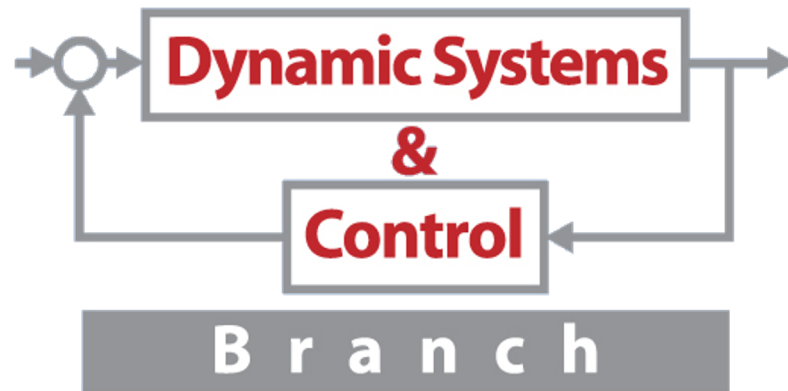


Modeling and Managing Uncertainty

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Overview

- Uncertainty modeling
- Set deformations
- Optimization under uncertainty
- Decision making

Uncertainty Classification

- **Aleatory uncertainty** (aka statistical uncertainty) is an inherent variation associated with the physical system: *irreducible, modeled as a random variable*
 - E.g., turbulence, measurement noise
- **Epistemic uncertainty** (aka as systematic uncertainty) is uncertainty caused by lack of knowledge of quantities or processes: *reducible, modeled as a fixed but unknown constant*
 - E.g., body mass, gravity constant

Uncertainty Classification

- **Example:**

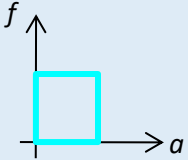
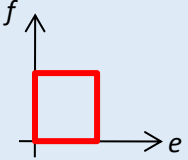
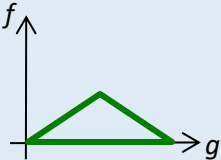
$y = M(\text{gravity constant of a planet, sun irradiance at its surface})$

- Physical values vs. their mathematical representation
- Shall we mix frequency-based randomness with belief-based randomness?

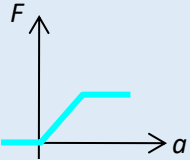
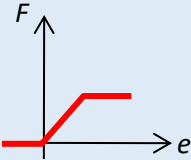
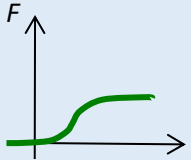
Implications of Uncertainty Modeling

	a in $[0,1]$	e in $[0,1]$	$g = a+e$	$E[g]$	$V[g]$	$P[g>3/2]$
Approach 1						
Approach 2						
Approach 3						

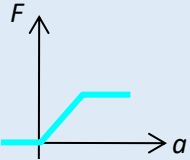
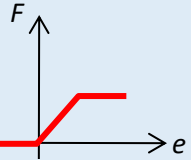
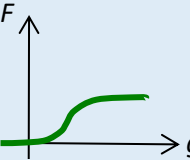
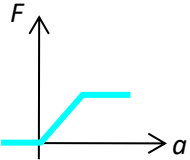
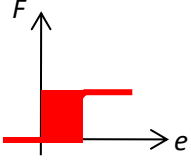
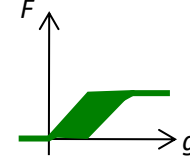
Implications of Uncertainty Modeling

	$a \text{ in } [0,1]$	$e \text{ in } [0,1]$	$g = a+e$	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16

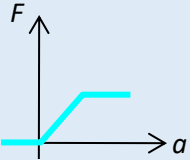
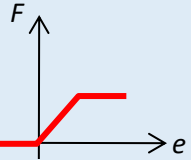
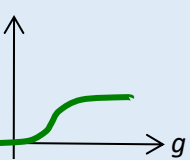
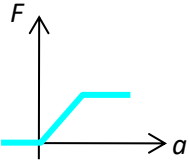
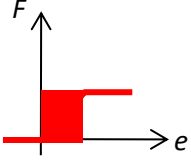
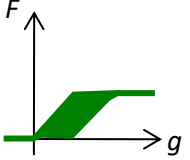
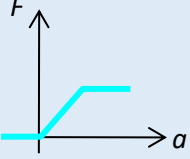
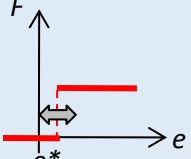
Implications of Uncertainty Modeling

	$a \text{ in } [0,1]$	$e \text{ in } [0,1]$	$g = a+e$	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16

Implications of Uncertainty Modeling

	a in $[0,1]$	e in $[0,1]$	$g = a+e$	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				$[1/2, 3/2]$	$[0, 13/16]$	$[0, 1/2]$

Implications of Uncertainty Modeling

	a in $[0,1]$	e in $[0,1]$	$g = a+e$	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				$[1/2, 3/2]$	$[0, 13/16]$	$[0, 1/2]$
Prob/UnkC				$[1/2, 3/2]$	Correct Answer 1/12	$[0, 1/2]$

Implications of Uncertainty Modeling

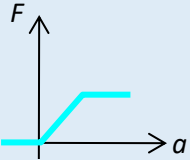
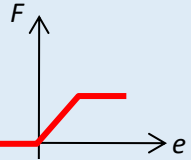
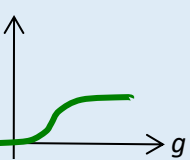
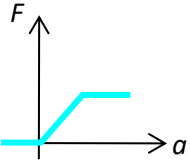
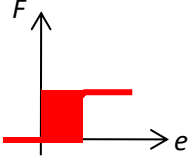
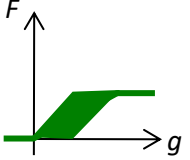
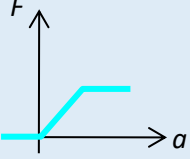
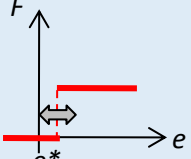
	a in $[0,1]$	e in $[0,1]$	$g = a+e$ <i>Random variable</i>	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				$[1/2, 3/2]$	$[0, 13/16]$	$[0, 1/2]$
Prob/UnkC				$[1/2, 3/2]$	1/12	$[0, 1/2]$

Not even conservative estimates

Implications of Uncertainty Modeling

	a in $[0,1]$	e in $[0,1]$	$g = a+e$ <i>P-Box</i>	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				$[1/2, 3/2]$	$[0, 13/16]$	$[0, 1/2]$
Prob/UnkC				$[1/2, 3/2]$	1/12	$[0, 1/2]$

Implications of Uncertainty Modeling

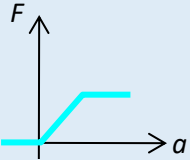
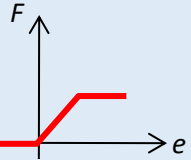
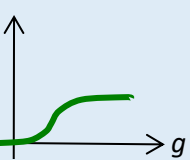
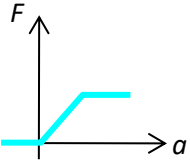
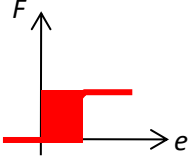
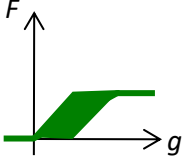
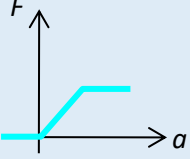
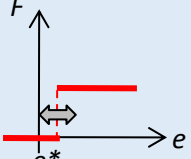
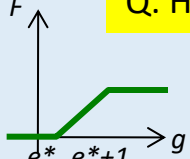
	a in $[0,1]$	e in $[0,1]$	$g = a+e$	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				Answers are right but formulations are wrong [1/2, 3/2]	[0, 13/16]	[0, 1/2]
Prob/UnkC				[1/2, 3/2]	1/12	[0, 1/2]

Implications of Uncertainty Modeling

	a in $[0,1]$	e in $[0,1]$	$g = a+e$ <i>Random process</i>	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				$[1/2, 3/2]$	$[0, 13/16]$	$[0, 1/2]$
Prob/UnkC				$[1/2, 3/2]$	1/12	$[0, 1/2]$

Q: How to do this systematically and accurately?

Implications of Uncertainty Modeling

	a in $[0,1]$	e in $[0,1]$	$g = a+e$ <i>Random process</i>	$E[g]$	$V[g]$	$P[g > 3/2]$
Prob/Prob				1	3/18	1/16
Prob/P-Box				$[1/2, 3/2]$	$[0, 13/16]$	$[0, 1/2]$
Prob/UnkC				Q: How to do this systematically and accurately? $[1/2, 3/2]$ 1/12 $[0, 1/2]$ A: Imprecise probabilities		

Application Settings

- Pure aleatory variables: sampling, FORM/SORM
- Pure epistemic variables: Interval analysis, sampling, optimization

Interval Analysis

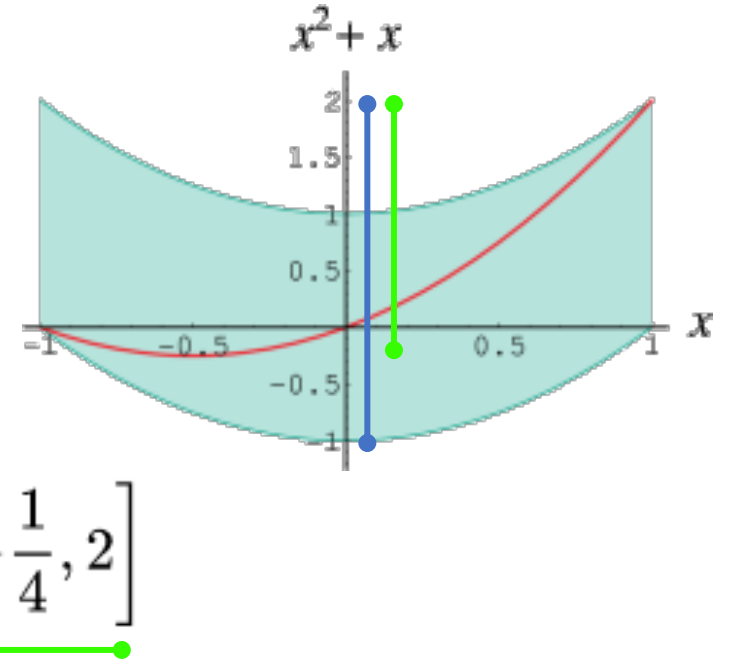
Q: What could go wrong with interval arithmetic?

$$f(x) = x^2 + x.$$

$$[-1, 1]^2 + [-1, 1] = [0, 1] + [-1, 1] = [-1, 2].$$

$$f(x) = \left(x + \frac{1}{2}\right)^2 - \frac{1}{4}$$

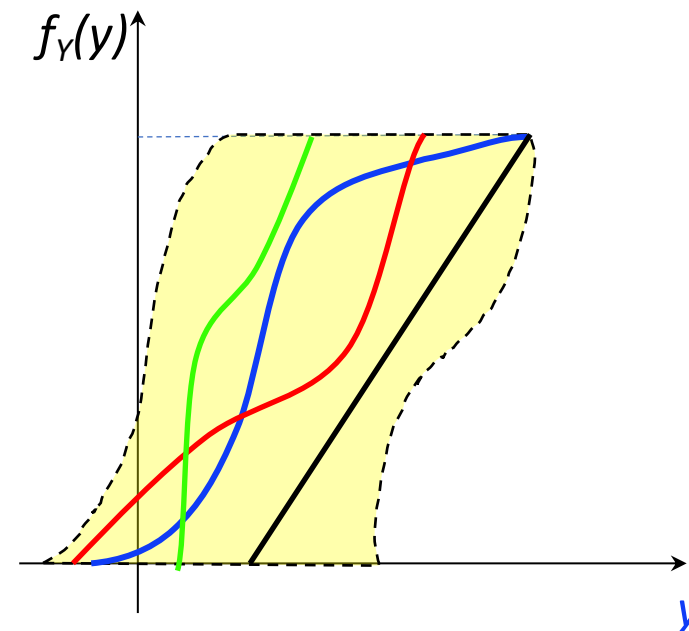
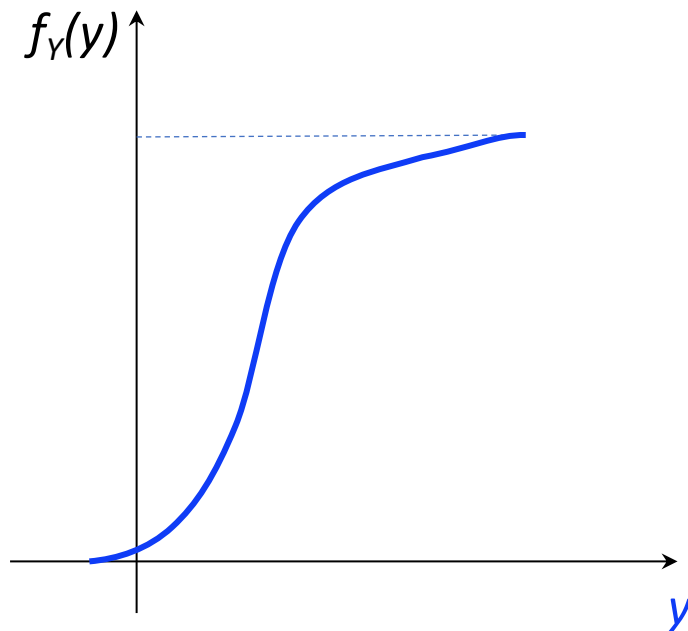
$$\left([-1, 1] + \frac{1}{2}\right)^2 - \frac{1}{4} = \left[-\frac{1}{2}, \frac{3}{2}\right]^2 - \frac{1}{4} = \left[0, \frac{9}{4}\right] - \frac{1}{4} = \left[-\frac{1}{4}, 2\right]$$



- Dependencies introduce conservatism: the exact range of values can be achieved if each variable appears only once and if the function is continuous. Otherwise, there will be overestimation/conservatism

Application Settings

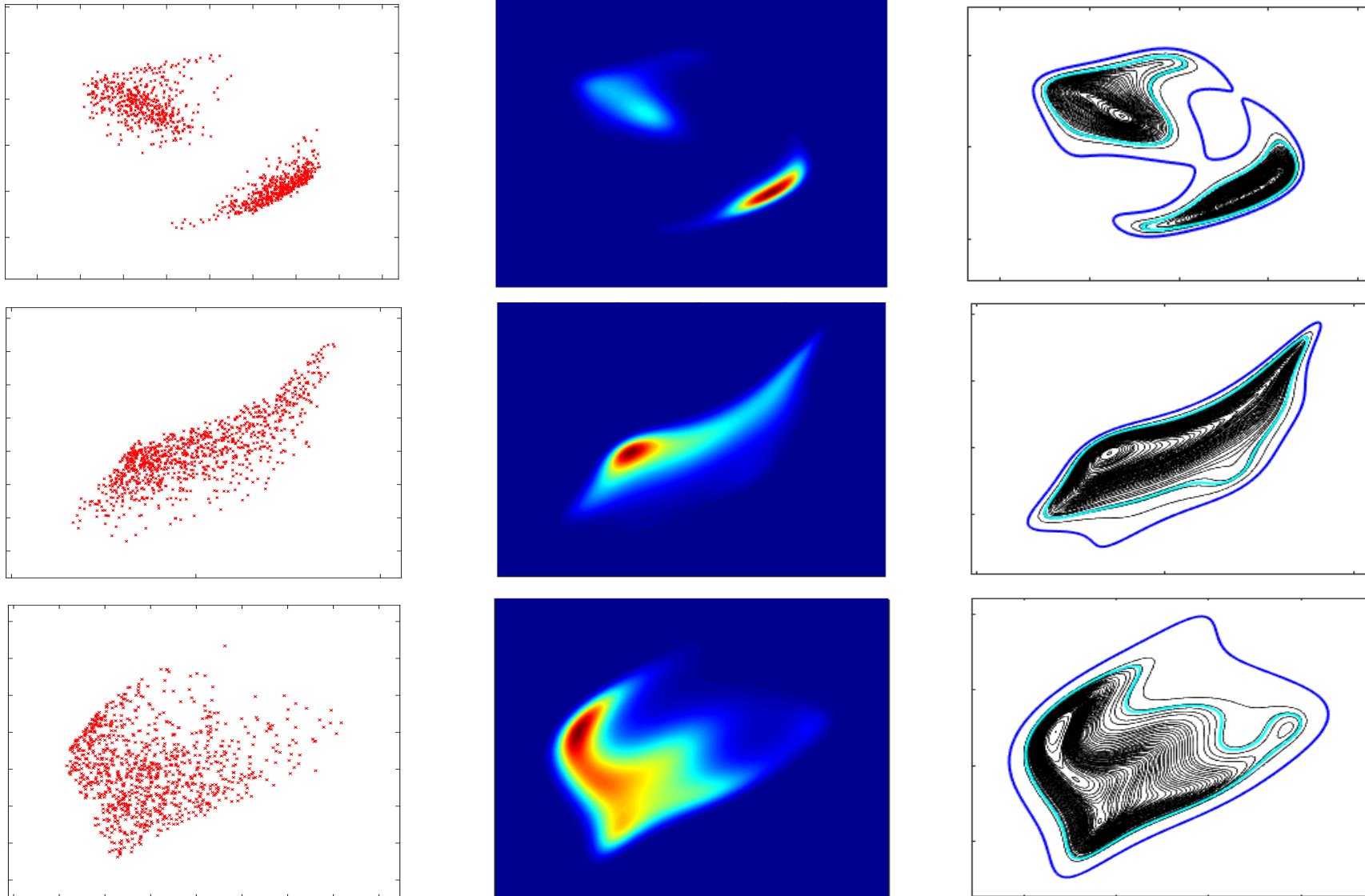
- Pure aleatory variables: sampling, FORM/SORM
- Pure epistemic variables: Interval analysis, sampling, optimization
- Mixed aleatory and epistemic variables: Pbox are calculated in a nested loop in which the outer loop correspond to the epistemic space and the inner loop to the aleatory space



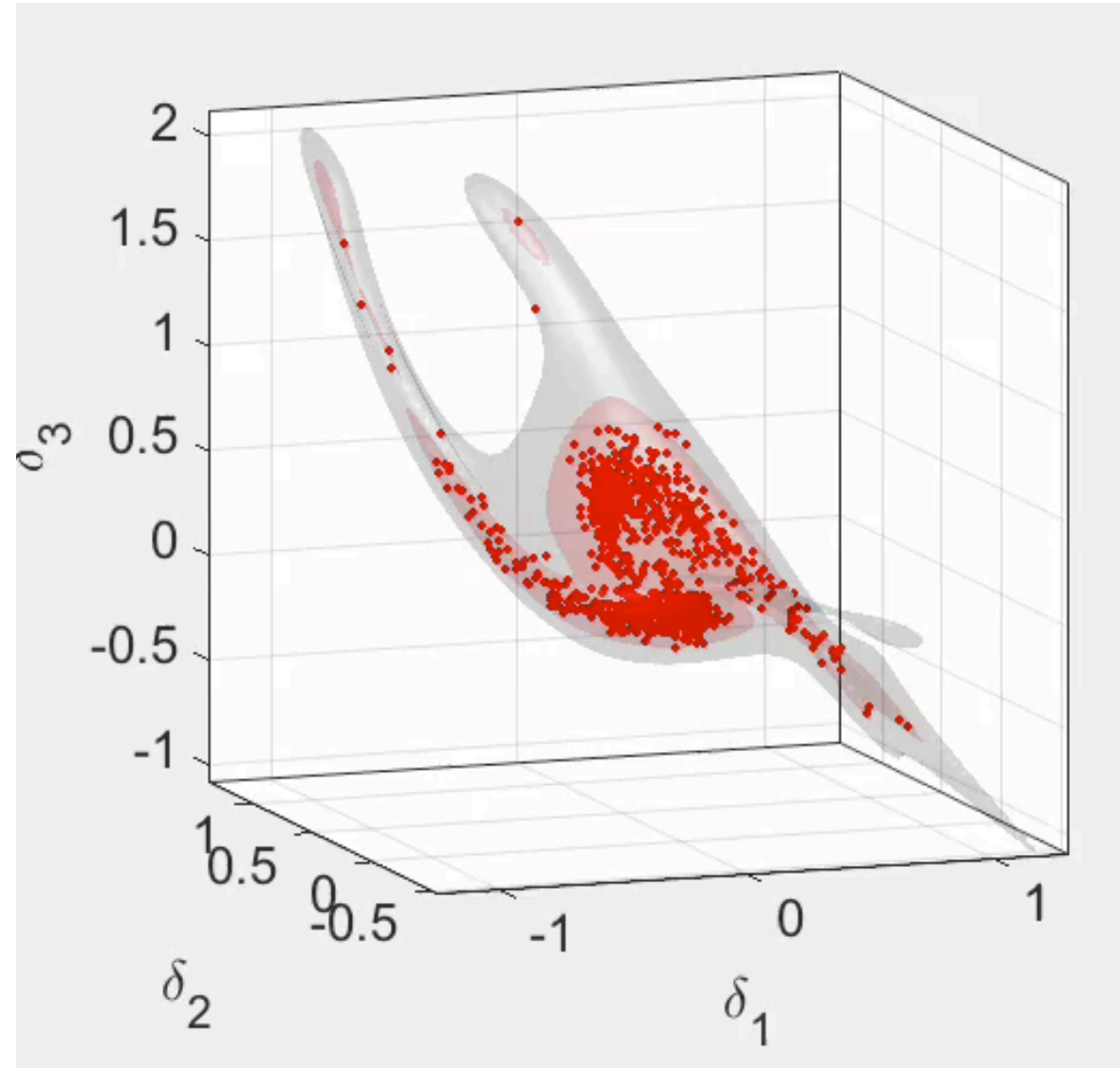
Uncertainty Modeling

- Data is instrumental in system identification, uncertainty quantification for robust control, V&V
- Poor characterizations of the data might lead to either overly conservative designs or insufficiently robust designs

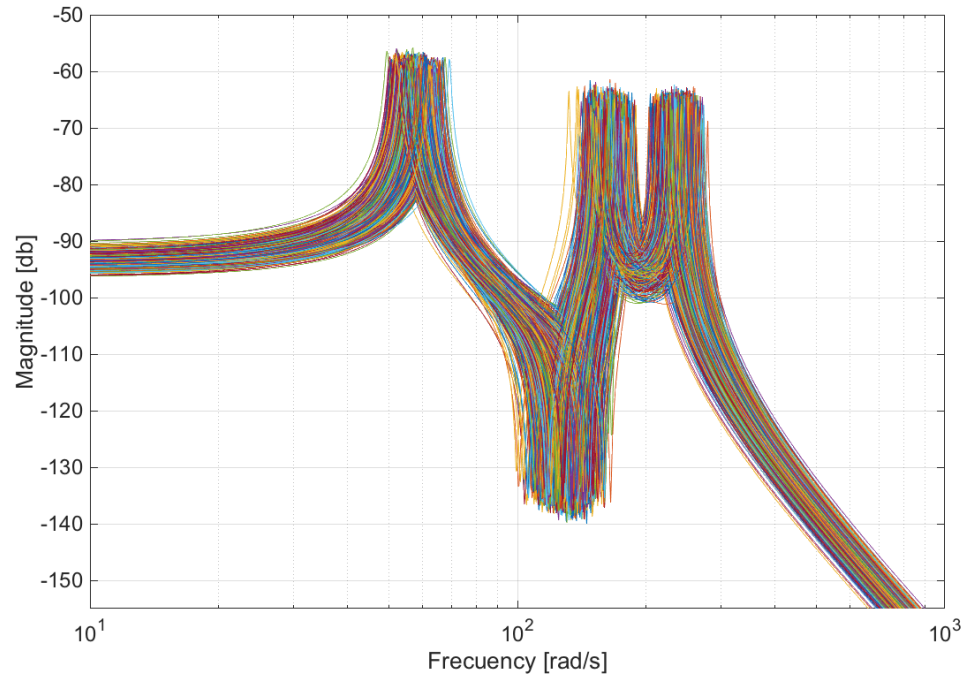
Uncertainty Modeling



Uncertainty Modeling: Sets

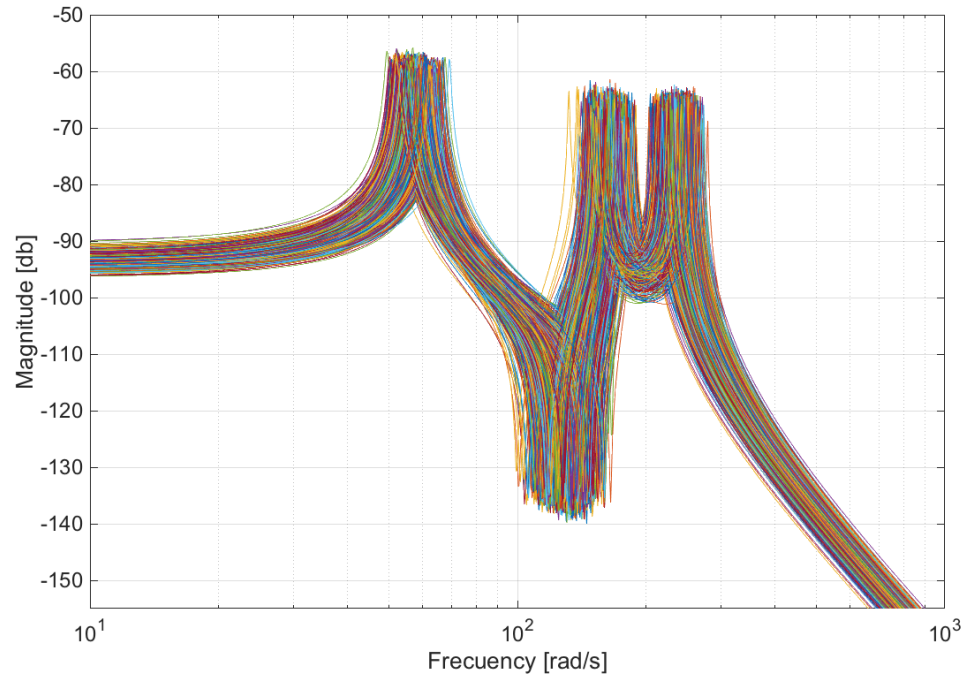


Uncertainty Modeling: Parameter Dependencies

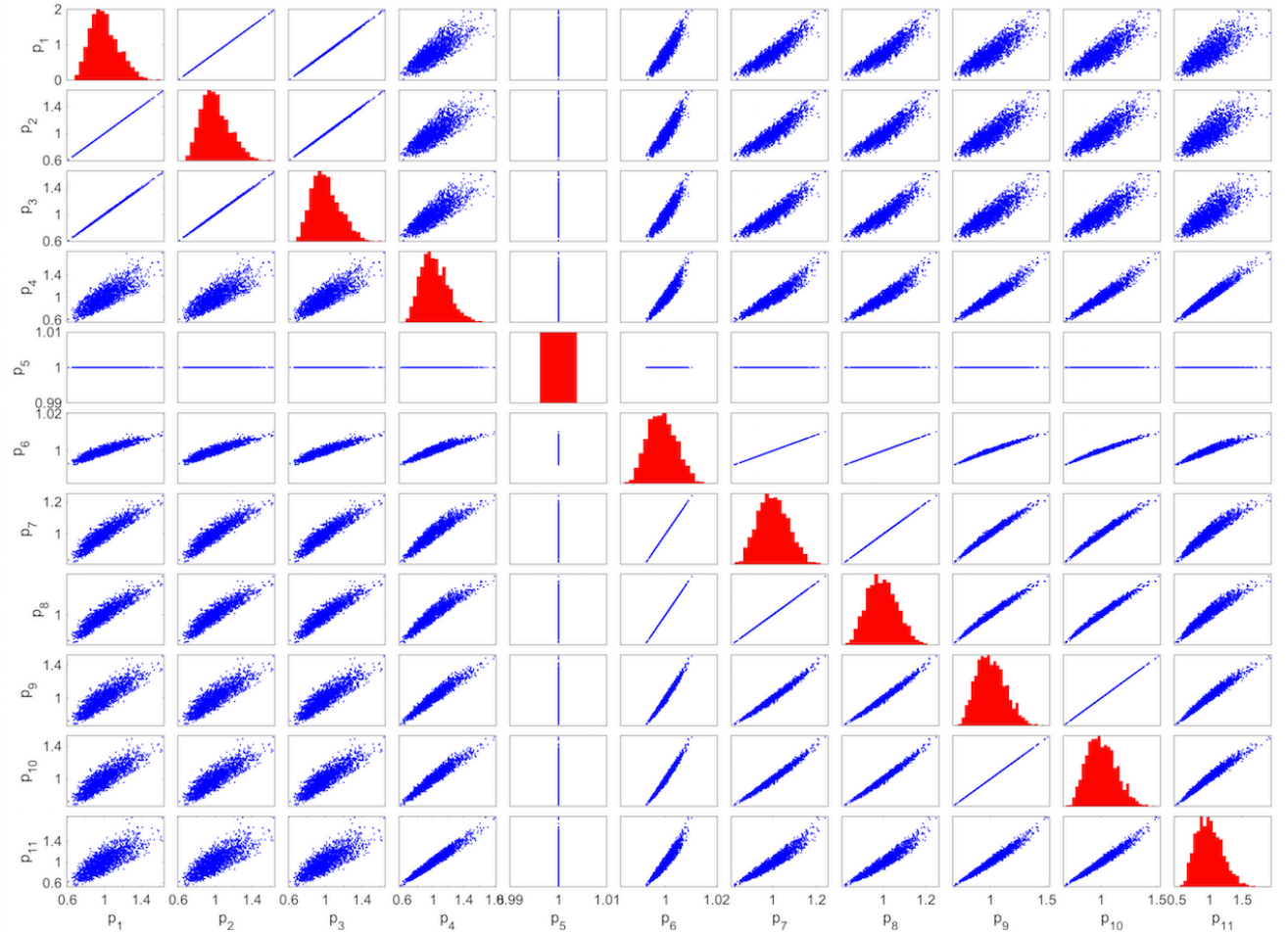


Uncertainty in masses and stiffnesses
are independent

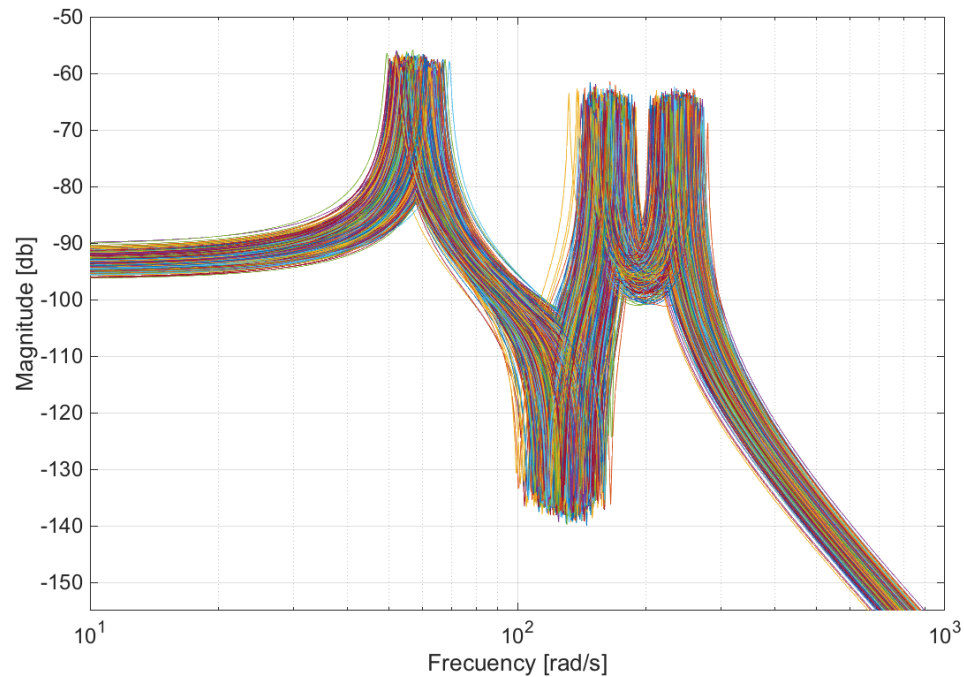
Uncertainty Modeling: Parameter Dependencies



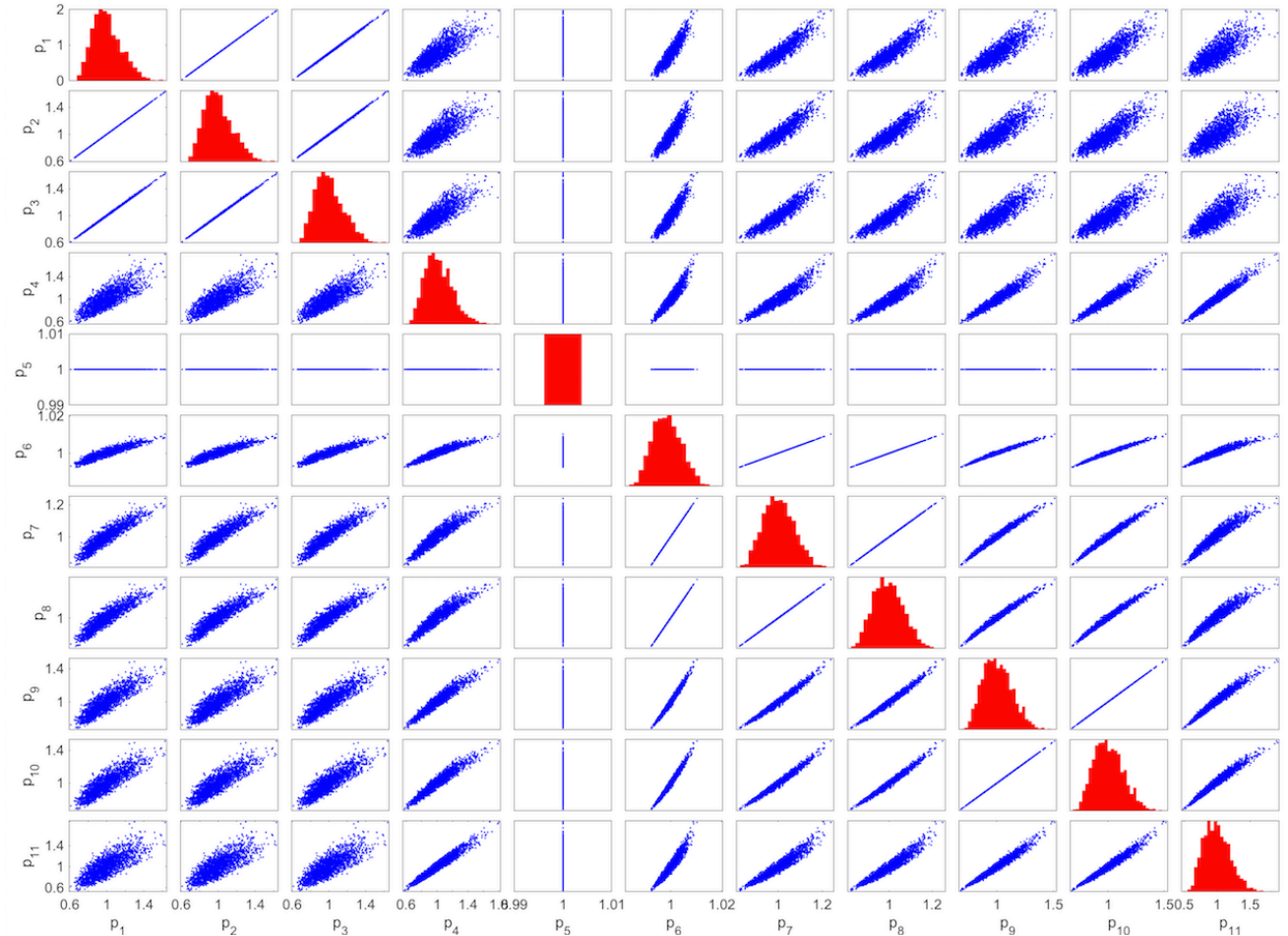
$$H(s) = \frac{p_1 s^4 + p_2 s^3 + p_3 s^2 + p_4 s + p_5}{p_6 s^5 + p_7 s^4 + p_8 s^3 + p_9 s^2 + p_{10} s + p_{11}}$$



Uncertainty Modeling: Parameter Dependencies

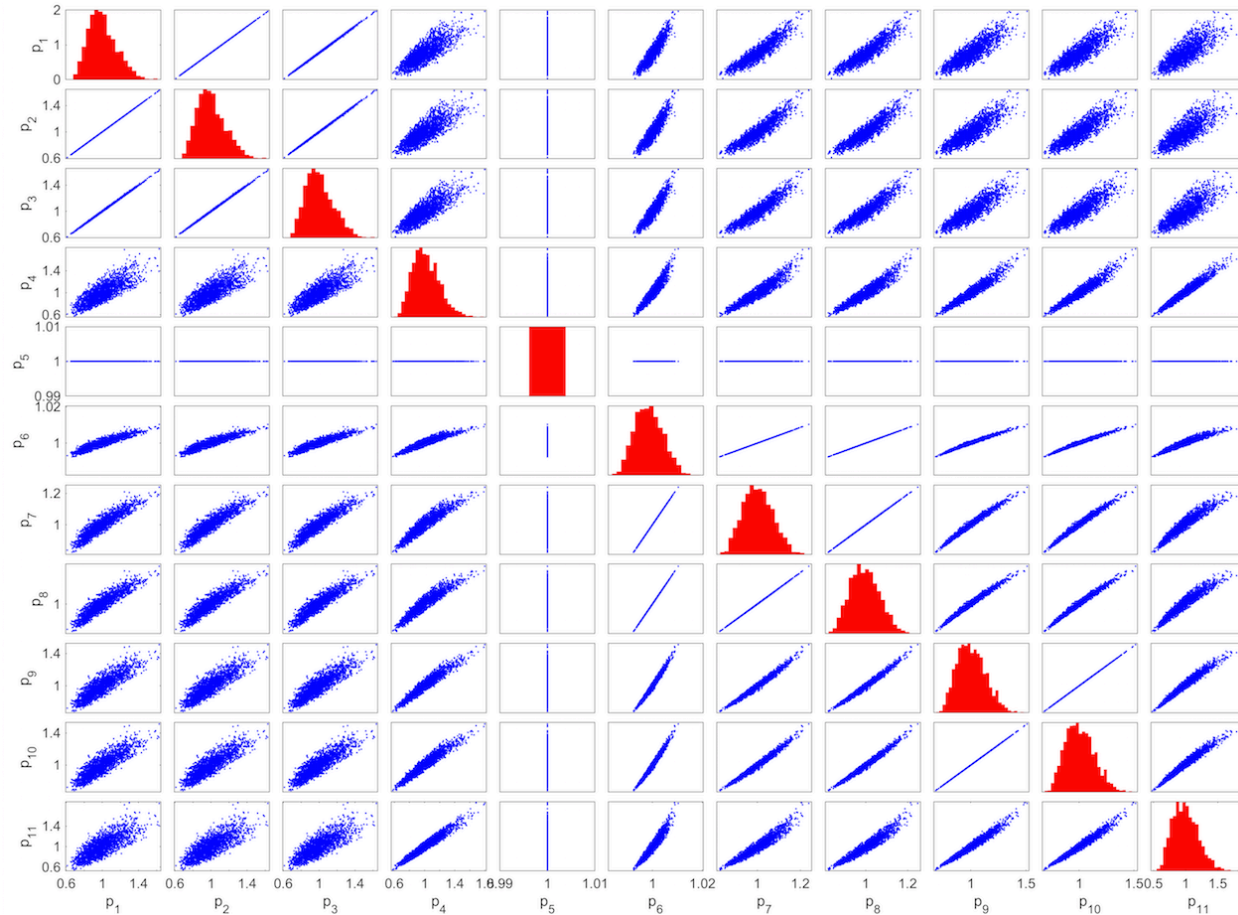


$$H(s) = \frac{p_1 s^4 + p_2 s^3 + p_3 s^2 + p_4 s + p_5}{p_6 s^5 + p_7 s^4 + p_8 s^3 + p_9 s^2 + p_{10} s + p_{11}}$$



- Independent parameters in physical space become dependent in the space of the transfer function parameters
- Assuming parameter independence introduces unnecessary conservatism

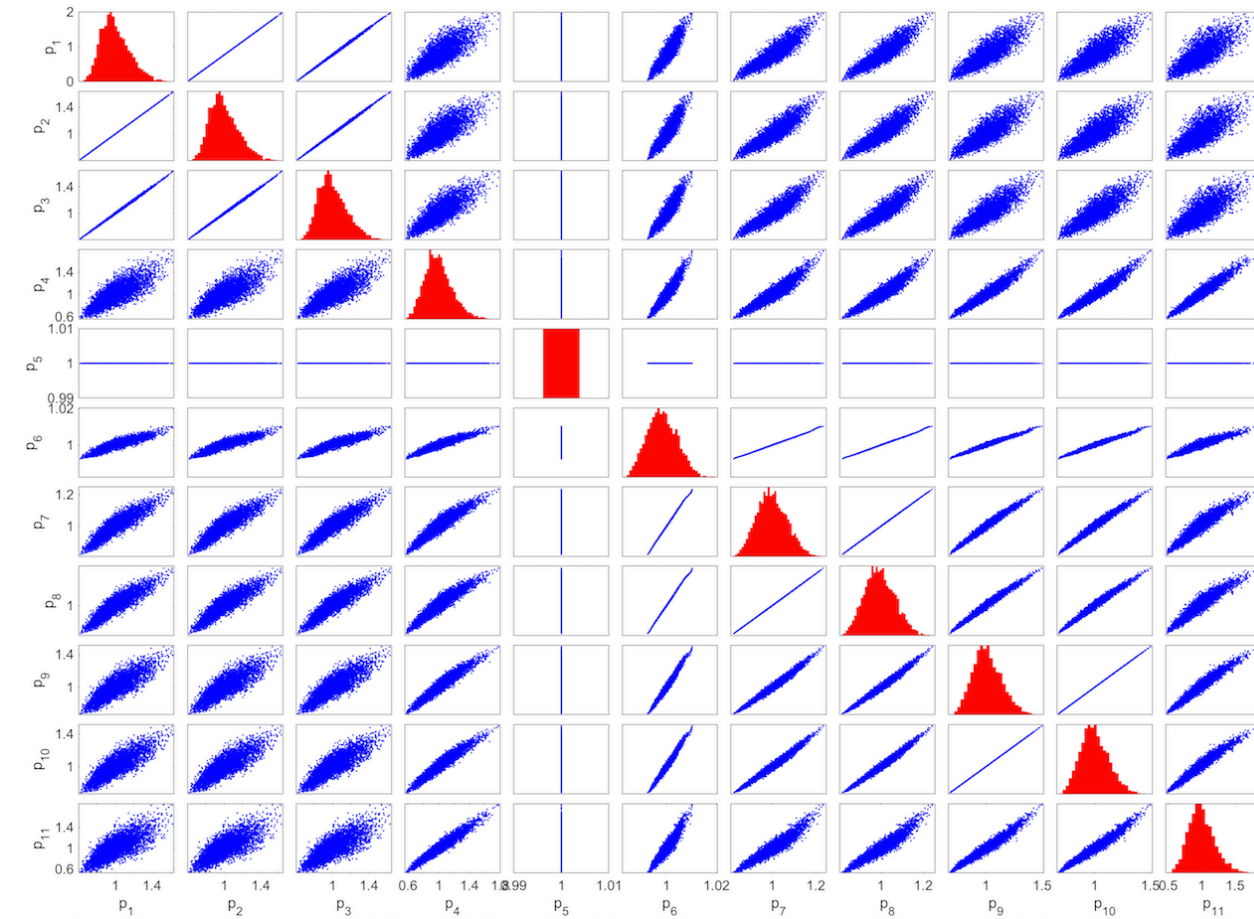
Uncertainty Modeling: Parameter Dependencies



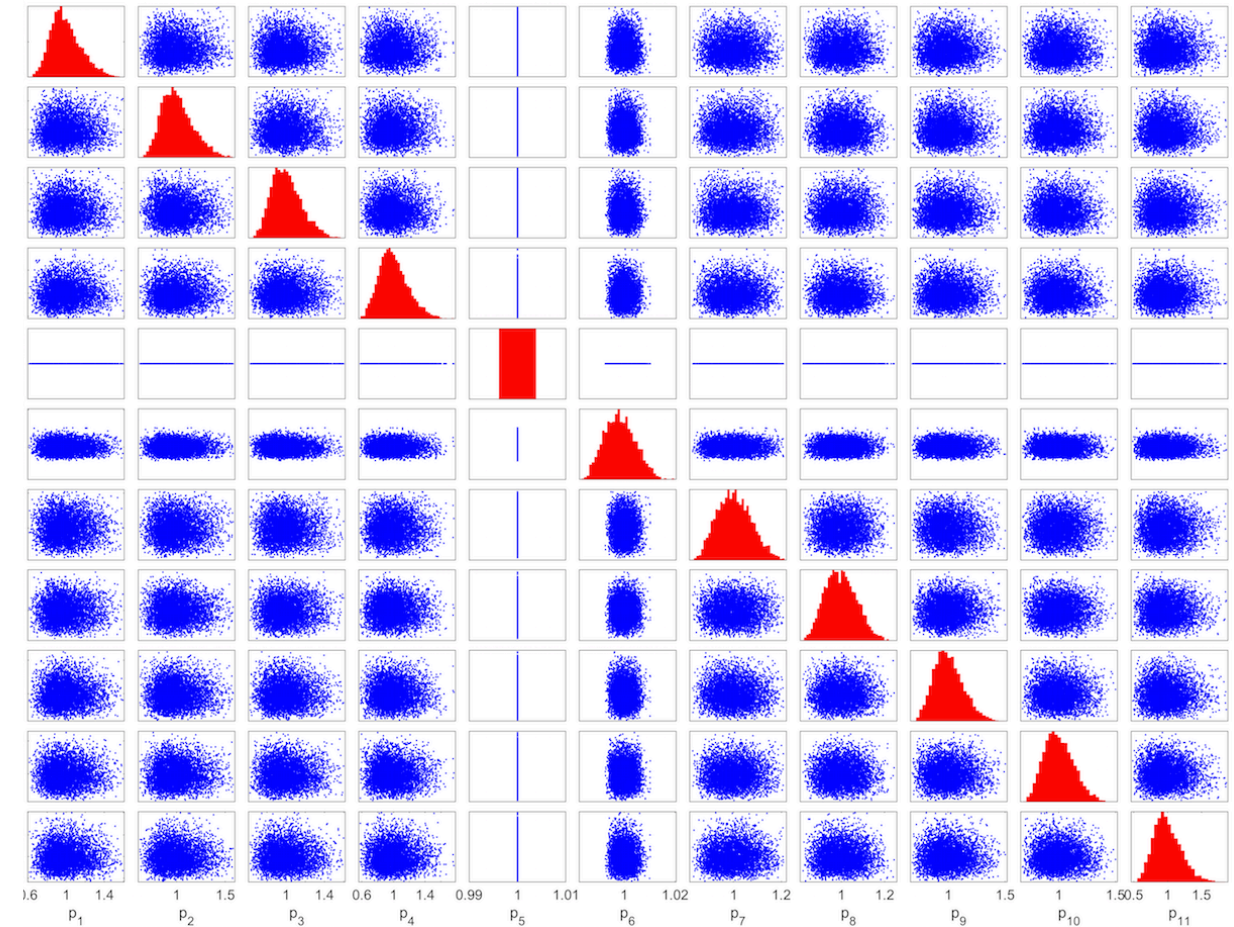
Consider two uncertainty models:

- Dependent parameters
- Independent parameters

Uncertainty Modeling: Parameter Dependencies

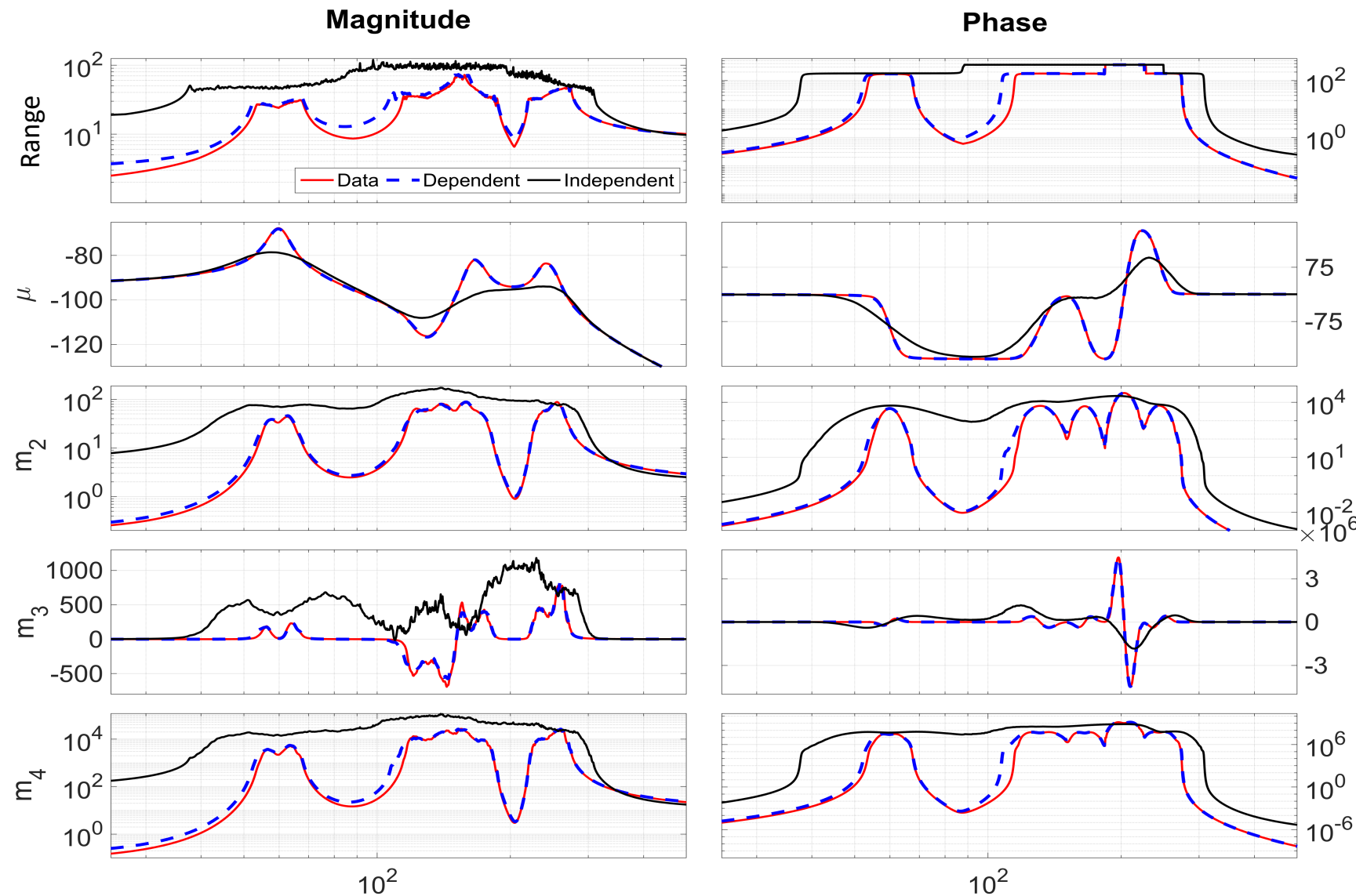


Samples of a distribution with **dependent** parameters



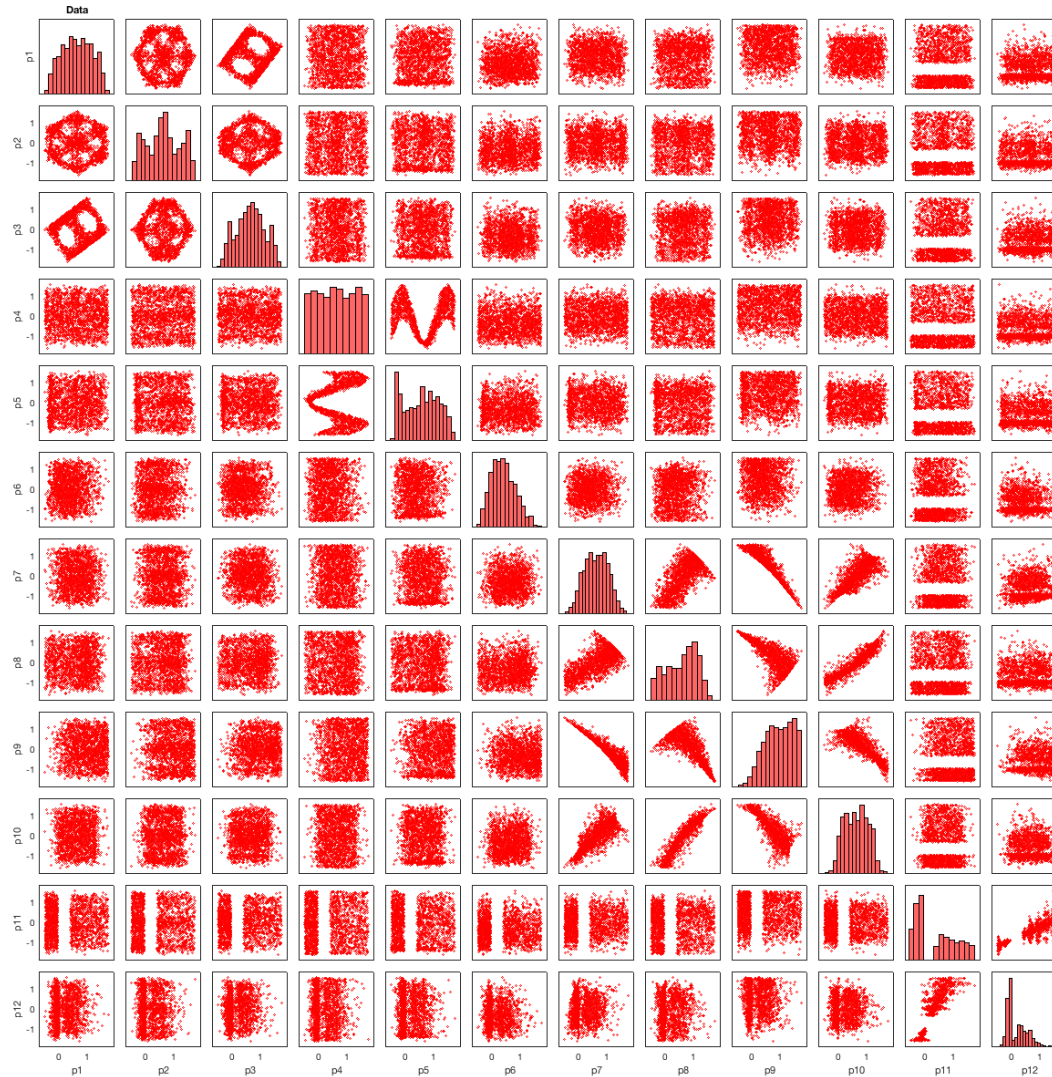
Samples of a distribution with **independent** parameters

Uncertainty Modeling: Parameter Dependencies



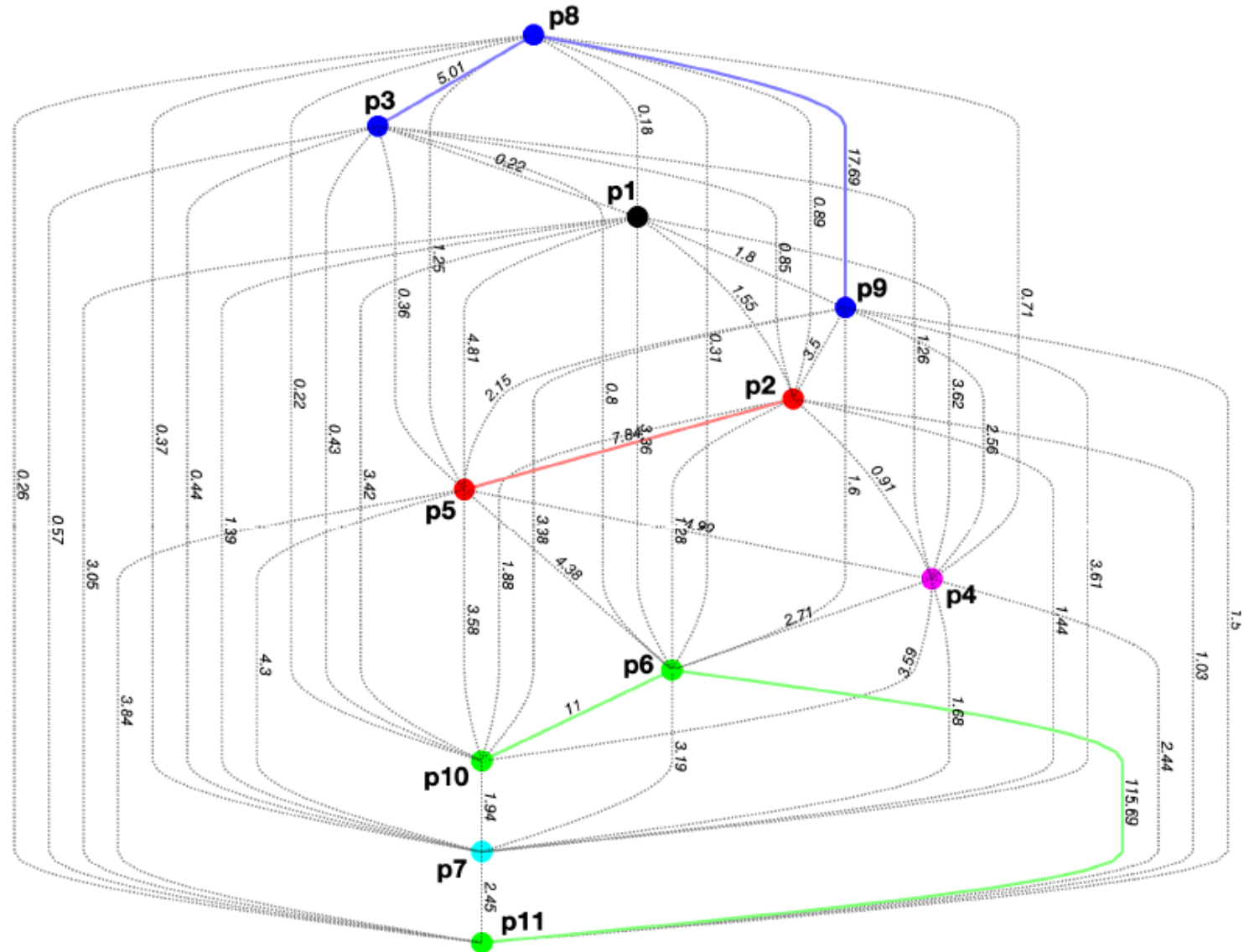
	P[instability]
Data	0
Model with dependent parameters	0
Model with independent parameters	0.23

How to Quantify Parameter Dependencies?

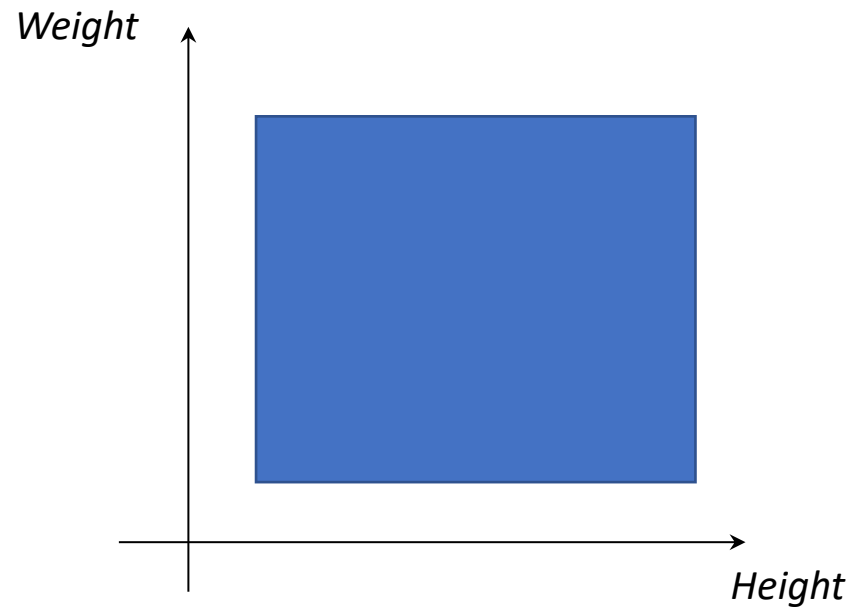


How to Quantify Parameter Dependencies?

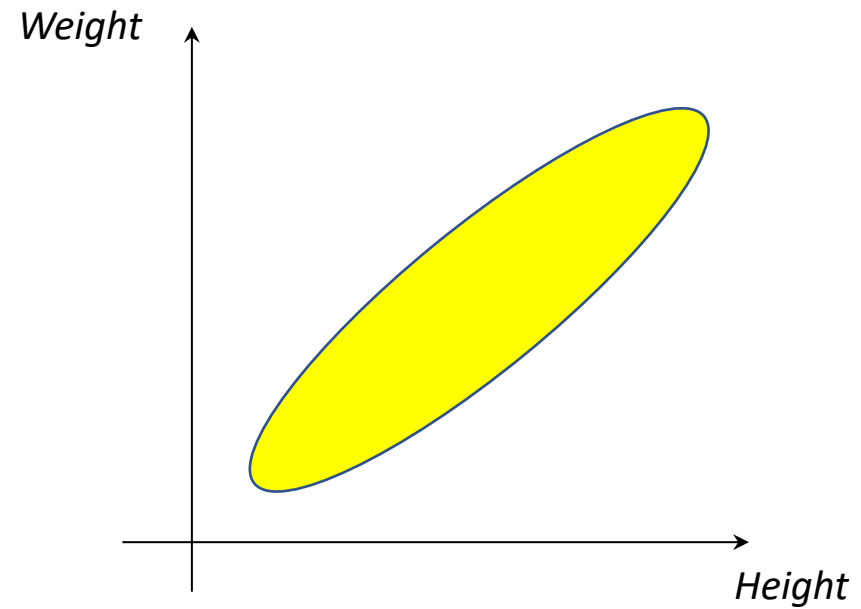
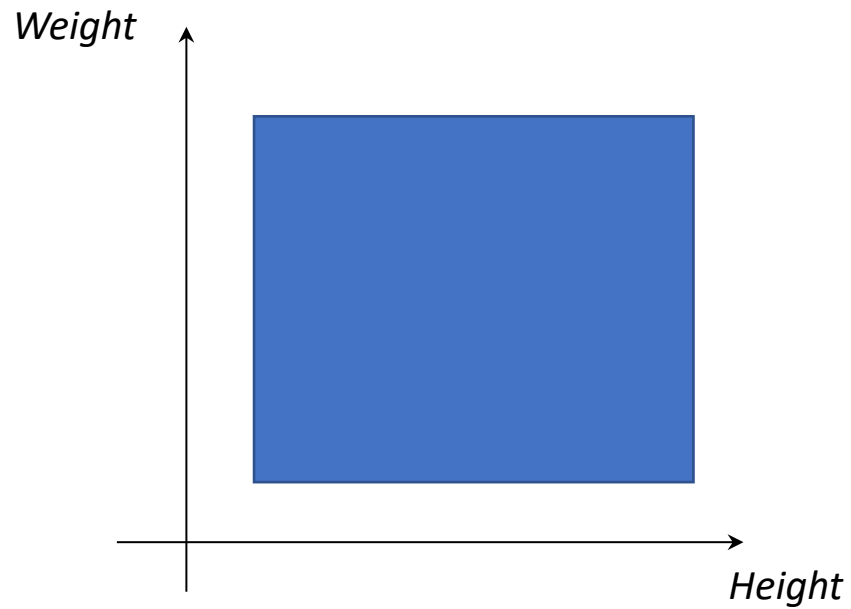
- Use copula-based metrics for quantifying the degree of dependency among variables



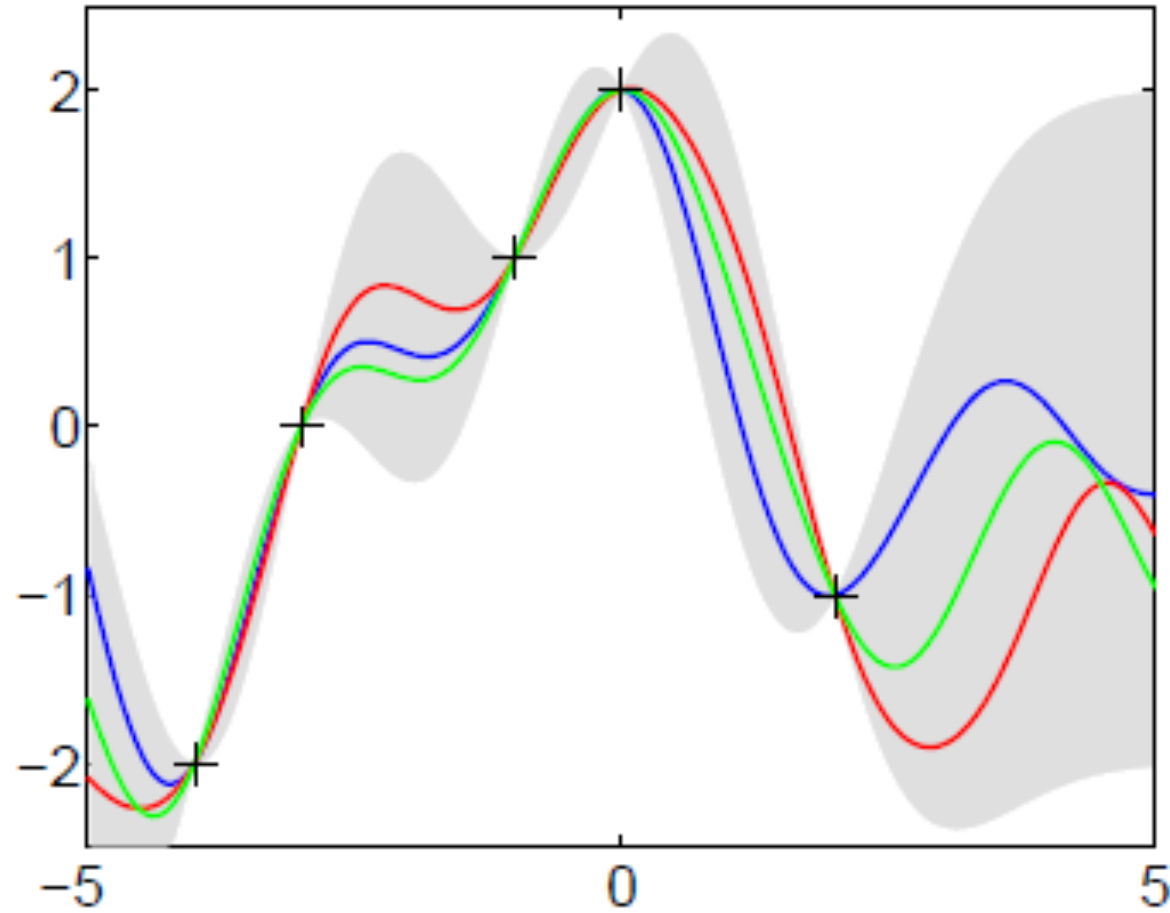
Parameter Dependencies for Epistemic Variables



Parameter Dependencies for Epistemic Variables



Quantifying Epistemic Uncertainty given Data



Overview

- Uncertainty modeling
- Set deformations
- Optimization under uncertainty
- Decision making

Set Deformations

- Quantify the largest deviation in the nominal value of x that leads to an unacceptable output

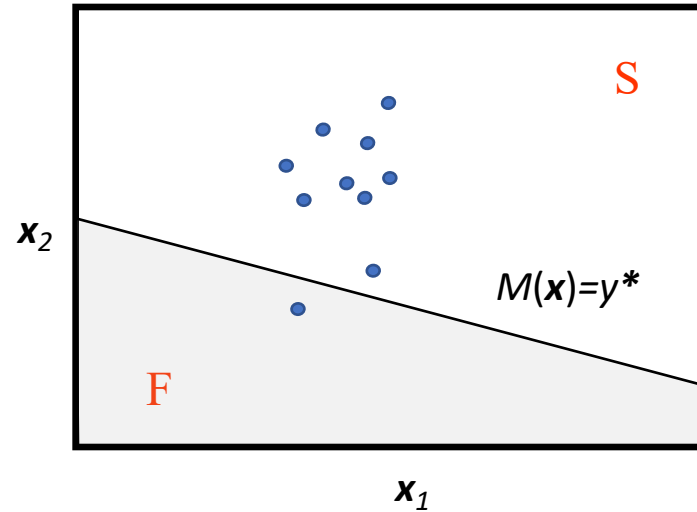
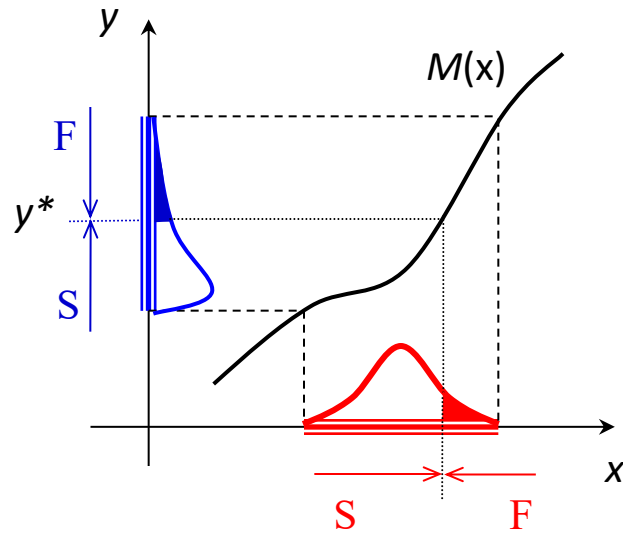
$$y = M(x)$$

Computational model

$$g(x) = y - y^* \leq 0$$

System requirements

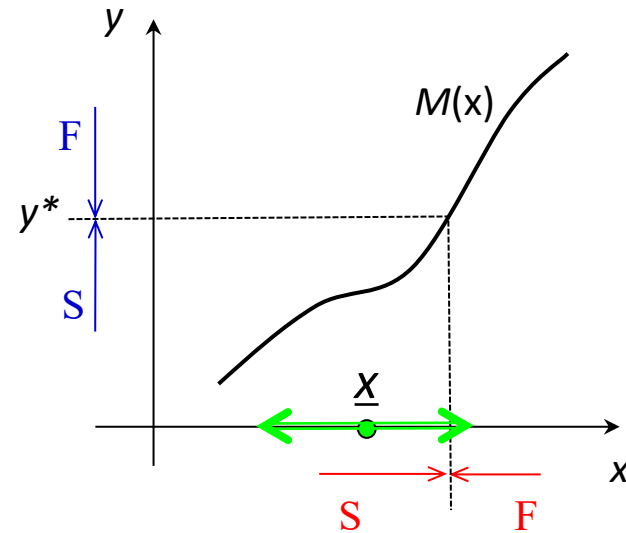
Set Deformations: Analysis



- Probability of failure by Monte Carlo simulations

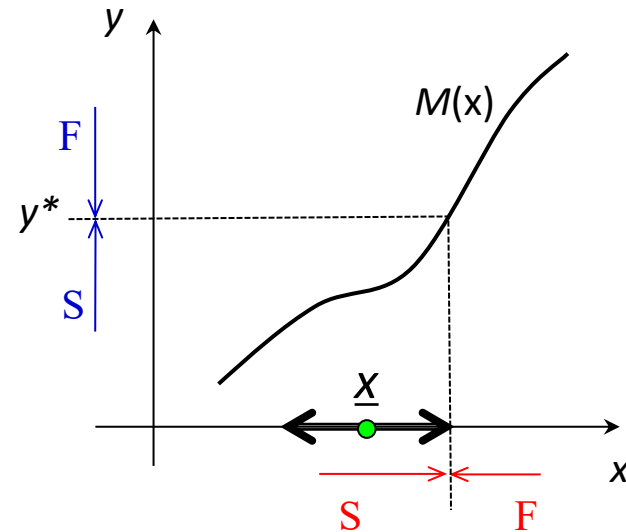
Set Deformations: Analysis

- What is the smallest deviation from $\underline{x}$ leading to a requirement violation?



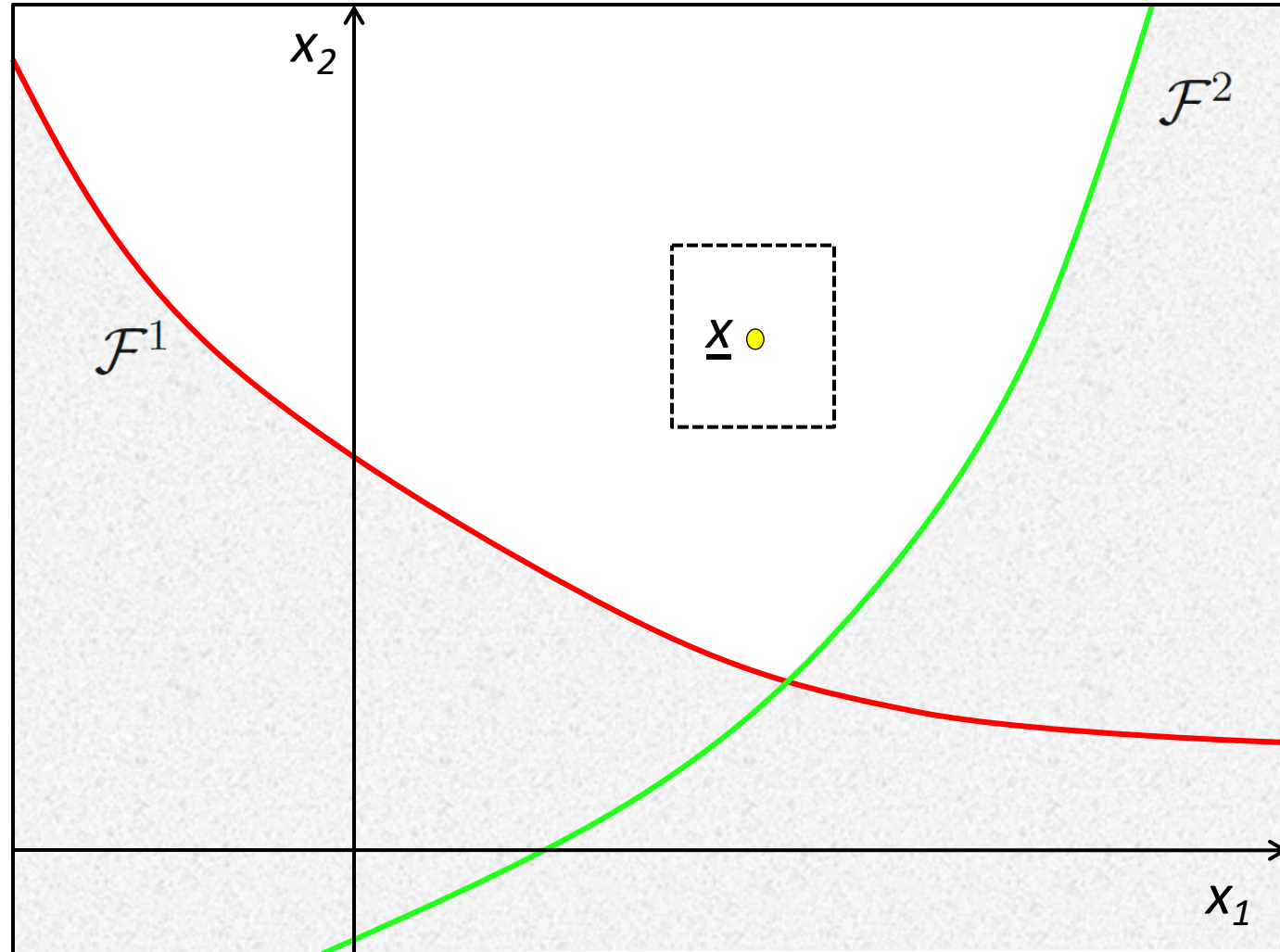
Set Deformations: Analysis

- What is the smallest deviation from $\underline{x}$ leading to a requirement violation?

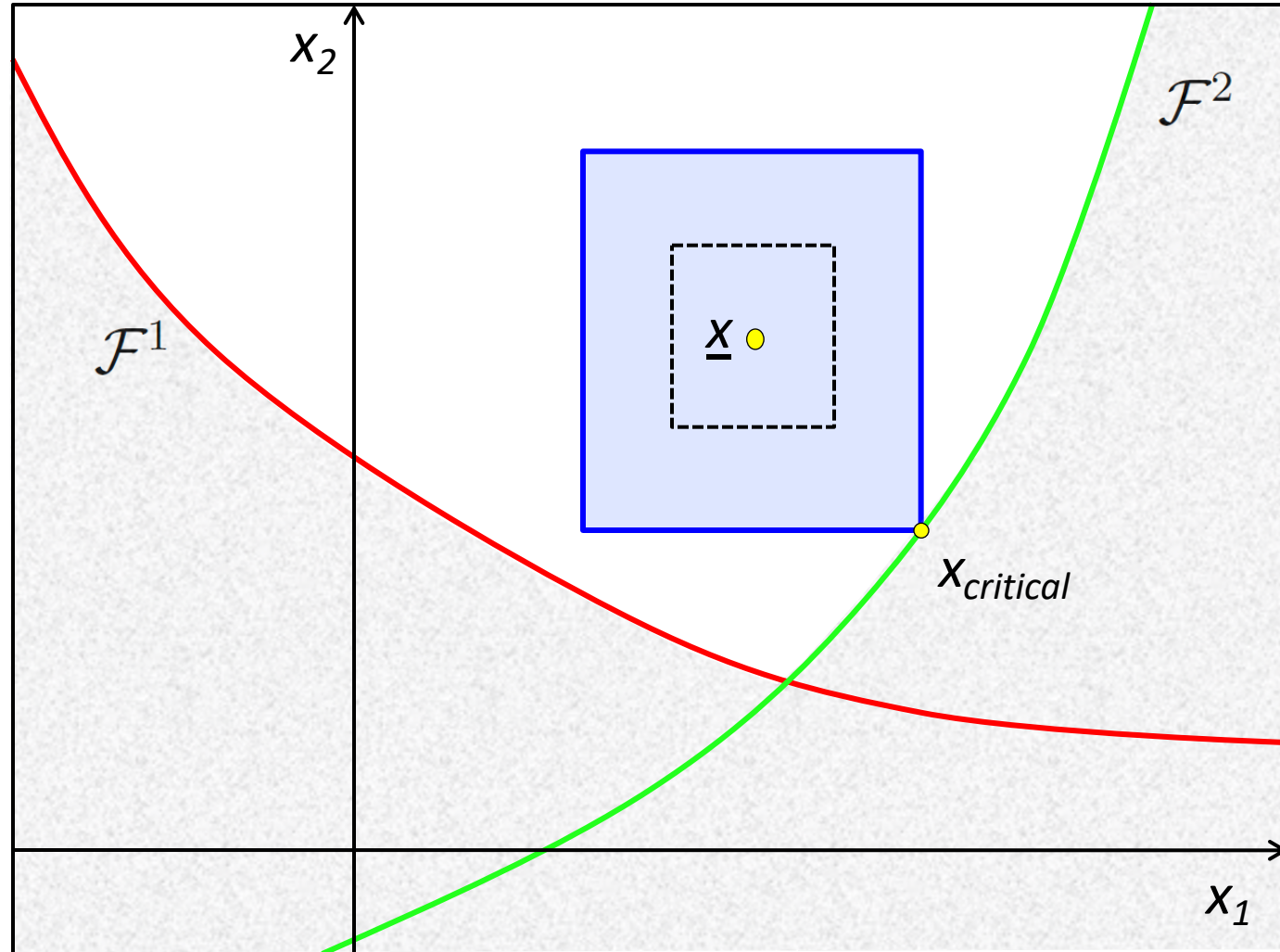


This notion of robustness is conceptually different from the probabilities of failure

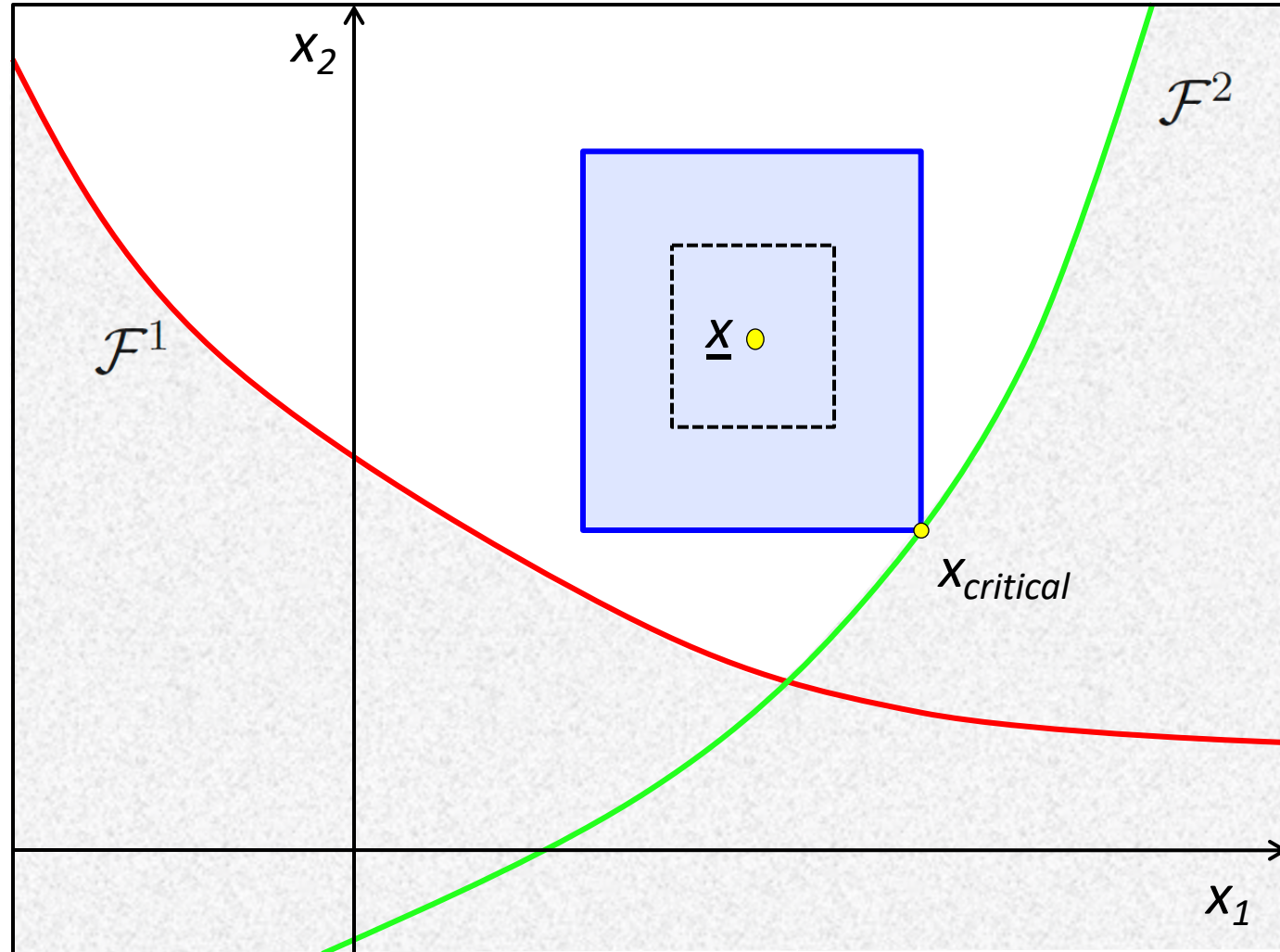
Set Deformations: Analysis



Set Deformations: Analysis



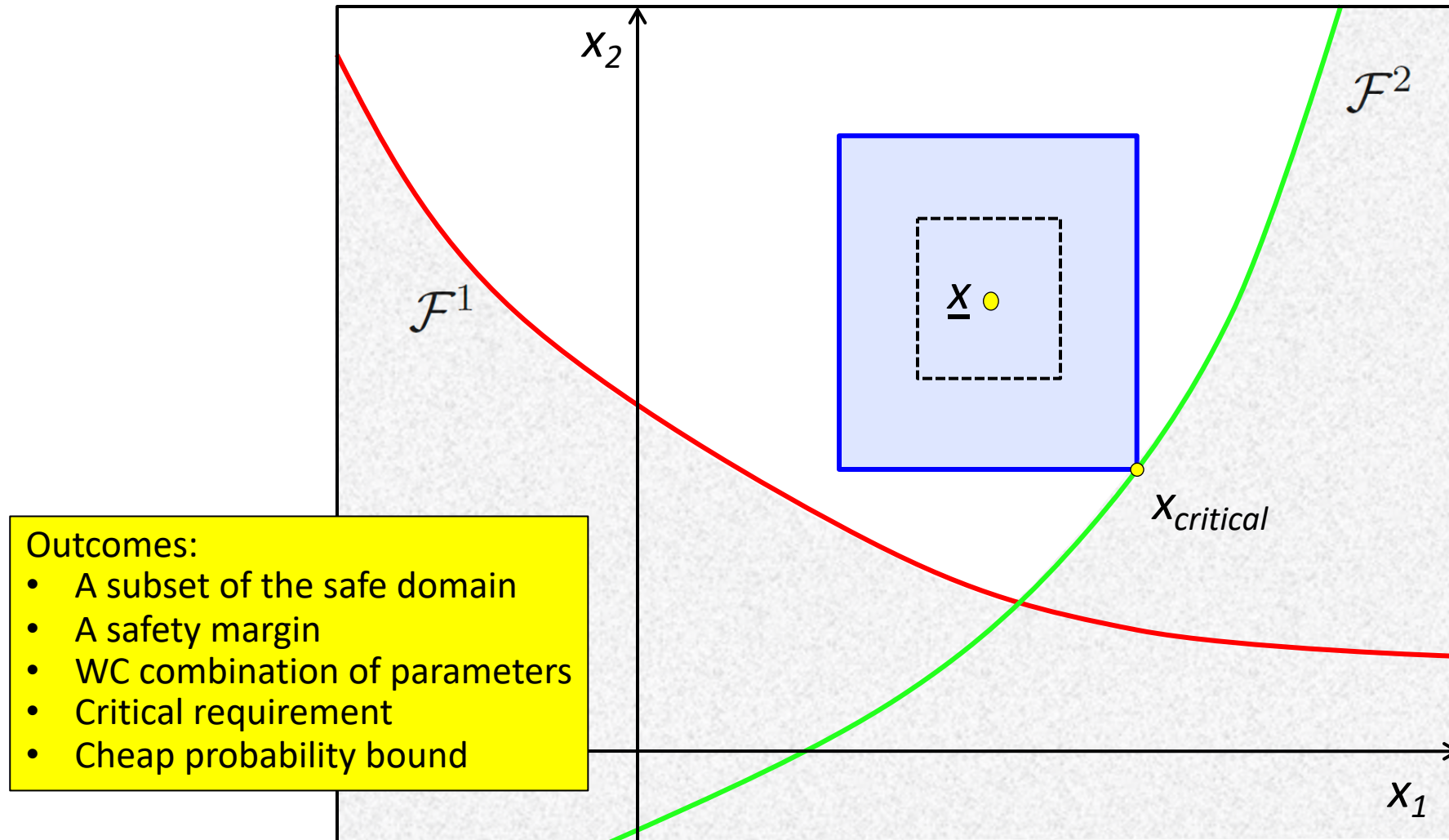
Set Deformations: Analysis



In contrast to MC, the outcome is a characterization of the uncertainty

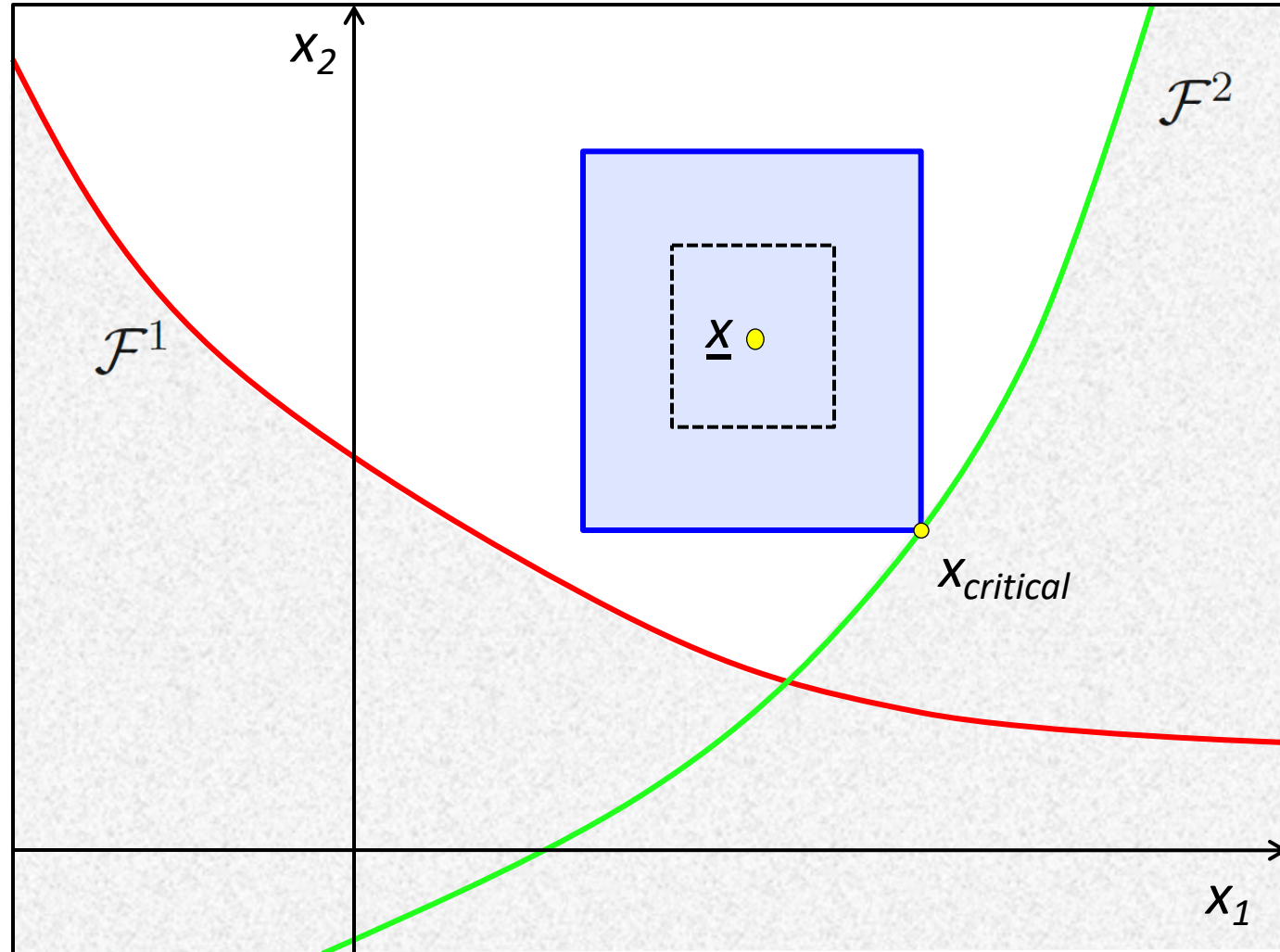
Set Deformations: Analysis

$$y = M(x)$$

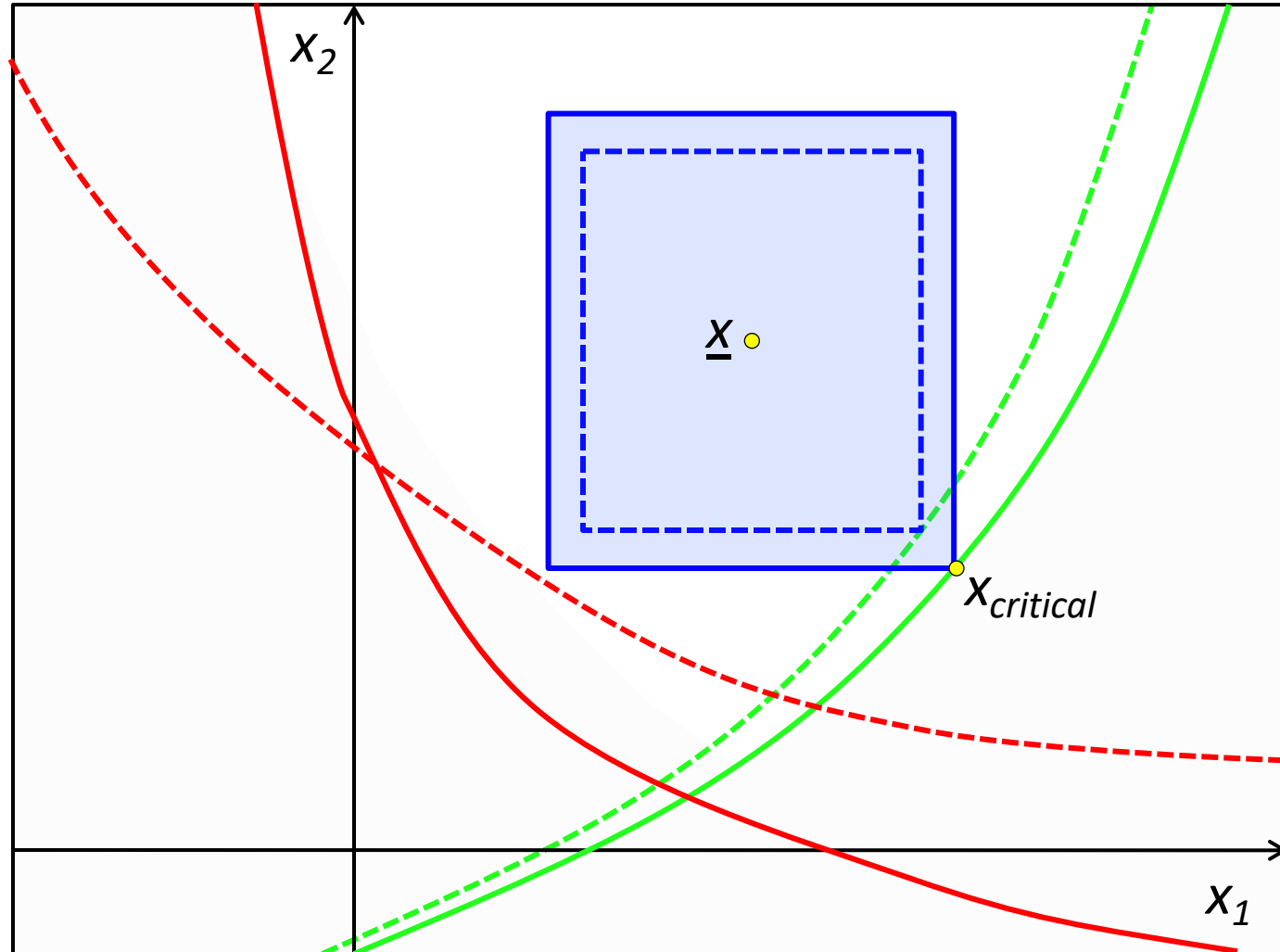


Set Deformations: Design

$$y = M(d, x)$$

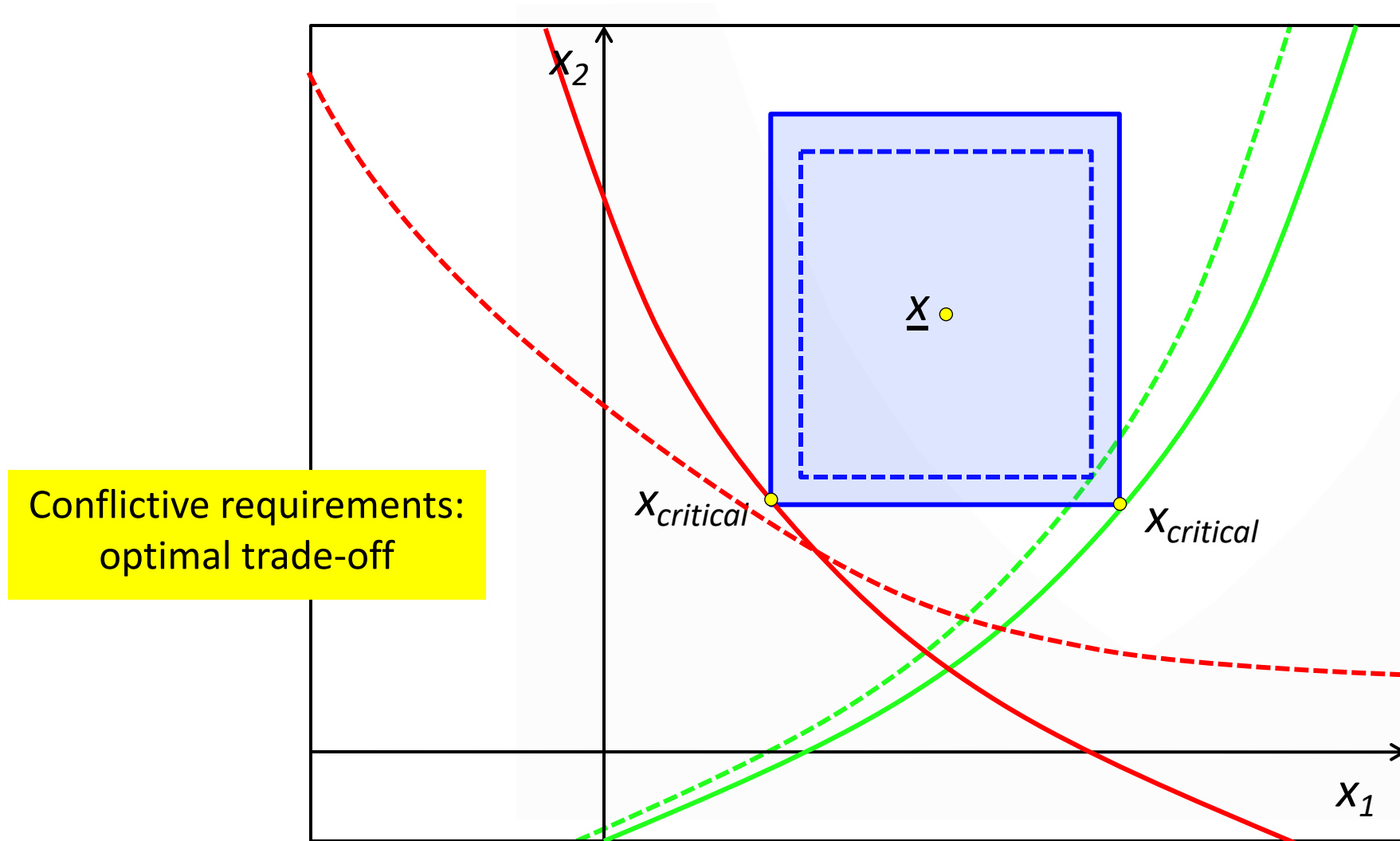


Set Deformations: Design



Set Deformations: Design

$$y = M(d, x)$$

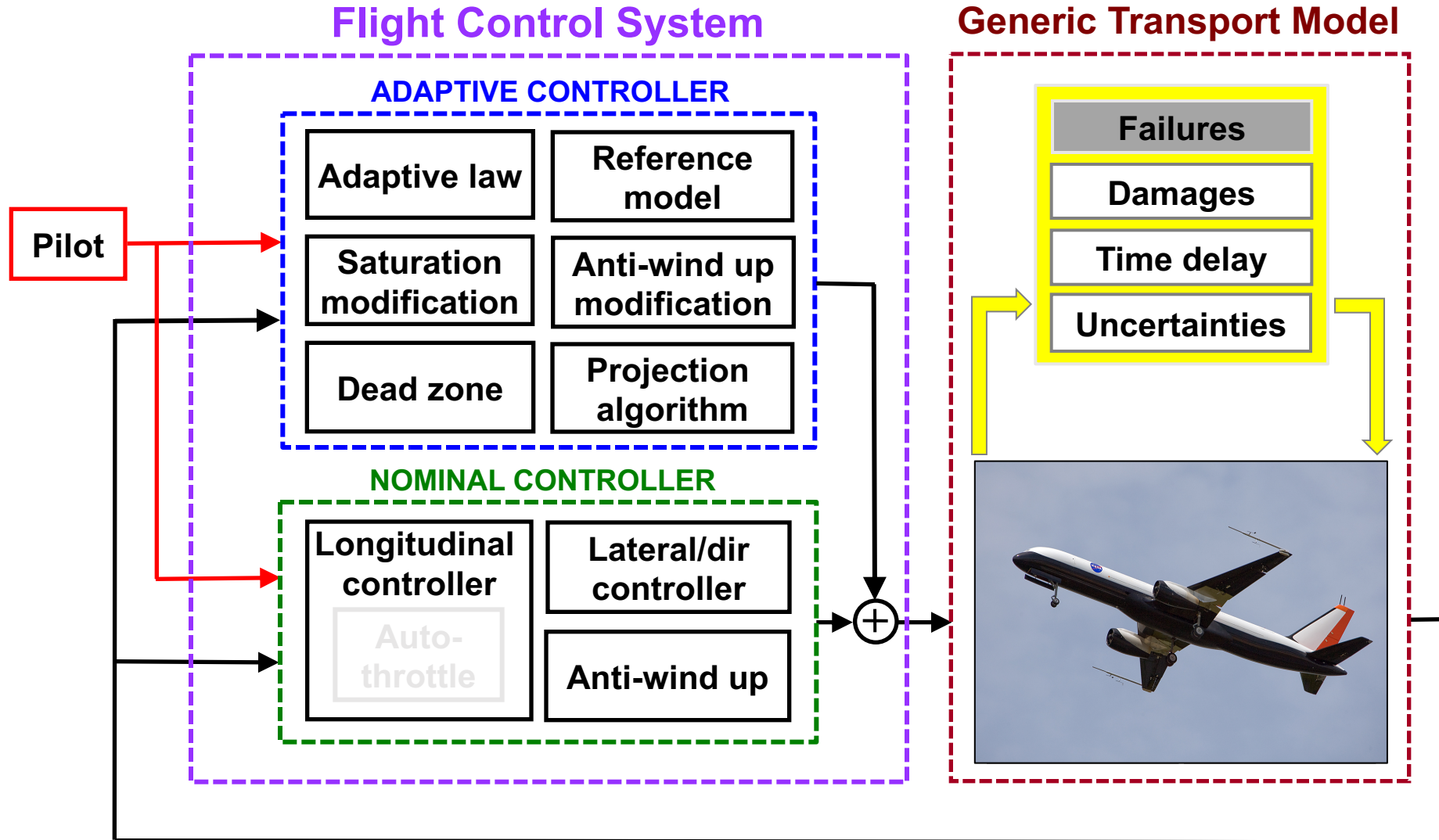


Set Deformations: GTM

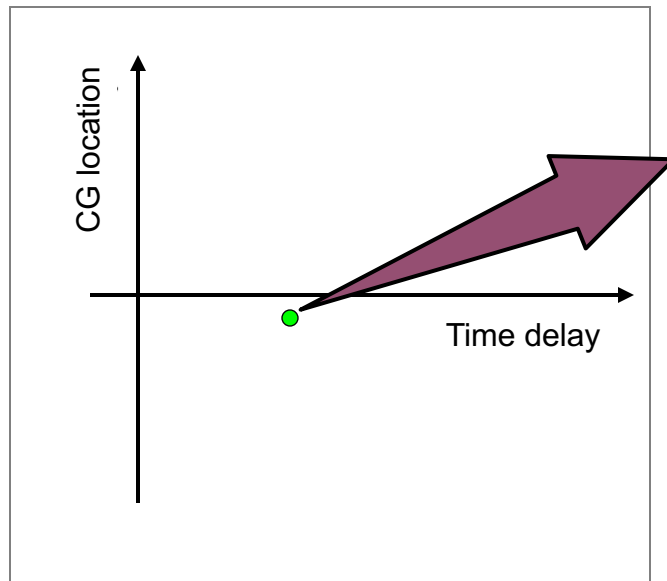


- Dynamically scaled flight test article
- High-fidelity mathematical model having non-linear aerodynamics, avionics, engine and sensor dynamics, atmospheric model, telemetry effects, etc. (278 states)
- **Analysis objective:** perform a robustness analysis of a flight controller
- **Design objective:** Improve controller's robustness with respect to key uncertainties

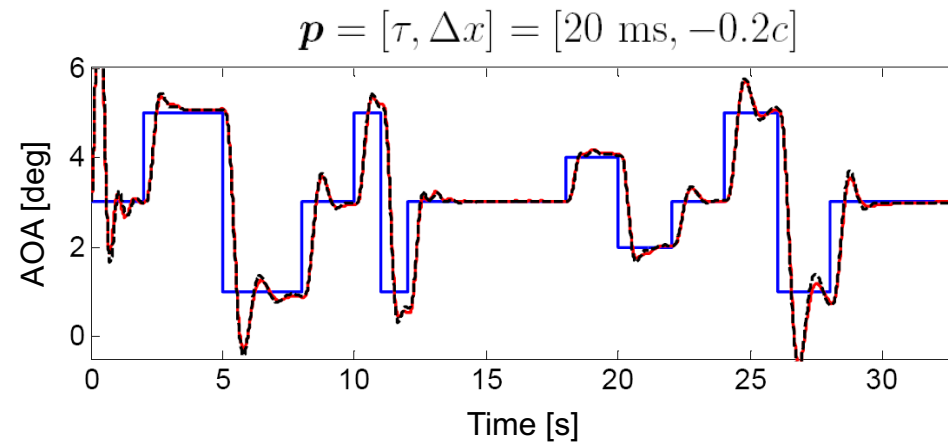
The GTM: Simulation Model



Set Deformations: GTM

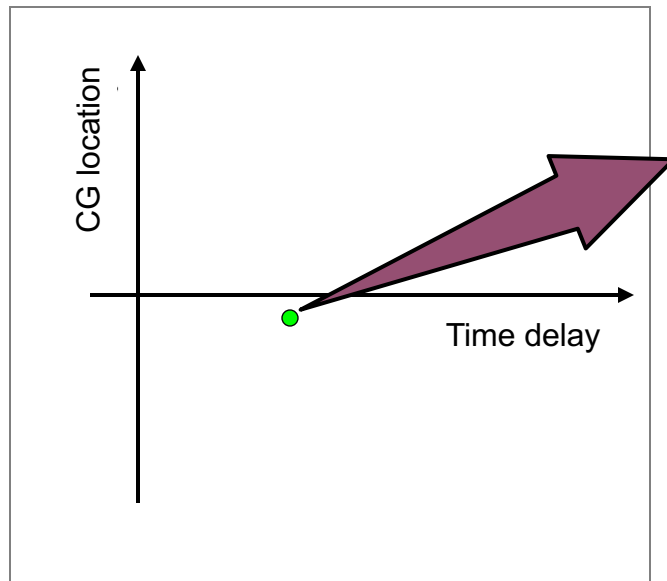


Parameter Space

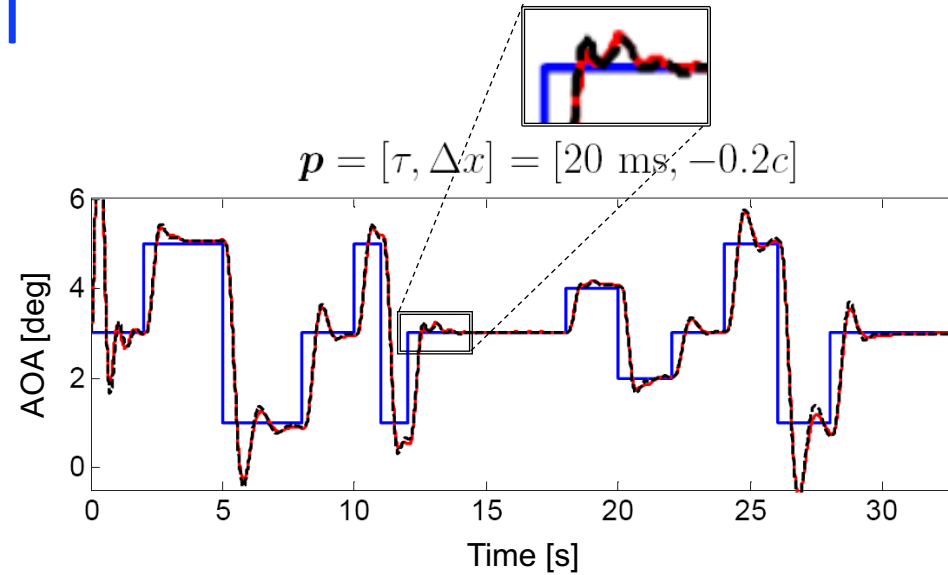


Reference command, Controller 1, Controller 2

Set Deformations: GTM

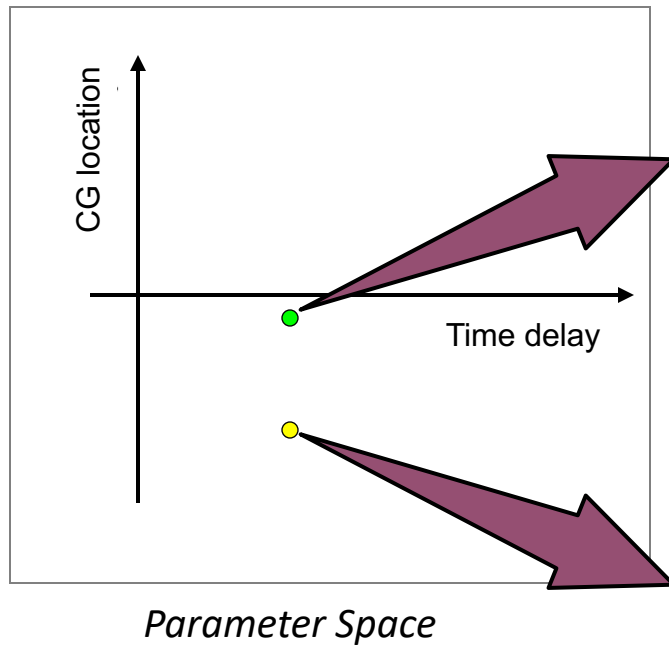


Parameter Space

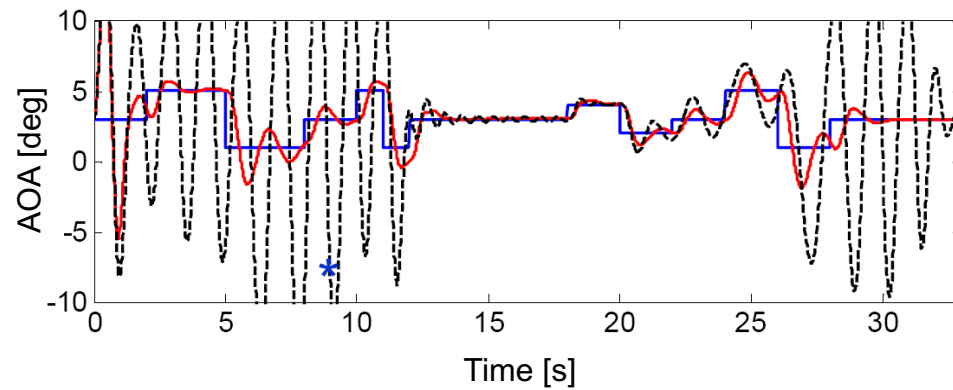
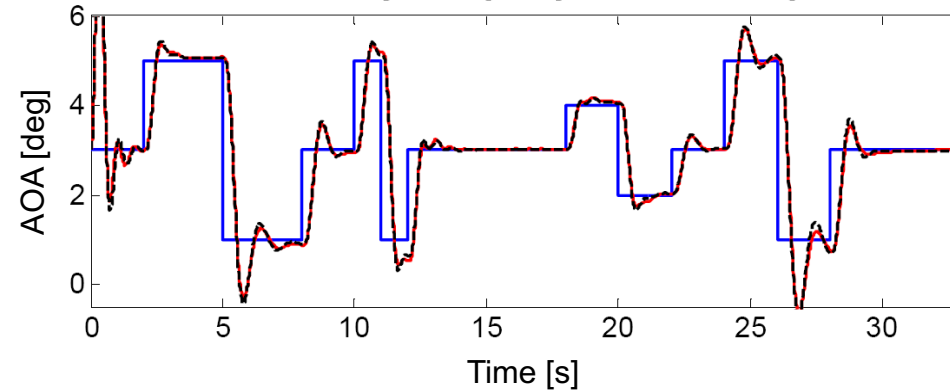


Reference command, Controller 1, Controller 2

Set Deformations: GTM

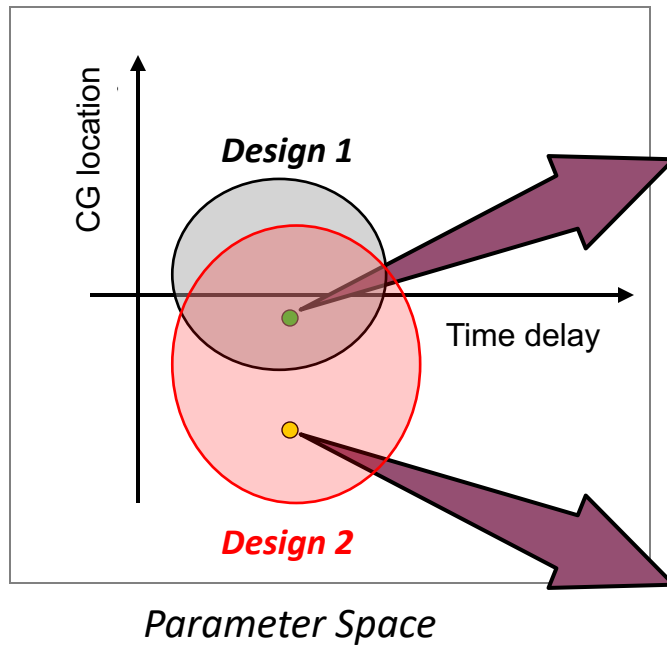


$$\mathbf{p} = [\tau, \Delta x] = [20 \text{ ms}, -0.2c]$$

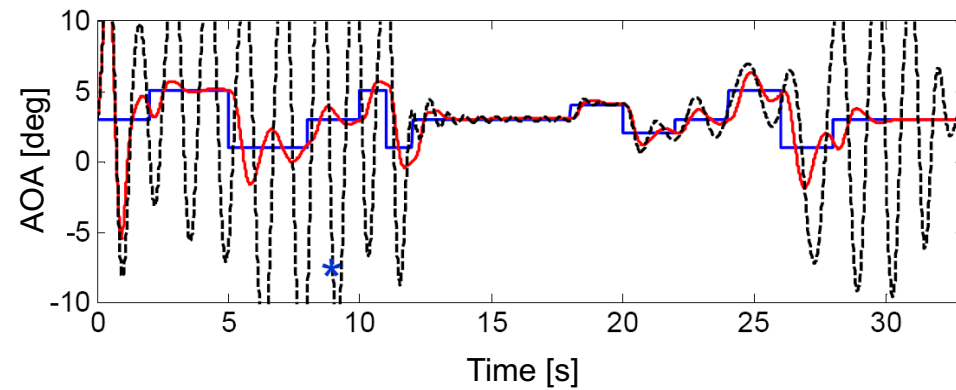
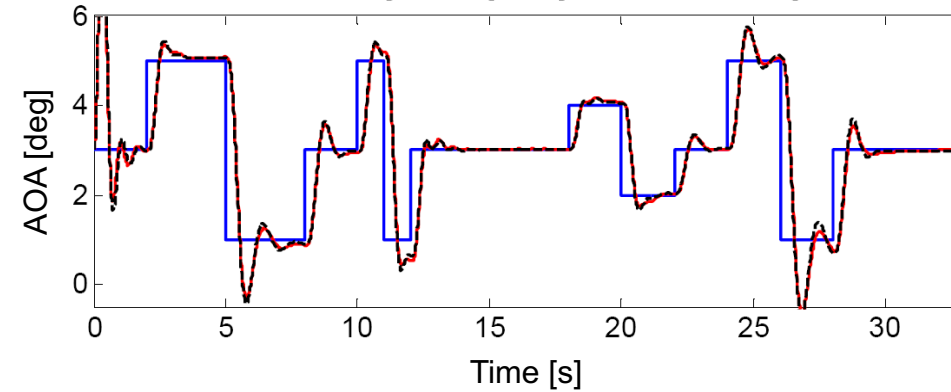


Reference command, Controller 1, Controller 2

Set Deformations: GTM



$$\mathbf{p} = [\tau, \Delta x] = [20 \text{ ms}, -0.2c]$$



Reference command, Design 1, Design 2

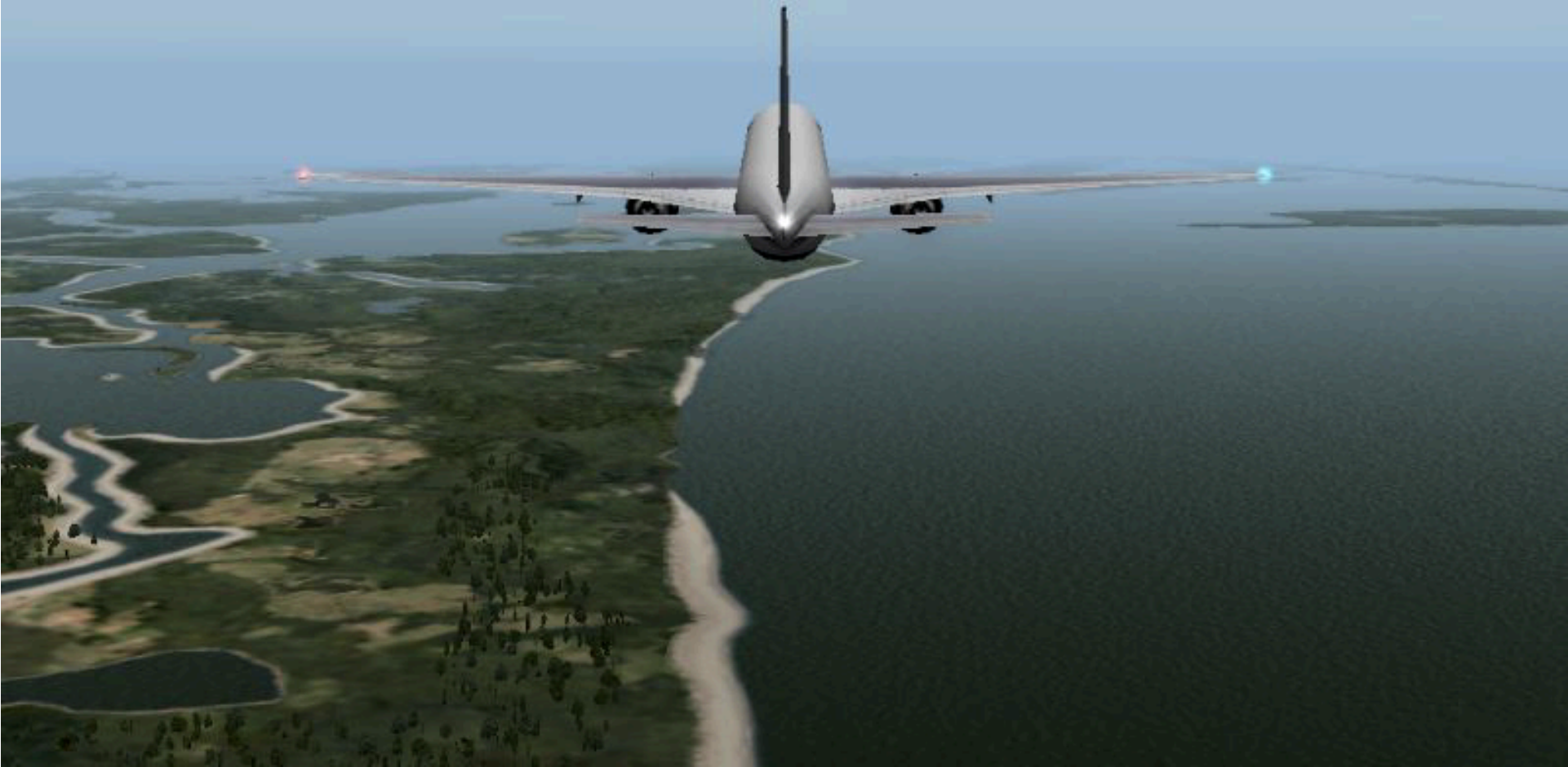
Design 1



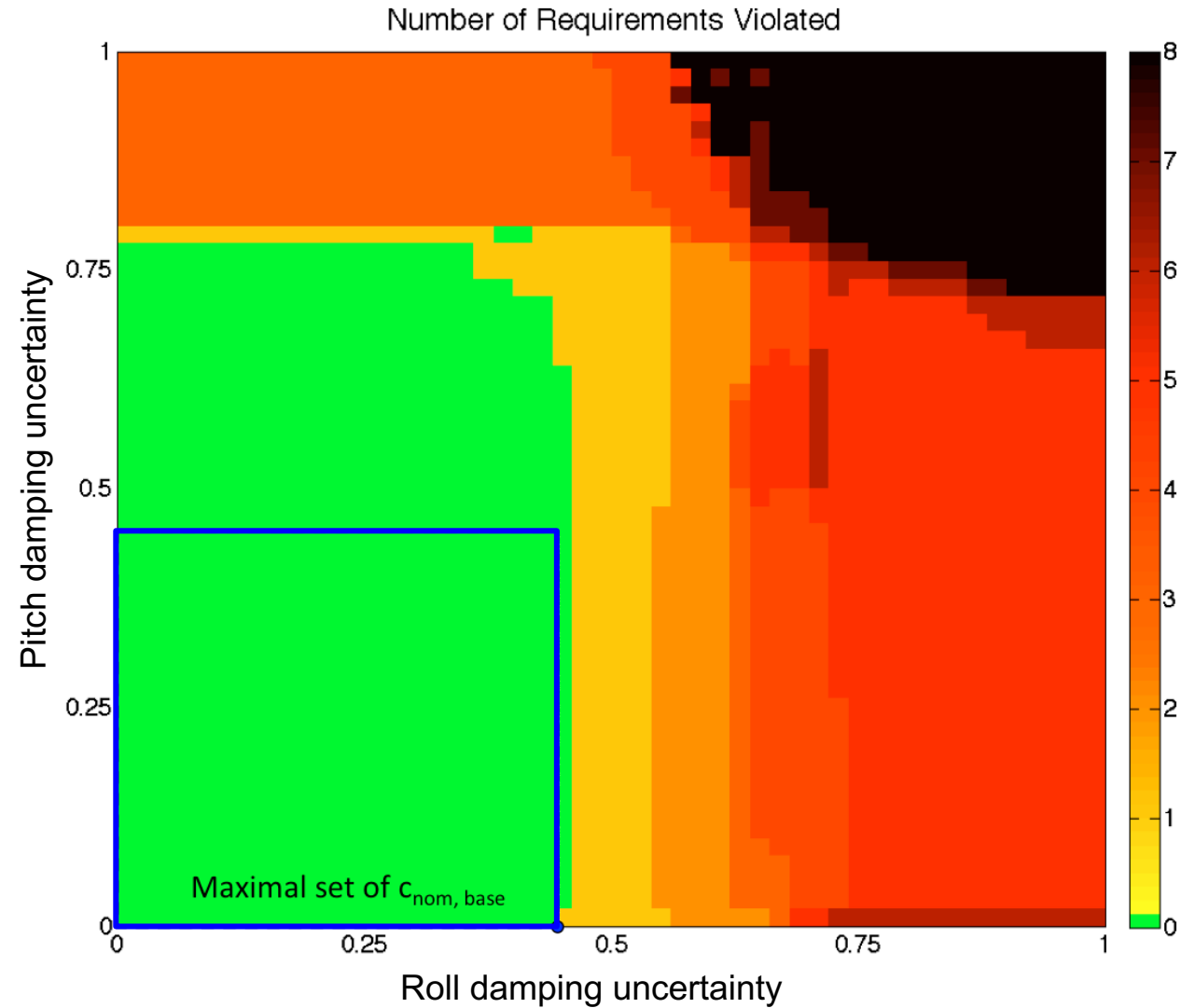
Design 2

Sim Data Overlay

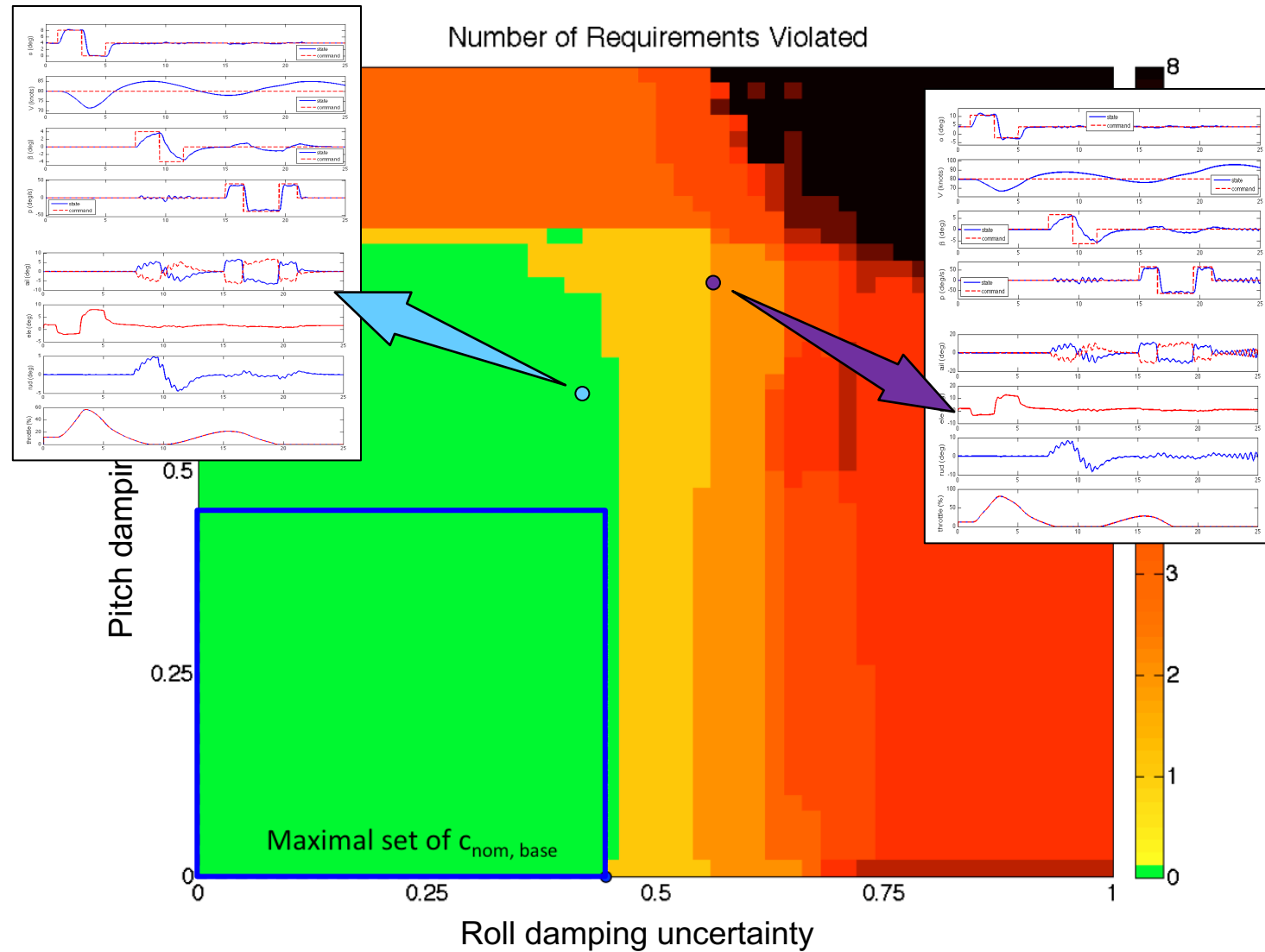
alpha:	4.38
beta:	-0.00
Velocity:	80.00
PAM.theta:	0.00
PAM.lambda:	0.00
SAM	0.00



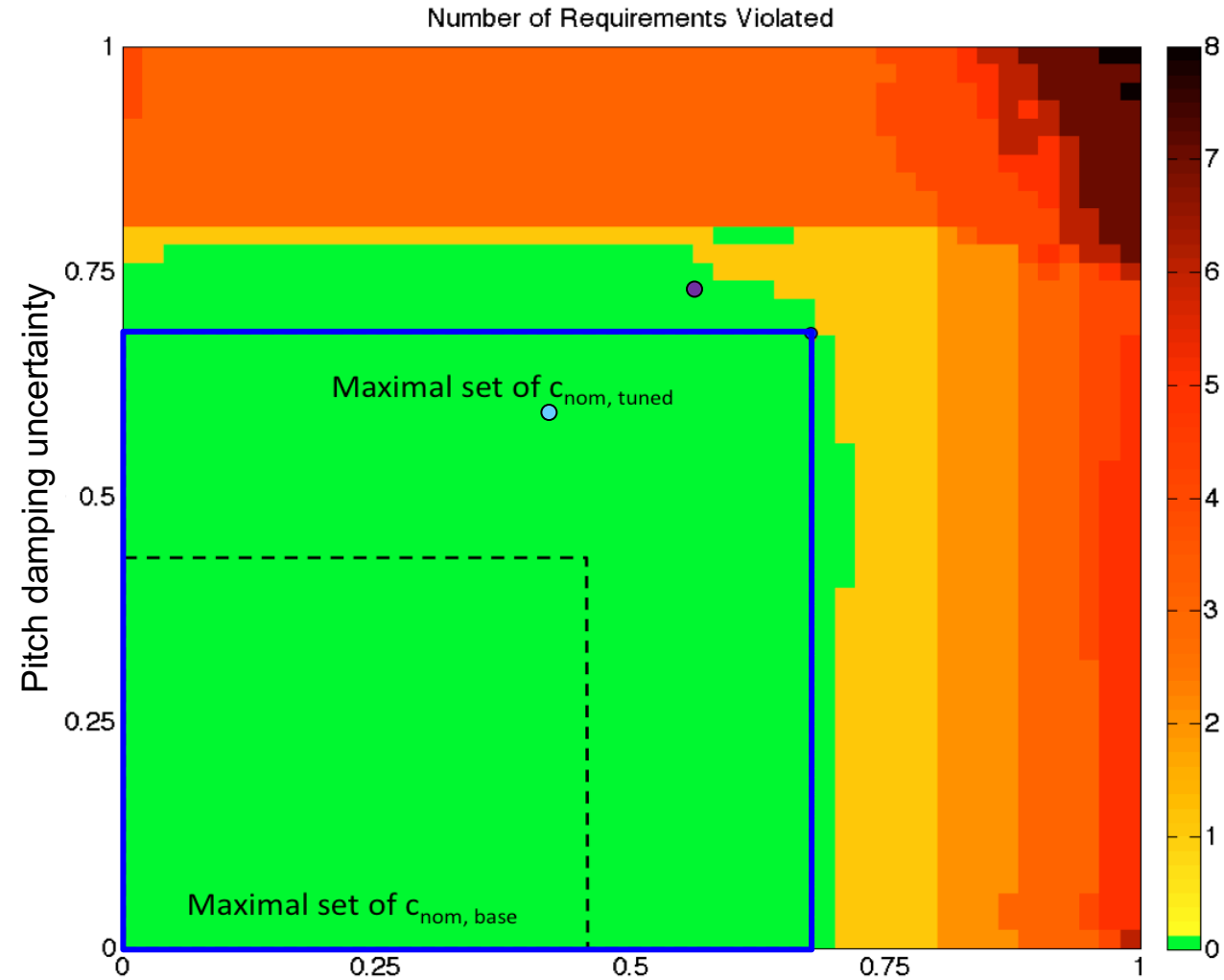
Set Deformations: GTM Analysis



Set Deformations: GTM Analysis

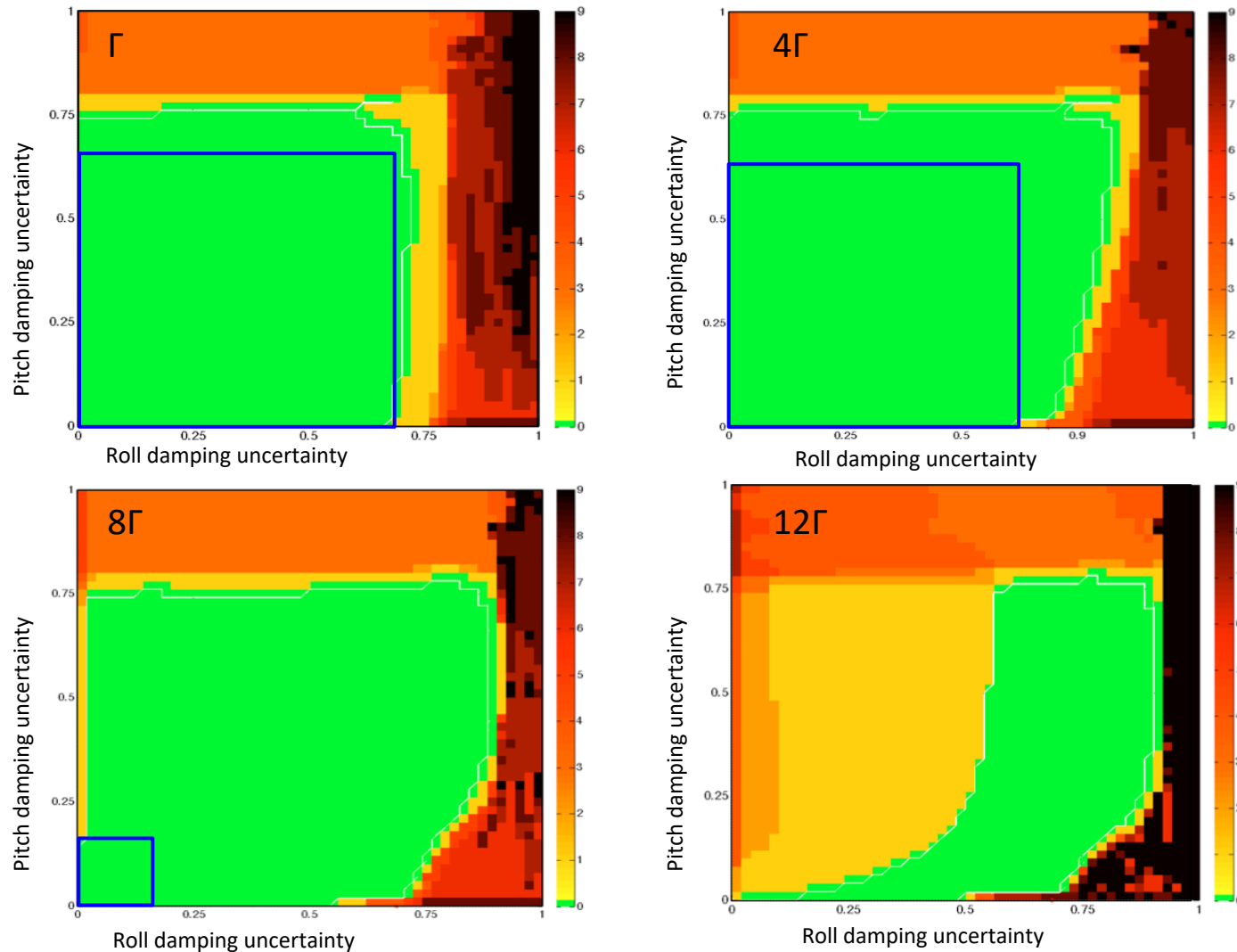


Set Deformations: GTM Design



The new controller tolerates 48% more roll damping uncertainty than the baseline controller

Set Deformations: PSM vs P[req-violation], Risk



Set Deformations

- ✓ Provides WC uncertainty combination and a robustness metric
- ✓ Uses a simple UM
- ✓ Sizes the uncertainty a given controller tolerates
- ✓ Can be naturally integrated into design
- × Complexity depends on the number of parameters and the manner by which the outputs depend on them
- × Requires convergence to a global optimum
 - What to do if parameters are dependent*?
 - What to do if data is available*?

*more on this presented later

Overview

- Uncertainty modeling
- Set deformations
- Optimization under uncertainty
- Decision making

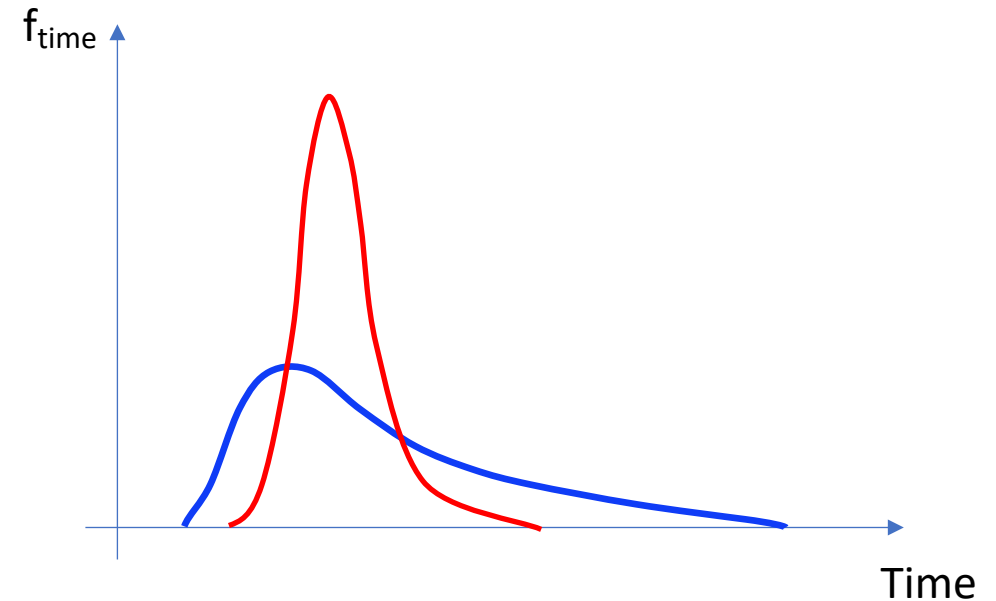
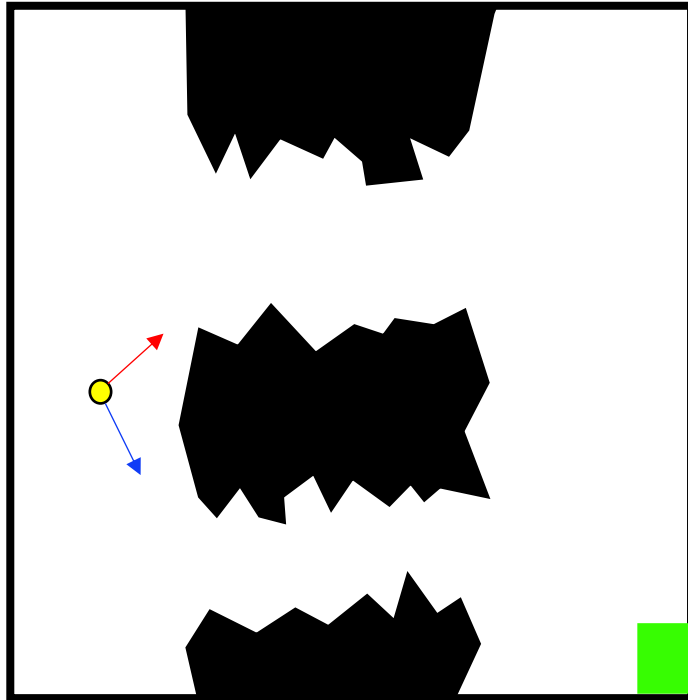
Optimization under Uncertainty

- Design processes are often performed by optimization
- Optimization is highly sensitive to what it is assumed to be known
- Plethora of applications: regression analysis, model calibration, system identification, control theory, machine learning, systems theory, autonomous systems, financial mathematics, decision making, etc.

Optimization under Uncertainty

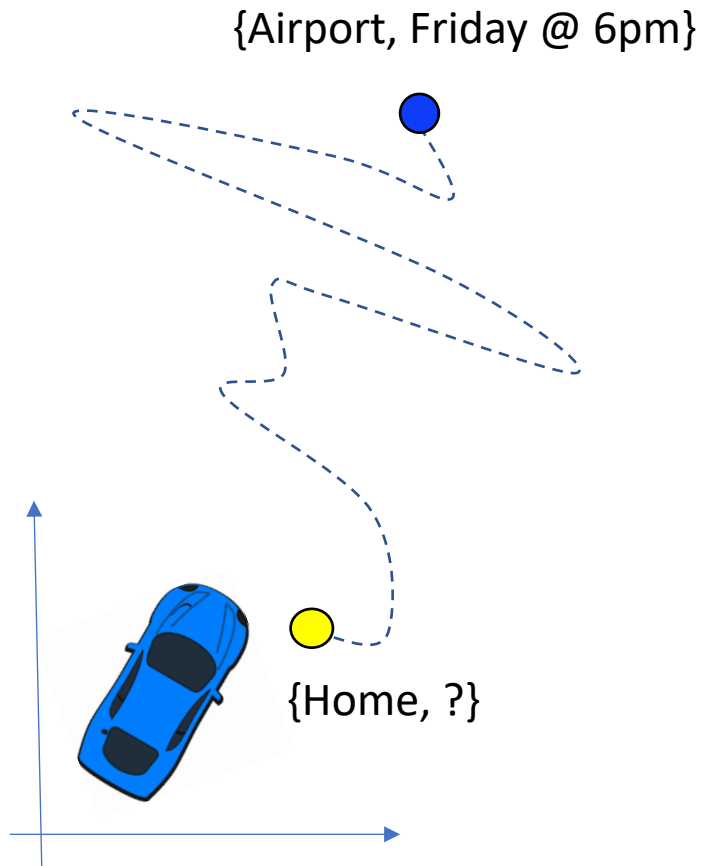
- Ideal properties: convexity & robustness to uncertainty: it is not only about reaching to the finish line fast, it is also about getting to the right finish line
- There is a price to pay for convexification
- The dependency of the program on the decision variables and on the uncertain parameters play a key role

Optimization under Uncertainty



- What policy should be chosen to score in minimum time?

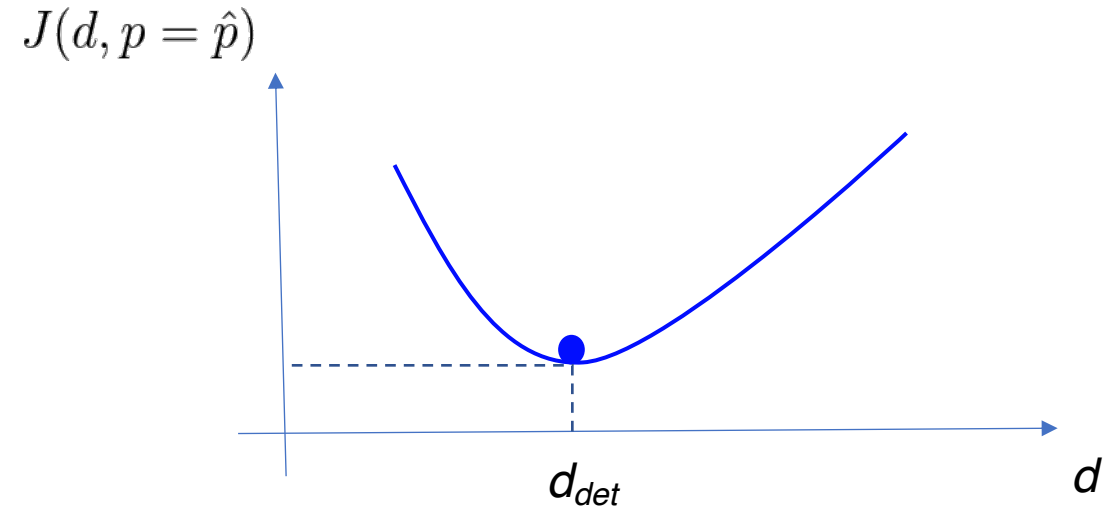
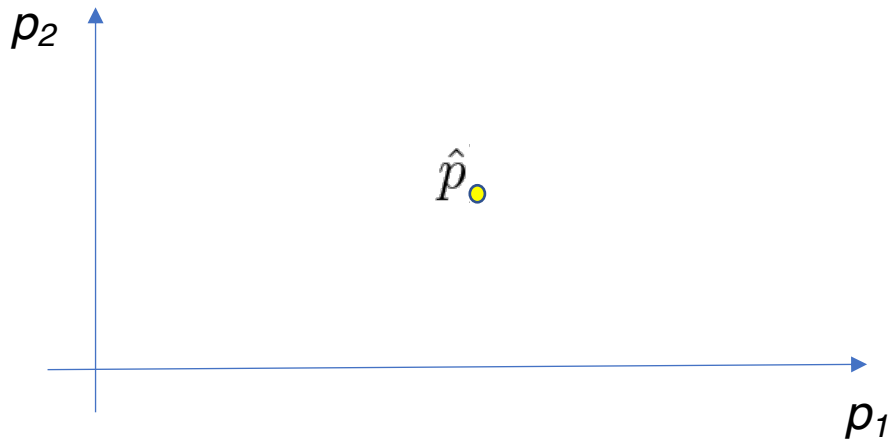
Motivational Example



- Cost J : time waiting in the airport
- Design variable d : time leaving home
- Uncertainty p : {traffic, road/lane closures}

Optimization under Uncertainty: $\min_d J(d, p)$

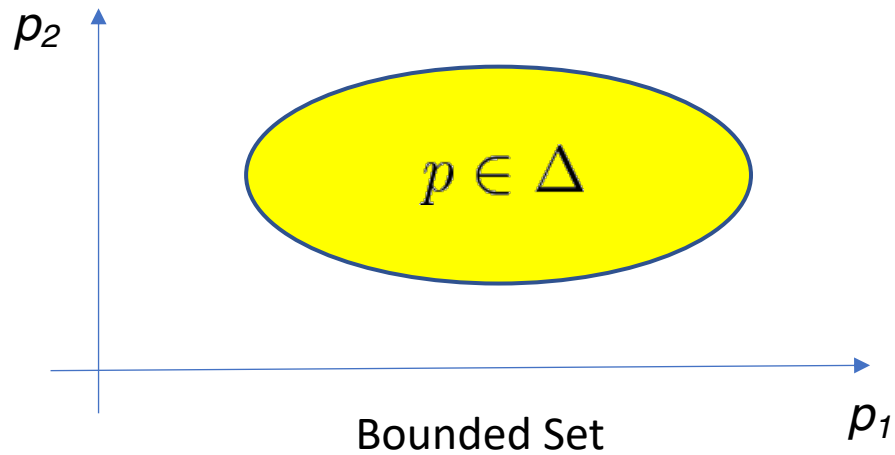
- The value of p is known



Epistemic Uncertainty

$$\min_d J(d, p)$$

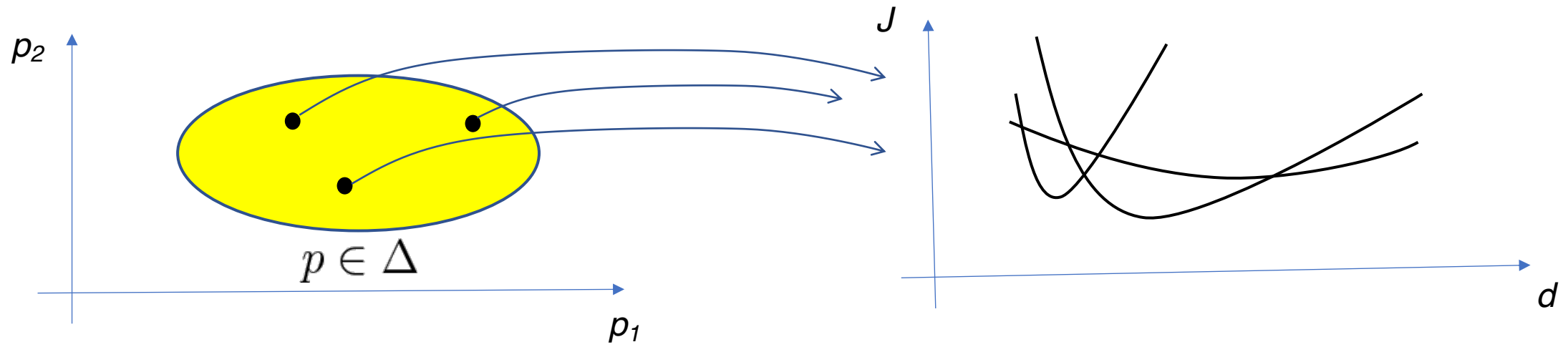
- The value of p is known
- The value of p is unknown



Epistemic Uncertainty

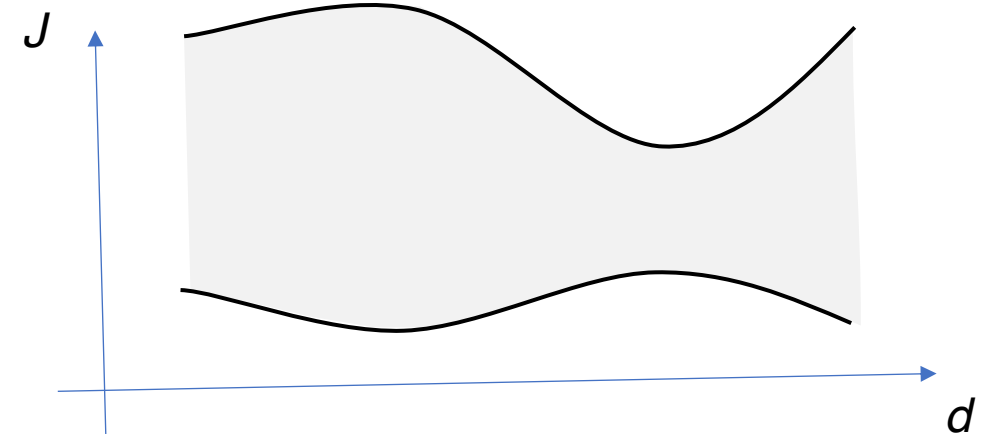
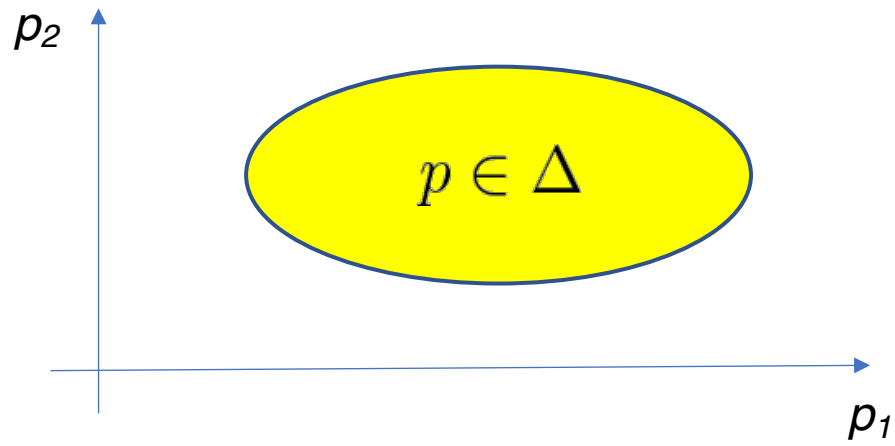
$$\min_d J(d, p)$$

- The value of p is known
- The value of p is unknown



Epistemic Uncertainty

- The value of p is known
- The value of p is unknown

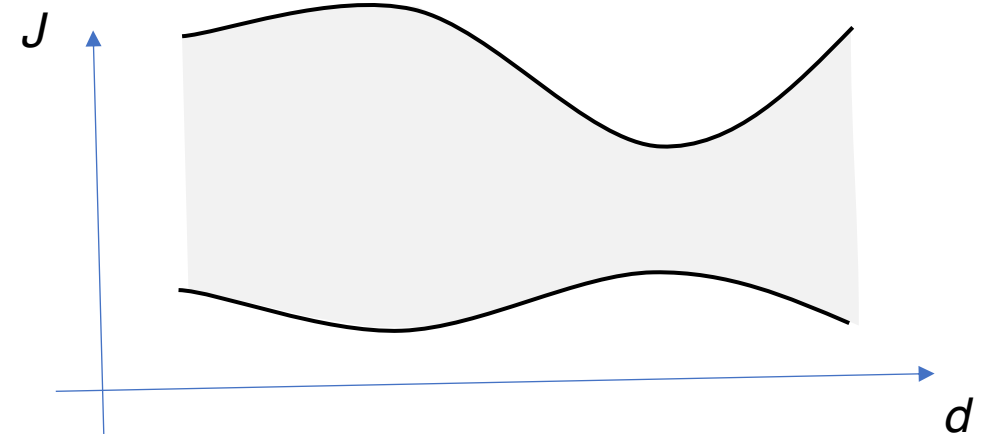
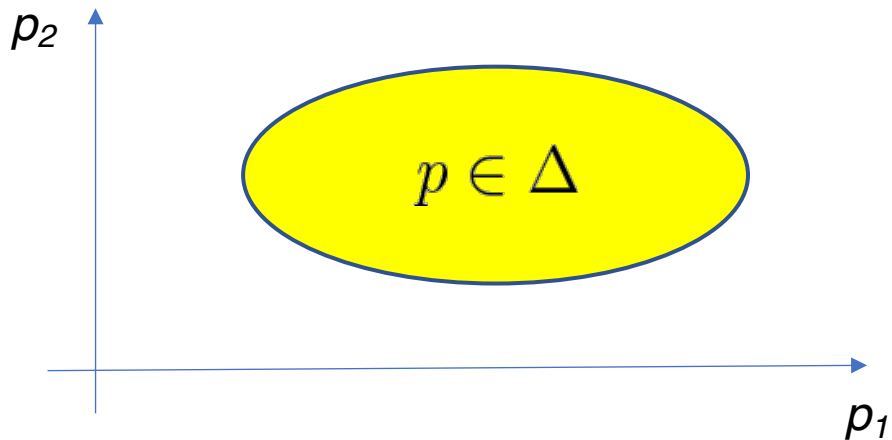


Interval-valued function

Epistemic Uncertainty

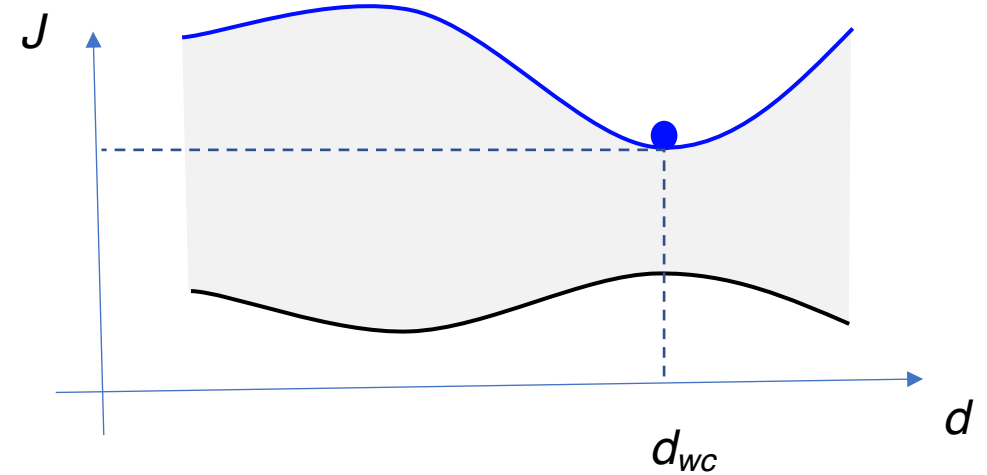
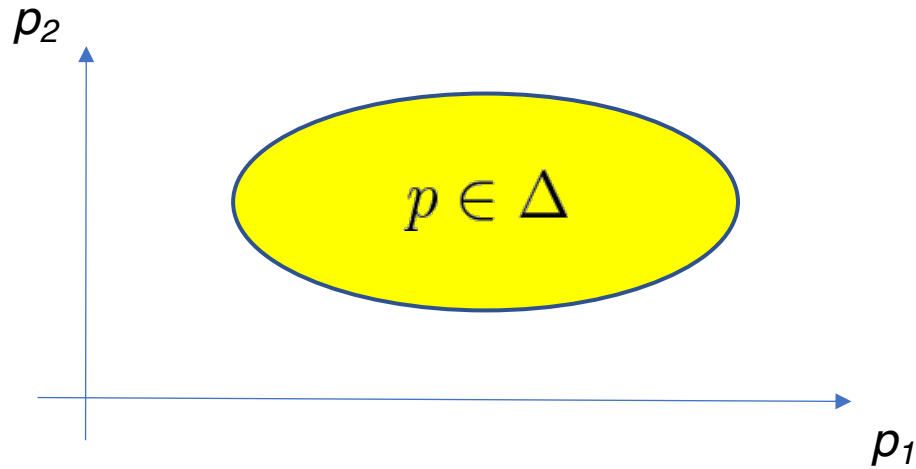
$$\min_d J(d, p)$$

- The value of p is known
- The value of p is unknown: worst-case optimization



Interval-valued function

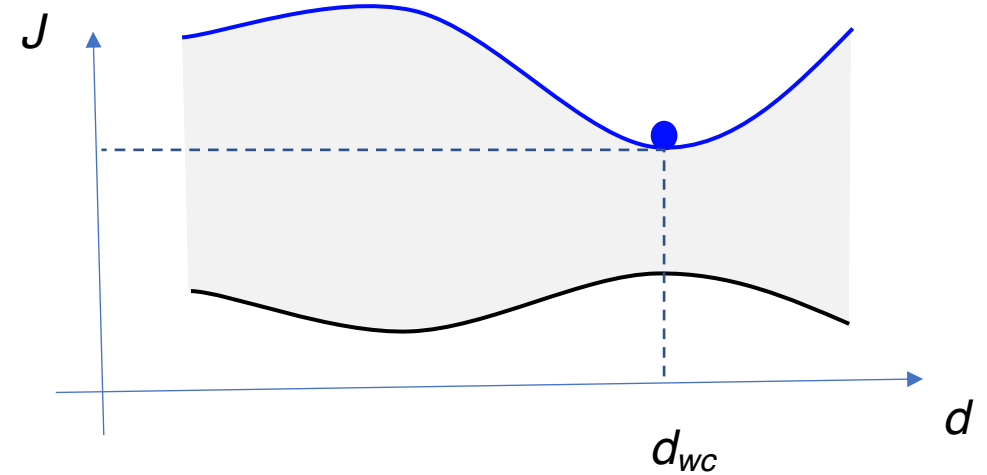
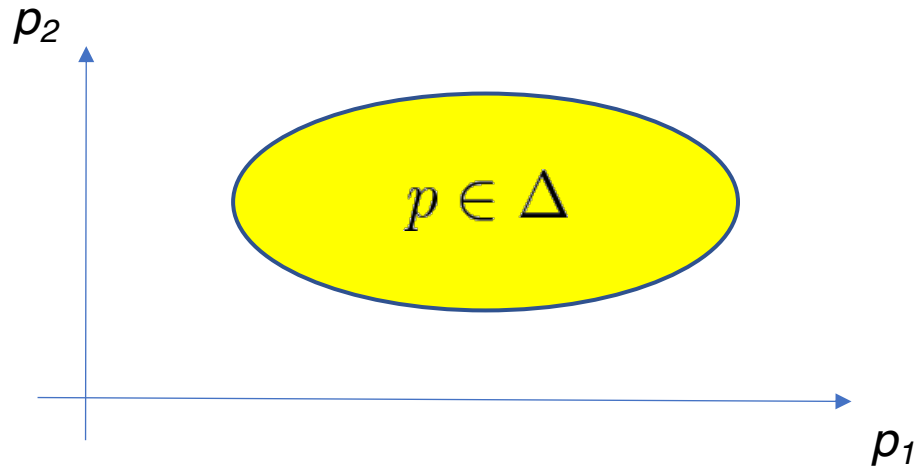
Worst-Case Optimization



- Formulation

$$\min_d \left[\max_{p \in \Delta} J(d, p) \right]$$

Worst-Case Optimization

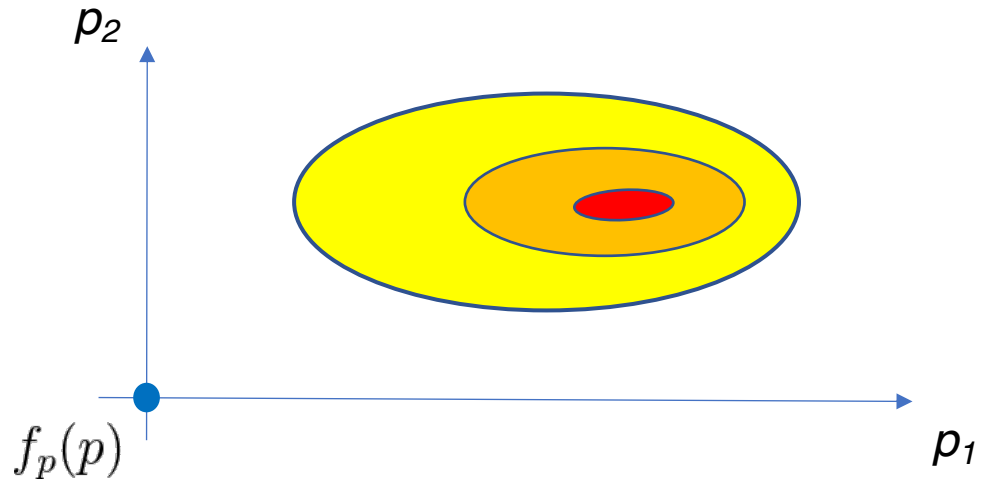


- Formulation

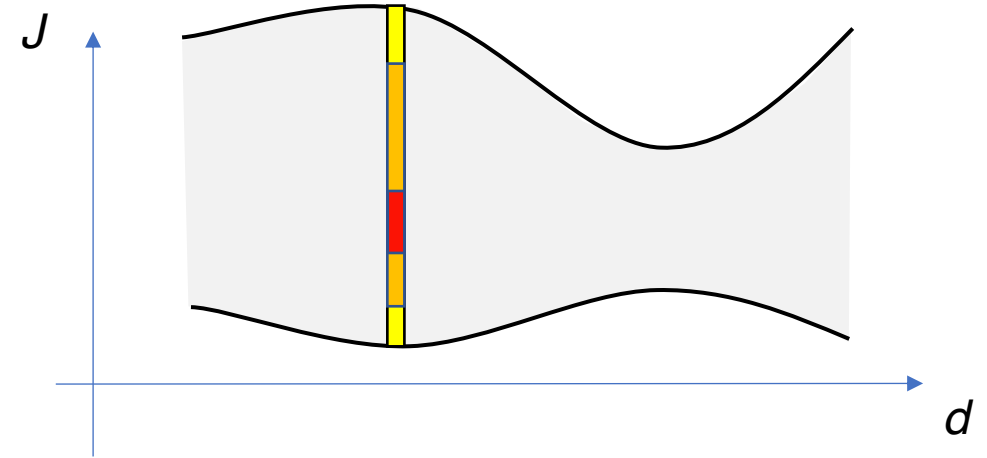
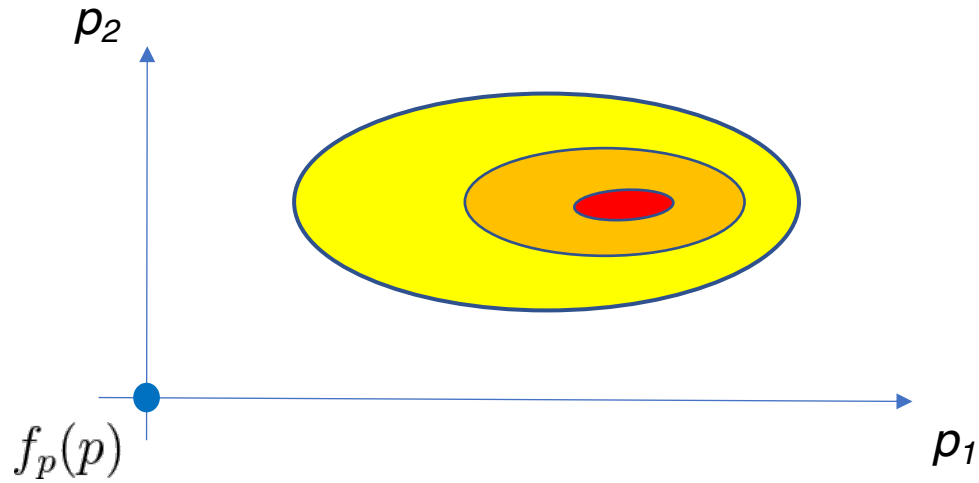
$$\min_d \left[\max_{p \in \Delta} J(d, p) \right]$$

- Performance guarantees, conservative policy

Probabilistic Uncertainty

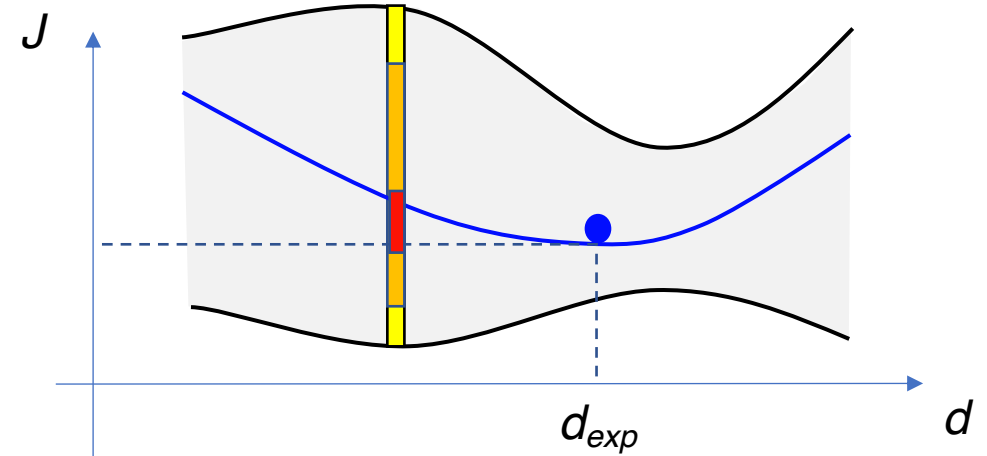
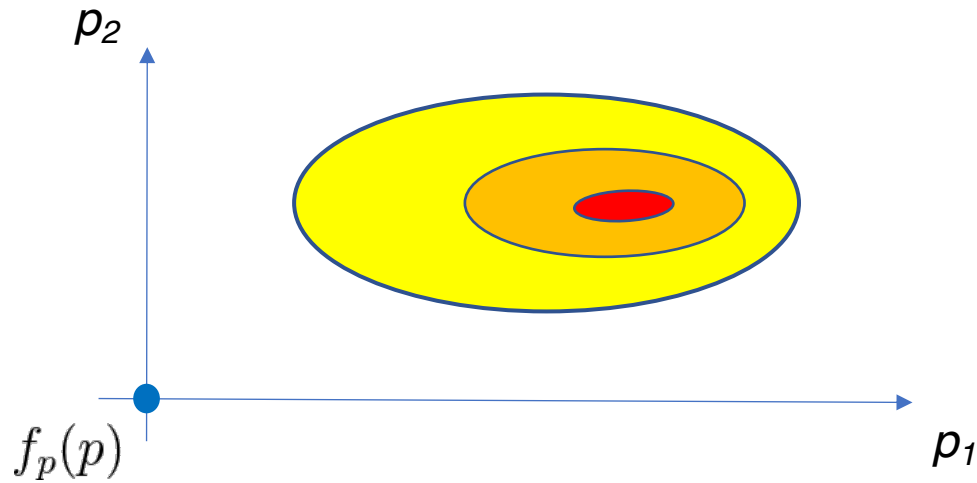


Probabilistic Uncertainty



Random Process

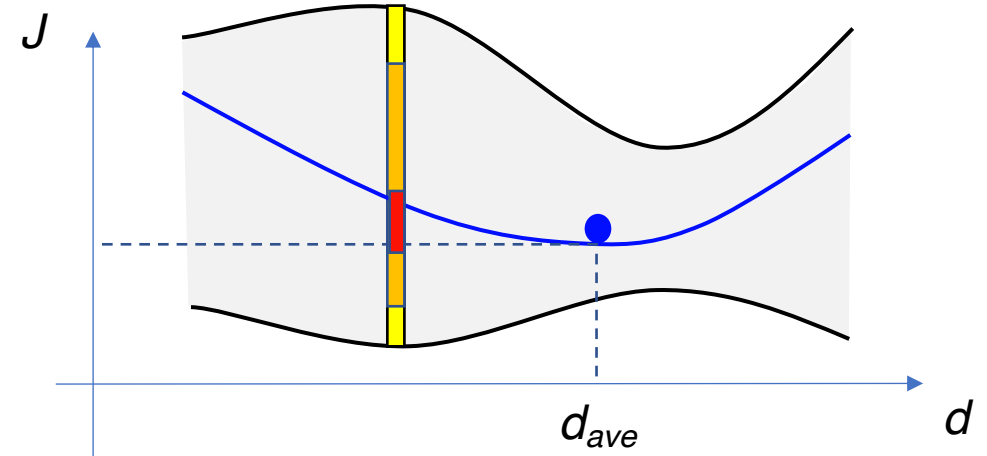
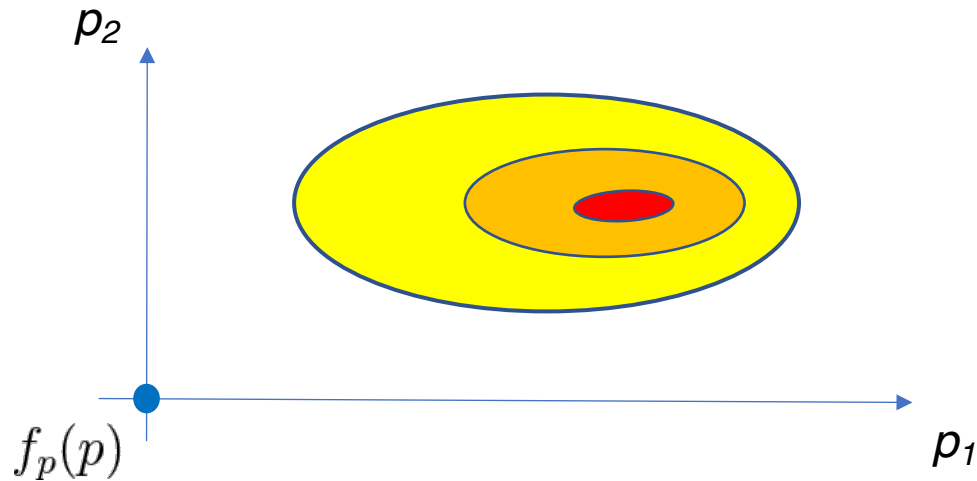
Optimal Expected Cost



- Formulation

$$\min_d \mathbb{E} [J(p, d)]$$

Optimal Expected Cost

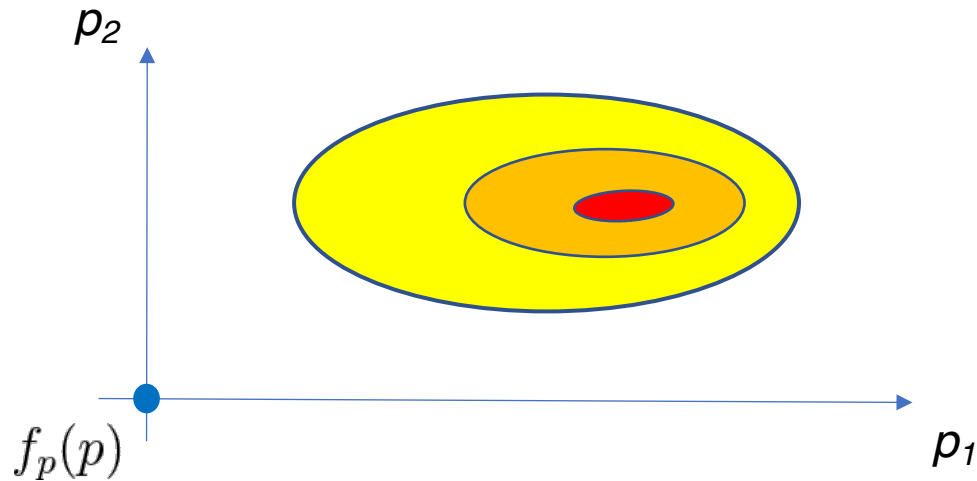


- Formulation

$$\min_d \mathbb{E} [J(p, d)]$$

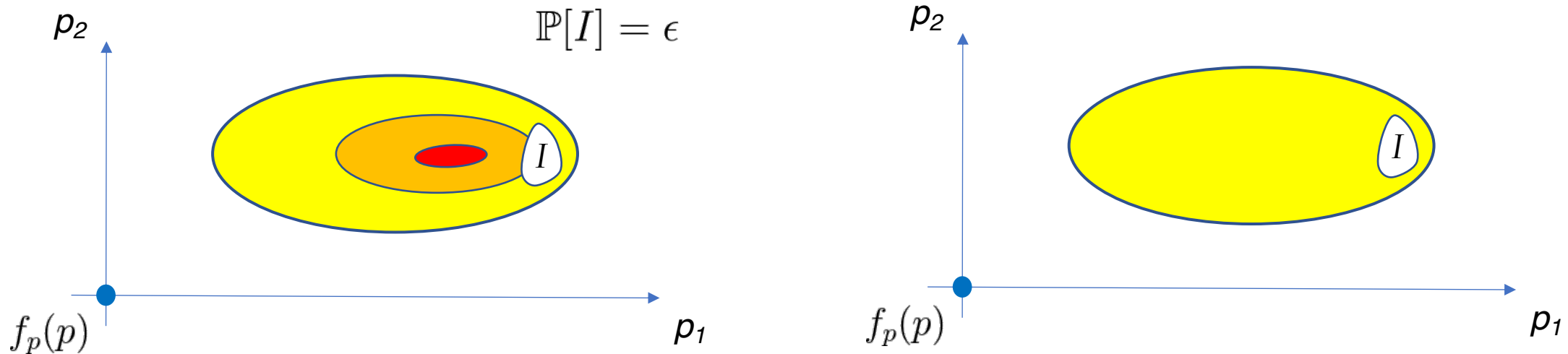
- Note $\mathbb{E}[J(p, d)] \neq J(\mathbb{E}[p], d)$ in general

Chance-Constrained Optimization



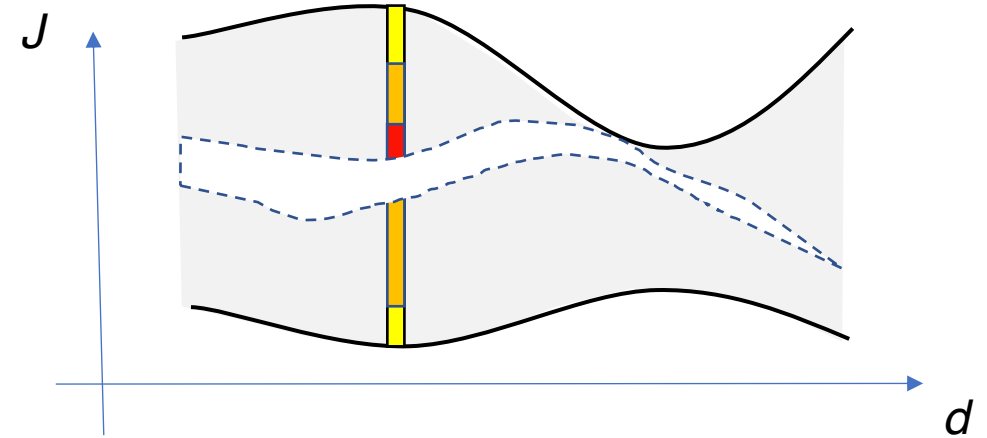
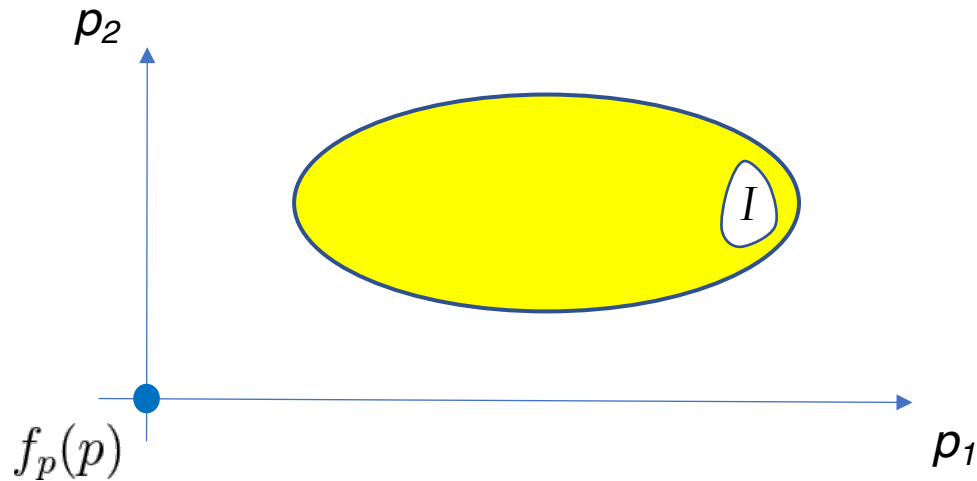
- Goal: to trade-off robustness for performance
- We need to accept some level of risk: science of gambling
- How can we do this?

Chance-Constrained Optimization



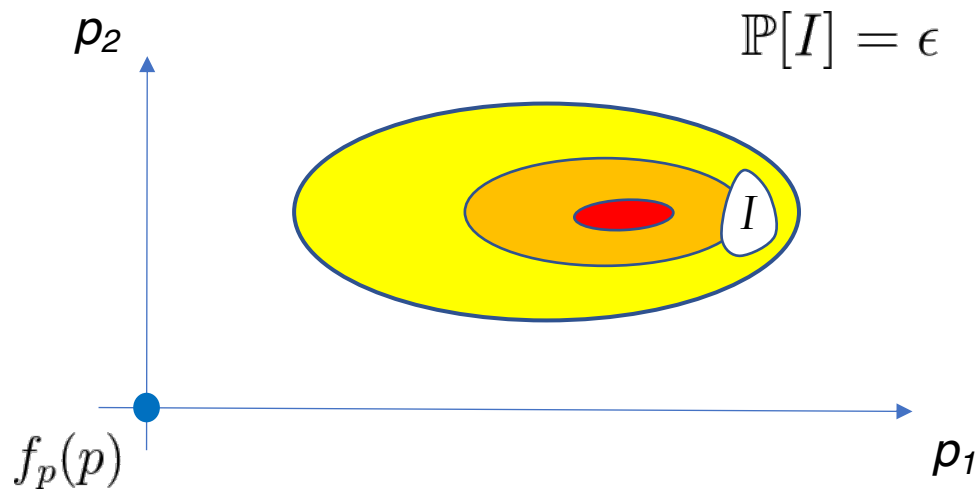
- Idea: find the WC solution after ignoring a low-probability region
- Option 1:
 - Make a “hole” in Δ of known probability
 - All the p 's within the hole will be ignored
 - Search for the optimal d

Chance-Constrained Optimization



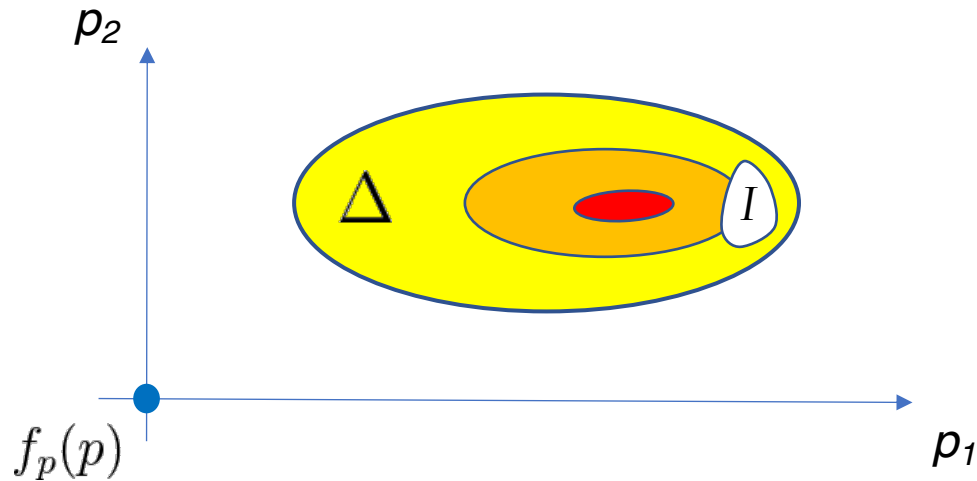
- The worst-performing cost depends on d
- The choice of I might make the optimal design subpar
- This is just another worst-case problem!!

Chance-Constrained Optimization



- Option 2:
 - Make a “hole” in Δ of known probability
 - All the p ’s within the hole will be ignored
 - d as well as the “best” hole are searched for

Chance-Constrained Optimization

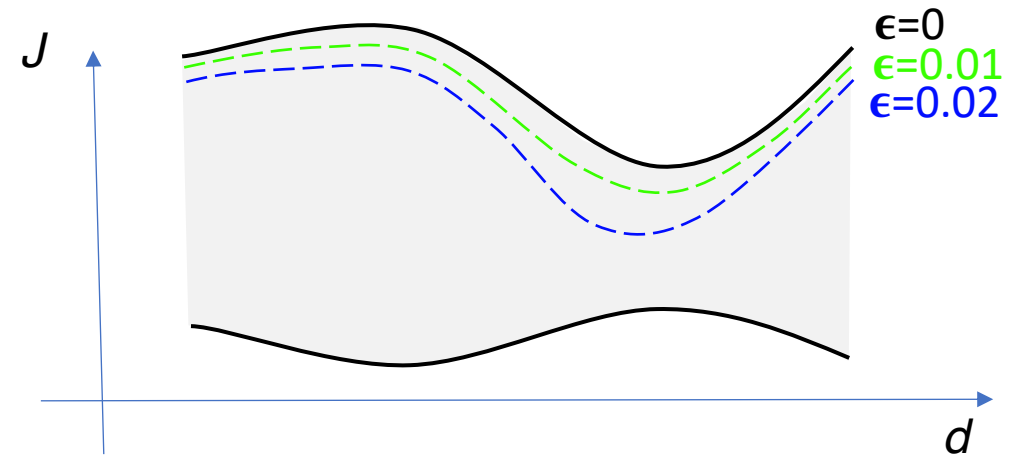
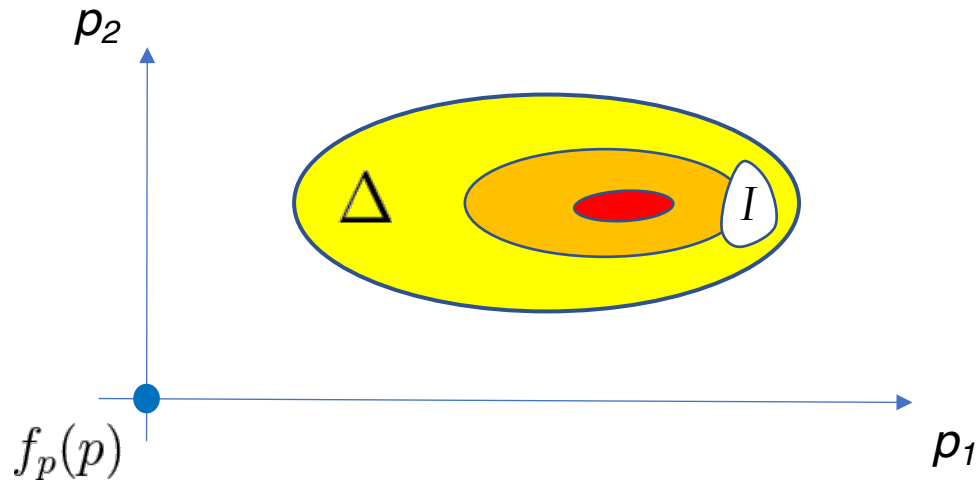


- Formulation

$$\min_{I, d} \left[\max_{p \in \Delta - I} J(d, p) : \mathbb{P}[I] = \epsilon \right]$$

Risk

Chance-Constrained Optimization

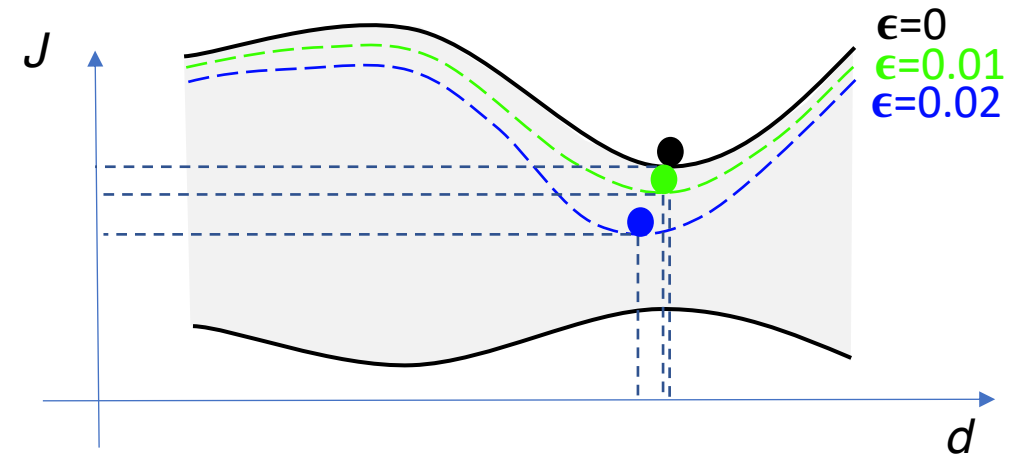
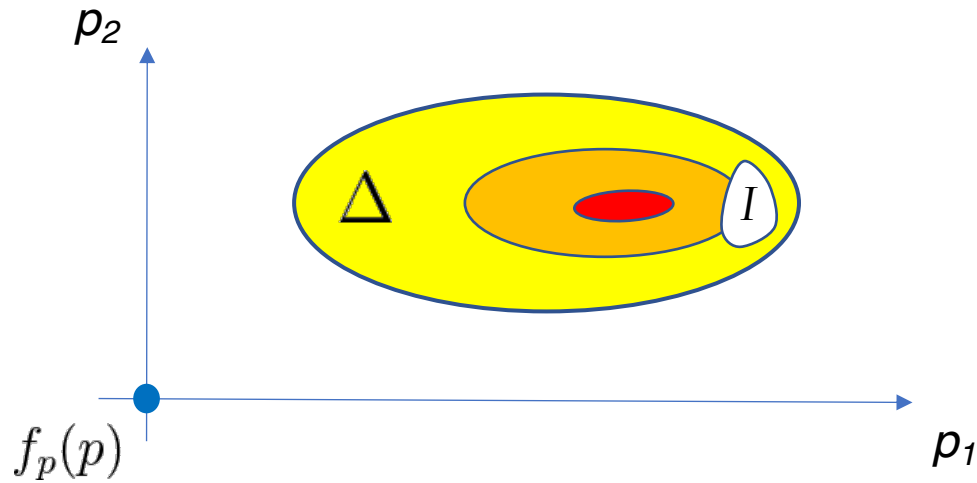


- Formulation

$$\min_{I, d} \left[\max_{p \in \Delta - I} J(d, p) : \mathbb{P}[I] = \epsilon \right]$$

Risk
↑

Chance-Constrained Optimization

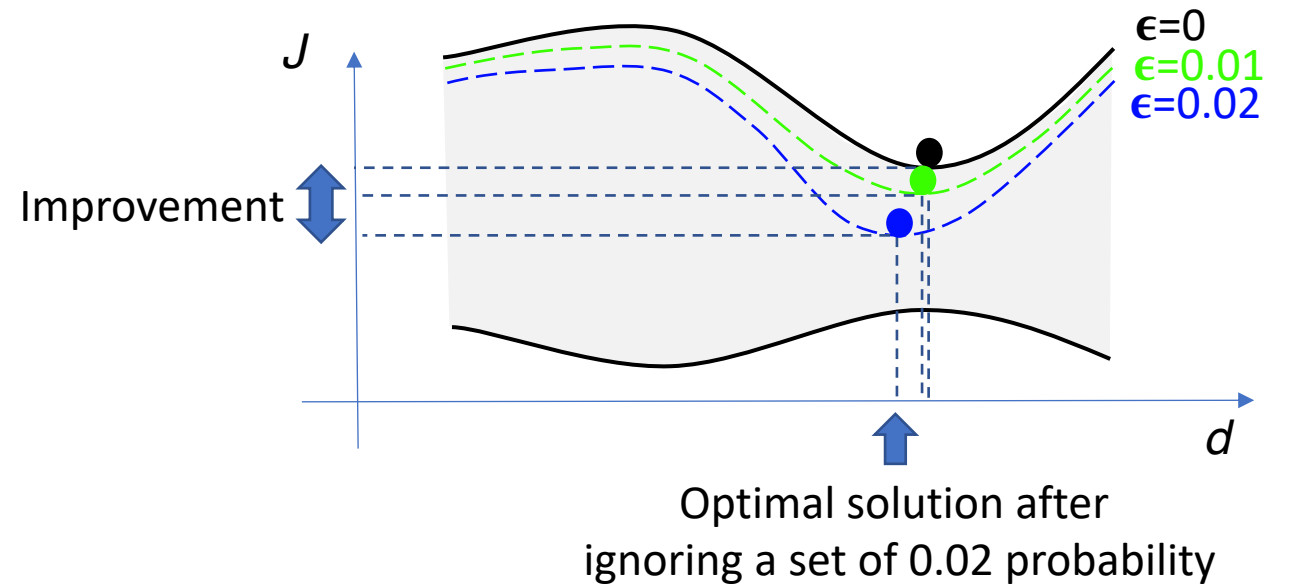
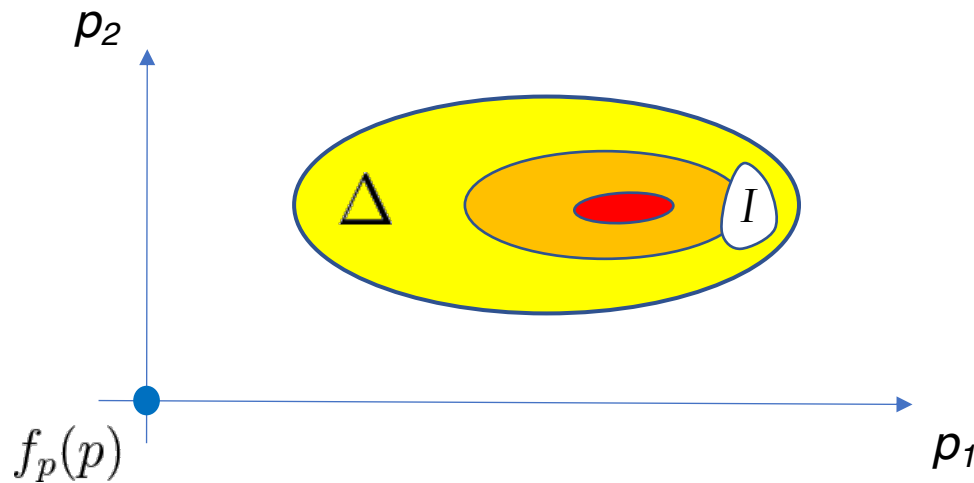


- Formulation

$$\min_{I, d} \left[\max_{p \in \Delta - I} J(d, p) : \mathbb{P}[I] = \epsilon \right]$$

Risk
↑

Chance-Constrained Optimization



- Formulation

$$\min_{I, d} \left[\max_{p \in \Delta - I} J(d, p) : \mathbb{P}[I] = \epsilon \right]$$

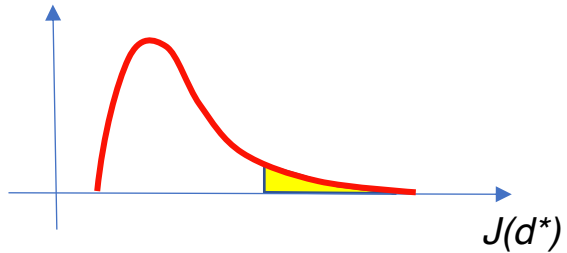
Risk

↑

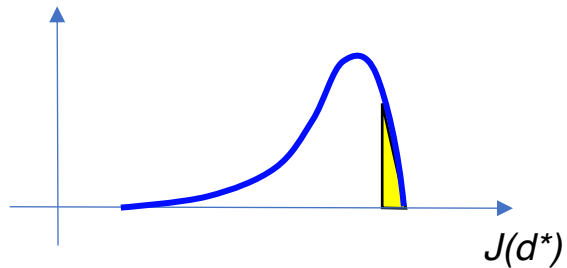
Chance-Constrained Optimization (CCO)

- When does CCO pay off?

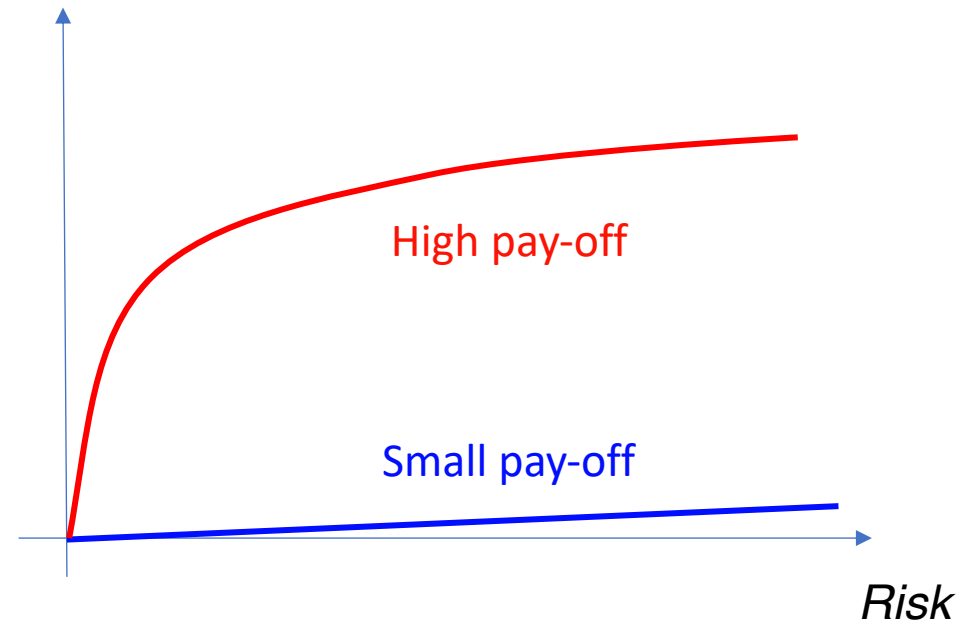
CASE 2



CASE 1



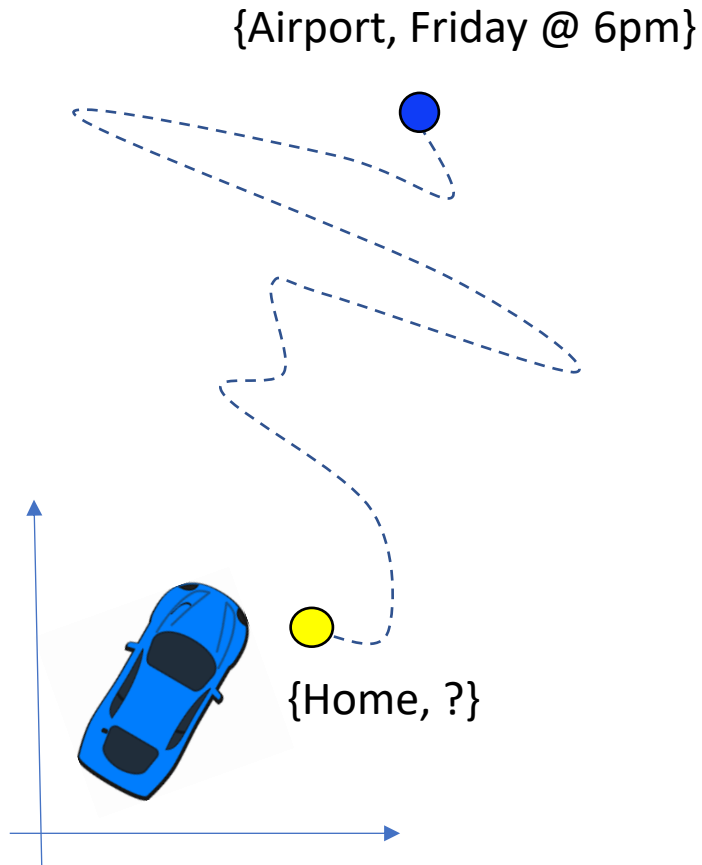
Cost
reduction



Optimization Under Uncertainty

- The computational complexity of WC is often exponential in the number of inputs and outputs whereas that of CCO is polynomial [Alamo, Automatica 52, 2015]

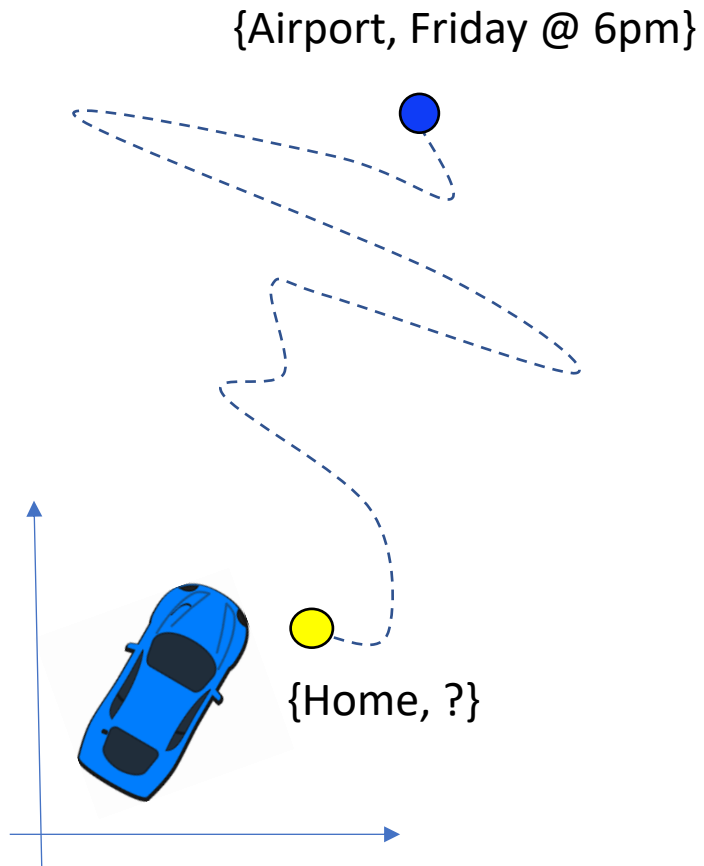
Example



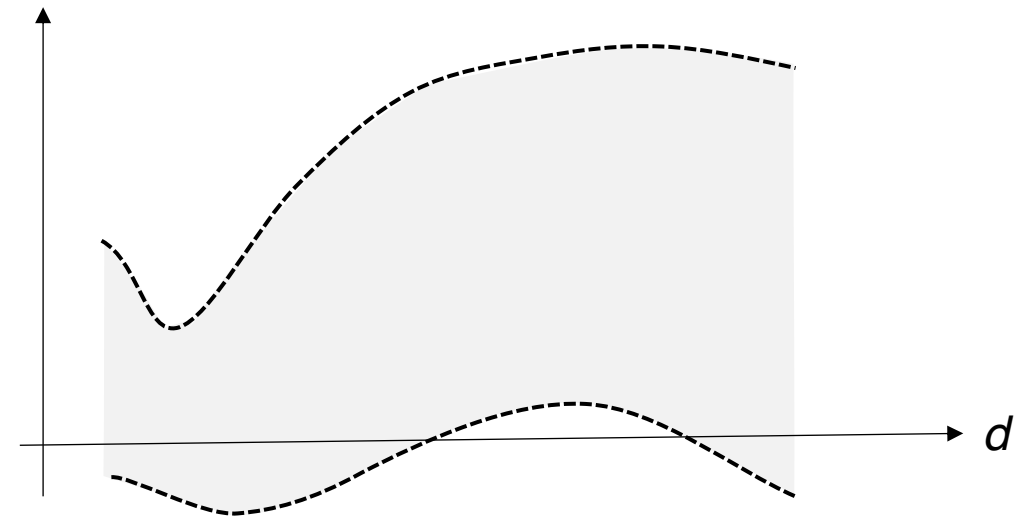
$$\min_d \{ J(d, p) : \mathbb{P}[J(d, p) < 0] \leq \epsilon \}$$

- Cost J : Time waiting in the airport
- Design variable d : Time leaving home
- Uncertainty p : {traffic, road/lane closures}
- Ill-posed problem as stated

Example: WC

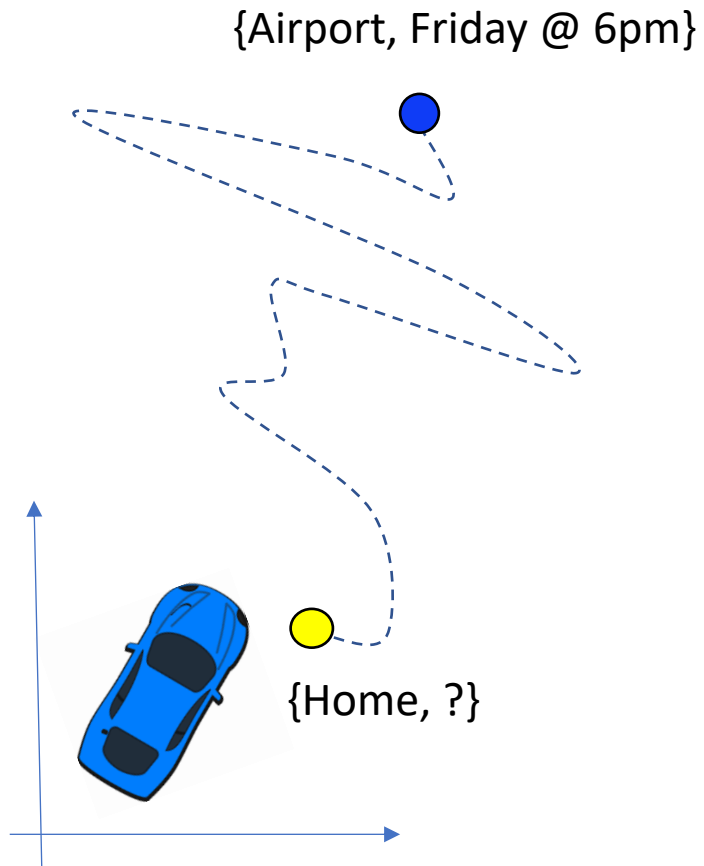


Time waiting
at the airport

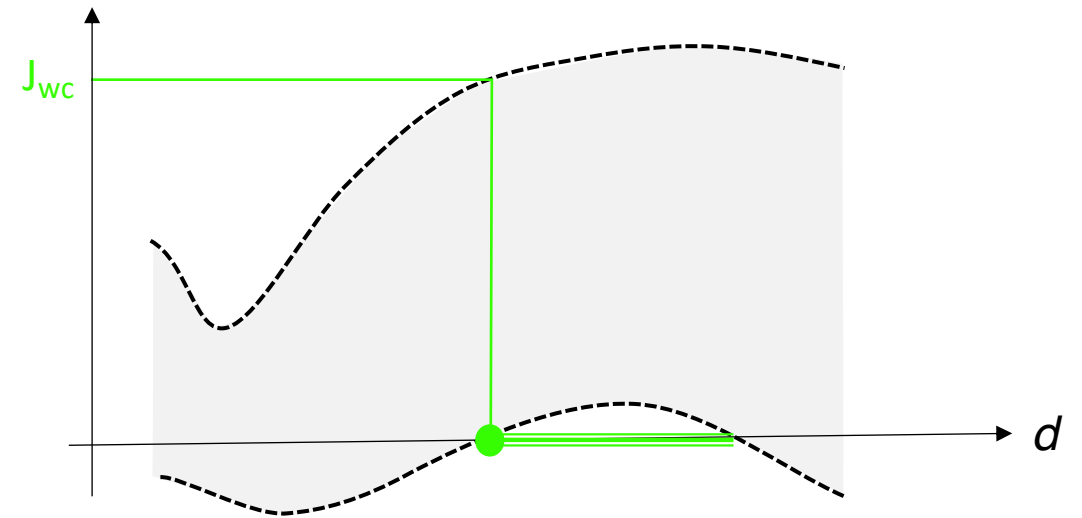


- What is the optimal worst case?
- Infinitely many options to not miss the flight

Example: WC

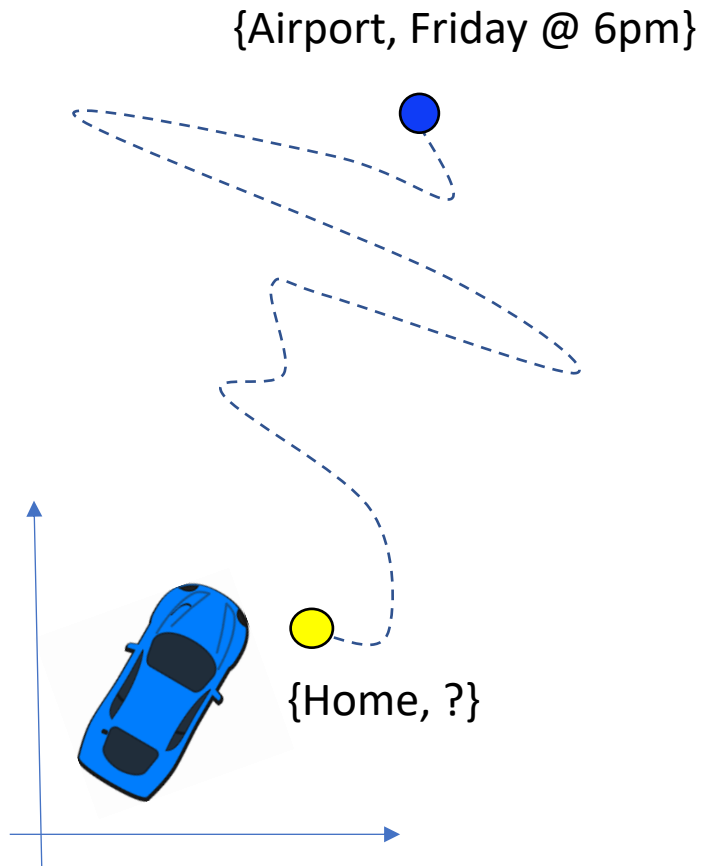


Time waiting
at the airport

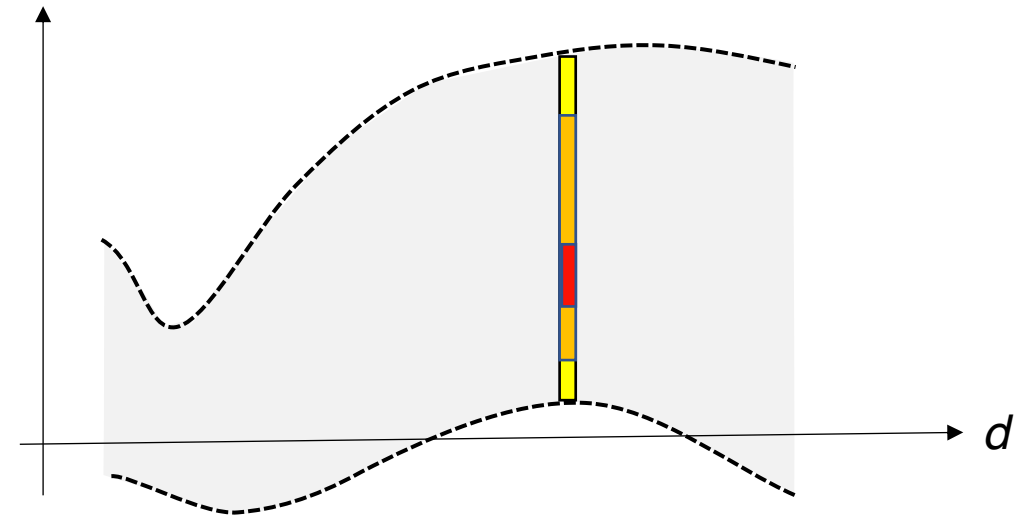


You will make the flight but might
have to wait a lot

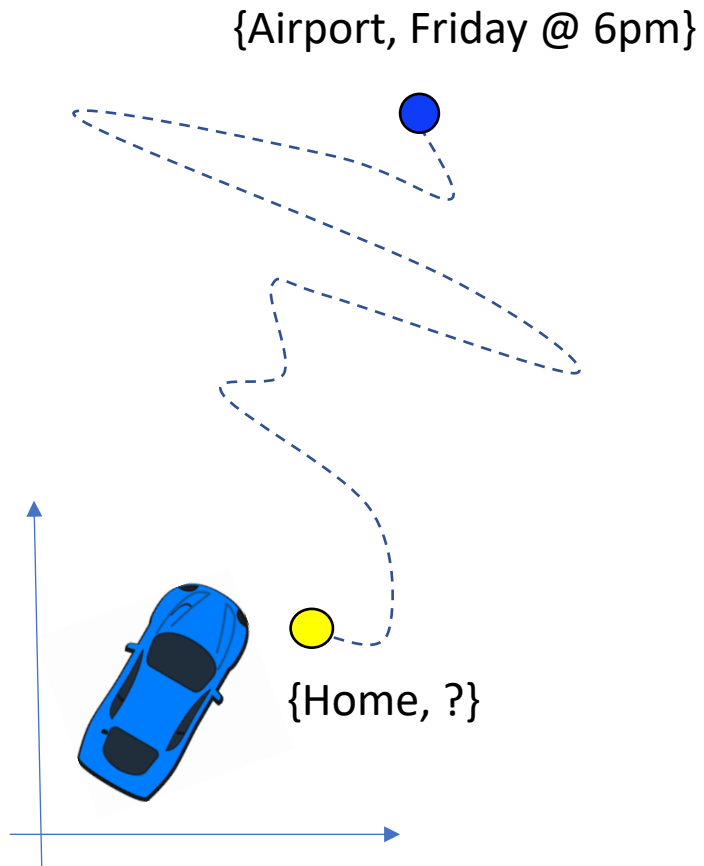
Example: WC vs. CCO



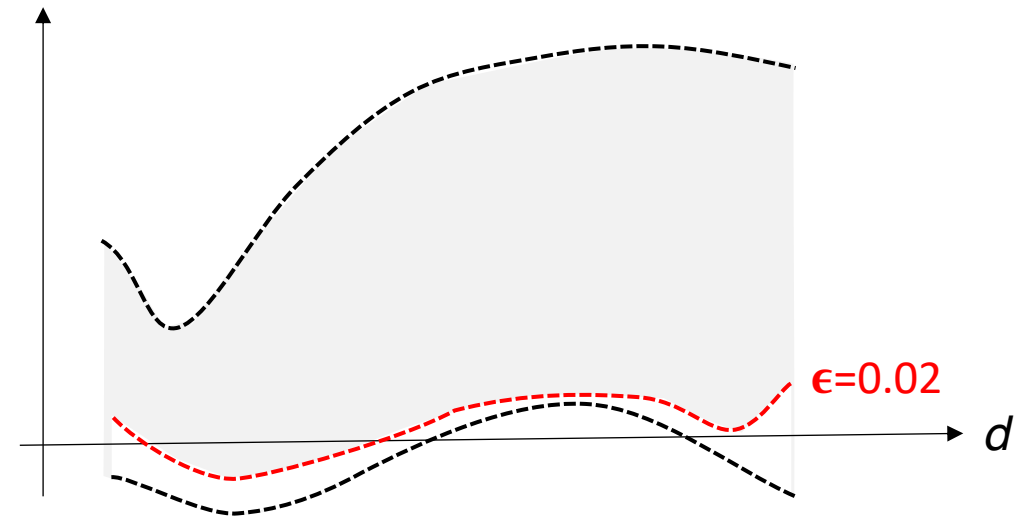
Time waiting
at the airport



Example: WC vs. CCO

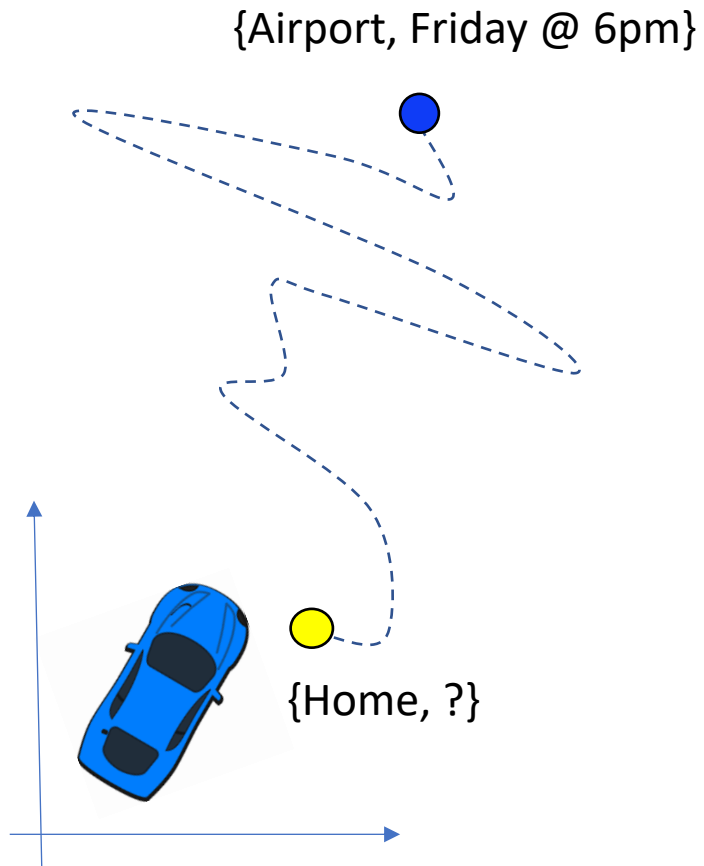


Time waiting
at the airport

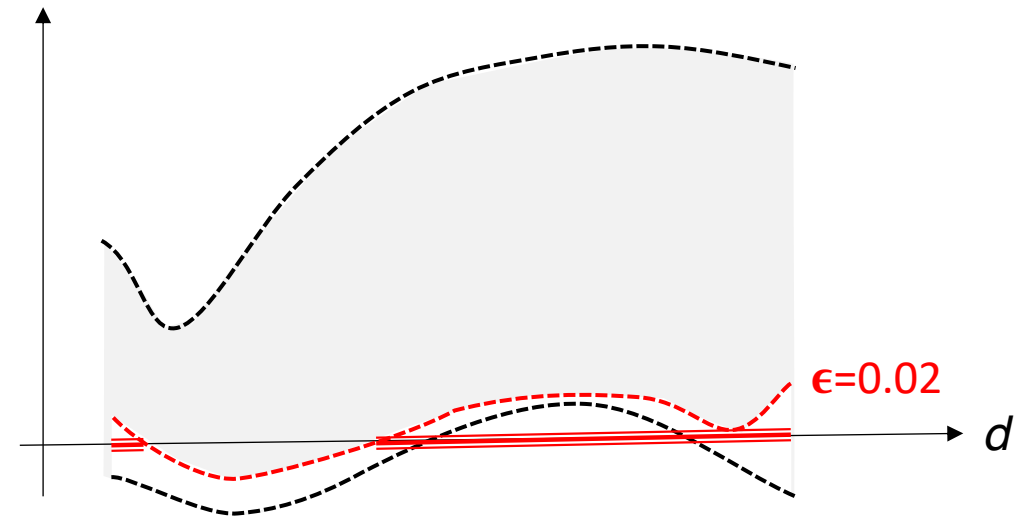


Which designs are feasible?

Example: WC vs. CCO

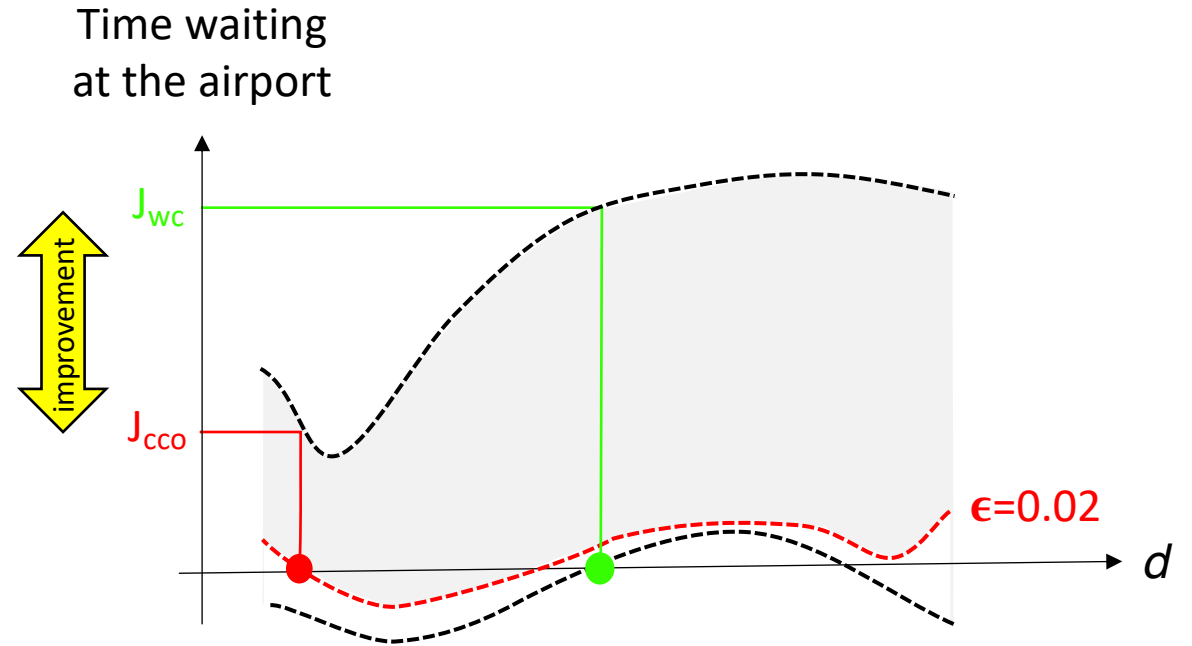
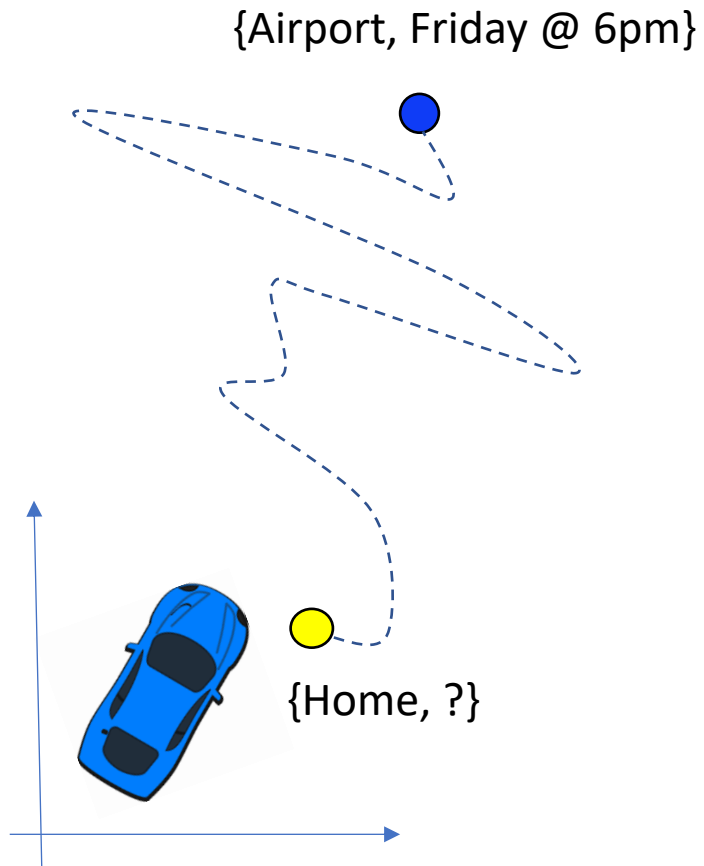


Time waiting
at the airport



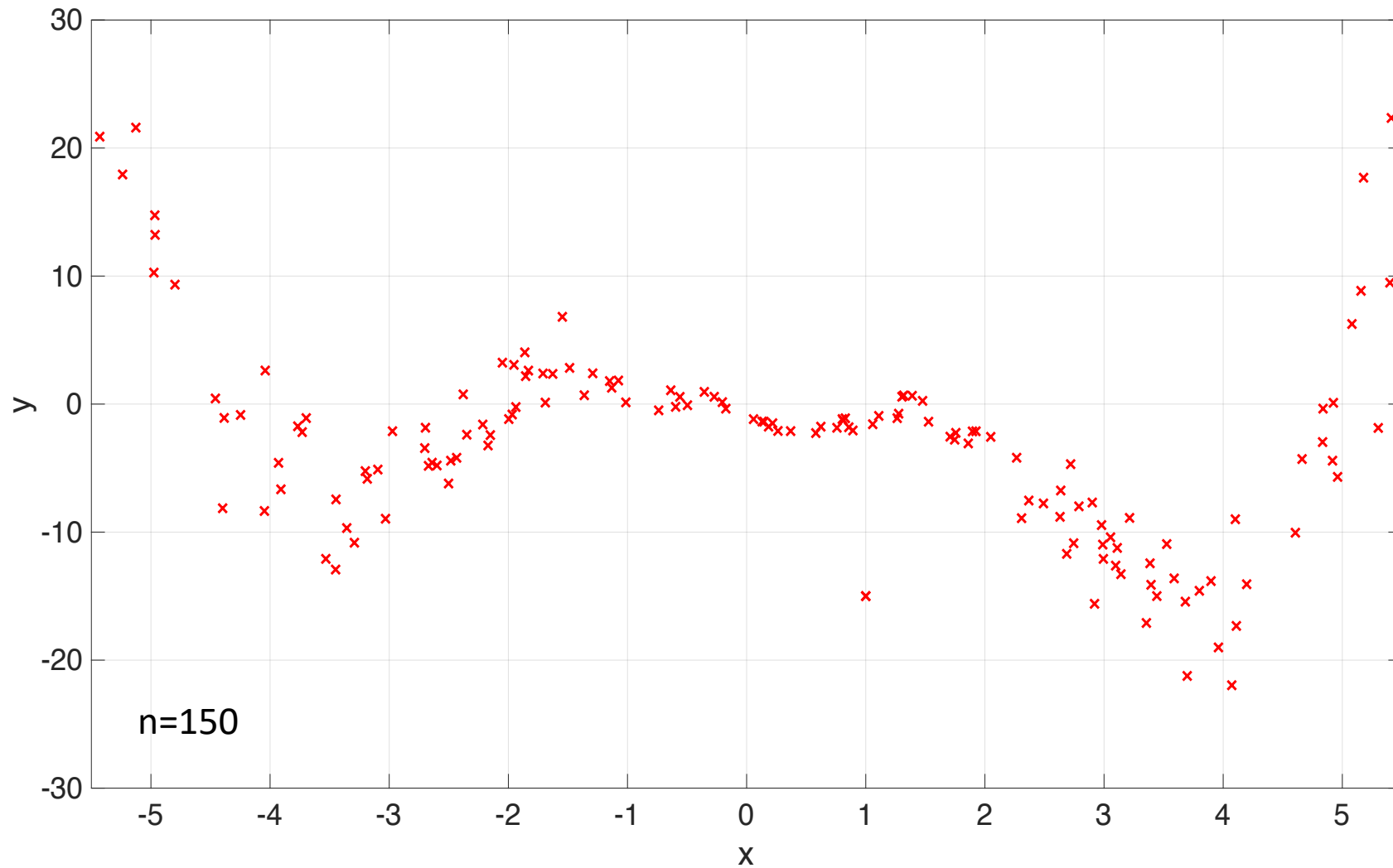
Which designs are feasible?

Example: WC vs. CCO

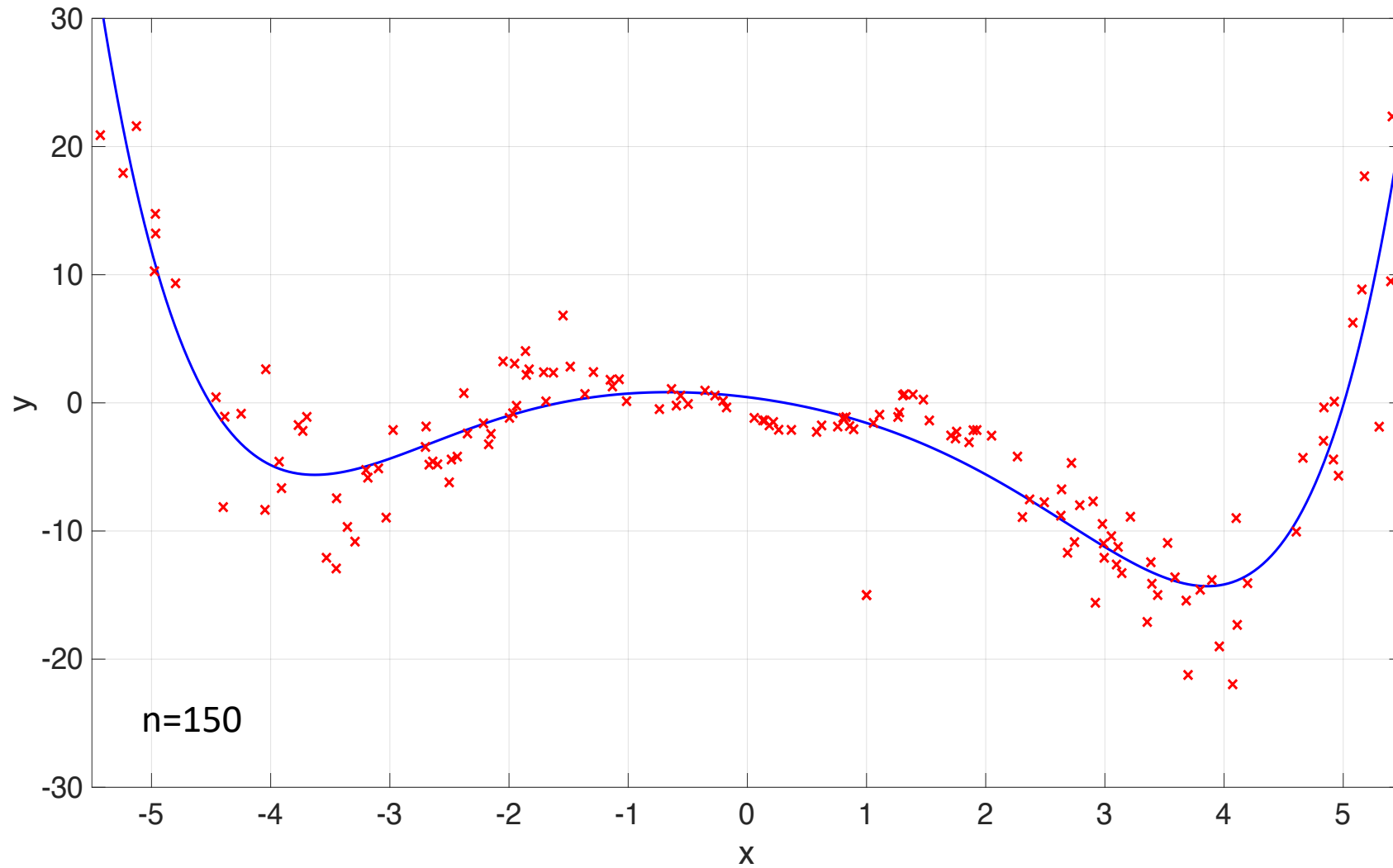


You might miss the flight with 0.02 probability
but will wait considerably less

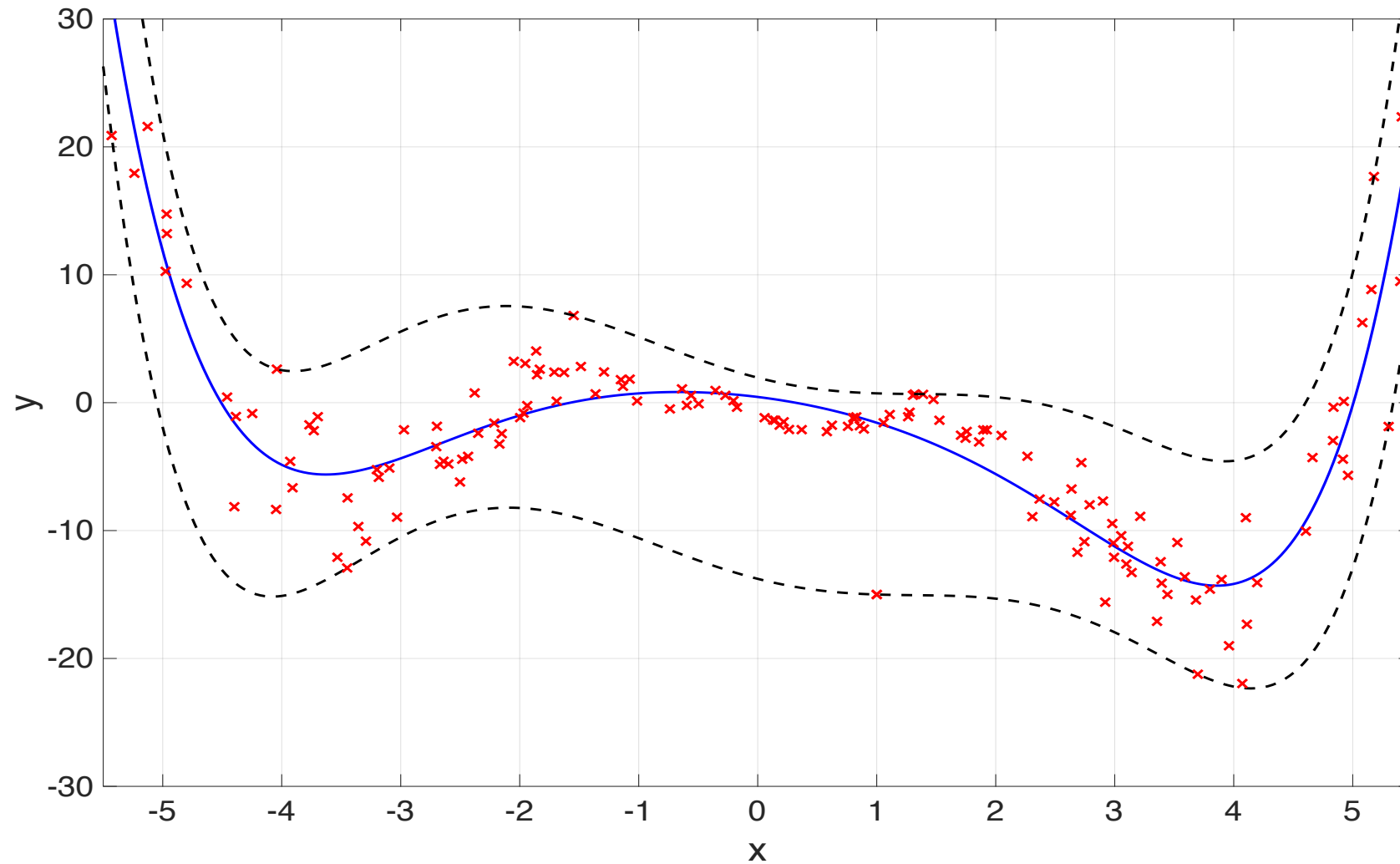
Example: Interval Predictor Models



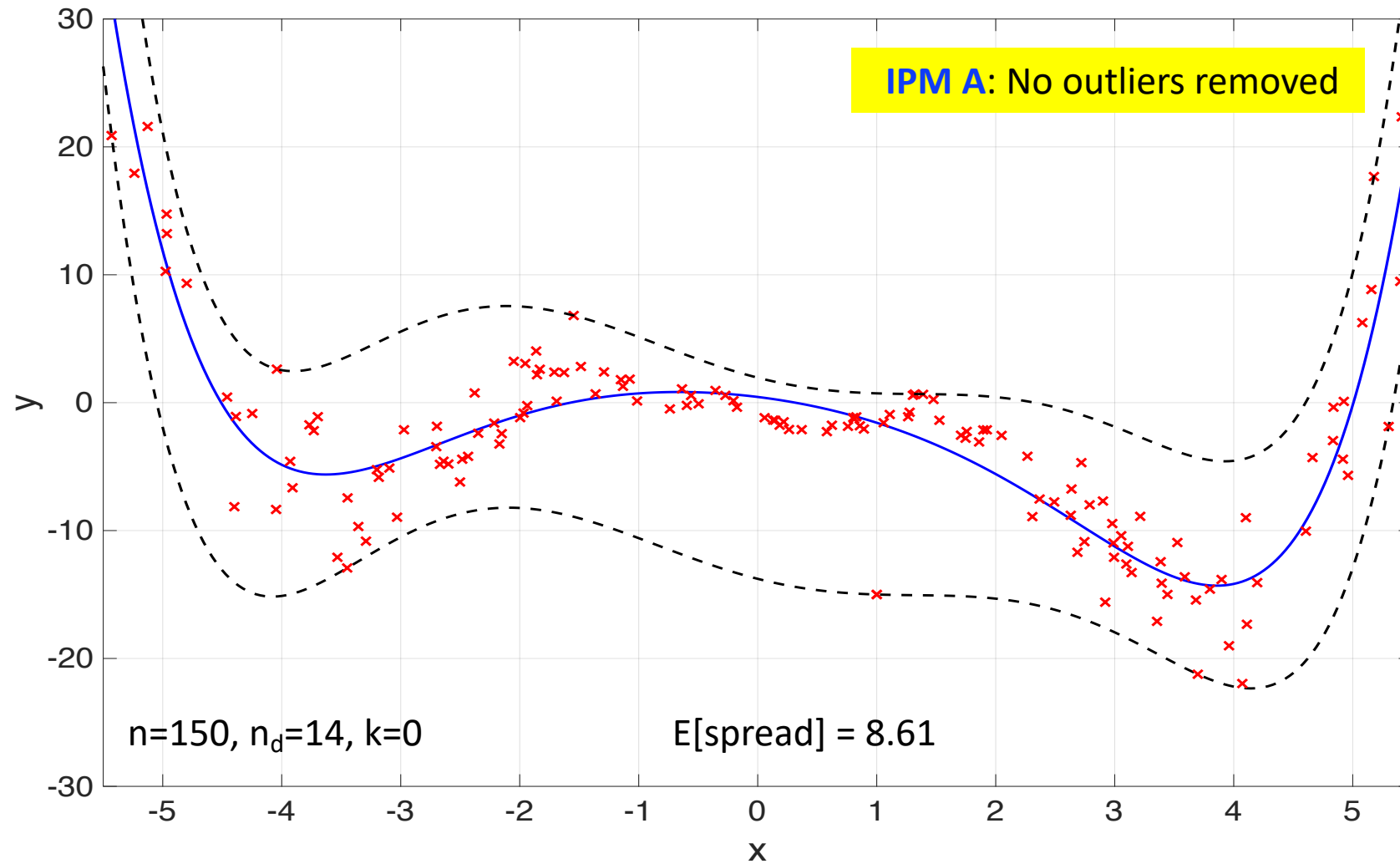
Example: Interval Predictor Models



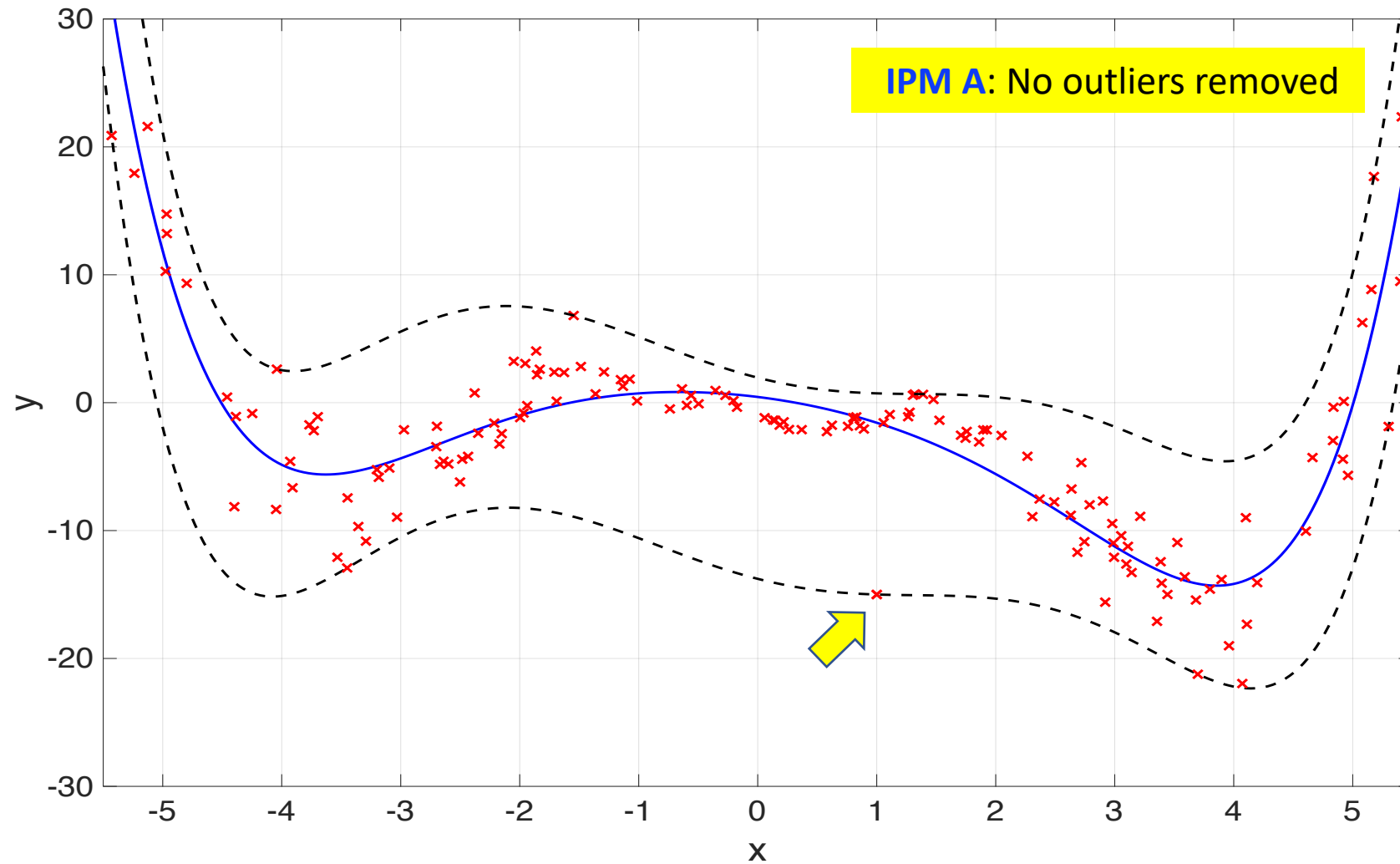
Example: Interval Predictor Models



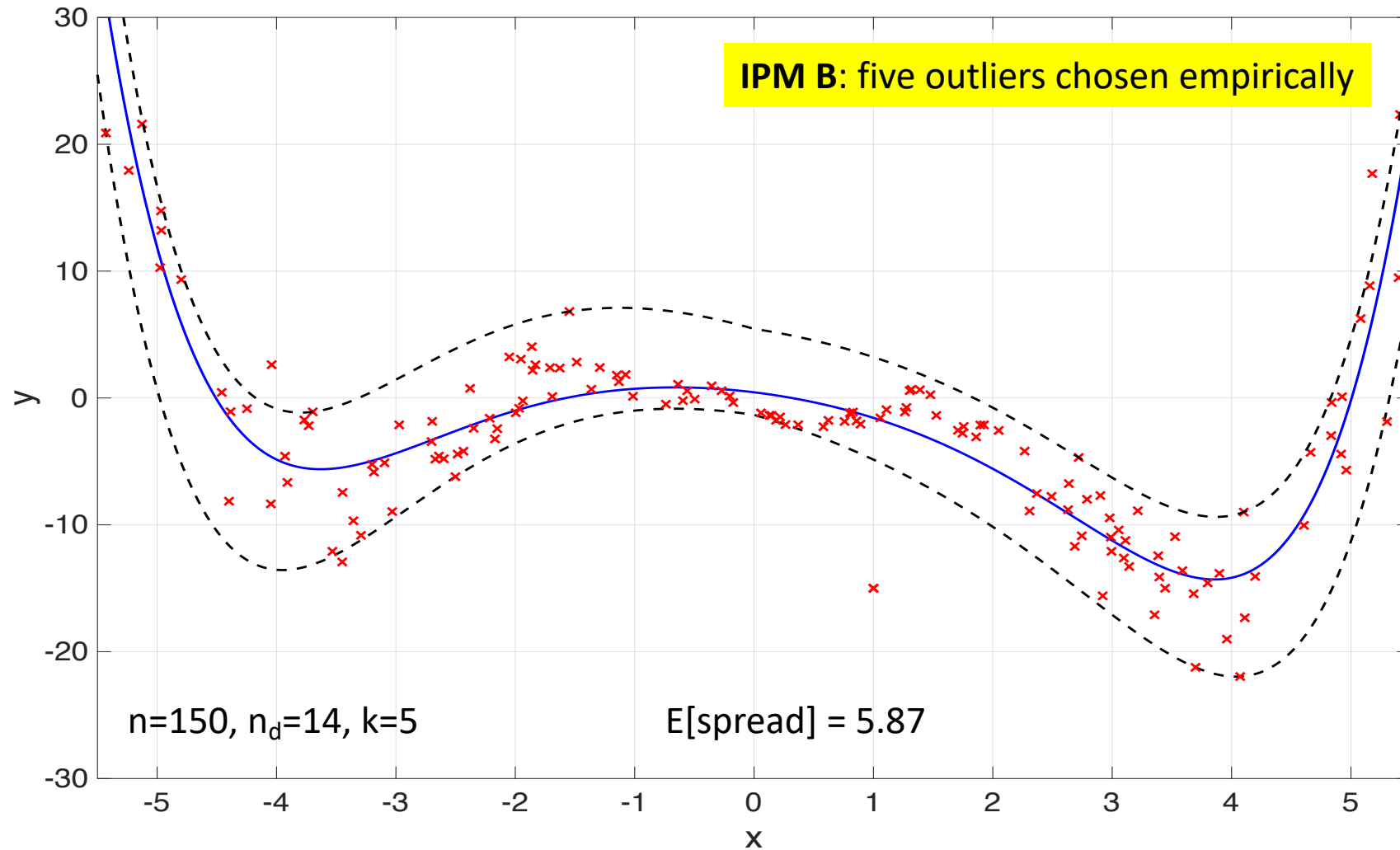
Example: Interval Predictor Models



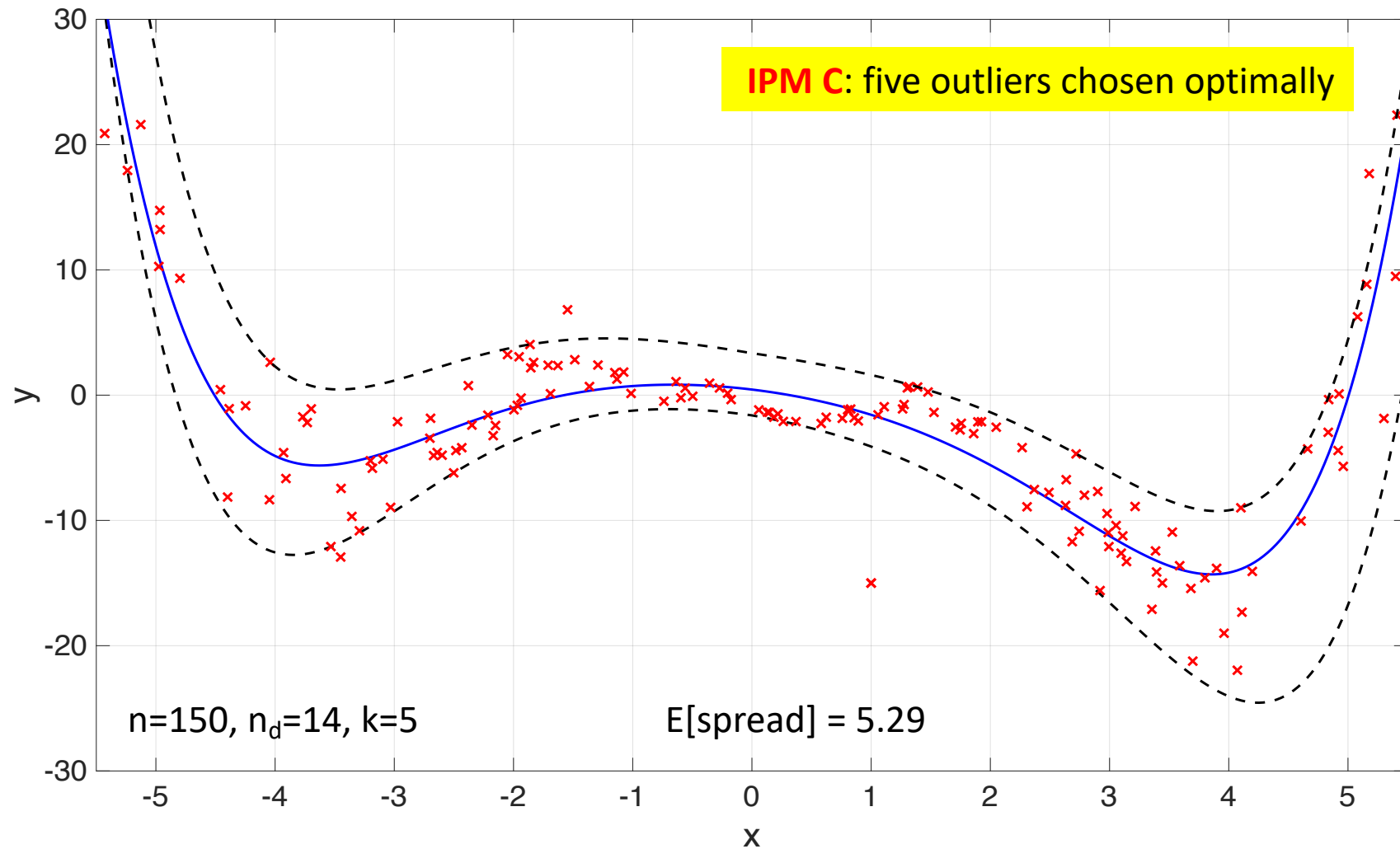
Example: Interval Predictor Models



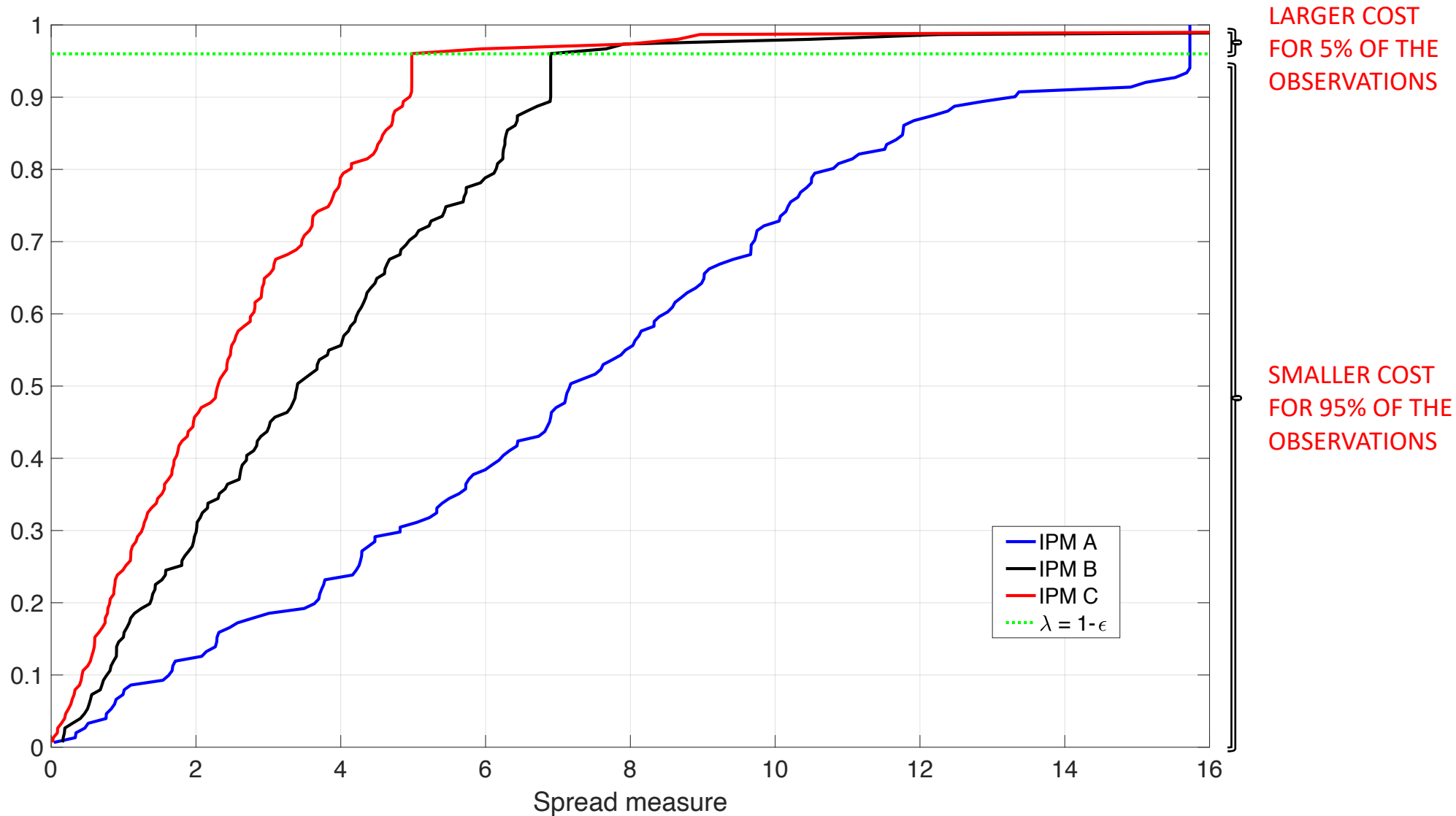
Example: Interval Predictor Models



Example: Interval Predictor Models



Example: Interval Predictor Models



Example: Interval Predictor Models Reliability

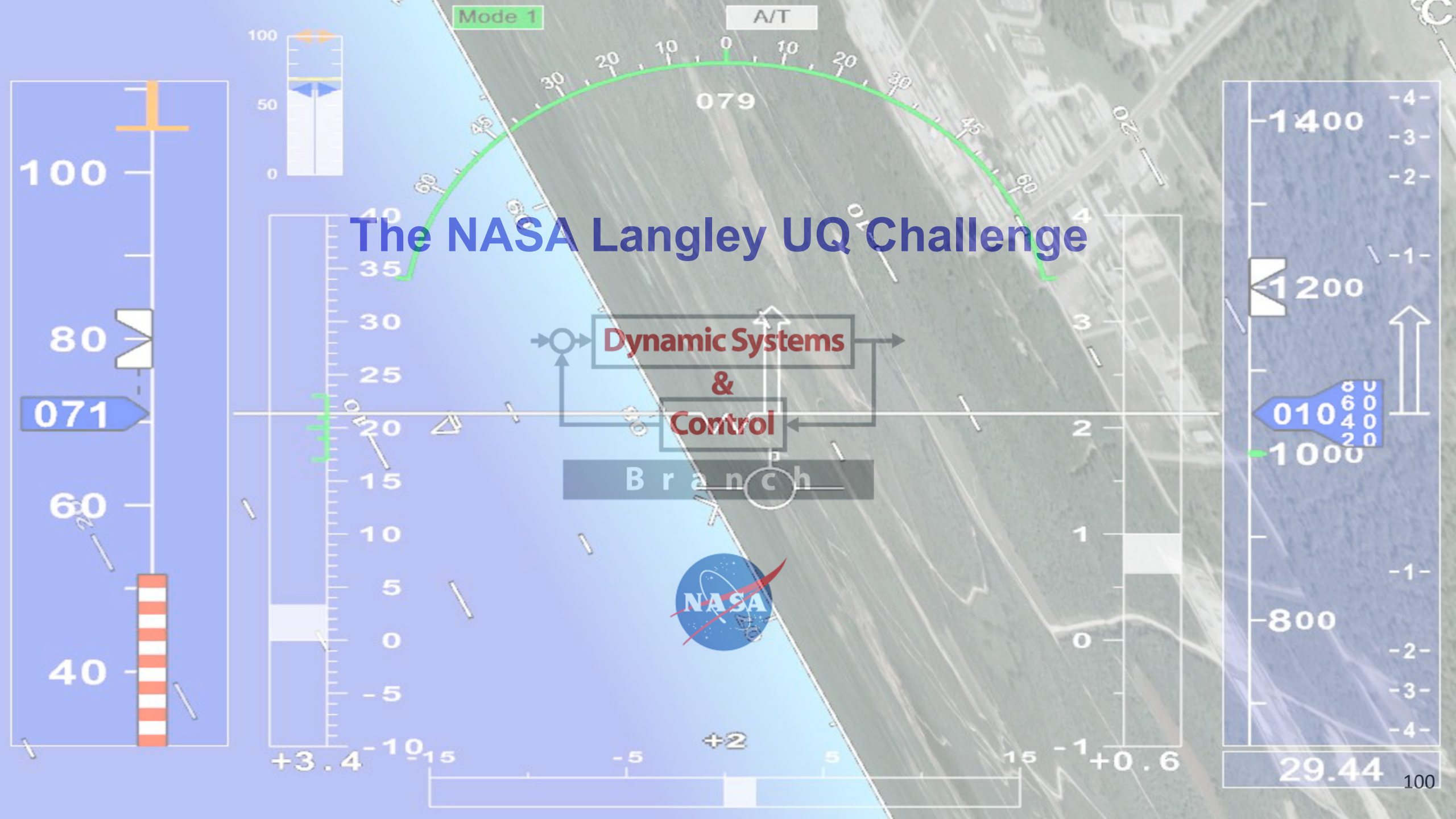
	PERFORMANCE	RISK ($\beta=0.99$)
IPM A (n=150, d=14, k=0)	8.61	$\epsilon < 0.155$
IPM B (n=150, d=14, k=5)	5.87	$\epsilon < 0.285$
IPM C (n=150, d=14, k=5)	5.29	$\epsilon < 0.285$

- A performance improvement of 32% increased the risk 0.13
- Bound is validated by using more data (we might not have it in practice)
 - Dividing the data set into a training set & a validation set is not needed

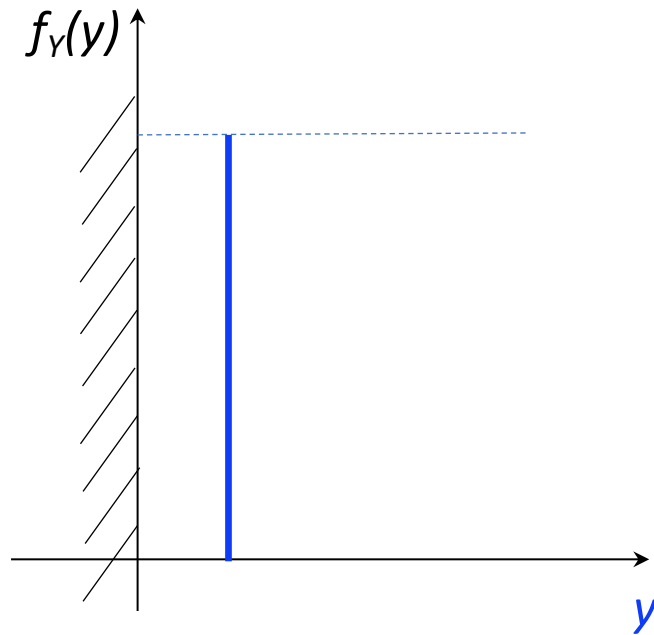
Overview

- Uncertainty modeling
- Set deformations
- Optimization under uncertainty
- **Decision making**

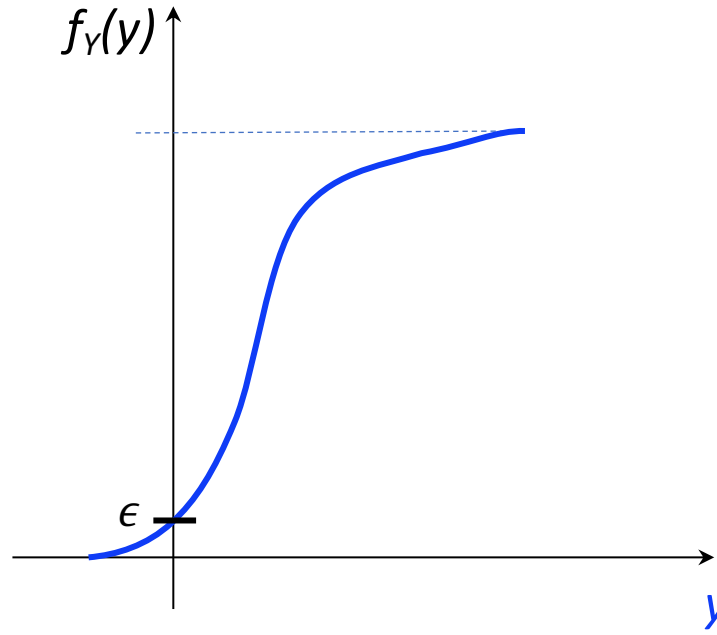
The NASA Langley UQ Challenge



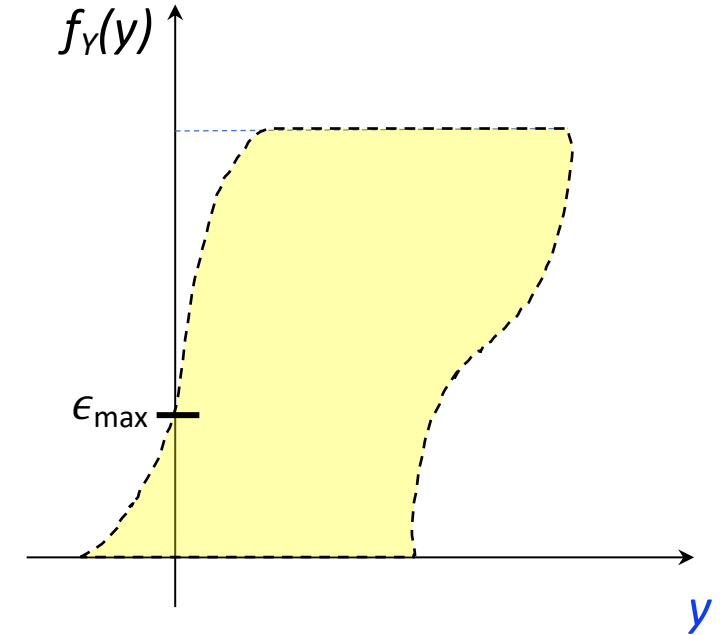
Decision Making under Epistemic Uncertainty



Deterministic Scenario:
no violation



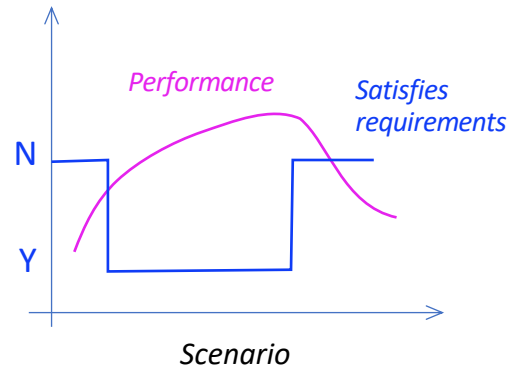
Probabilistic Scenario:
 $P[\text{violation}] = \epsilon$



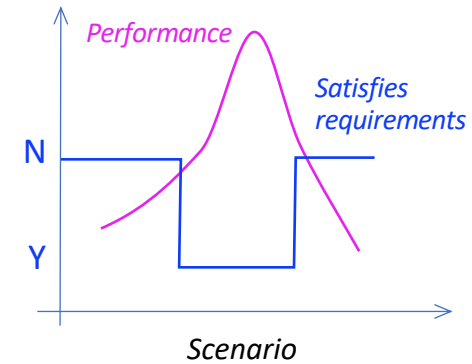
P-Box Scenario:
 $P[\text{violation}] = [0, \epsilon_{\max}]$

Practicalities

- A good robust design might not perform well any given day



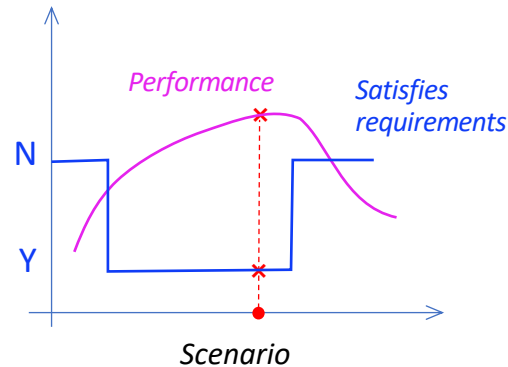
Design 1 (+ complex, + robust, - performance)



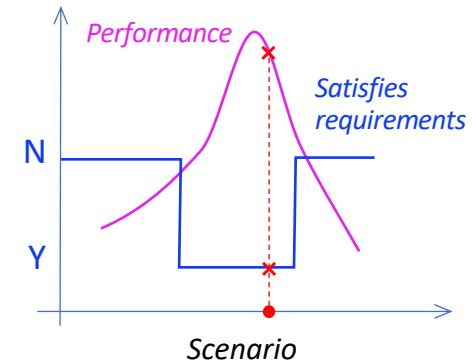
Design 2 (- complex, - robust, + performance)

Practicalities

- A good robust design might not perform well any given day



Design 1 (+ complex, + robust, - performance)

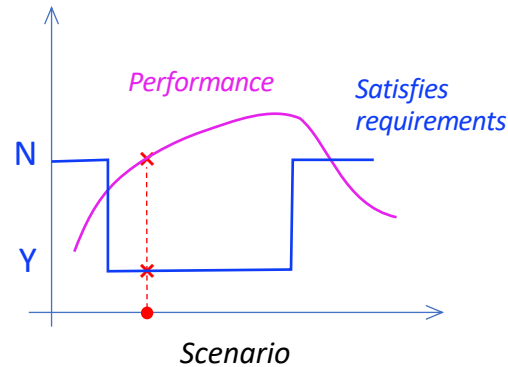


Design 2 (- complex, - robust, + performance)

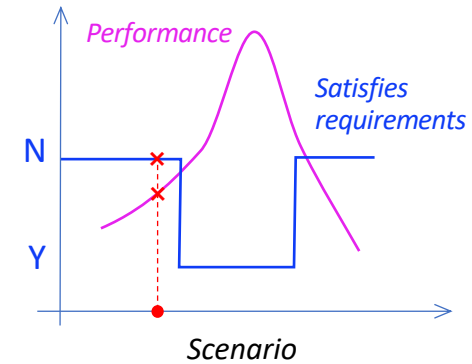
Design 1 is more robust/reliable than design 2 but the latter design performed better that day

Practicalities

- A good robust design might not perform well any given day



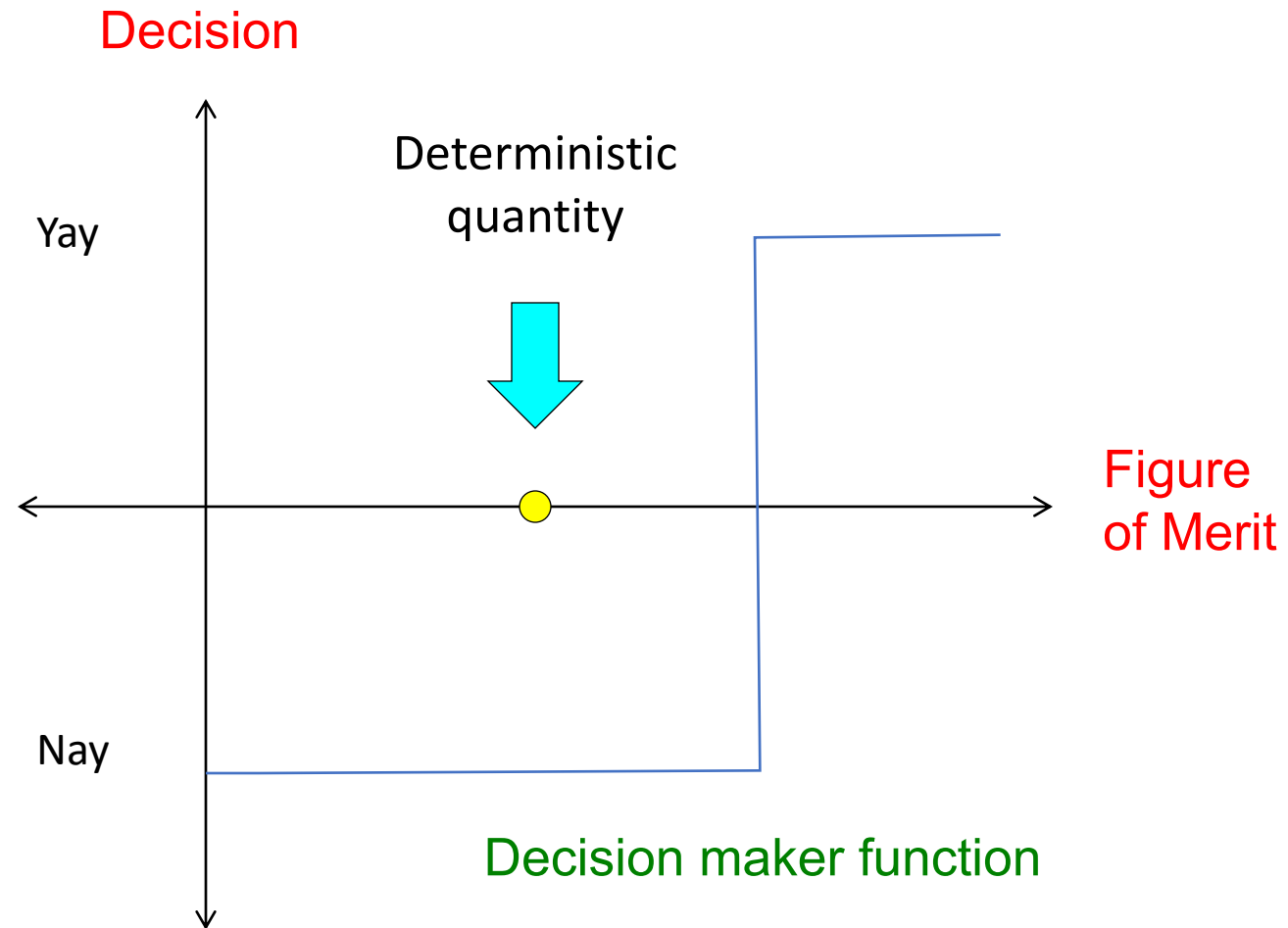
Design 1 (+ complex, + robust, - performance)



Design 2 (- complex, - robust, + performance)

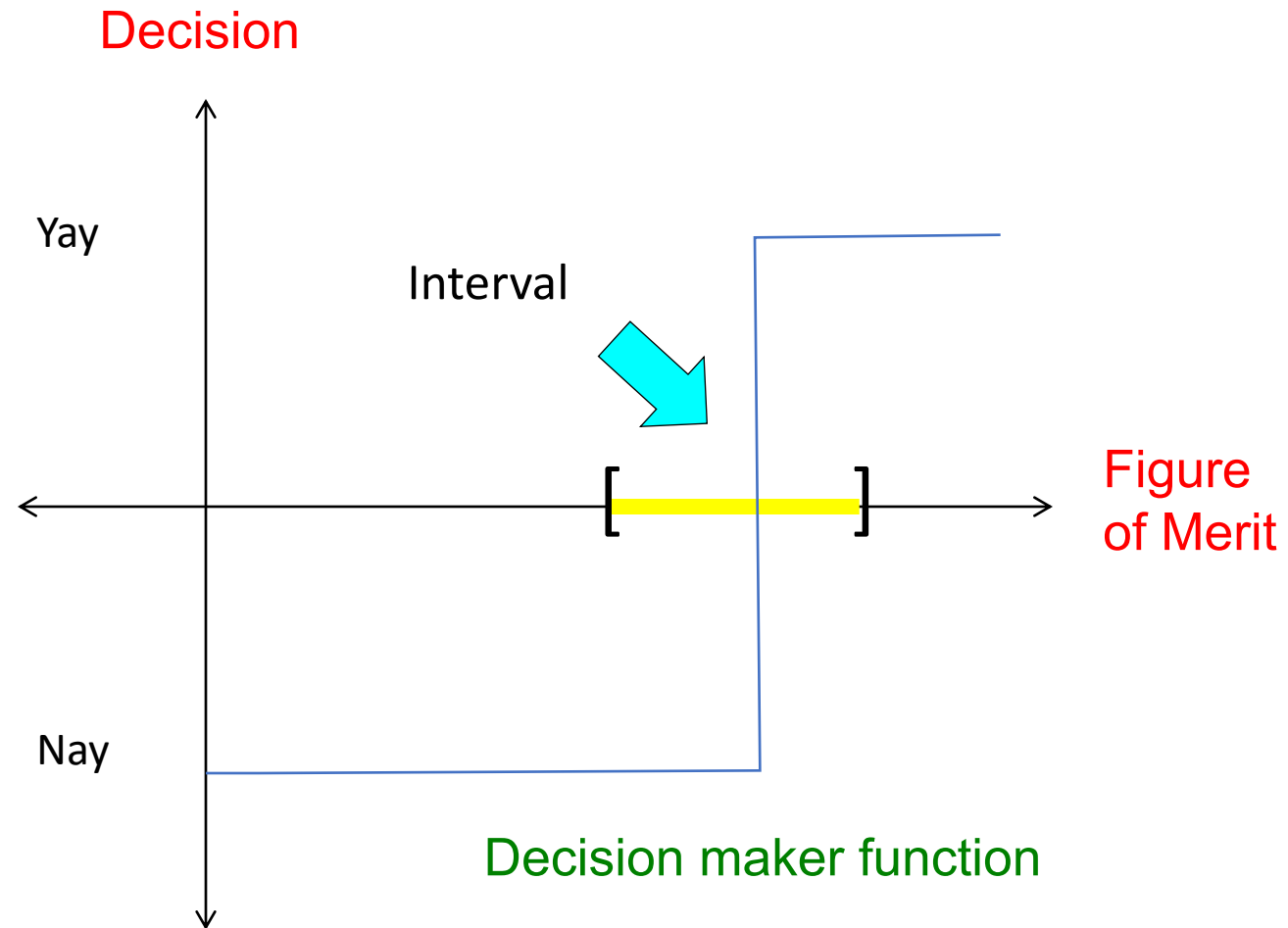
Design 1 is more robust/reliable than design 2 but the latter design performed better that day

Decision Making without Epistemic Uncertainty

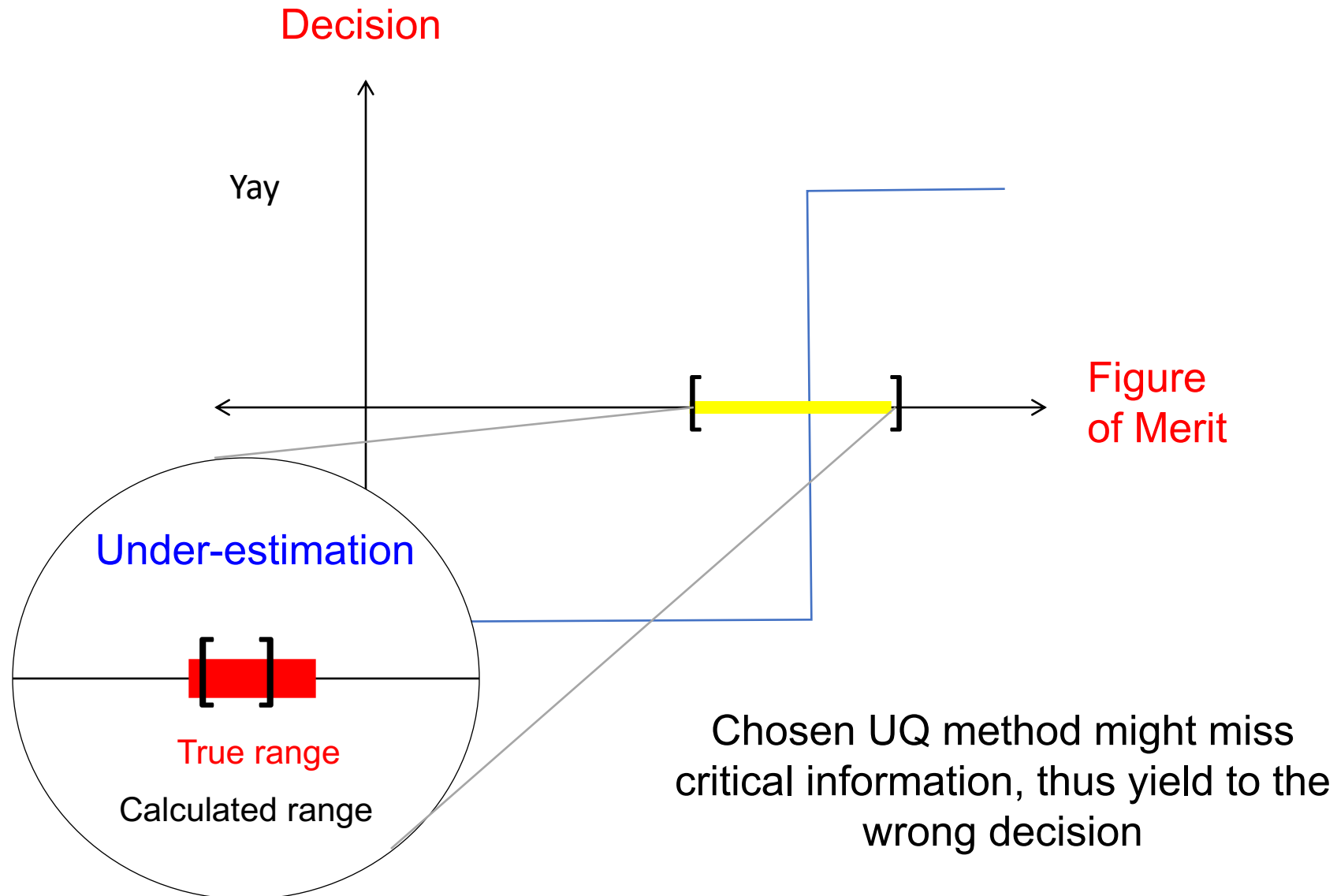


Decision maker objective vs. UQ analyst objective

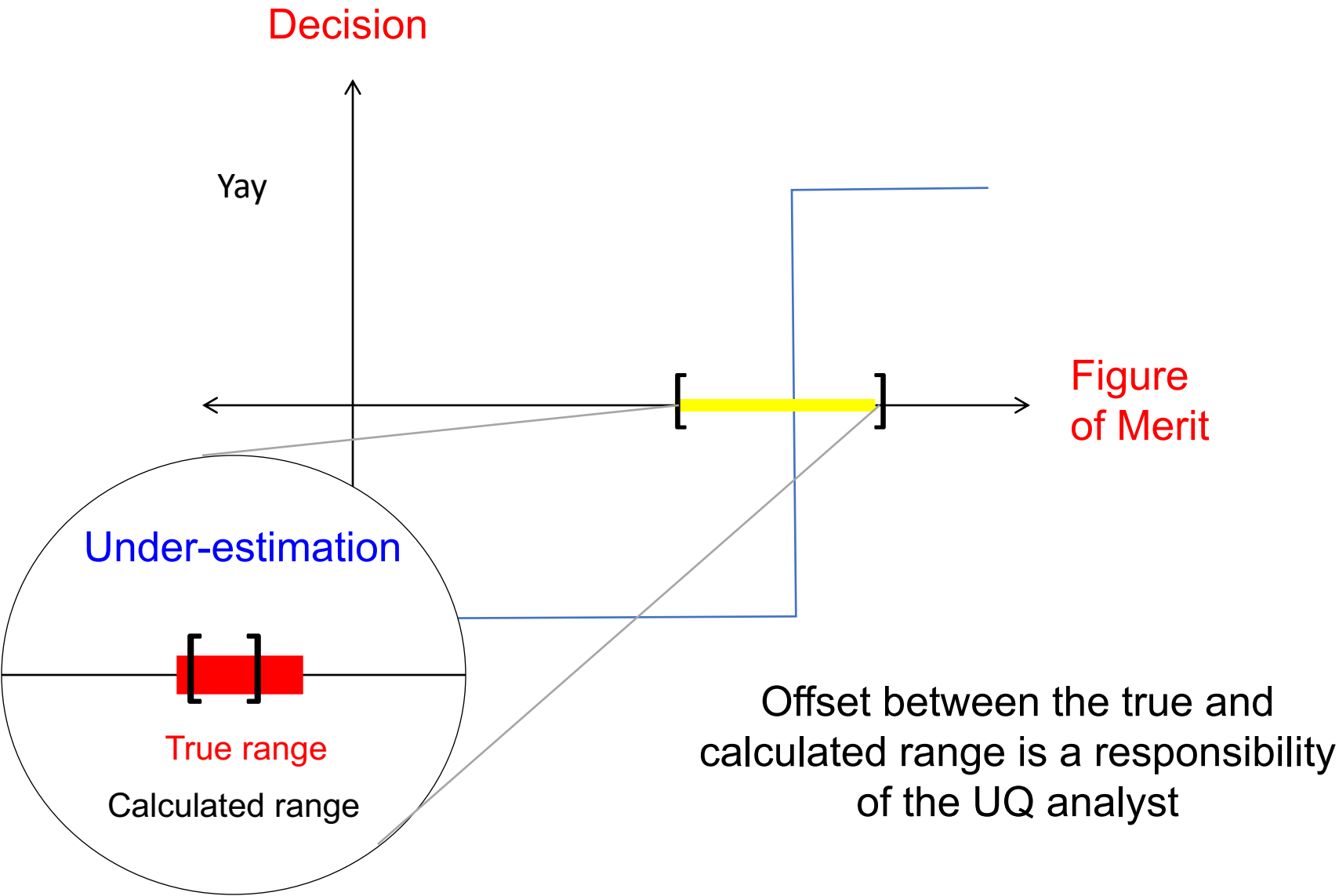
Decision Making with Epistemic Uncertainty



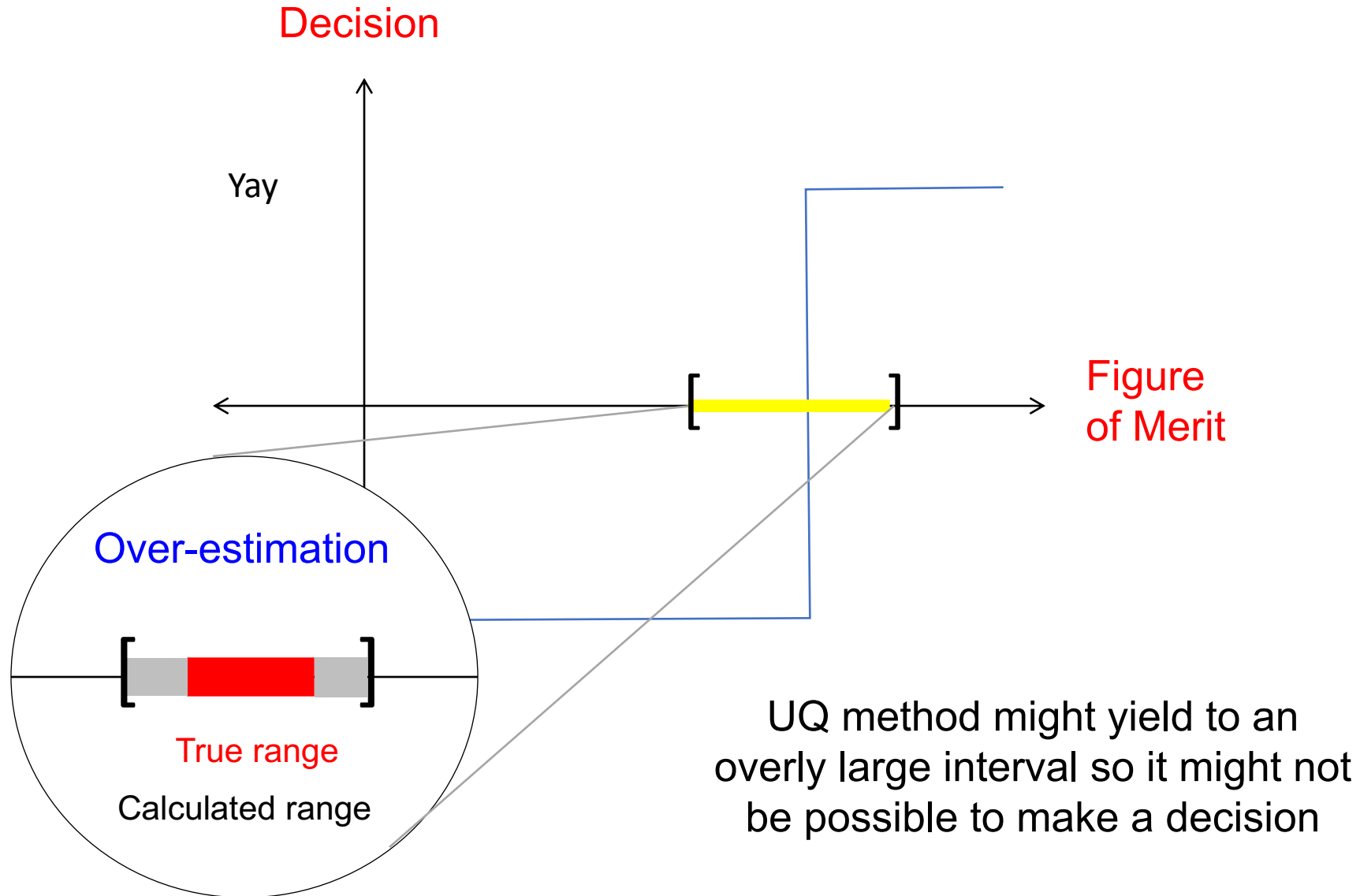
Decision Making with Epistemic Uncertainty



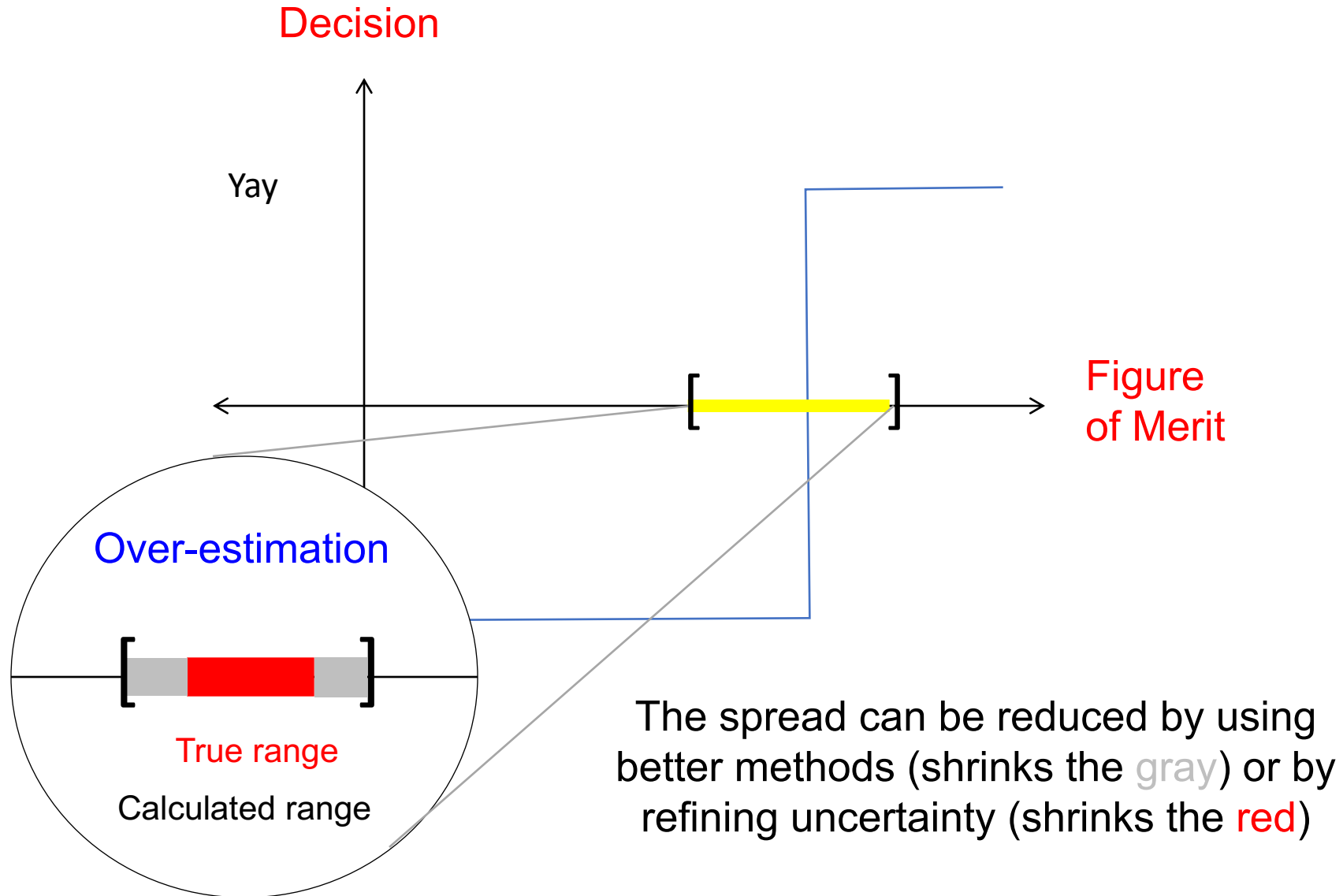
Decision Making with Epistemic Uncertainty



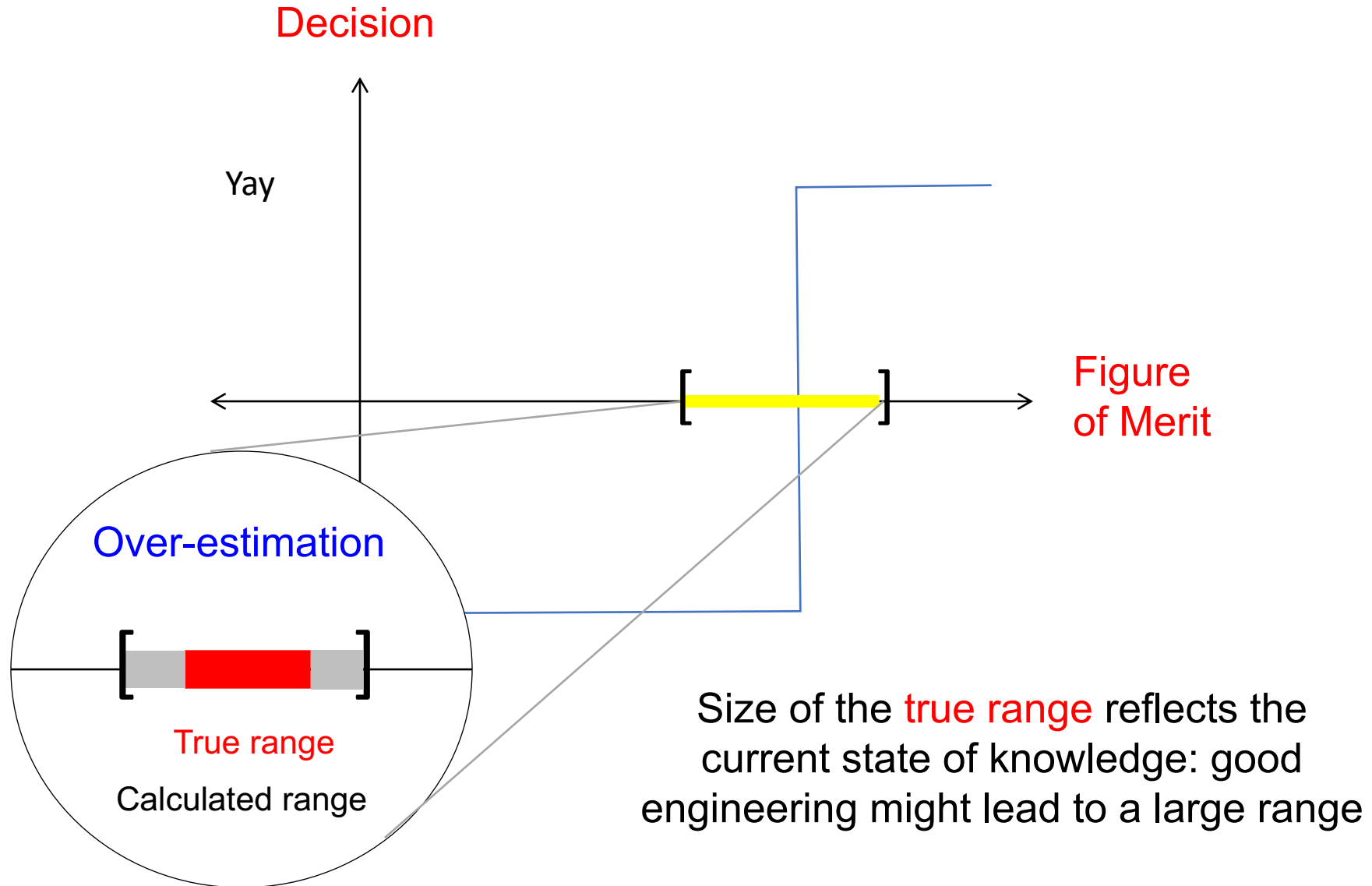
Decision Making with Epistemic Uncertainty



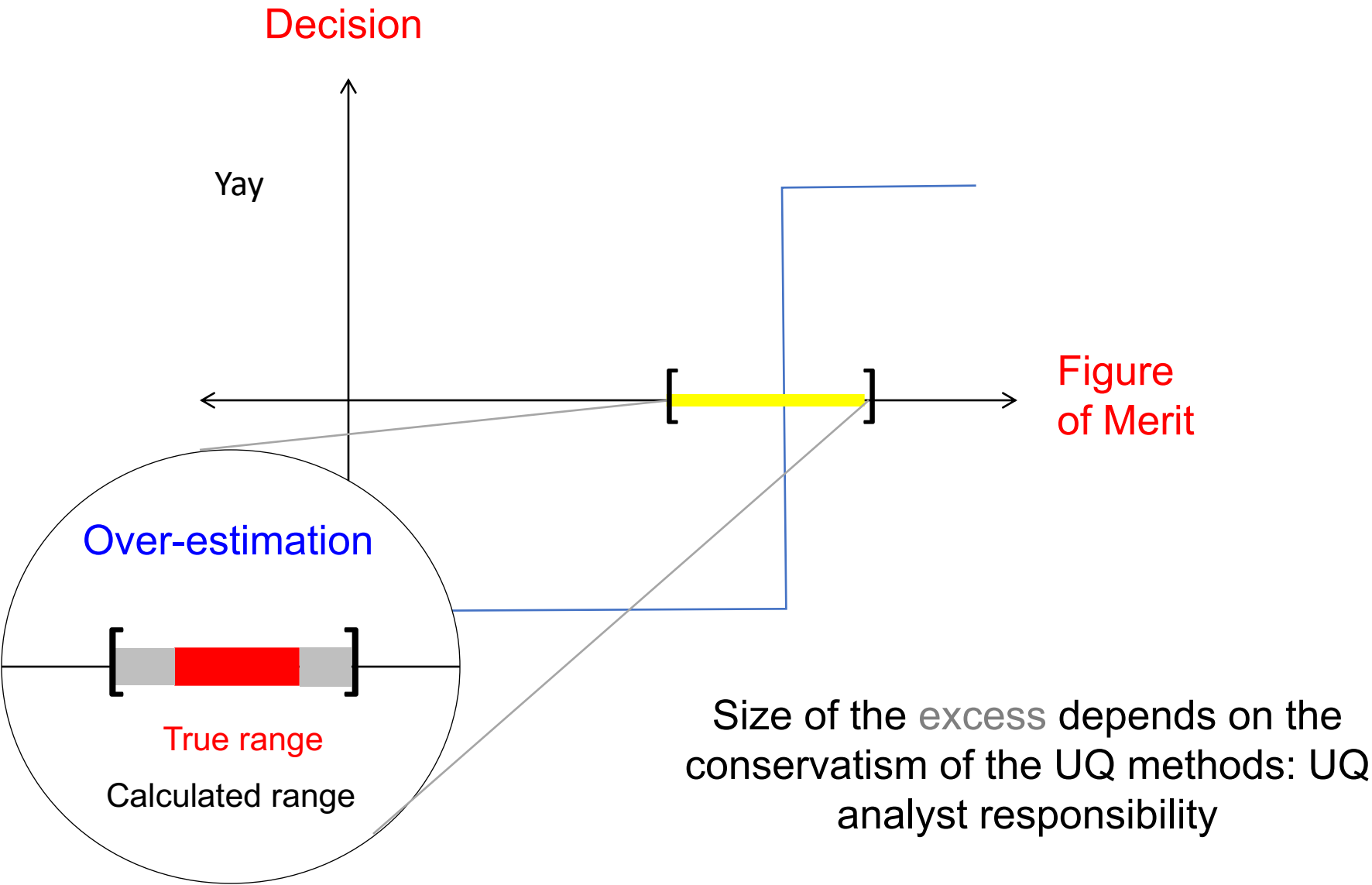
Decision Making with Epistemic Uncertainty



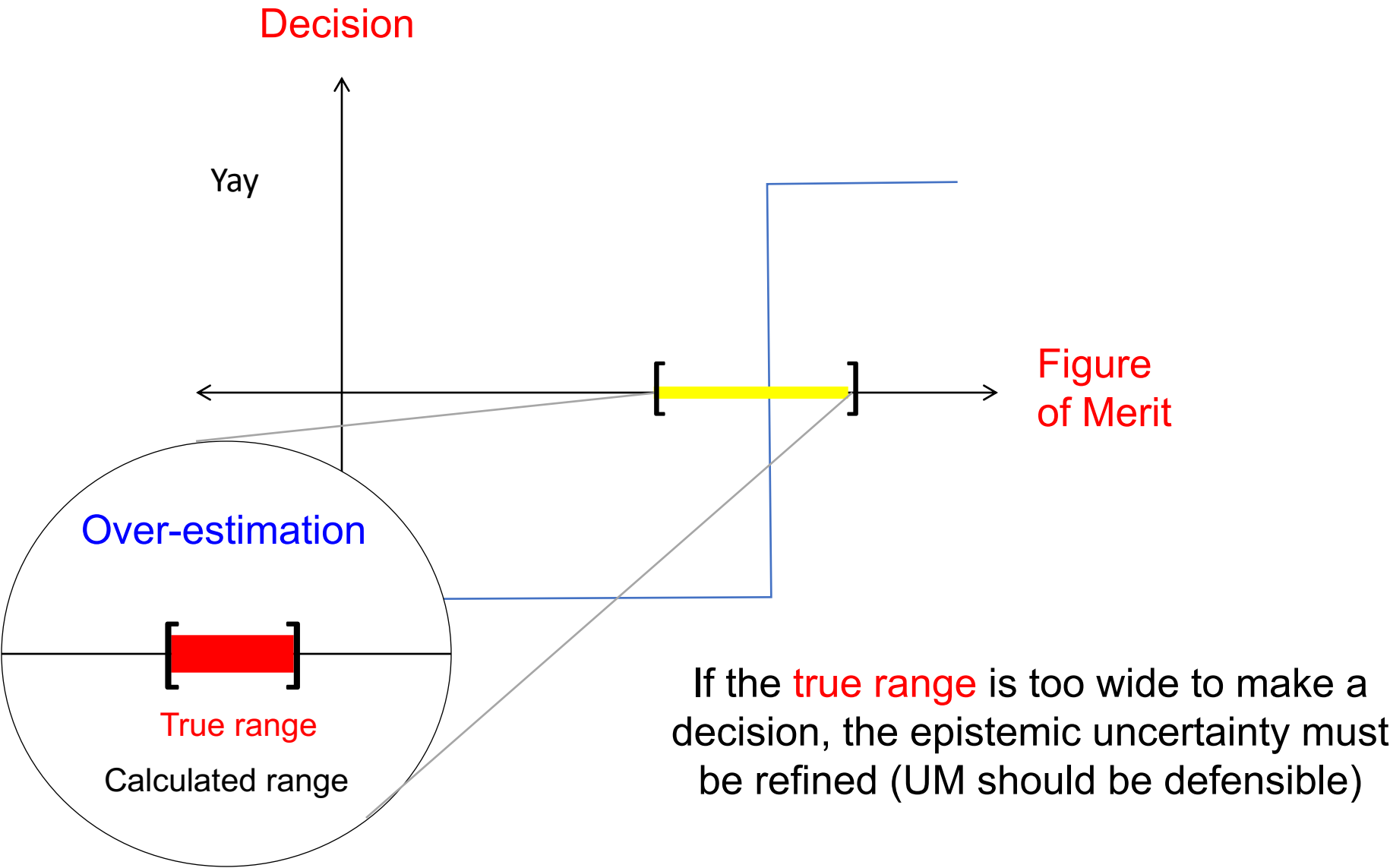
Decision Making with Epistemic Uncertainty



Decision Making with Epistemic Uncertainty



Decision Making with Epistemic Uncertainty



UQ Challenge: Lessons Learned

- State of knowledge might prevent making a decision
 - The UQ analyst does not need to apologize!!
- Not making a decision is preferable to making the wrong decision
- Decision maker should accept an inconclusive state
- Means to narrow the predicted uncertainty ranges
 - Refine epistemic uncertainty: resources required
 - Use optimization under uncertainty
- UQ analyst vs. decision maker
 - UQ analyst: ask for resources, know what to ask for
 - Decision maker: challenge the uncertainty model, provide resources

UQ Challenge: Lessons Learned

- UQ Method verification: perform convergence studies
- Global sensitivities for different metrics are different
- **Refine epistemic uncertainties in series not in parallel**
- Beware of the cascading effect: results depend on the methods, numerical setup and decisions made upstream: calibration strategy, sensitivities, chosen parameters, and propagation method, etc.
- **The error resulting from using a surrogate model should be quantified**
- A considerable spread in the results observed
- Key UQ needs are not fully addressed by existing methods
- Further R&D needed

UQ Challenge: Lessons Learned

- Underestimation of the uncertainty might lead to the wrong decision
- Overestimation of the uncertainty might prevent making a decision
- State of knowledge might lead to wide predicted ranges
 - The UQ analyst does not need to apologize for such an outcome!!
- Decision maker should accept an inconclusive state
- Means to narrow the predicted uncertainty ranges
 - Refine epistemic uncertainty: resources required
 - Use optimization under uncertainty
- Good engineering must have precedence over the urgency of making a decision