_{zz} = \frac{1}{12}ML^2$ and $L_C = \frac{1}{2}L$, $L_Q = -\frac{1}{6}L$ in the $\hat{\mathbf{b}}_1$ direction from G . The reader can refer to the Appendix on the analogous center of percussion of a baseball bat.

III. Stability Boundary for a Single Tank Configuration

In this section, the classic slosh stability boundary of a space vehicle with TVC and a single propellant tank is re-derived from first principles using Lagrange's method.^{5,6} The slosh mass is represented by a pendulum particle with a length of L_s and mass of M_s as shown in Fig. 4. L represents the distance to the slosh mass hinge point, P , from the center of mass, G . The mass and transverse moment of inertia of the vehicle about G (without the slosh particle) are denoted by M and I_{zz} . $\hat{\mathbf{b}}$ denotes the body frame and $\hat{\mathbf{n}}$ denotes the inertial frame. θ describes the attitude of the rigid body relative to $\hat{\mathbf{n}}_1$, while θ_1 describes the angle of the slosh mass relative to $\hat{\mathbf{b}}_1$. The generalized coordinates are: θ , θ_1 , and Y .

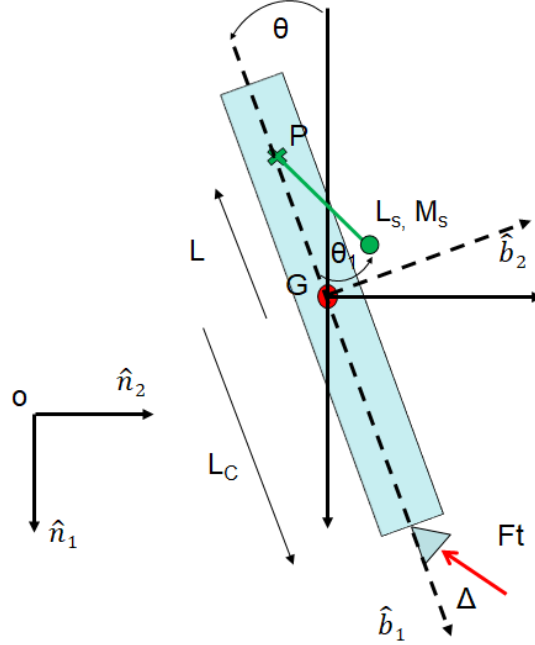


Figure 4. Mechanical Analog

The inertial position (X_s and Y_s) of the pendulum particle is as follows:

$$\begin{aligned} X_s &= -L \cos \theta + L_s \cos(\theta + \theta_1) \\ Y_s &= Y - L \sin \theta + L_s \sin(\theta + \theta_1) \end{aligned} \quad (6)$$

The total kinetic (T) and potential (V) energies of the system can be expressed as:

$$T = \frac{1}{2}I_{zz}\dot{\theta}^2 + \frac{1}{2}M\dot{Y}^2 + \frac{1}{2}M_s[L\dot{\theta} \sin \theta - L_s(\dot{\theta} + \dot{\theta}_1) \sin(\theta + \theta_1)]^2 + \frac{1}{2}M_s[\dot{Y} - L\dot{\theta} \cos \theta + L_s(\dot{\theta} + \dot{\theta}_1) \cos(\theta + \theta_1)]^2 \quad (7)$$

$$V = M_s g [-L \cos \theta + L_s \cos(\theta + \theta_1)] \quad (8)$$

To derive the full nonlinear equations of motion, substitute T and V into Eq. 9

$$\begin{aligned} \mathcal{L} &= T - V \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} &= Q \end{aligned} \quad (9)$$

here $\mathcal{L}$ is the Lagrangian, q represents the generalized coordinates: $[\theta \ \theta_1 \ Y]^T$, and Q represents the generalized forces. Subsequently, one can linearize the nonlinear system about: $[\theta = 0, \theta_1 = 0, \dot{\theta} = 0, \dot{\theta}_1 = 0]$ and

re-write in $\dot{x} = Ax + Bu$ format:

$$\begin{aligned}
 & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{zz} + M_s L^2 & 0 & M_s L_s L & 0 & M_s L \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & M_s L_s L & 0 & M_s L_s^2 & 0 & M_s L_s \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & M_s L & 0 & M_s L_s & 0 & (M + M_s) \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{Y} \\ \ddot{Y} \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -M_s g(L + L_s) & 0 & -M_s g L_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -M_s g L_s & 0 & -M_s g L_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \theta_1 \\ \dot{\theta}_1 \\ Y \\ \dot{Y} \end{bmatrix} + \begin{bmatrix} 0 \\ -T_C L_C \\ 0 \\ 0 \\ 0 \\ -T_C \end{bmatrix} \Delta
 \end{aligned} \tag{10}$$

Assuming the contribution of the slosh mass is negligible on the rigid body, $M_s g(L + L_s) \approx 0$, $M_s g L \approx 0$, and the amplitude of the slosh pendulum is much greater than the rocket itself ($\theta_1 \gg \theta$), one can obtain the 4th order transfer function from Δ to θ :

$$\frac{\theta}{\Delta} = -\frac{T_C}{s^2} \left[\frac{s^2 + \left(\frac{L_C M g - L g M_s + L_C g M_s}{L_C L_s M} \right)}{s^2 + \left(\frac{L^2 M M_s g + I_{zz} M g + I_{zz} M_s g}{I_{zz} M L_s} \right)} \right] \tag{11}$$

In the absence of damping, the slosh pole-zero pair are located on the imaginary axis of the s-plane. As discussed in Sec. I, for the slosh dipole to be phase stable, $\Omega_p^2 > \Omega_z^2$ is required:

$$\frac{L^2 M M_s g + I_{zz} M g + I_{zz} M_s g}{I_{zz} M L_s} > \frac{L_C M g - L g M_s + L_C g M_s}{L_C L_s M} \tag{12}$$

Simplifying Eq. 12 yields the final inequality:

$$L(LL_C M + I_{zz}) > 0 \tag{13}$$

Equation 13 indicates that if $L > 0$ or if the slosh mass is below the vehicle center of mass, G , then it is phase stable. If $L < 0$, then the following condition needs to be satisfied:

$$(LL_C M + I_{zz}) < 0 \tag{14}$$

Solving for L :

$$\boxed{L < -\frac{I_{zz}}{L_C M}} \tag{15}$$

The term $\frac{I_{zz}}{L_C M}$ is the location of the center of percussion as shown in Eq. 1 and 5. Equation 13 indicates that if $L < 0$ or if the slosh mass is ahead of G then in order for it to be phase stable it needs to be ahead of the center of percussion. This conclusion is consistent with the classic result described in Refs. [2] and [4].

IV. Stability Boundary for Tandem Centerline Tank Configuration

In this section, slosh stability boundary for a tandem centerline tank configuration as shown in Fig. 5 is investigated. The primary slosh mass is represented by a pendulum particle with a length of L_{s1} and mass of M_{s1} . L_1 represents the distance to the hinge point of the primary slosh mass relative to the center of mass, G . The subscript, 2, represents the parameters for the secondary slosh particle. The secondary slosh mass is of higher frequency and lower mass compared to the primary. θ describes the attitude of the rigid body relative to the vertical, while θ_1 and θ_2 represent the angle of the slosh masses relative to $\hat{\mathbf{b}}_1$. The generalized coordinates are: θ , θ_1 , θ_2 , and Y .

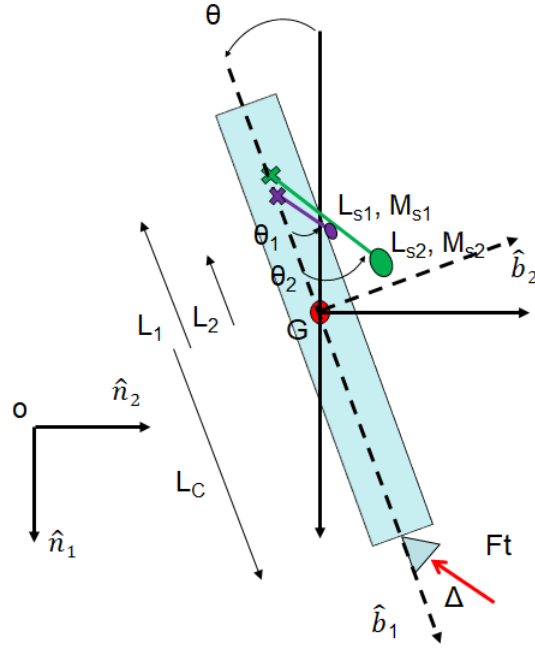


Figure 5. Mechanical Analog

The inertial positions of the pendulum particles #1 and #2 are denoted by the following equations:

$$\begin{aligned} X_{s1} &= -L_1 \cos \theta + L_{s1} \cos(\theta + \theta_1) \\ Y_{s1} &= Y - L_1 \sin \theta + L_{s1} \sin(\theta + \theta_1) \end{aligned} \quad (16)$$

$$\begin{aligned} X_{s2} &= -L_2 \cos \theta + L_{s2} \cos(\theta + \theta_2) \\ Y_{s2} &= Y - L_2 \sin \theta + L_{s2} \sin(\theta + \theta_2) \end{aligned} \quad (17)$$

Following the same procedure and assumptions for the single tank configuration outlined in Sec. III and linearizing about: $[\theta = 0, \theta_1 = 0, \theta_2 = 0, \dot{\theta} = 0, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0]$, the state-space representation of the system is as follows:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{zz} + M_{s1}L_1^2 + M_{s2}L_2^2 & 0 & M_{s1}L_1L_{s1} & 0 & M_{s2}L_2L_{s2} & 0 & (M_{s1}L_1 + M_{s2}L_2) \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{s1}L_1L_{s1} & 0 & M_{s1}L_{s1}^2 & 0 & 0 & 0 & M_{s1}L_{s1} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & M_{s2}L_2L_{s2} & 0 & 0 & 0 & M_{s2}L_{s2}^2 & 0 & M_{s2}L_{s2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & (M_{s1}L_1 + M_{s2}L_2) & 0 & M_{s1}L_{s1} & 0 & M_{s2}L_{s2} & 0 & M + M_{s1} + M_{s2} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \\ \dot{Y} \\ \ddot{Y} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -M_{s1}g(L_1 + L_{s1}) - M_{s2}g(L_2 + L_{s2}) & 0 & -M_{s1}gL_{s1} & 0 & -M_{s2}gL_{s2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -M_{s1}gL_{s1} & 0 & -M_{s1}gL_{s1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -M_{s2}gL_{s2} & 0 & 0 & 0 & -M_{s2}gL_{s2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \\ Y \\ \dot{Y} \end{bmatrix} + \begin{bmatrix} 0 \\ -T_C L_C \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -T_C \end{bmatrix} \Delta \end{aligned} \quad (18)$$

Assuming the contribution of the slosh mass is negligible on the rigid body, $M_{s1}gL_1 + M_{s2}gL_2 \approx 0$, $M_{s1}gL_{s1} \approx 0$, $M_{s2}gL_{s2} \approx 0$, and the amplitude of the slosh pendulum is much greater than the rocket itself ($\theta_1 \gg \theta$), The 6th order transfer function from Δ to θ can be expressed as:

$$\frac{\theta}{\Delta} = -\frac{T_C}{s^2} \left\{ \frac{ML_CL_{s1}L_{s2}s^4 + [(M + M_{s2})gL_CL_{s1} + (M + M_{s1})gL_CL_{s2} - M_{s1}gL_1L_{s2} - M_{s2}gL_2L_{s1}]s^2 + M_Tg^2L_C - M_{s1}L_1g^2 - M_{s2}L_2g^2}{IML_{s1}L_{s2}s^4 + [IgL_{s1}(M + M_{s2}) + IgL_{s2}(M + M_{s1}) + Mg(M_{s1}L_1^2L_{s2} + M_{s2}L_2^2L_{s1})]s^2 + IM_Tg^2 + Mg^2(M_{s1}L_1^2 + M_{s2}L_2^2) + (L_1 - L_2)^2M_{s1}M_{s2}g^2} \right\} \quad (19)$$

where M_T is the total mass of the system. If $L_{s1} = M_{s1} = 0$ or $L_{s2} = M_{s2} = 0$, Eq. 19 reduces to the single tank solution shown in Eq. 11. As discussed in Sec. III, $\Omega_p^2 > \Omega_z^2$ is required for stability. To obtain the stability boundaries, L_1 and L_2 would need to be solved in terms of the remaining system parameters. However, due to the drastic increase in the complexity of the transfer function, obtaining a closed-form solution is no longer tractable. Hence, numerical analysis is required to provide insight on the impact the secondary slosh mass has on the stability boundary of the primary slosh mass and vice versa. Table 1 shows notional parameters for a space vehicle with two centerline slosh tanks. Assuming uniform density, the center of percussion is $\frac{1}{6}L$ ahead of G . M_{s2} and L_{s2} are intentionally left blank to assess the impact of slosh mass fraction ratio ($\frac{M_{s2}}{M_{s1}}$) and frequency ratio ($\frac{\omega_{s2}}{\omega_{s1}}$) on the stability boundaries.

Table 1. Nominal System Parameters

parameter	value	units
M	16000	kg
M_{s1}	1000	kg
M_{s2}	—	kg
L	50	m
I	3.333e6	kg m ²
L_{s1}	3	m
L_{s2}	—	m
L_C	$\frac{1}{2}L$	m
T_C	3e5	N
CP	$-\frac{1}{6}L$	m

Figure 6 illustrates the stability regions when coupling between the tanks is at a minimal ($\omega_{s2} \gg \omega_{s1}$ and $M_{s2} \ll M_{s1}$). The x and y axes represent all possible locations for L_1 and L_2 along $\hat{\mathbf{b}}_1$. The region shaded in orange represents conditions where M_{s1} is unstable ($\Omega_{z1} > \Omega_{p1}$), while the blue region represents conditions where M_{s2} is unstable ($\Omega_{z2} > \Omega_{p2}$). The yellow region indicates where both slosh masses are stable. The single tank danger zone solution is characterized by the green and red dashed lines, respectively. For small mass fraction ratios and large frequency separations, the coupling effects are at a minimal, and the stability boundaries closely resemble the single tank solution. Figure 7 illustrates the case where the mass fraction ratio remains small but with reduced frequency separation. The stability region for the primary tank still resembles that of the single tank solution; however, the stability region for the secondary tank deviates substantially compare to the single tank solution. It shrinks as L_1 moves towards the aft or front ends of the vehicle. Figure 8 illustrates the case with a significant increase in mass fraction ratio and reduction in the frequency separation. The stability region of the primary tank grows as L_2 moves toward the aft end of the vehicle and shifts forward as L_2 moves toward the front of the vehicle. Figure 9 illustrates the limiting case in which the slosh tanks are nearly identical in mass and frequency. The stability regions for both tanks deviate significantly from the single tank solution.

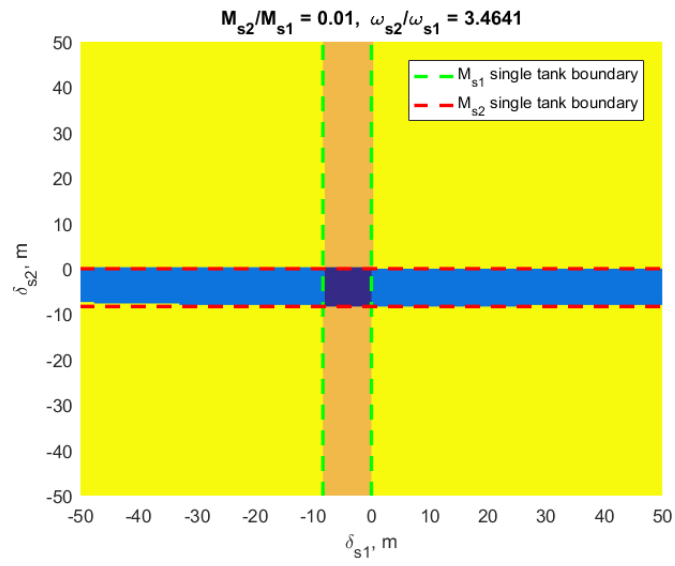


Figure 6. Tandem Tank Danger Zone $\frac{M_{s1}}{M_{s2}} = 0.01, \frac{\omega_{s1}}{\omega_{s2}} = 3.46$

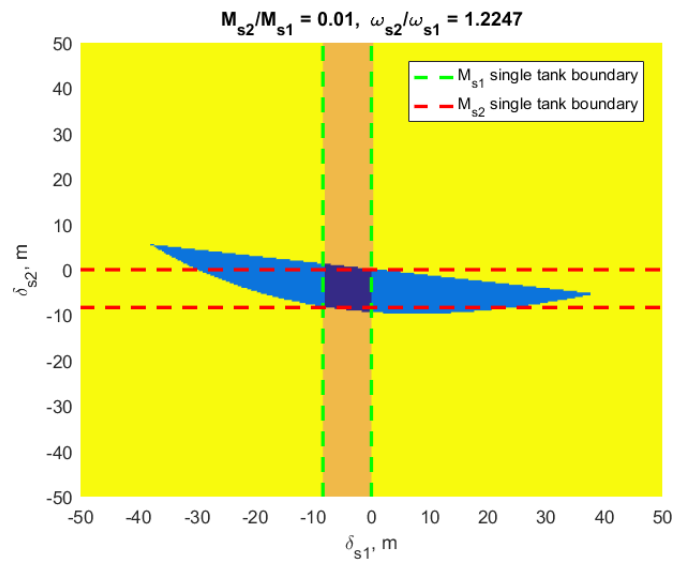


Figure 7. Tandem Tank Danger Zone $\frac{M_{s1}}{M_{s2}} = 0.01, \frac{\omega_{s1}}{\omega_{s2}} = 1.22$

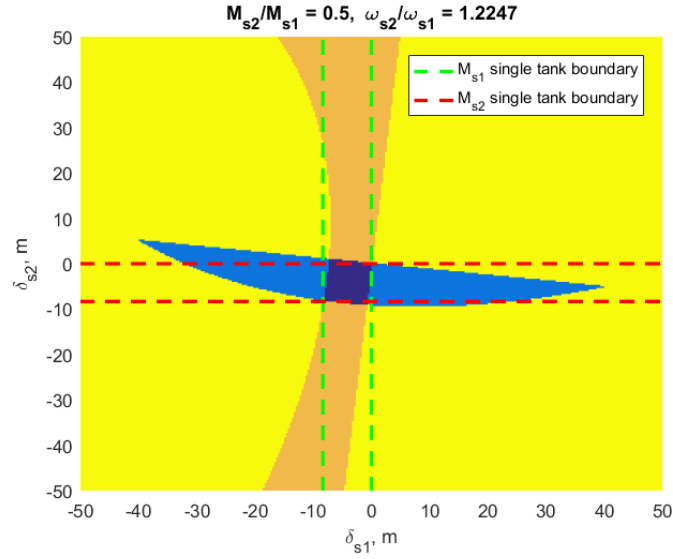


Figure 8. Tandem Tank Danger Zone $\frac{M_{s1}}{M_{s2}} = 0.5, \frac{\omega_{s1}}{\omega_{s2}} = 1.22$

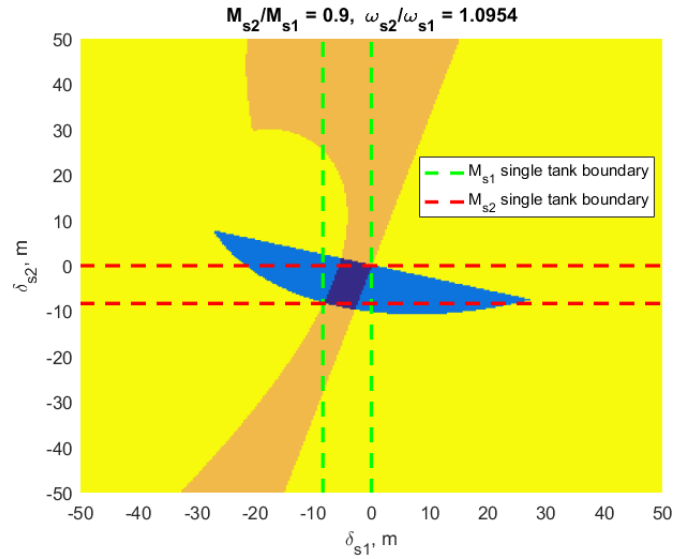


Figure 9. Tandem Tank Danger Zone $\frac{M_{s1}}{M_{s2}} = 0.9, \frac{\omega_{s1}}{\omega_{s2}} = 1.09$

Unlike for the single tank configuration, there does not exist a closed-form analytical solution for L_1 and L_2 for the tandem centerline tank configuration. The stability regions shown in Figs. 6 through 9 are vehicle dependent. It is prudent for the flight controls designer to use the classic single-tank solution with caution as the relative mass and frequency of the slosh tanks approach unity.

V. Stability Boundary for Two Off-Centerline Tank Configuration

In this section, the slosh stability boundary of an Apollo-like lander configuration with two identical off-centerline tanks is investigated. As illustrated the in Fig. 10, the two slosh masses are denoted by M_{s1} , L_{s1} and M_{s2} , L_{s2} respectively. L and a represent displacements of the slosh hinge points away from the center of mass, G along $\hat{\mathbf{b}}_1$ and $\hat{\mathbf{b}}_2$. θ describes the attitude of the rigid body relative to the vertical, while θ_1 and θ_2 represents the angle of the two slosh masses relative to $\hat{\mathbf{b}}_1$.

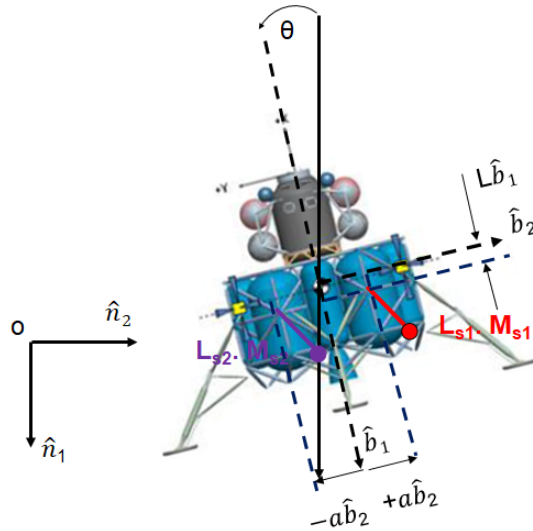


Figure 10. Configuration with Two Tanks Offset from Centerline

Following the same procedures as outlined in Sec. III and V and assuming the tank parameters are identical (removing the subscripts 1 and 2), one can solve for L , the location of the slosh masses along $\hat{\mathbf{b}}_1$:

$$L(LL_CM + I_{zz} + 2M_s a^2) > 0 \quad (20)$$

Equation 20 indicates that if $L > 0$ (slosh masses are below G), then they are phase stable. If $L < 0$, then following condition needs to be satisfied:

$$(LL_CM + I_{zz} + 2M_s a^2) < 0 \quad (21)$$

Solving for L :

$$L < -\frac{I_{zz} + 2M_s a^2}{L_CM} \quad (22)$$

Equation 22 suggests that the upper bound of the danger zone shifts beyond the center of percussion by an additional $\frac{2M_s a^2}{L_CM}$ compared to Eq. 15, where $M_s a^2$ is the inertia of the slosh mass about $\hat{\mathbf{b}}_1$ and 2 is the number of tanks.

VI. Slosh Stability Boundary for Single Tank with Outboard Engines

Thus far, the configuration of interest has been a space vehicle with a single aft gimbal main engine for attitude control. In this section, the slosh stability boundary of a space vehicle with a single centerline tank and coupled outboard reaction jets for attitude control such as the Altair ascent module concept⁹ is investigated. The Altair ascent module main engine does not have gimbal capabilities hence pairs of jets are used for attitude control. Figure 11 is an illustration of the configuration. Thruster pairs 1a/2b produces a positive moment about $\hat{\mathbf{b}}_3$ while thruster pairs 1b/2a produces a negative moment about $\hat{\mathbf{b}}_3$.

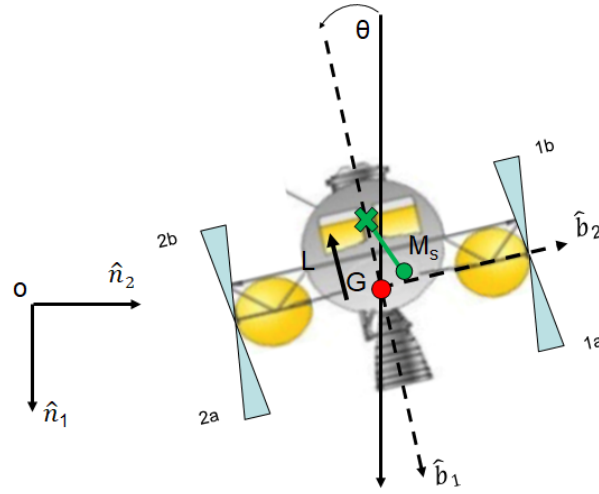


Figure 11. Notional Altair Ascent Module Configuration

Following the same procedure outlined in Section III, obtain the transfer function from Δ to θ ; however, here, Δ represents a pseudo control in terms of on-times for the thruster pairs.

$$\frac{\theta}{\Delta} = -\frac{T_C}{s^2} \left[\frac{s^2 + \left(\frac{L_C M g + L_C g M_s - \epsilon L g M_s}{L_C L_s M} \right)}{s^2 + \left(\frac{L^2 M M_s g + I_{zz} M g + I_{zz} M_s g}{I_{zz} M L_s} \right)} \right] \quad (23)$$

Ideally, the on-times for the thruster pairs are equal and resulting in a pure moment about $\hat{\mathbf{b}}_3$. However, on-time discrepancies in thruster pairs or center of mass offset would lead to a lateral acceleration in $\hat{\mathbf{n}}_2$. ϵ in Eq. 23 captures the effect of unwanted lateral acceleration. In the extreme case, $\epsilon = 1$, only one thruster in the pair fires, producing both a moment about $\hat{\mathbf{b}}_3$ and acceleration in $\hat{\mathbf{n}}_2$. Following the same procedures in Sec. III, $\Omega_p^2 > \Omega_z^2$ is required for stability:

$$\frac{L^2 M M_s g + I_{zz} M g + I_{zz} M_s g}{I_{zz} M L_s} > \frac{L_C M g + L_C g M_s - \epsilon L g M_s}{L_C L_s M} \quad (24)$$

Simplifying and yielding the final inequality:

$$L(LL_C M + \epsilon I_{zz}) > 0 \quad (25)$$

Equation 25 indicates that if $L > 0$, or if the slosh mass location is below the vehicle center of mass, G , then it is phase stable. If $L < 0$, then the following condition needs to be satisfied:

$$(LL_C M + \epsilon I_{zz}) < 0 \quad (26)$$

Solving for L :

$$L < -\epsilon \frac{I_{zz}}{L_C M} \quad (27)$$

if $\epsilon = 1$, the original solution shown in Eq. 13 is recovered. As $\epsilon \rightarrow 0$ (pair of coupled thrusters fire, producing a pure torque), $L \rightarrow 0$, which indicates that the slosh mass location can only be below the G for stability.

VII. Aerodynamic and Axial Acceleration Effects on Stability Boundary

In this section, the effects of aerodynamics and axial acceleration on the slosh danger zone of a single tank configuration is investigated. The mass-spring-damper model^{1,10} for the slosh mass is used as instead of the pendulum model. As shown in Fig. 12, the slosh mass is located at a distance L away from the vehicle center of mass, G along $\hat{\mathbf{b}}_1$. η_s represents the lateral displacement of the slosh mass along $\hat{\mathbf{b}}_2$. L_a is the center of pressure.

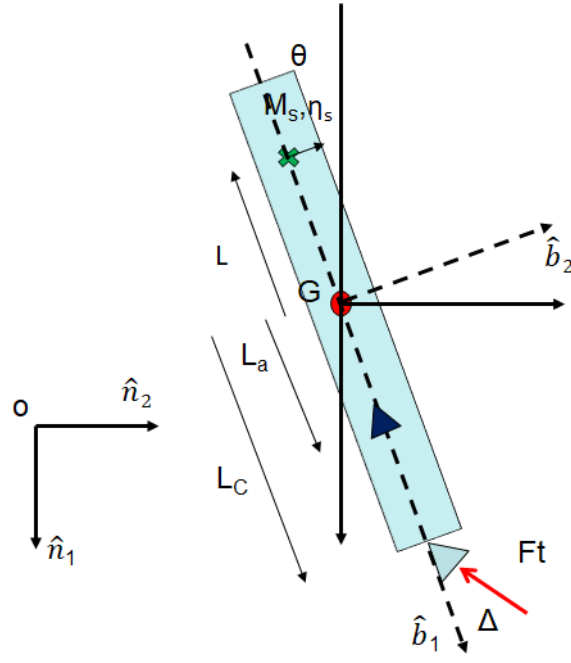


Figure 12. Frosh and Valley Model

The rigid body rotational and translational equations along with the 2nd order slosh dynamics are shown below:

$$I_{zz}\ddot{\theta} = -k_4 L_a \theta - T_C L_C \Delta - M_s L \ddot{\eta}_s - M_s k_3 \eta_s \quad (28)$$

$$M\ddot{Y} = k_4 \theta - T_C \Delta - M_s \ddot{\eta}_s - M k_3 \theta \quad (29)$$

$$\ddot{\eta}_s + 2\zeta_s \omega_s \dot{\eta}_s + \omega_s^2 \eta_s = -\ddot{Y} - L\ddot{\theta} - k_3 \theta \quad (30)$$

$k_4 = C_{N_\alpha} q_\infty S_{ref}$ and $k_3 \approx \frac{T_C}{M}$. Where C_{N_α} is the aerodynamic normal force coefficient with respect to angle of attack, α , q_∞ is the freestream dynamic pressure, and S_{ref} is the vehicle reference area. ζ_s and ω_s are the damping and natural frequency of the slosh mass, respectively. Re-writing Eq. 28 to 30 in $\dot{x} = Ax + Bu$ format:

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{zz} & 0 & 0 & 0 & M_s L \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M & 0 & M_s \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & L & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{Y} \\ \ddot{Y} \\ \dot{\eta}_s \\ \ddot{\eta}_s \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k_4 L_a & 0 & 0 & 0 & -M_s k_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_4 - M k_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_3 & 0 & 0 & 0 & -\omega_s^2 & -2\zeta_s \omega_s \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ Y \\ \dot{Y} \\ \eta_s \\ \dot{\eta}_s \end{bmatrix} + \begin{bmatrix} 0 \\ -T_C L_C \\ 0 \\ 0 \\ 0 \\ -T_C \end{bmatrix} \Delta \end{aligned} \quad (31)$$

Obtaining the transfer function from Δ to θ :

$$\frac{\theta}{\Delta} = - \frac{T_C [(L_C M + L M_s - L_C M_s) s^2 + (K_3 M_s + L_C M \omega_s^2)]}{(I_{zz} M - I_{zz} M_s - M M_s L^2) s^4 + (k_4 L_a M - k_4 L_a M_s - k_4 M_s L + I M \omega_s^2 - L K_3 M M_s) s^2 + (k_4 L_a M \omega_s^2 - K_3 K_4 M_s)} \quad (32)$$

Ignoring aerodynamics ($k_4 = 0$) and assuming an unaccelerating flight ($k_3 = 0$), Eq. 32 reduces to:

$$\frac{\theta}{\Delta} = -\frac{T_C (L_C M + L_s M_s - L_C M_s) s^2 + L_C M \omega_s^2}{s^2 (I_{zz} M - I_{zz} M_s - M M_s L_s^2) s^2 + I M \omega_s^2} \quad (33)$$

Equating the slosh mode pole-zero pair as done in Sec. III, the slosh danger zone reduces to the classic solution shown in Eq. 13 and 15:

$$\boxed{-\frac{I_{zz}}{L_C M} < L < 0} \quad (34)$$

To assess the lone effect of aerodynamics on the danger zone, set k_3 to 0 in Eq. 32 and solve for L by equating the slosh pole-zero pair:

$$L = \begin{cases} \frac{L_C [I_{zz} M \omega_s^2 - (M - M_s) L_a k_4 - (M - M_s) L_C k_4]}{-L_C^2 M^2 \omega_s^2 + M_s L_a k_4 + M_s L_C k_4} \\ 0 \end{cases} \quad (35)$$

Further assuming $M \gg M_s$, the first solution in Eq. 35 can be reduced to:

$$L = -\frac{I_{zz}}{L_C M} + \frac{L_a k_4 + L_C k_4}{L_C^2 M \omega_s^2} \quad (36)$$

The slosh stability boundary with the inclusion of aerodynamics can be approximated as:

$$\boxed{-\frac{I_{zz}}{L_C M} + \frac{(L_a + L_C) k_4}{L_C^2 M \omega_s^2} < L < 0} \quad (37)$$

For an aerodynamically stable vehicle ($L_a > 0$), Eq. 37 suggests that the upper bound of the danger zone would shift closer to the center of mass, G , while the lower bound remains at the G . Hence a reduction in the range of the danger zone compared to the classic solution shown in Eq. 34. The upper bound of the danger zone could exceed the classic solution if $L_a < -L_C$.

To assess the effect of axial acceleration on the danger zone, set k_4 to 0 in Eq. 32 and solve for L by equating the slosh pole-zero pair:

$$L = \begin{cases} -\frac{I_{zz}}{L_C M} \\ \frac{K_3 (M - M_s)}{M \omega_s^2} \end{cases} \quad (38)$$

The slosh stability boundary with axial acceleration effect can be expressed as:

$$\boxed{-\frac{I_{zz}}{L_C M} < L < \frac{k_3 (M - M_s)}{M \omega_s^2}} \quad (39)$$

The upper bound of the danger zone stays fixed at the center of percussion while the lower bound shifts to somewhere below the center of mass by $\frac{k_3 (M - M_s)}{M \omega_s^2}$. Generally, $M \gg M_s$ and the lower bound reduces to $\frac{k_3}{\omega_s^2}$. Equation 39 suggests that as acceleration along $\hat{\mathbf{b}}_1$ increases, the lower bound of the danger zone would shift further away from the center of mass towards the engines. The result is consistent with the derivation shown in Ref. [10]. If the effects of axial acceleration and aerodynamics are both included:

$$L = \begin{cases} -\frac{I_{zz} M_s k_3 - L_C^2 M k_4 + L_C^2 M_s k_4 + I_{zz} L_C M \omega_s^2 - L_a L_C M k_4 + L_a L_C M_s k_4}{-L_C^2 M^2 \omega_s^2 - M M_s L_C k_3 + M_s L_C k_4 + L_a M_s k_4} \\ \frac{K_3 (M - M_s)}{M \omega_s^2} \end{cases} \quad (40)$$

It is apparent from Eq. 40 that the lower bound of the danger zone is solely dependent on k_3 while the upper bound is dependent on both k_3 and k_4 . However, if both k_3 and k_4 are small, the upper bound simplifies to the center of percussion.

VIII. Conclusion

In this work, the methodology behind the classic stability boundary for a single slosh tank onboard a space vehicle is extended to the tandem tank configuration where the tanks are either located along the vehicle centerline or symmetrically offset from the centerline. In the case of the tandem centerline tank configuration, the slosh danger zone deviates significantly compared to the classic single-tank configuration when the mass and frequency ratios of the slosh masses approach unity. In the case of the off-centerline tank configuration, the upper bound of the danger zone shifts by a distance proportional to the square of the lateral offset distance, beyond the center of percussion. Secondly, the slosh stability boundary for a vehicle with a single centerline tank and pair of outboard reaction jets, rather than thrust vectoring, for attitude control is studied. In the limit that the jets fire such that a pure angular acceleration is produced, the upper bound of the danger zone collapses to the vehicle center of mass. On the other hand, if only one of the thrusters fire and produce both angular and linear acceleration, then the danger zone reduces to the classic solution. Finally, the impacts of aerodynamics and vehicle axial acceleration on the classic danger zone solution are analyzed. Aerodynamics primarily affects the upper bound of the stability boundary, while axial acceleration shifts the lower bound away from the center of mass towards the aft end of the vehicle. The latter is consistent with findings published in recent literature.

Appendix: Baseball Bat Analogy

As an analogy to Sec. II, the center of percussion is derived for a baseball bat. The center of mass is denoted by the point G . The handle of the bat is denoted by point A . The center of mass is located at a distance L_G away from A . The center of percussion (often noted as the “sweet spot”) is located at a distance L_Q from A . $I_{zz/A}$ is the transverse moment of inertia of the bat with respect to A and M is the mass. F represent force acting on the bat as it strikes the ball. $\hat{\mathbf{b}}$ denotes the body frame and $\hat{\mathbf{n}}$ denotes the inertial frame.

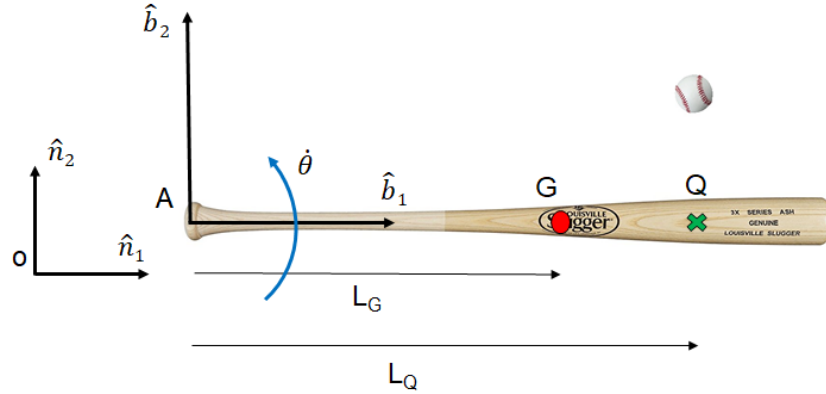


Figure 13. Baseball Bat

The moment equation about $\hat{\mathbf{b}}_3$ with respect to A (assumed to be fixed in the inertial frame):

$$I_{zz/A} \ddot{\theta} = -FL_Q \quad (41)$$

Acceleration of G in the body frame:

$$a_{G/o} = L_G \ddot{\theta} \hat{\mathbf{b}}_2 - L_G \dot{\theta}^2 \hat{\mathbf{b}}_1 \quad (42)$$

Equating force and acceleration along $\hat{\mathbf{b}}_2$:

$$ML_G \ddot{\theta} = R_y - F \quad (43)$$

where R_y is the reaction force experienced by the batter. If the ball strikes the bat at Q , $R_y \approx 0$. Solving

Eq. 41 for F , substituting into Eq. 43, and solving for L_Q :

$$L_Q = \frac{I_{zz/A}}{ML_G} \quad (44)$$

For a uniform rod, $I_{zz/A} = \frac{1}{3}ML^2$, $L_G = \frac{1}{2}L$. L_Q is located at $\frac{2}{3}L$ from the base or $\frac{1}{6}L$ from the center of mass in the $\hat{\mathbf{b}}_1$ direction. This result is consistent with result shown in Eq. 5.

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