



Adjoint-based Sensitivities of Flutter Predictions based on the Linearized Frequency-domain Approach

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Background

Motivation 1: Flutter certification by analysis

- Parametric uncertainty quantification of flutter boundary

Motivation 2: Flutter-constrained multidisciplinary optimization

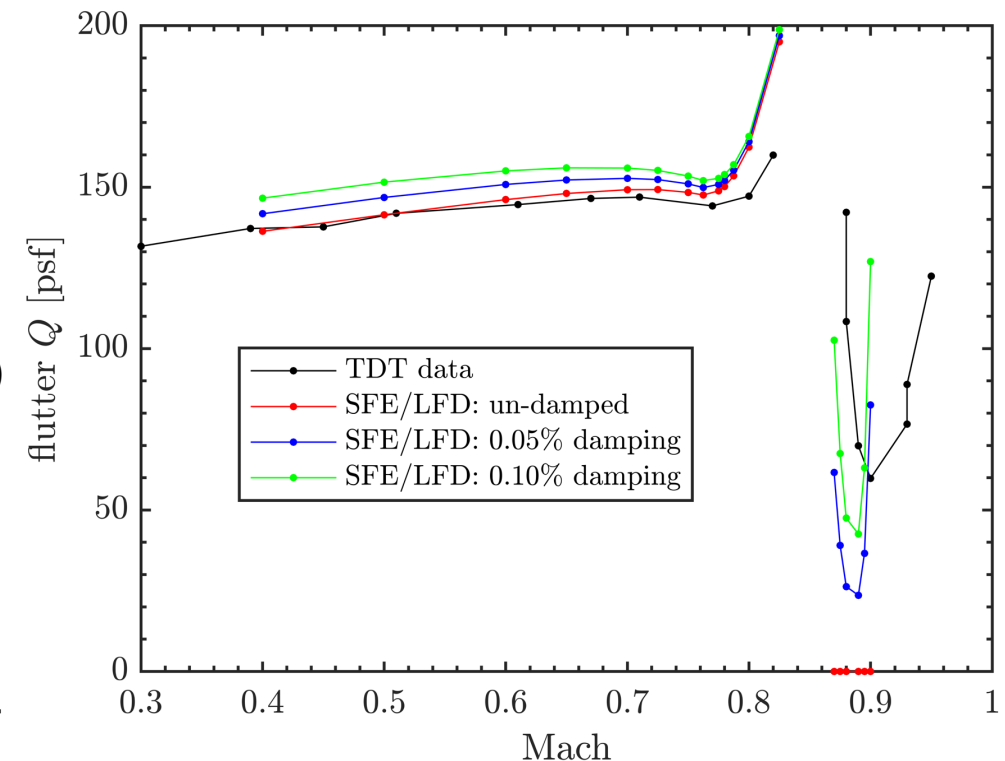
- Avoid infeasible optimization results by accounting for unsteady aeroelasticity in high-fidelity MDAO

Requirements for both:

- Fast turn-around times with CFD-level accuracy
- Efficient computation of sensitivity information

Approach:

- In SciTech 2020¹, added a Linearized Frequency-Domain (LFD) solver to the Stabilized Finite-Element discretization of FUN3D (FUN3D/SFE)
 - LFD solver ~10x faster than standard FUN3D time domain solver for transonic Benchmark NACA 0012 Wing
- In this work, adding adjoint-based sensitivities to the LFD solver



Transonic flutter boundary of Benchmark NACA 0012



¹K. Jacobson, B. Stanford, S. Wood, and W.K. Anderson, "Flutter Analysis with Stabilized Finite Elements based on the Linearized Frequency-domain Approach"

Linearized Aeroelastic Problem

Flutter is a linear stability problem

Start from linearized aeroelastic governing equation, write in a modally reduced matrix form:

$$\overline{\mathbf{M}}\ddot{\hat{\xi}} + \overline{\mathbf{C}}\dot{\hat{\xi}} + \overline{\mathbf{K}}\hat{\xi} = q_{\infty}\overline{\mathbf{Q}}(i\omega)\hat{\xi}$$

Vector of modal displacements from equilibrium

$Q_{m,n}(i\omega_k)$ GAFs - Generalized Aerodynamic Force on mode m due to harmonic motion of mode n at frequency ω_k

Writing in nondimensional state space form and assuming solution of form $\hat{\xi} = \bar{\xi}e^{\nu t}$ leads to p-k eigenproblem:

$$\left(\overline{\mathbf{A}}p - \overline{\mathbf{B}}(ik)\right)\overline{\mathbf{R}} = 0$$

$$b\nu/U_{\infty} = g + ik$$

$$\omega b/U_{\infty}$$

$$\overline{\mathbf{A}} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \left(\frac{U_{\infty}}{b}\right)^2 \overline{\mathbf{M}} \end{bmatrix}$$

$$\overline{\mathbf{B}}(ik) = \begin{bmatrix} 0 & \mathbf{I} \\ -\overline{\mathbf{K}} + q_{\infty}\overline{\mathbf{Q}}(ik) & -\left(\frac{U_{\infty}}{b}\right)\overline{\mathbf{C}} \end{bmatrix}$$

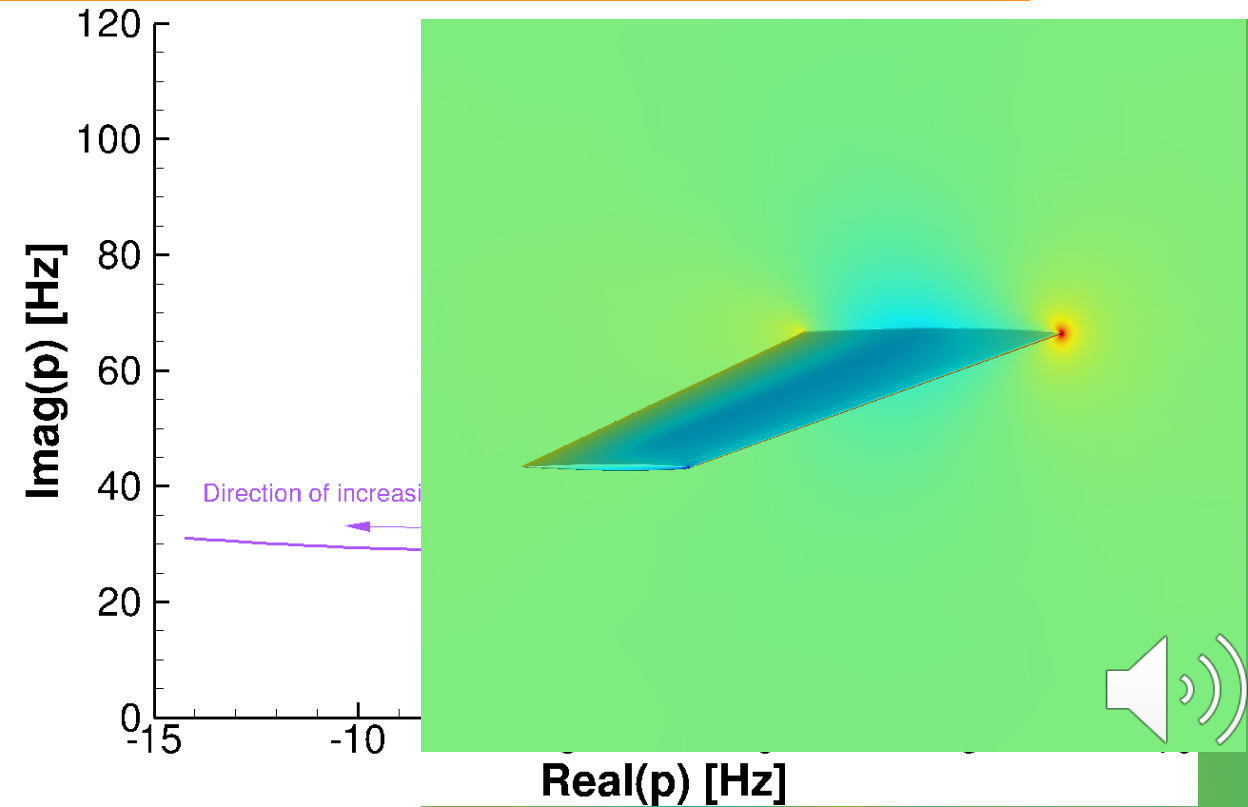
Valid flutter point when:

$$\text{real}(p) = 0$$

$$k - \text{imag}(p) = 0$$

The Full Problem

1. Compute steady state solution
 - Assuming symmetric bodies in this presentation (no static aeroelastic deformation)
2. Compute LFD aerodynamics
 - Aerodynamic response to harmonic motion of structural modes linearized about the steady state solution
3. Compute linearized frequency-domain aerodynamic forces and GAFs
4. Solve the p-k eigenproblem
 - Sensitivities solved as two adjoint subproblems
 - The **eigenproblem adjoint** serves at the cost function term for the **aerodynamic adjoint problem**



Complications in the Adjoint

In this presentation:

- **Aerodynamic adjoint formulation** accounting for complex state variables

See the paper for details on:

- **Linearized forward problem** requires second derivatives of the RANS equations
 - Complex step directional derivatives of first order linearizations

$$\left[\frac{\partial}{\partial \mathbf{q}} \left(\frac{\partial \mathbf{R}}{\partial \mathbf{x}_G} \bigg|_{\mathbf{x}_{G0}, \mathbf{q}_0} \hat{\mathbf{x}}_{G,j} \right) \right]^T \psi_{ALFD,j,k} = \frac{\text{imag} \left(\frac{\partial \mathbf{R}}{\partial \mathbf{q}} \bigg|_{\mathbf{x}_{G0} + ih \text{real}(\hat{\mathbf{x}}_{G,j}), \mathbf{q}=\mathbf{q}_0}^T \text{real}(\psi_{ALFD,j,k}) \right)}{h} - \frac{\text{imag} \left(\frac{\partial \mathbf{R}}{\partial \mathbf{q}} \bigg|_{\mathbf{x}_{G0} + ih \text{imag}(\hat{\mathbf{x}}_{G,j}), \mathbf{q}=\mathbf{q}_0}^T \text{imag}(\psi_{ALFD,j,k}) \right)}{h}$$

- **Eigenproblem adjoint formulation**

$$\frac{\partial p}{\partial y} = \left[\bar{\mathbf{L}}^T \left(\frac{\partial \bar{\mathbf{B}}}{\partial y} - p \frac{\partial \bar{\mathbf{A}}}{\partial y} \right) \bar{\mathbf{R}} \right] / (\bar{\mathbf{L}}^T \bar{\mathbf{A}} \bar{\mathbf{R}})$$

Complex Adjoint Derivation

Cauchy-Riemann equations for complex differentiability:

$$z := x + iy \quad f(z) := u + iv \quad \longrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

For the flutter cost functions, the Cauchy-Riemann equations are not satisfied since $v = 0$

- Need to modify typical adjoint derivation
 - State vector: $\mathbf{y} = \mathbf{a} + i\mathbf{b}$
 - Residual: $\mathbf{W}(\mathbf{y}) = 0$
- Design variable derivative of cost function:

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial \mathbf{a}} \frac{d\mathbf{a}}{dx} + \frac{\partial f}{\partial \mathbf{b}} \frac{d\mathbf{b}}{dx} \\ &= \frac{\partial f}{\partial x} + \text{real} \left[\left(\frac{\partial f}{\partial \mathbf{a}} - i \frac{\partial f}{\partial \mathbf{b}} \right) \left(\frac{d\mathbf{a}}{dx} + i \frac{d\mathbf{b}}{dx} \right) \right] \\ &= \frac{\partial f}{\partial x} + \text{real} \left[\left(\frac{\partial f}{\partial \mathbf{a}} - i \frac{\partial f}{\partial \mathbf{b}} \right) \frac{d\mathbf{y}}{dx} \right] \end{aligned}$$

Complex Adjoint Derivation

Design variable derivative:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \text{real} \left[\left(\frac{\partial f}{\partial \mathbf{a}} - i \frac{\partial f}{\partial \mathbf{b}} \right) \frac{d\mathbf{y}}{dx} \right]$$

$$\mathbf{y} = \mathbf{a} + i\mathbf{b}$$

$$\mathbf{W}(\mathbf{y}) = 0$$

From the residual definition:

$$\frac{d\mathbf{W}}{dx} = \frac{\partial \mathbf{W}}{\partial x} + \frac{\partial \mathbf{W}}{\partial \mathbf{y}} \frac{d\mathbf{y}}{dx} = 0 \quad \Rightarrow \quad \frac{d\mathbf{y}}{dx} = - \left[\frac{\partial \mathbf{W}}{\partial \mathbf{y}} \right]^{-1} \frac{\partial \mathbf{W}}{\partial x}$$

Plug into design variable derivative:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \text{real} \left[- \left(\frac{\partial f}{\partial \mathbf{a}} - i \frac{\partial f}{\partial \mathbf{b}} \right) \left[\frac{\partial \mathbf{W}}{\partial \mathbf{y}} \right]^{-1} \frac{\partial \mathbf{W}}{\partial x} \right]$$

Complex Adjoint Equation

Define the adjoint vector:

$$\psi^T = - \left(\frac{\partial f}{\partial \mathbf{a}} - i \frac{\partial f}{\partial \mathbf{b}} \right) \left[\frac{\partial \mathbf{W}}{\partial \mathbf{y}} \right]^{-1} \quad \Rightarrow$$

$$\left[\frac{\partial \mathbf{W}}{\partial \mathbf{y}} \right]^T \psi = - \left(\frac{\partial f}{\partial \mathbf{a}} - i \frac{\partial f}{\partial \mathbf{b}} \right)^T$$

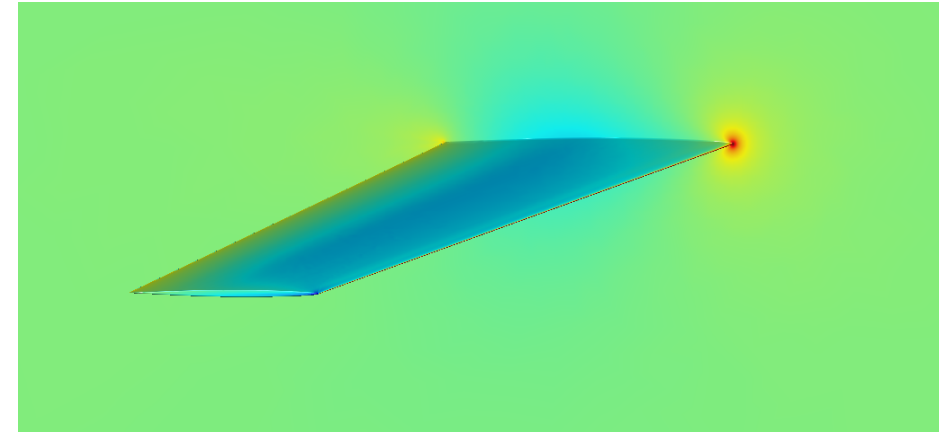
Final design variable derivative:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \text{real} \left[\psi^T \frac{\partial \mathbf{W}}{\partial x} \right]$$

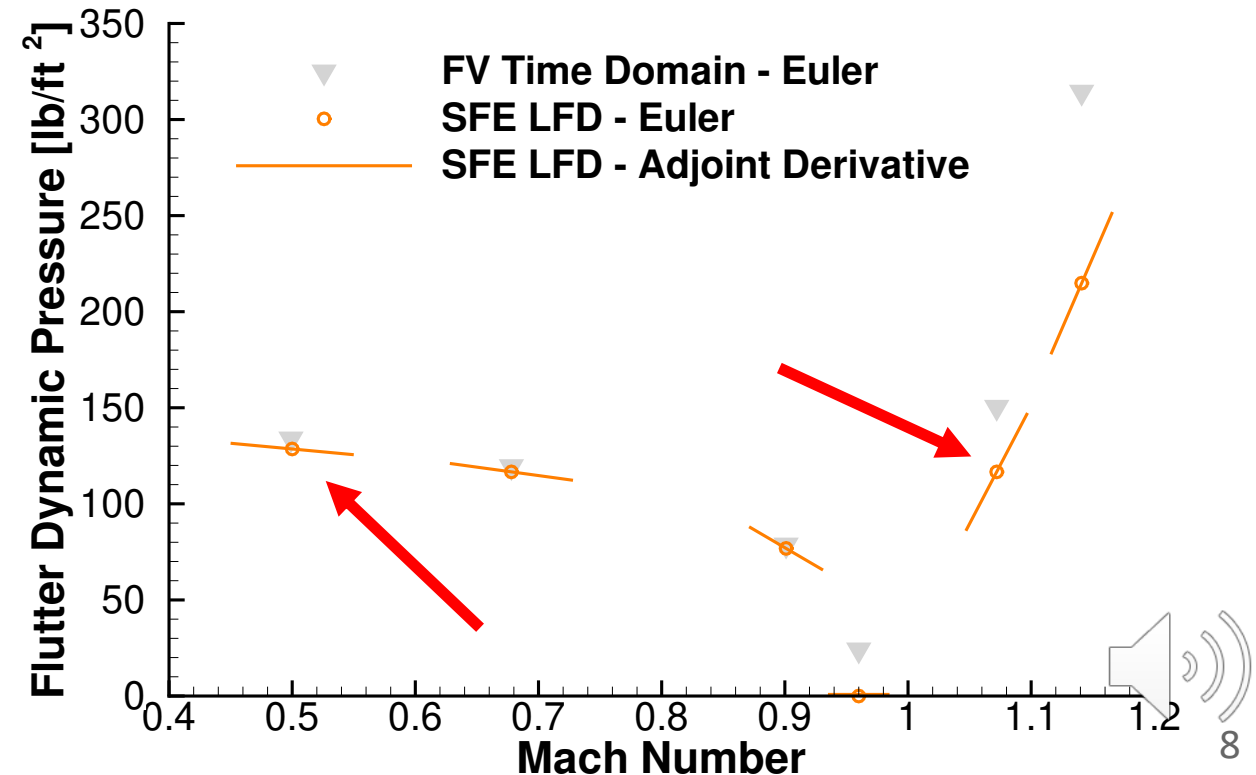
AGARD 445.6 – Flutter Boundary

Problem Definition:

- Modal structure: 4 modes
- CFD: Euler analysis with 292,638 node mesh
- LFD: 10 frequencies ($k = 0.01 - 2.0$)
- Mach number range (0.5 – 1.141)
- Derivative of flutter dynamic pressure with respect to Mach number



	$M_\infty = 0.5$	$M_\infty = 1.072$
Adjoint	-59.6929941	1225.945250
Finite Difference (10^{-4})	-59.7222326	1248.034599
Finite Difference (10^{-5})	-59.6930351	1245.346810
Finite Difference (10^{-6})	-59.6926180	1226.022198
Finite Difference (10^{-7})	-59.6874500	1225.946330
Finite Difference (10^{-8})	-59.6376992	1225.944700



AGARD 445.6 – Computational Cost

Computational time for $M_\infty=0.5$ on 1 dual-socket Skylake node (40 cores)

Solution phase	Walltime (min)	Percentage of total time
Steady flow solution	87.9	18.7
Volume mode shapes	4.7	1.0
LFD linear problems	226.2	48.2
LFD adjoint solver and terms	143.5	30.6
Steady flow adjoint	7.0	1.5
Total	469.3	100.0

LFD flutter point and adjoint-based sensitivities 5.3x the cost of the steady forward analysis

- Only 2 frequencies required for LFD adjoint (10 frequencies for forward)
- Adjoint = 32.1% of total cost

Conclusions

Implemented the adjoint equations for flutter cost functions with the LFD method in FUN3D/SFE

- Presented an LFD adjoint formulation accounting for complex variables
- Compute second derivatives of flow residual with complex step directional derivatives
- Linearized p-k eigenproblem for flutter cost functions

AGARD 445.6 wing derivatives of flutter dynamic pressure with respect to Mach number

- 5-6 digits agreement with finite difference derivatives

Future Work

- Extend to linearization about static aeroelastic solution
 - All required LFD terms verified, but need to add LFD adjoint terms to static aeroelastic adjoint equations

Acknowledgments

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