National Aeronautics and Space Administration

Notes on Digital-Data Analysis

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A Discussion of Concepts Normally Encountered During Digital-Signal Processing

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RULE # 1

FORGET EVERYTHING YOU KNOW ABOUT CONTINUOUS FOURIER TRANSFORMS!!!

Continuous Fourier And Laplace Transforms Are Great For:

- Understanding Analog Electronic Circuitry
- Some Aspects of Control Theory

But:

- Continuous Concepts Are More Likely Than Not To Lead You Astray When Dealing With Sampled Time Series or Trying to Understand Associated Issues

OUTLINE

Why Today's Discussion

- There Are Many Misconceptions Out There
- Some Are Caused by 'Outdated' Engineering Practices (Somewhat Incorrect Too)
- Some Are Caused by Intellectual Laziness

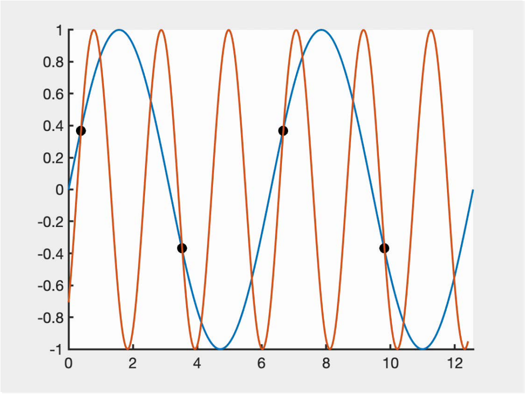
What Are We Going to Discuss

- Digital Sampling and Representation
- Aliasing or Frequency Folding
- Power Spectrum
- Windowing and Spectral Spreading
- Scalloping
- Best Practices

NYQUIST-SHANNON SAMPLING THEOREM

Sufficient condition for a sample rate, f_s, that permits a discrete sequence of equidistant samples to capture all the information from a continuous-time signal of finite bandwidth, B.

- $B < f_s / 2 = f_c$ – signal bandwidth must be less than the Nyquist frequency
- At least two sample points per period of highest frequency content
- The blue curve represents the highest pure harmonic that can be detected at the sample rate indicated by the black dots
- If present, a higher harmonic, red curve, could also pass through the sample points
- In that case, its 'power' will be misidentified as belonging to the blue harmonic – this is aliasing!



ALIASING – FREQUENCY FOLDING

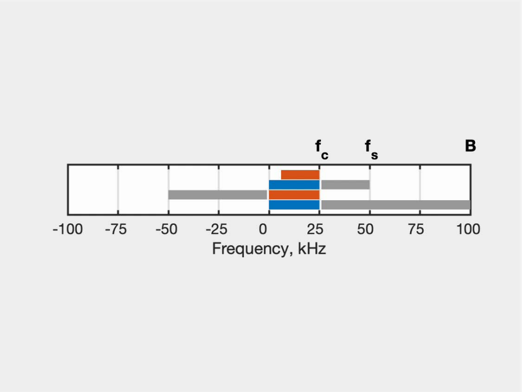
Stationary random signal u'(t) sampled at N equally spaced points

$$u'_m = u'(t_m)\ ,\quad t_m = m \triangle t\ ,\quad m = 0, 1, \ldots, N-1$$

- Sampling frequency $f_s = 1/ \triangle t$
- Frequency resolution $\triangle f = f_s / N$
- Nyquist, or cut-off, frequency $f_c = 1/ (2\triangle t)$
- Any frequencies $> f_c$ folds into $0 \leq f < f_c$

Frequency-Folding Illustration

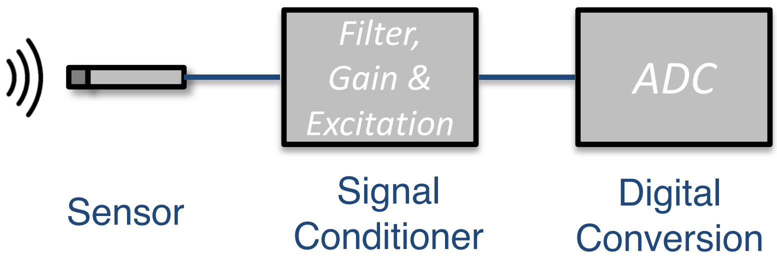
- A signal with 100 kHz bandwidth
- Incorrect 50 kHz sampling rate -> f_c = 25 kHz
- Grey portion of spectrum is folded and refolded
- Folding occurs both at $f = f_c$ and $f = 0$



ANTI-ALIASING ANALOG LOWPASS FILTERING

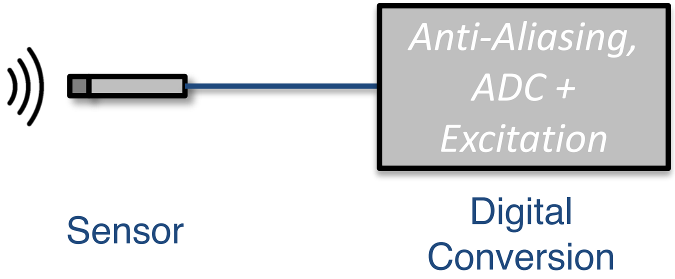
Analog Lowpass Filtering Must Be Applied Before A/D Conversion!!!

Old-School Approach



- External Signal Conditioning
- Lowpass-Filter Choice Difficult
  - Butterworth, Elliptical, or Bessel?
  - Gain transition, ripples, phase delay?
- Sample at $2.5 \times B$
- Disregard Top 20% of Spectrum

Modern Approach



- Internal Signal Conditioning
- Delta-Sigma ADC Architecture
- Large Oversampling
  - Analog-filter design is less crucial!
- Decimation with Sharp Digital Filters*
- 24-bit A/D Conversion

*in hardware

RULE # 2

MAKE SURE YOU UNDERSTAND ALIASING!!!

- Sometimes You Hear Dangerous Statements Like “the Microphone Preamplifiers Limits the Bandwidth So It’s Taken Care of”
  - This Is OK If Your Sample Rate Is Larger or Equal to Twice That Bandwidth
  - Or, If You Have Modern ADC Equipment That Automatically Filters Based on Requested Sampling Rate
- Time-Sequence Decimation Must Be Preceded by Digital Filtering
- CFD Produced Time Series Are a Bit of a Tricky Issue
  - The Numerical Dissipation of the Scheme, or the Eddy-Viscosity If Used, Normally Limits Aliasing – But Not Fully Eliminates It

DISCRETE FOURIER TRANSFORM (DFT)

- It Is Generally Considered Good Practice to First Remove the Mean

$$u_m \equiv u(t_m) = u'(t_m) - \frac{1}{N}\sum_{m=0}^{N-1} u'(t_m)$$

- Discrete Fourier Transform (or Finite Fourier Series) – Don’t Call It FFT!!!

$$U_n \equiv DFT\{u\}_n = \sum_{m=0}^{N-1} u_m \mathrm{e}^{\mathrm{i}2\pi nm/N}, \quad n = 0, 1, \ldots, N-1$$

- Inverse DFT

$$u_m = IDFT\{U\}_m = \frac{1}{N}\sum_{n=0}^{N-1} U_n \mathrm{e}^{-\mathrm{i}2\pi nm/N}, \quad m = 0, 1, \ldots, N-1$$

- Note that

$$U_{N+n} = U_n \quad \text{and} \quad U_{N-n} = U_n^* \quad \text{where} \quad {}^* = \text{complex conjugation}$$

PARCEVAL'S THEOREM AND POWER SPECTRUM

- It Is Pretty Straightforward to Obtain Parceval's Relation for the DFT – Respect It!

$$\sum_{m=0}^{N-1} u_m^2 = \frac{1}{N} \sum_{n=0}^{N-1} |U_n|^2$$

- Signal 'Power'

$$u_{rms}^2 = \left\langle \frac{1}{N} \sum_{m=0}^{N-1} u_m^2 \right\rangle = \frac{1}{N^2} \sum_{n=0}^{N-1} < |U_n|^2 > , \qquad < \ldots > = \text{ ensemble average}$$

- Power Spectrum (one-sided Auto-Spectrum)

$$G_0 = \frac{< |U_0|^2 >}{N^2} , \quad G_n = \frac{2 < |U_n|^2 >}{N^2} \quad \text{for} \quad n = 1, 2, \ldots , N/2-1 , \quad G_{\frac{N}{2}} = \frac{< |U_{\frac{N}{2}}|^2 >}{N^2}$$

- Note that

$$\sum_{n=0}^{N/2} G_n = u_{rms}^2$$

POWER SPECTRAL DENSITY (PSD)

- The Power-Spectrum Level Changes As the Number of Components Is Varied – The Total Power (Fixed) Is Re-Distributed Over a Different Set of Components
- For This Reason, the Power Spectral Density (PSD) Is Often Introduced
- The PSD Is Really a Concept That Belongs in the Continuous Realm – There, All Spectra Are Actually Spectral Densities
- In the Discrete World, the PSD Should Be Thought of As a Power Spectrum Renormalized to 1-Hz Binwidth
- Discrete Power Spectral Density:

$$PSD_n = \frac{2 < |U_n|^2 >}{N f_s} \quad \text{for} \quad n = 0, 1, \ldots, N/2$$

- Note that

$$\Delta f \left(\frac{1}{2} PSD_0 + \sum_{n=1}^{N/2-1} PSD_n + \frac{1}{2} PSD_{\frac{N}{2}} \right) = u_{rms}^2$$

RULE # 3

NEVER ASK COMMERCIAL SOFTWARE TO PRODUCE A POWER SPECTRUM!!!

- You Can NEVER Be Sure of What You Get! – I Will Explain Later
- Instead Set The Flag to Obtain the PSD
  - Convert to Power Spectrum Yourself
  - Check that Parseval's Relation Is Satisfied

GENERAL REMARKS ON POWER SERIES AND PSD

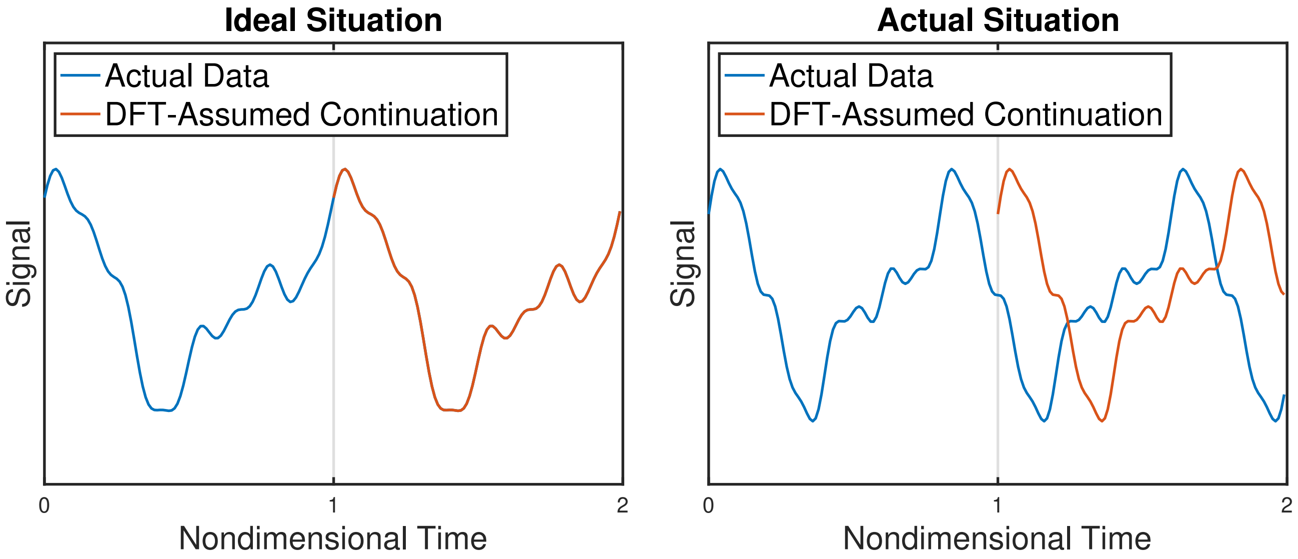
- Discrete PSD Is a Binwidth-Average Estimate of Continuous PSD

Dimensions/Units?

- In Signal Processing the Square of a Signal Is Traditionally Referred to as the Signal 'Power' – That Is, of Course, Not Strictly Correct.
- The Dimension of the Power Spectrum Is That of the Signal Squared
  - Voltage^2, (Length/Time)^2, Pressure^2, etc. – units: V^2, $(m/s)^2$, Pa^2, etc.
- PSD Units
  - V^2s, m^2/s, Pa^2s, etc.
  - dB/Hz is Incorrect Usage – It Describes a Slope in a Narrowband Spectrum

SPECTRAL LEAKAGE

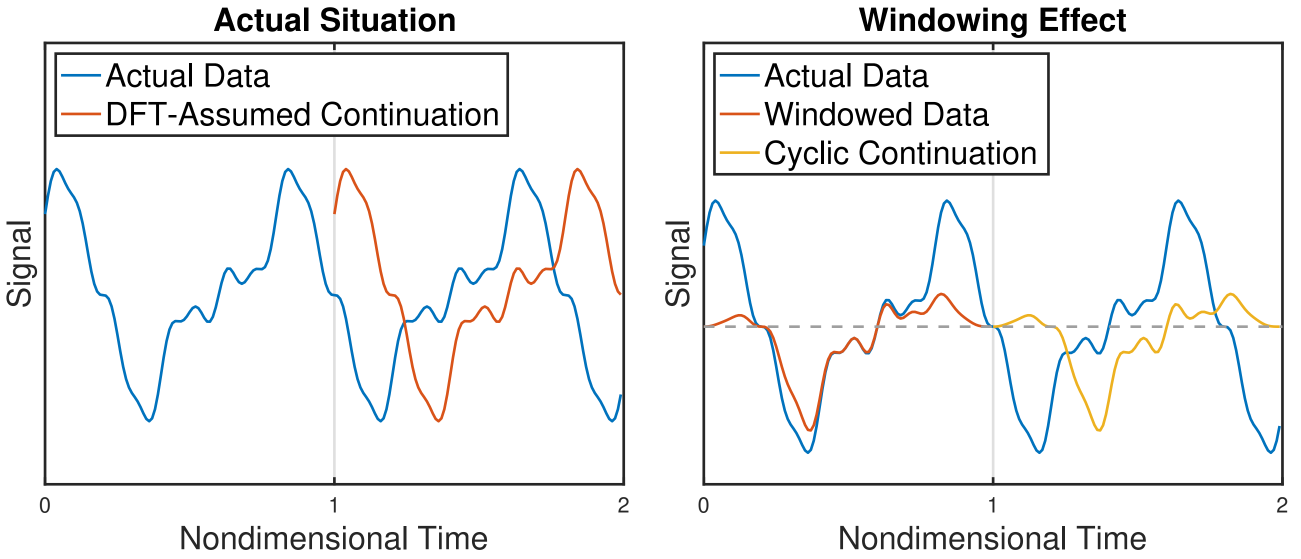
DFT Assumes That the Signal Is Periodic Over the Sampling Time



- Actual Signals Rarely Have the Required Periodicity
- The DFT Cyclic Continuation Then Has a Step Discontinuity
- Artificial Frequencies Then Fold Into the DFT-Frequency Range

WINDOWING

Spectral Leakage Caused By Finite-Time Records Can Be Reduced By Windowing



- DFT of the Product of Window and Original Sequence

$$V_n \equiv DFT\{wu\}_n = \frac{1}{N}\sum_{k=0}^{N-1} W_k U_{n-k}$$

- Equivalent to Filtering in the DFT Space

WINDOWING EFFECTS ON THE SIGNAL POWER

- Windowing Reduces the Signal Amplitude -> Power Is Reduced

$$v_{rms}^2 \equiv \frac{1}{N^2} \sum_{n=0}^{N-1} < |V_n|^2 >= \left(\frac{1}{N} \sum_{m=0}^{N-1} w_n^2 \right) u_{rms}^2$$

- The Factor (…) Is the Window Incoherent-Power Gain, $IPG_w \leq 1$
- The PSD of the Original Time Sequence Is Then (Account for Gain)

$$PSD_n = C_w \frac{2 < |V_n|^2 >}{N f_s}, \qquad n = 0, 1, \ldots, N/2, \qquad \text{where} \qquad C_w = \frac{1}{IPG_w}$$

- Parseval's Relation is Satisfied!
- Commercial Software Usually Does This Power Correction Internally

WINDOW EXAMPLES – COSINE BASED

$$w_m = \alpha - (1-\alpha)\cos\left(\frac{2\pi m}{N}\right), \qquad m = 0, 1, \ldots, N-1, \qquad IPG_w = \alpha^2 + \tfrac{1}{2}(1-\alpha)^2$$

- Box Car (No Tapering): $\boldsymbol{\alpha}$ = 1 –> IPG_w = 1
- Hann (Julius von Hann): $\boldsymbol{\alpha}$ = ½ –> IPG_w = 3/8 = 0.375
- Hamming (Richard W. Hamming): $\boldsymbol{\alpha}$ = 0.54 –> IPG_w = 0.3974
- Hann and Hamming Windows – Weighted Sum of the Original-Sequence DFT Coefficient and Its Two Nearest Neighbors

$$V_n = \alpha U_n - \tfrac{1}{2}(1-\alpha)(U_{n-1} + U_{n+1})$$

- There Is No Such Thing As a *Hanning* Window!!!

WINDOW EXAMPLES – KAISER-BESSEL

$$w_m = \mathrm{I}_0 \left[\pi\alpha \sqrt{1 - (2m/N - 1)^2} \right] \Big/ \mathrm{I}_0(\pi\alpha) \qquad \text{for} \qquad m = 0, 1, \ldots, N-1$$

- I_0 Is the Zeroth-Order Modified Bessel Function of the First Kind
- The Parameter $\boldsymbol{\alpha}$ Allows Tuning For Specific Situations
- As an Example, $\boldsymbol{\alpha} = 3$ –> IPG_w = 0.2909087

<u>MATLAB Window Functions (v. 2019b)</u>

Box-Car: *w* = rectwin(*N*)

Hann: *w* = hann(*N*, ‘*periodic*’), Hamming: *w* = hamming(*N*, ‘*periodic*’)

Kaiser-Bessel: *w* = wsym(*1:N*) where *wsym* = kaiser(*N+1*,$\boldsymbol{\alpha}$)

TRULY-PERIODIC PURE-TONE CONTRIBUTIONS

- Pure Tone With (Integer) Index *k* – Periodic Over Sampling Time

$$u_m \equiv u(t_m) = A\cos(2\pi km/N) \qquad m = 0, 1, \ldots, N-1 \qquad 0 < k < N/2$$

$$U_n = \tfrac{1}{2}AN\delta_{nk}$$

$$V_n = \tfrac{1}{2}AW_{n-k} \qquad n = 0, 1, \ldots, N-1$$

- One-Sided Power Spectrum (Satisfying Parseval's Relation)

$$G_n = \frac{1}{2} < A^2 > \frac{|W_{n-k}|^2}{N\sum_{m=0}^{N-1} w_m^2} \qquad n = 1, 2, \ldots, N/2-1 \qquad \text{where} \qquad \frac{1}{2} < A^2 > = u_{rms}^2$$

- At the index k

$$G_k = \frac{1}{2} < A^2 > \frac{\left(\frac{1}{N}\sum_{m=0}^{N-1} w_m\right)^2}{\frac{1}{N}\sum_{m=0}^{N-1} w_m^2} \qquad \text{where} \qquad (\ldots)^2 = CPG_w$$

Coherent-Power Gain

COSINE-BASED-WINDOW EFFECTS ON PURE TONES

- Cosine-Based Window Functions and Discrete Tones
- Discussion Only For Tones Periodic Over Sampling Time
- One-Sided Power Spectrum

$$G_k = \tfrac{1}{2} C_w \alpha^2 < A^2 > \qquad G_{k\pm1} = \tfrac{1}{8} C_w (1-\alpha)^2 < A^2 > \qquad \text{all other } G_n = 0$$

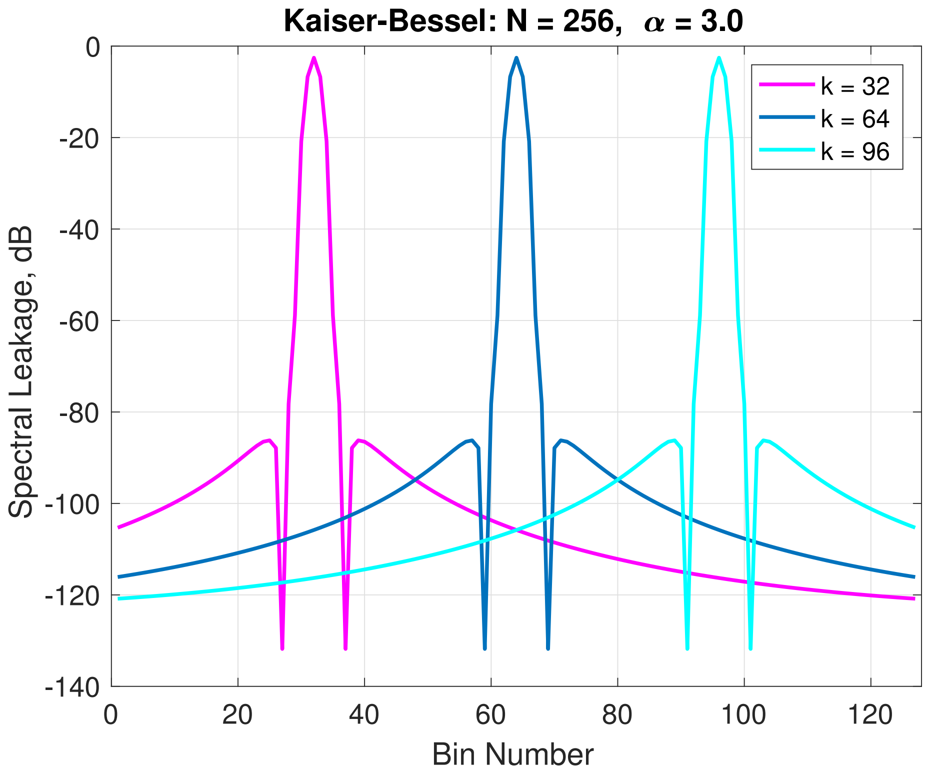
- Parceval's Relation Is Satified

$$G_{k-1} + G_k + G_{k+1} = \tfrac{1}{2} < A^2 > \equiv u_{rms}^2$$

- Here, Spectral Leakage Into the Two Nearest Side Frequencies
- Summing Relevant Power-Spectrum Components Recovers the Power

KAISER-BESSEL WINDOW EFFECT ON PURE TONES

- Again, Result Applies Only For Tones Periodic Over Sampling Time
- Spectral Leakage Into All Frequencies



‘TONE-CORRECTED’ POWER SPECTRUM

The Basic Idea That Engineers Came Up With Was:

- For a Given Window, the Power Loss for Any Tone Is Known
- Why Not Just Multiply the Power Spectrum With a Correction Factor

$$G_n^{(tc)} = B_w G_n \qquad n = 0, 2, \ldots, N/2 \qquad \text{where} \qquad B_w = \frac{IPG_w}{CPG_w}$$

Ratio of Incoherent to Coherent Gain

- B_w Can Be Thought of As an Effective Binwidth

Problems:

- Parseval’s Relation Is Violated – Power Is Not Preserved
- Broadband Level Is Bumped Up
- Leaked Tonal Power Is Not Removed But Increased Instead
- Commercial Software Often Applies This Correction to Power Spectra

SCALLOPING

- Tones Are Almost Never Perfectly Periodic Over The Data Segment

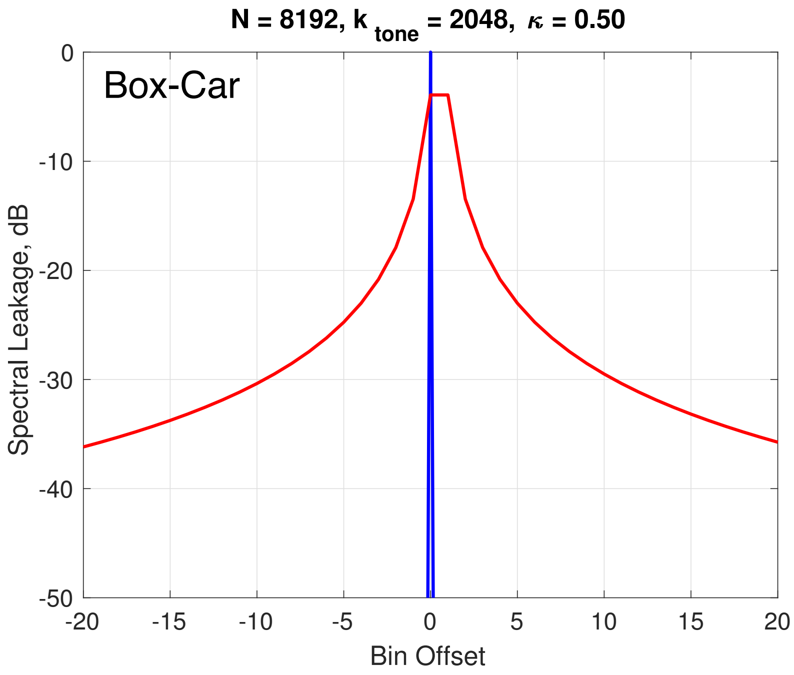
$$u_m \equiv u(t_m) = A\cos[2\pi(k+\kappa)m/N) \qquad m = 0, 1, \ldots, N-1 \qquad 0 < k < N/2 \qquad |\kappa| \le \tfrac{1}{2}$$

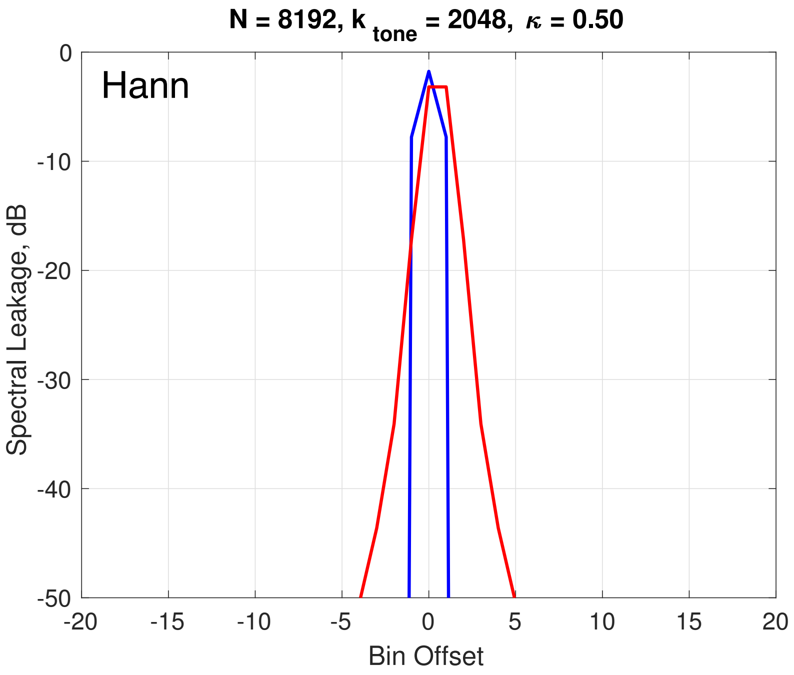
$$U_n \equiv DFT\{u\}_n = \tfrac{1}{2}A\left[D_N(x_+) + D_N(x_-)\right] \qquad n = 0, 1, \ldots N-1$$

$$D_N(x) = \frac{\sin(Nx/2)}{\sin(x/2)}\,\mathrm{e}^{\mathrm{i}(N-1)x/2} \qquad x_\pm = 2\pi\left\{[n \pm (k+\kappa)]/N\right\}$$

- Contributions to All Frequency Bins (or DFT Components)
- This Negates the Whole Concept of Simple Tone Correction Using B_w
- Scalloping Loss of Tone Amplitude Power Is Just Spectral Leakage
- Largest Tone Scalloping Loss for $|\boldsymbol{\kappa}| = ½$

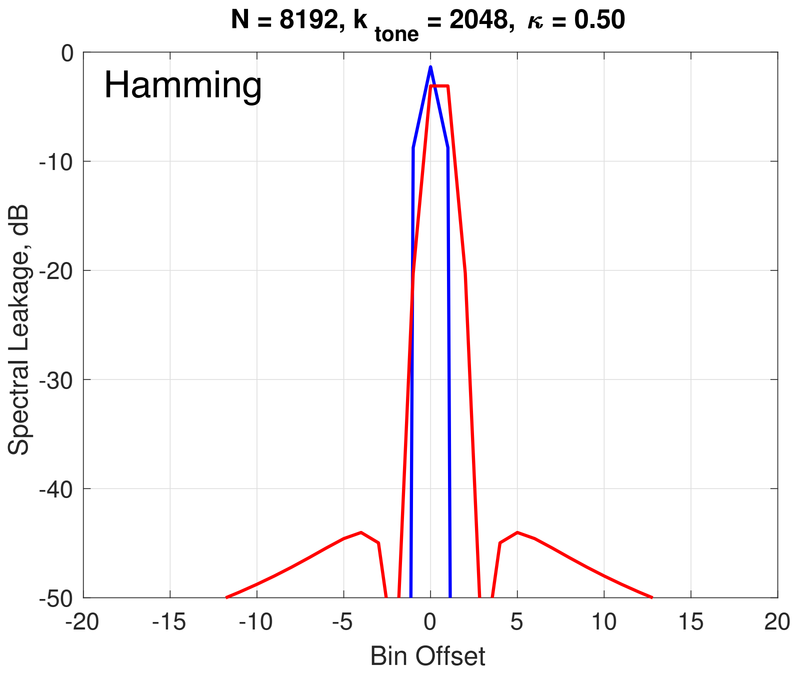
SCALLOPING EXAMPLES

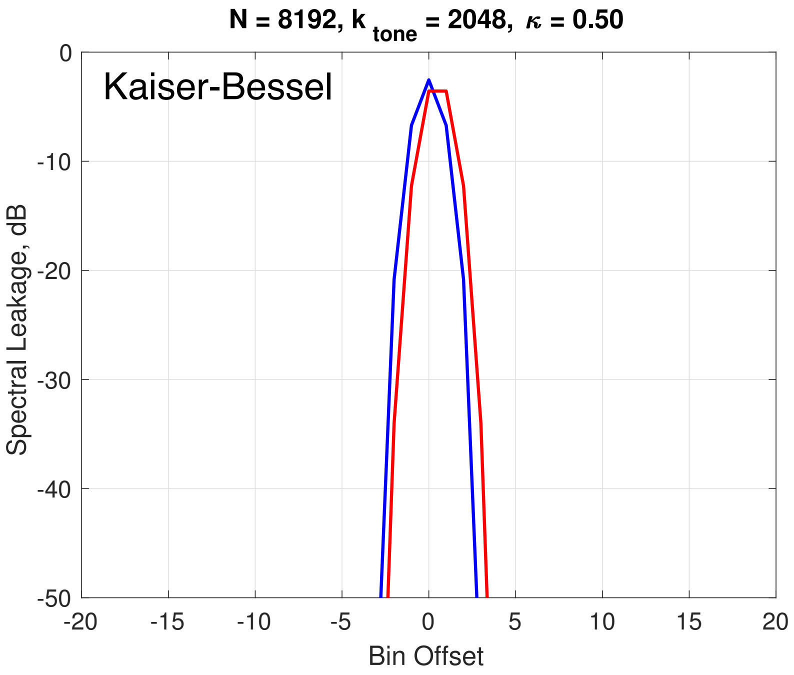




- Box-Car and Hann Window Scalloping for ***κ*** = 0 (none) and ***κ*** = ½ (worst case)

SCALLOPING EXAMPLES





- Hamming and K-B (***α*** = 3) Scalloping for ***κ*** = 0 (none) and ***κ*** = ½ (worst case)

WORST-CASE PROCESSING LOSS (TONES)

- Maximum Scalloping Power Loss for Tones

$$SL_{max} = |V_{n=k}^{(\kappa=\frac{1}{2})}|^2 \Big/ |V_{n=k}^{(\kappa=0)}|^2$$

- Worst-Case Processing Loss for Tones – Combined B_w and SL_{max}

Window	PSD Correction C_w	Tone-Effective Binwidth B	$10\log_{10} B$, dB	Max Scalloping Loss SL_{max}	$-10\log_{10} SL_{max}$, dB	Worst-Case Processing Loss $10\log_{10}(B/SL_{max})$
Box-Car	1	1	0	0.405285	3.9224	3.9224
Hann	8/3	1.5	1.7609	0.720506	1.4236	3.1845
Hamming	1/0.3974	1.362826	1.3444	0.668124	1.7514	3.0958
Kaiser-Bessel, $\alpha = 3$	3.437505	1.795235	2.5412	0.790198	1.0226	3.5639

- Box-Car: The Largest Scalloping Loss & Worst-Case Processing Loss
- Kaiser-Bessel: The Smallest Scalloping Loss
  - If Only Interested In a Single Well Separated Tone
    -> Use Kaiser-Bessel and Apply Tone Correction
- Otherwise Hamming or Hann Are Better Choices for Window Function

SOME ADDITIONAL OBSERVATIONS

- Generally a Window Function With A Good Dynamic Range Is Needed
  - Detect Weak Tones In the Presence of a Strong Tone When No Tones Are Too Close
  - Hamming, Hann and Kaiser-Bessel Are Good Choices
- Good Spectral Resolution Is Needed If Some Tones Are Close in Frequency
  - A Wider Main Processing Spectral Lobe Is Worse
  - Hamming, Hann Are Reasonable Choices – Kaiser-Bessel Possibly Can Be Optimized

OPINION

- The Hamming Window Presents a Good Compromise for Real-Data Requirements – Particularly If Broadband Signals Are Of Interest
- An Efficient and Practical Way of Estimating a Tone Level Might Simply Be To Add Next Nearest Values to the Peak Value

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