

Stability Analysis of the Flow over a Swept Forward-Facing Step using PIV Base Flows in a Nonorthogonal Coordinate System

Koen J. Groot^{*,1} and Jenna L. Eppink^{†,2}

¹*Texas A&M University, College Station, TX 77843*

²*NASA Langley Research Center, Hampton, VA 23681*

Understanding the flow of a crossflow-vortex-dominated boundary layer over a forward-facing step excrescence is necessary in order to mitigate an early transition scenario on the wings and tail of commercial aircraft. By performing BiGlobal stability analysis on the flowfield measured with high-resolution, stereographic Particle Image Velocimetry (PIV), previous work has shown that a family of unstable disturbances exists in the direct downstream vicinity of a supercritical, forward-facing step. The goal of the present paper is to improve the previously used stability approach on two fronts: 1) the stability problem is formulated in a nonorthogonal coordinate system and 2) the contribution of the out-of-plane base-flow derivatives is accounted for locally. The resulting eigensolutions feature significantly stronger growth rates and eigenfunctions that are localized above regions of reverse flow. The changes in the problem formulation furthermore establish a significant improvement in the comparison of the stability solutions with the results from the Spectral Proper Orthogonal Decomposition (SPOD) of a time-resolved measurement of the perturbation content. A large destabilizing effect by the out-of-plane base-flow derivatives is determined to be justified, despite the assumption that these derivatives are small, because the effect can be reconstructed upon using an eigenvalue-correction formula that assumes the responsible terms are infinitesimally small. The unstable perturbation mechanisms are demonstrated to have a convective nature by assessing the relation between the frequency and the out-of-plane wavenumber, which indicates that their group speed does not approach zero. The last 2 facts, that 1) the perturbations are convective and 2) that the out-of-plane base-flow-derivative terms are small, remove any qualitative suspicions that the perturbation problem is not conducive to parabolization. This opens the path to analyzing this problem with a plane-marching approach.

I. Introduction

SURFACE roughness can yield a large contribution to the drag of commercial aircraft by inciting laminar-turbulent transition. A particular kind of roughness is the step excrescence, which results if the junction between skin panels is not perfectly flush. In flowing over a step, the boundary layer is strongly influenced and, if the step height exceeds the critical value for the given conditions, laminar-turbulent transition is induced at the chordwise location of the step. The literature [1, 2] indicates that the critical step height for backward-facing steps is smaller than that for forward-facing steps in subsonic flow, due to the former causing a larger laminar separation length. Hence, a forward-facing step permits a larger manufacturing tolerance.

Commercial aircraft have swept wings and tail surfaces in order to reduce wave drag. If the sweep angle is sufficient, stationary crossflow vortices emerge as a consequence of the (primary) crossflow instability to the undisturbed laminar base flow. This instability starts to produce these vortices onward from a position very close to the leading edge [3]. Hence crossflow vortices can attain a large amplitude at the location where the flow encounters steps. As a consequence, the flow of crossflow vortices over a forward-facing step is a typical configuration that must be understood in order to mitigate an early transition scenario on a commercial aircraft's wings and tail.

Previous work by the community has yielded significant progress for this particular flow configuration, the crossflow-dominated boundary-layer flow over a forward-facing step. Upon testing a curved airfoil in flight and the Klebanoff-Saric wind-tunnel at Texas A&M, Tufts et al. [4] identified a main culprit driving forward-facing-step-induced transition on

^{*}Assistant Research Scientist, Department of Aerospace Engineering, Member AIAA

[†]Research Aerospace Engineer, Flow Physics and Control Branch, M.S. 170, Member AIAA

swept wings. They discovered a correlation between the critical step height and the height of the vortex core of the crossflow vortices. In particular, transition was found to be caused if the step height exceeds the height of the vortex core of the incoming crossflow vortex. This is referred to as the ‘Tufts-Reed criterion’. Further investigations have led to both further refinements and the identification of limitations of the criterion. In the flat-plate experiments by Eppink [5] in the 2-Foot by 3-Foot Low Speed Boundary-Layer Channel at NASA Langley Research Center, the critical step height predicted by this criterion was observed to be approximately 15% higher than the actual critical step height found in the experiment, i.e., demonstrating the criterion is conservative under the considered conditions. This same campaign demonstrated that an increased crossflow-vortex amplitude resulted in premature transition for a step height that the Tufts-Reed criterion would otherwise predict to be subcritical. Considering a curved airfoil in the Low-Turbulence Tunnel at TU Delft, Rius Vidales et al. [6] show that the limit between the subcritical- and critical-step-height regime agreed well with the Tufts-Reed criterion. Expanding their study, Rius Vidales & Kotsonis [7] indicate that local stability properties of the crossflow vortices have a further contribution that needs to be accounted for in correlations.

Despite the increasing body of insights surrounding the Tufts-Reed criterion, a question that remains open is *how* the step-induced distortion of the crossflow vortices encourages transition in the portion of the flow downstream of the step if the step height is supercritical. Eppink [5, Fig. 21] observes the emergence of high-frequency perturbations downstream of the step. This unsteady content is observed to be concentrated around the region of maximal shear [8, Figs. 25 & 26], which hints that these perturbations emerge as a consequence of an instability mechanism.

The overarching aim of the present study is to characterize the stability characteristics of the flow immediately downstream of a forward-facing step subject to a boundary layer with crossflow vortices. The authors’ previous work, see Groot and Eppink [9], indicates that a reasonable match can be established between the unsteady perturbation content observed in the experiment and the results of the spanwise BiGlobal stability analysis of the measured base flow, which accounts for all linear perturbation dynamics in a plane. In the present paper, the formulation of the stability approach is improved by:

- 1) making use of a nonorthogonal coordinate system, which reduces the magnitude of the out-of-plane base-flow derivatives; and
 - 2) (locally) accounting for the contribution of the out-of-plane base-flow derivatives.
- Both aspects reduce the modeling error committed in the formulation of the stability problem.

II. Measured base flow

This study focuses on the experimental measurement data by Eppink [5, 8]. A summary of the experimental setup is shown in Fig. 1. The model represents a flat plate at a sweep angle $\Lambda = 30^\circ$. A three-dimensional pressure body was used to introduce a favorable pressure gradient in order to activate the crossflow instability mechanism. To fix the crossflow vortices’ receptivity, Discrete Roughness Elements (DREs) were applied with a spanwise spacing of $\lambda_r = 11$ mm parallel to the leading edge. It was carefully demonstrated that the model supports a constant pressure distribution in the direction parallel to the leading edge [8, Fig. 2]. Hence the infinite-swept-wing assumption can be adopted for the undistorted base flow, i.e., the boundary-layer flow without the presence of crossflow vortices. In order to comply with the imposed spanwise invariance, the crossflow vortices are constrained to be approximately periodic in the direction parallel to the leading edge. In what follows, the steady flow *distorted by the crossflow vortices* will be referred to as the base flow. Quantities that specifically relate to the base flow are denoted with an overbar.

The model consists of two pieces that can be moved with respect to each other to form the step excrescence. The step is parallel to the leading edge and located 354 mm downstream of the leading edge as measured in the direction orthogonal from the leading edge. The present paper is focused on the case with a step height of $h = 1.7$ mm, which was observed to be supercritical. In Fig. 1, all streamwise locations and distances are indicated in the direction parallel to the freestream, $\bar{Q}_\infty$.

Several coordinate systems are defined in order to describe the flowfield. The origin for overbarred coordinates (i.e., the $\bar{x}\bar{z}$ - and $\bar{x}_\perp\bar{z}_{||}$ -combinations) is located at the leading edge, while coordinates without an overbar (i.e., the xz - and $x_\perp z_{||}$ -combinations and y) have their origin at the location of the step, see the symbol O in Fig. 1. The y -coordinate is normal to the plate. The coordinates z , $z_{||}$, and $\bar{z}_{||}$ are all parallel to the step (and the leading edge) and point outboard, while only $x_\perp$ and $\bar{x}_\perp$ are orthogonal to the step. The x -coordinate points in the direction of the freestream: $x \parallel \bar{Q}_\infty$. It is important to note that the xyz -system is not orthogonal. The difference between z and $z_{||}$ is that the former equals 0 along the x -axis, while the latter is zero along the $x_\perp$ -axis, see Fig. 1. Please see Fig. 10 in Appendix A for a detailed illustration of the relations between x , z , $x_\perp$, and $z_{||}$ and related partial derivatives.

The flow fields are measured through stereographic particle image velocimetry (stereo-PIV). In particular, all ($x_\perp$, y ,

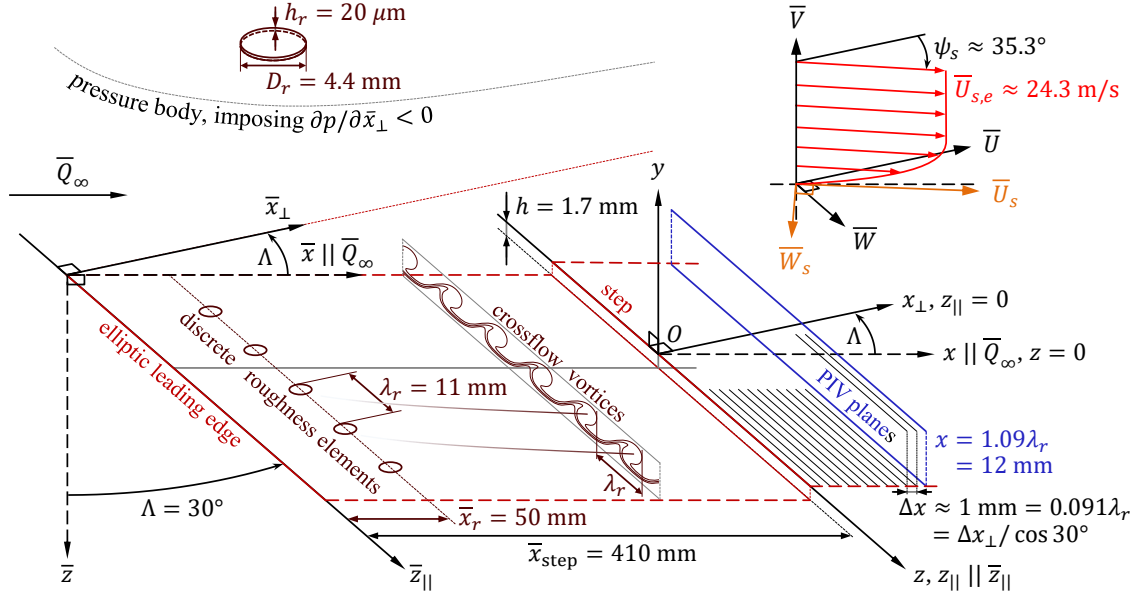


Fig. 1: Overview of the experimental swept-flat-plate setup (not to scale).

and $z_{||}$) velocity components are obtained in the zy -planes indicated with the blue lines in Fig. 1. That is, in moving the measurement plane in the freestream direction, it is moved along the ‘freestream’ x -direction, along lines of constant z . The $z_{||}$ -coordinate increases when tracing lines of constant z downstream. The xyz -system yields the most natural description of the flow field, because the crossflow vortices are 1) approximately periodic in the z -direction and 2) approximately aligned with the x -direction. As a consequence of the alignment of the x -coordinate, the variation of the base-flow quantities in the x -direction is relatively small; especially as opposed to the variation in the $x_{\perp}$ -direction. In the direction parallel to the step, the measurement domain spans more than 5 crossflow vortices. In the x -direction, the distance from one measurement plane to the next is approximately $\Delta x \approx 1$ mm ($\Delta x_{\perp} \approx 0.866$ mm).

Lengths will be scaled with $\lambda_r = 11$ mm and velocities with 24.3 m/s, where the latter approximates the local velocity of the inviscid flow in the direction of the inviscid streamline, $\bar{U}_{s,e}$, directly downstream of the step. Around that same location, the angle between the inviscid streamline and the leading-edge-orthogonal direction is approximately $\psi_s = 35.3^\circ$. The Reynolds number based on the length and velocity scale, $\bar{U}_{s,e}\lambda_r/\nu$, equals 1.66×10^4 .

The present paper focuses on the case where 16×16 pixels were used per interrogation window for the cross-correlation algorithm used to obtain the raw velocity vectors. The resolution of the resulting raw velocity fields in both the wall-normal and spanwise direction equals $\Delta y = \Delta z = 7.12\lambda_r \times 10^{-3} = 0.0783$ mm. Note that the resolution in the zy -plane is very high in proportion to the 1 mm-resolution in x . All other relevant parameters related to the PIV measurement and the conversion of the mean velocity fields to the input for the stability problem (the treatment of the wall, the differentiation approach, etc.) are detailed by Groot and Eppink [9]; the same parameters are used here.

The typical shape of the shear layer is shown in Fig. 2(a) in terms of the $\bar{U}_s$ -component, which is the flow velocity in the direction of the inviscid streamline. Isocontours of $\bar{U}_s$ are shown as the solid gray-scale lines; the dashed black contour represents $\bar{U}_s = 0$. These contours do not resemble the typical shape of crossflow vortices. Usually, those vortices are recognized in the shape of the 80% to 90% contours with respect to $\bar{U}_{s,e}$. Their shape is sketched in Fig. 1. Here, instead, the contours close to the wall are curved the most and follow the outline of localized regions of reverse flow ($\bar{U}_s < 0$ below the dashed black contours). The primary features, the largest shear magnitude and the shape of the shear layer, are confined to the region below δ_{50} . High-amplitude crossflow vortices themselves typically reach to about $2\delta_{99} > 4\delta_{50}$. As the localized regions of reverse flow are traced downstream, the shear layer evolves into localized pockets above the reverse-flow regions.

III. Stability Approach

Stability analysis relies on using an accurate representation of the steady base flow, because it is used as the input for the stability problem. Due to the use of a measured base flow, the access is presently limited to the mean of instantaneous vector fields. This is allowable under the hypothesis that the difference between the base and mean flow becomes negligible as the ensemble size for the mean is increased; the ‘mean = base flow’-hypothesis. It has been demonstrated

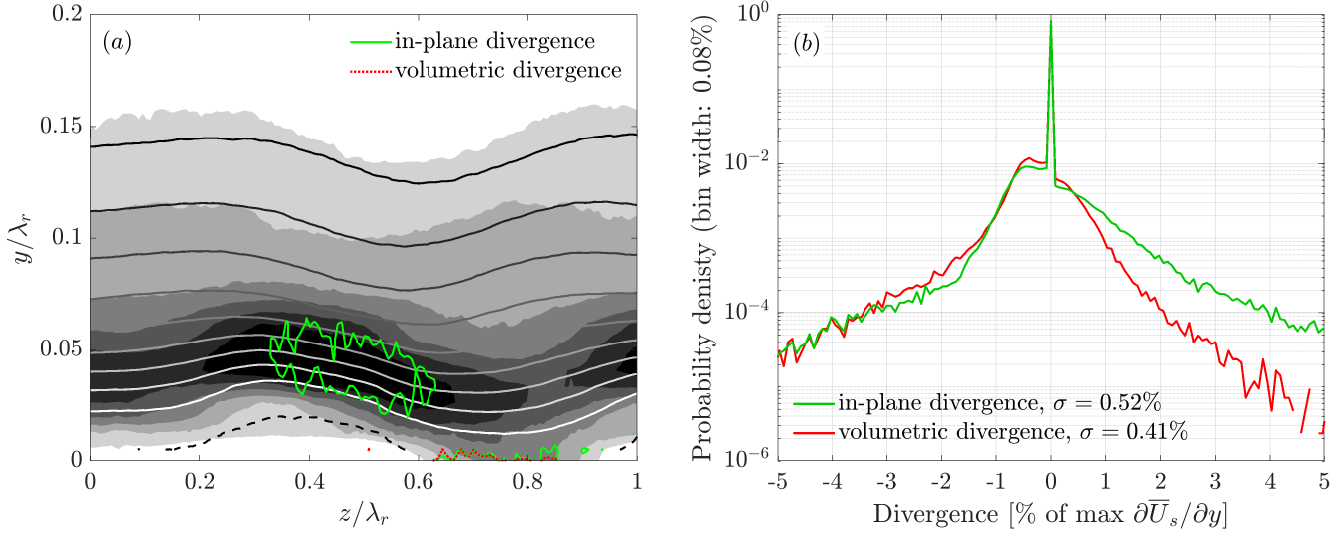


Fig. 2: Divergence of the base-flow velocity field (in-plane: ∇_{zy} ; full (volumetric): ∇ , see Eq. (3)) at $x = 4$ mm. (a) Distribution in zy -plane (red and green contours, $+5\%$ of $\max \partial \bar{U}_s / \partial y$); properties of inviscid-streamline velocity component, $\bar{U}_s$: values (gray solid contour lines, 0.1, 0.2, ..., 0.9 of $\bar{U}_{s,e}$; $\bar{U}_s = 0$ is indicated by the black dashed contour) and wall-normal derivative (filled contours, 2, 4, ..., 12 units of $\bar{U}_{s,e} / \lambda_r$). (b) Occurrence histogram of divergence values (evaluated for $y / \lambda_r < 0.2$ and for all z in the measurement domain, σ : standard deviation).

before that this assumption can yield success [9], even if the amplitude of unsteady perturbations is larger than what can reasonably be taken as representative of linear dynamics. What is most important is that all relevant mean-flow features are properly accounted for [10–12]. In the present case, the experimentally observed perturbation amplitude is relatively large in the region of interest. The root-mean-square of the velocity component in the direction parallel to the step, $w'_{||}$, attains the largest values. In the region of interest, $w'_{||}$ is bounded above by 10% of the edge velocity, $\bar{W}_{||,e}$. This circumstance is quite similar to that considered by Groot et al. [12].

In representing the perturbations by formulating a stability problem, the unsteady, three-dimensional perturbation equations can be simplified with the following 3 assumptions. First, the disturbances are assumed to have an infinitesimally small amplitude as they evolve through the considered flow field, leading to the formulation of a linear perturbation problem. Second, the base flow is approximately periodic in the direction parallel to the step, the z -coordinate, which justifies the use of periodic boundary conditions. Third, because the crossflow vortices are approximately aligned with the freestream x -direction, the variation of the base-flow quantities in the freestream x -direction is much smaller than that in the direction orthogonal to the step, $x_{\perp}$. This third aspect is important, because it permits assuming that the evolution of the perturbation quantities is negligible in the x -direction as a first approximation.

The x -direction deviates from the direction orthogonal to the step, $x_{\perp}$, by the 30° -sweep angle. Hence, to simultaneously impose periodic boundary conditions in z and slow evolution in x , it is required to formulate the stability equations in the nonorthogonal xyz -system. In secondary instability analyses of crossflow vortices without a step [12], the plane is usually reoriented such that it is orthogonal to the overall direction of minimal change in the base-flow quantities, i.e., it would span a finite distance in the direction orthogonal to the leading edge. This is permissible if no step is involved, because the undisturbed base flow evolves slowly in that case and the extent of the domain orthogonal to the leading edge is very small. This argument cannot be used in the present case, because 1) the variation of the base flow is relatively large in the direction orthogonal to the step, which sheds doubt on the use of periodic boundary conditions, and 2) the base flow data are available at the highest resolution in zy -planes.

A. Spanwise BiGlobal Stability Approach

In order to account for the strong shear and all linear perturbation dynamics in the zy -plane, the spanwise BiGlobal stability approach is used, which, without further corrections, omits the variation of the base flow and the perturbation shape, wavenumber and growth rate in x [13, 14]. This is encoded in the following perturbation ansatz:

$$q'(x, y, z, t) = \tilde{q}(z, y) e^{i(\alpha x - \omega t)} + c.c., \quad (1)$$

where $\tilde{q}$ is the eigenfunction equivalent of the perturbation quantity q' , ω the angular frequency, α the wavenumber in the freestream x -direction, and $c.c.$ represents the complex conjugate such that the left-hand side is real. The angular

frequency ω is related to the cycle frequency as $\omega = 2\pi f$. Perturbations are assumed to have a convective nature, i.e., they are amplified as they are advected downstream by the flow. This means that the perturbations are assumed to be purely oscillatory in time, which constrains ω to be real. One is particularly interested in which disturbance frequency is amplified the most in the x -direction. To this end, one looks to find the negative of the imaginary part of α , $-\alpha_i$, which represents the exponential growth rate of the disturbance in x .

The equations that govern $\{\alpha, \tilde{q}\}$ -combinations form an eigenvalue problem and are referred to as the spanwise BiGlobal stability equations. They are obtained by substituting ansatz (1) into the linearized Navier-Stokes equations and dropping terms that are small based on the aforementioned assumptions. Expressing this system in a nonorthogonal coordinate system is nonstandard, hence the transformation rules and the resulting equations are given in Appendix A. From now on, the use of different coordinate systems will be indicated with the variable θ , which denotes the angle of the out-of-plane coordinate with respect to the direction orthogonal to the step, i.e., $\theta = 0^\circ$ represents the orthogonal coordinate system and $\theta = \Lambda = 30^\circ$ the nonorthogonal coordinate system.

The system of equations is completed by the prescription of boundary conditions. Periodic conditions are used in the z -direction for all eigenfunctions. The length of the domain in the z -direction is limited to unit multiples of λ_r ; in this paper a single such unit will be used. At the wall, the no-slip condition is imposed to the perturbation velocities and the y -momentum equation is used as a compatibility condition for the pressure perturbation. At the top boundary, placed at $1.5\lambda_r \approx 7.5\delta_{99}$, all perturbation quantities are zeroed.

Similarly as in the previous work [9], Chebyshev collocation is used to discretize the domain, see Groot [14]. A BiQuadratic mapping [12] is used in z with $(z_{i1}, z_{i2}) = (1, 2)\lambda_r/3$ in order to yield an approximately uniform spacing in the middle-third of the domain. The same mapping is used in the wall-normal direction with the parameters $(y_{i1}, y_{i2}) = (0.045, 0.15)\lambda_r$, which yield a high resolution in the region of interest. It was shown before that a resolution of $N_z \times N_y = 50 \times 50$ nodes in the zy -plane yields reasonably converged results, especially for the most unstable modes, whose eigenfunctions have the simplest spatial structure [9]. Despite a slight change in the eigenvalues, an inspection of the (presently) computed eigenfunctions at higher resolutions showed they are practically invariant.

B. Accounting for the out-of-plane derivatives of the base flow

As mentioned, the perturbation problem is simplified by dropping small terms from the most general form of the linear perturbation equations; these terms are highlighted in Eqs. (14a-f). In dropping these terms, one obtains a set of stability equations that are fully consistent with the imposed assumptions.

Several approaches exist that aim to eliminate the model error committed by dropping these terms. While restricting the attention to the amplification of linear, slowly-evolving, and convective disturbances, the method with a higher fidelity for the present problem is a parabolized, plane-marching approach [15]. Before such an approach can be justified, however, the following criteria need to be checked:

- 1) The perturbation mechanisms should indeed be convective, i.e., perturbations should propagate downstream, away from the source of excitation after removing the source.
- 2) The assumption that the presently omitted terms are small should be appropriate, e.g., the terms involving x -derivatives of the base-flow quantities.

If one of the above criteria is not satisfied, this is a precursor that parabolization may not work.

The satisfaction of the first criterion is not trivial for the present case, because the flow supports a region of reverse flow, which can advect perturbations in the upstream direction. If this is the case, the perturbation problem cannot be solved by a marching approach; the parabolized system would be ill-posed. This criterion is checked by establishing that the real part of the disturbances' group speed:

$$c_g = \frac{\partial \omega}{\partial \alpha} \quad (2)$$

is positive for the frequencies of interest. From one-dimensional stability analyses, including only the wall-normal direction, it is known that unstable solutions with $c_{g,r} = 0$ can be supported if the inflection point of the boundary-layer profile lies inside the reverse-flow region [16]. This is presently not the case: Fig. 2 shows that the maximum shear (the darkest, filled contour level) lies above the region of reverse flow (black dashed line). This is the first hint that $c_{g,r} > 0$ for the solutions of interest; this will be checked with the present two-dimensional stability data in order to be certain.

The second criterion mentioned above will be checked by way of two approaches that are elaborated upon as follows. Both involve the subset of the terms that are dropped, which specifically contains partial derivatives of the base-flow quantities in the x -direction; i.e., the orange terms in Eqs. (14a-f). These base-flow derivatives are available from the measurement data (they are determined with second-order, central finite differences), while the terms otherwise do not

introduce new unknowns, like x -derivatives of the unknown variables. That is, these terms ‘fit’ in the eigenvalue-problem formulation. The first approach to check the second criterion will therefore be to include these terms in the eigenvalue problem.

The second approach relies on constructing a correction to the eigenvalue as a consequence of the dropped terms by using the adjoint eigenfunctions corresponding to the eigenvalue problem that does not include the terms. The definition of the problem for the adjoint eigenfunctions and the general formula for the correction (of the operator matrices) are given in Appendix B. This second approach allows justifying that the found correction in the eigenvalue is small enough upon establishing a match with the result from the first approach, because this second approach relies upon the assumption that the change in the operator is infinitesimally small.

Besides reducing the model error of the stability problem, there are reasons with a physical background that favor the inclusion of the x -derivatives of the base-flow. Reinstating $\partial \bar{U}_\perp / \partial x$, for example, reduces the divergence of the flow, as is required by the satisfaction of the equations for the steady base flow at practically-incompressible conditions. To this end, the following shorthands are introduced for the full (volumetric) divergence and the in-plane contribution to the divergence:

$$\begin{aligned} \text{divergence of the} \\ \text{base-flow velocity field} \end{aligned} = \nabla = \sec \theta \frac{\partial \bar{U}_\perp}{\partial x} + \underbrace{\frac{\partial \bar{V}}{\partial y} + \frac{\partial \bar{W}_\parallel}{\partial z} - \tan \theta \frac{\partial \bar{U}_\perp}{\partial z}}_{\nabla_{zy}}. \quad (3)$$

Figure 2(a) shows that, when dropping the contribution: $\sec \theta \partial \bar{U}_\perp / \partial x$ from the expression for the full divergence, the resulting in-plane contribution (green contour) has the largest contribution in the region where the shear is largest and, hence, where the eigenfunctions are dominant. Reinstating this term identically removes the contour in the region of elevated shear.* The largest contribution is then observed at the point closest to the wall, where it is hard to obtain an accurate representation of the base flow with PIV. Whether or not $\sec \theta \partial \bar{U}_\perp / \partial x$ is accounted for, however, the divergence is argued to be negligible versus the principal shear contribution. This is quantified in Fig. 2(b): the probability density of values of both ∇ and ∇_{zy} sharply drops upon considering divergence magnitudes larger than 0.04% of the maximum value of $\partial \bar{U}_s / \partial y$. Root-mean-square magnitudes (see the σ -values) are furthermore recorded that are more than 2 orders of magnitude smaller than the maximum value of $\partial \bar{U}_s / \partial y$.

IV. Results

Eigensolutions were computed for the range of frequencies of interest. The present results are restricted to considering the plane at $x = 4$ mm, because the largest growth rates were observed around this location across all considered cases. Furthermore, the domain is limited to a single crossflow-vortex wavelength in the spanwise direction: $z \in [0, \lambda_r]$.

A. Overview

An overview of the growth rates is shown in Fig. 3. The quantity $-\alpha_i \sec \theta$ corresponds to the growth rate (in rad/m) in the $x_\perp$ -direction, i.e., the direction *orthogonal* to the step, whether the orthogonal coordinate system is used or not. In making this quantity nondimensional with λ_r , the quantity $-\alpha_i \lambda_r \sec \theta$ becomes interpretable as the change in the N -factor for the corresponding disturbance if the growth rate would persist for the short distance $\Delta x_\perp = \lambda_r = 11$ mm in the direction orthogonal to the step. The modes at the most unstable frequency are highlighted with a red cross. In turn, these crosses are labelled from the most unstable to the fourth-most unstable case.

Previous findings, based on the stability problem formulated in the orthogonal coordinate system ($\theta = 0^\circ$) and dropping the $\partial \bar{Q} / \partial x$ -terms, established that disturbances undergo explosive growth rates [9]. These results are given here in Fig. 3(a). The growth rates are judged to be explosive upon comparing their magnitude to the N -factors computed for the crossflow vortices. Eppink [17, Fig. 2.4] indicates that $N \approx 5$ is attained at the location of the step, as calculated with linear parabolized stability equations, where the disturbance could grow over a leading-edge-orthogonal distance of 354 mm = 32.2 λ_r .

The growth rates obtained while using the nonorthogonal coordinate system with $\theta = 30^\circ$, while still dropping the $\partial \bar{Q} / \partial x$ -terms, are shown in Fig. 3(b). The largest growth rate has approximately doubled with respect to the solutions in the orthogonal coordinate system. In addition, the upper limit of the unstable-frequency range is extended from 2.7 up to about 3.5 kHz. The largest growth rate is observed at $f = 2.05$ kHz for this case.

*The derivative $\partial \bar{U}_\perp / \partial x$ is locally negative in this region (decelerating flow). Adding $\sec \theta \partial \bar{U}_\perp / \partial x$ to ∇_{zy} reduces its locally positive value.

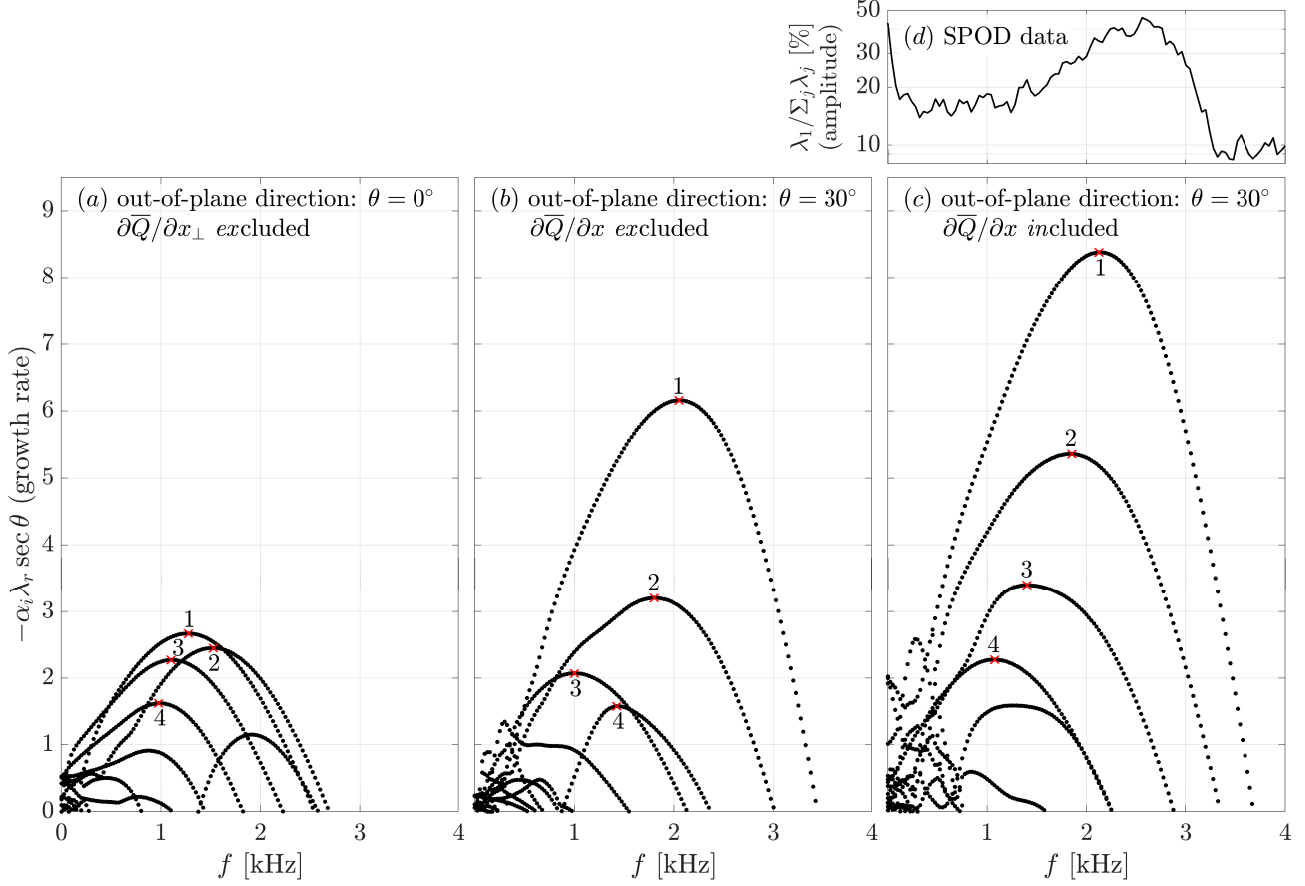


Fig. 3: (a-c) Spatial growth rates versus frequency obtained from the stability analysis at $x = 4$ mm while accounting for different effects as indicated. Crosses indicate local maxima, labels indicate the subpanel in Fig. 4 showing the corresponding eigenfunction. (d) largest SPOD eigenvalue normalized by the total energy per frequency at $x = 10$ mm.

In switching to the nonorthogonal coordinate system, one would expect that, *if the represented instability mechanism does not change*, smaller growth rates would be retrieved. This follows from the logic that a larger step in x is needed in order to reach the same distance in $x_\perp$, i.e., the same orthogonal distance from the step, see Fig. 10. It is found, however, that the represented mechanism depends on the angle θ . This becomes apparent when inspecting the eigenfunctions in Fig. 4, which correspond to all labelled modes in Fig. 3. The $\tilde{w}_\parallel$ -function is shown in particular, which represents the perturbation velocity in the direction parallel to the step (as represented in the orthogonal coordinate system).[†] The contour plots show both the distribution of the real part (filled contours) and the absolute value (red dashed) of $\tilde{w}_\parallel$. Whenever the real part of an eigenfunction is shown, the (otherwise random) phase is oriented such that the real part locally attains the maximum absolute amplitude of the eigenfunction ($\tilde{w}_{\parallel,r} = \max |\tilde{w}_\parallel|$).

When computed in the orthogonal coordinate system, the $\tilde{w}_\parallel$ -eigenfunctions are located on the right side of the reverse-flow region or otherwise throughout the entire spanwise extent of the domain. The behavior changes quite markedly when using the nonorthogonal coordinate system. In that case, these eigenfunctions are either located on top of the region of reverse flow or on its left. The real part of $\tilde{w}_\parallel$ still displays alternating positive and negative values in the wall-normal direction, which could be indicative of the (traveling) crossflow mechanism. While the $\tilde{w}_\parallel$ -component is the largest velocity component for the modes labelled 1 and 3 in Fig. 4(a) and is comparable to $\tilde{u}_\perp$ for the other labelled modes, this is no longer the case for the most unstable mode when using the nonorthogonal coordinate system. For the modes labelled 1, 2, and 4 in Fig. 4(b), the $\tilde{u}_\perp$ -component is the largest by a factor of at least 1.34 with respect to $\tilde{w}_\parallel$. Only for mode 3, $\tilde{w}_\parallel$ is larger than the $\tilde{u}_\perp$ -component. The real part of $\tilde{w}_\parallel$ for mode 3 still shows the typical slanted shape that is usually associated to the (traveling) crossflow instability.

[†]Equations (15b) show that the velocity in the direction parallel to the step depends on θ .

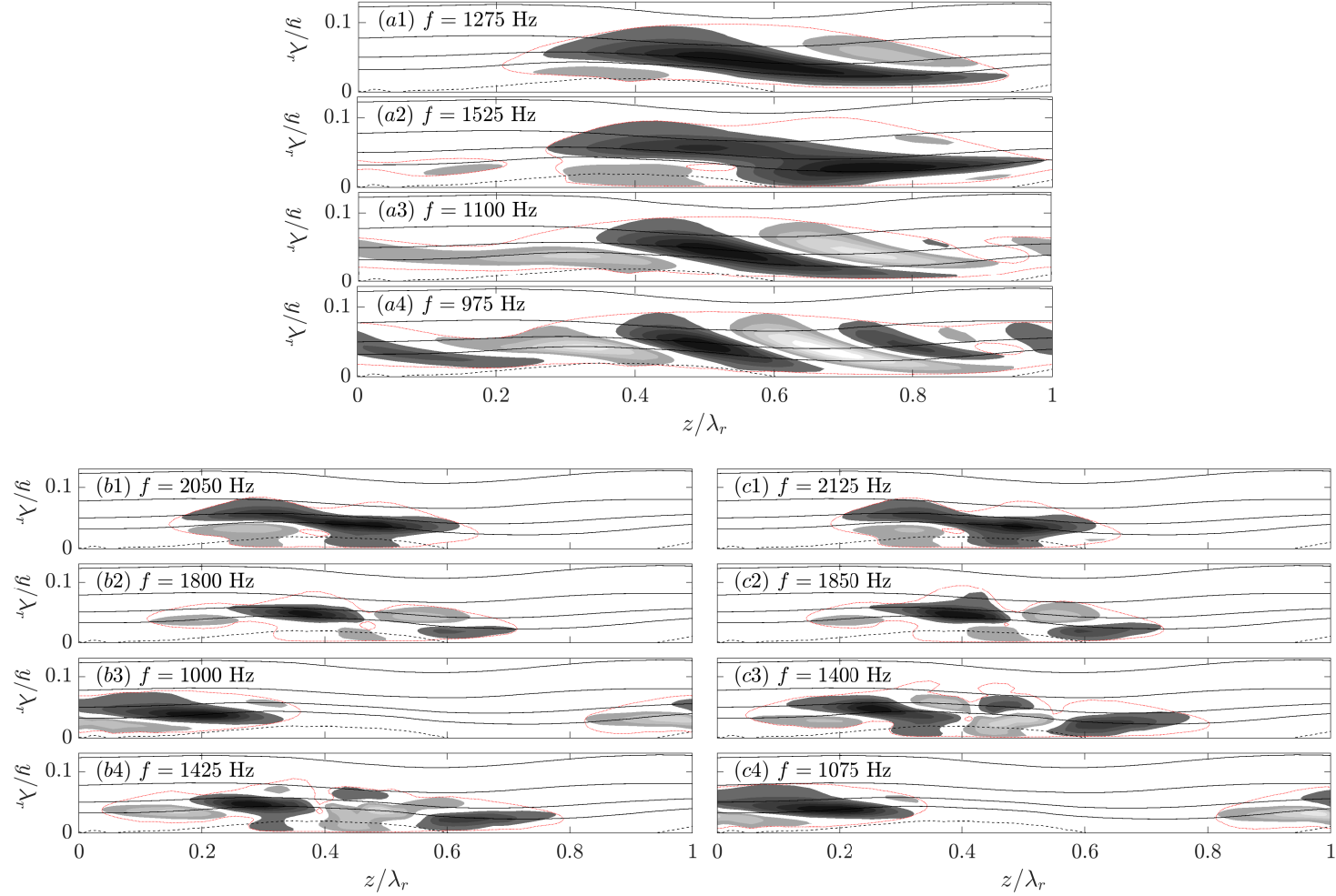


Fig. 4: $\tilde{w}_{||}$ -Eigenfunctions (filled contours: real part, levels: $\pm[1, 5]/6$ with $\Delta = 1/6$, lighter contours indicate negative values; red dashed contours: absolute value, level: $1/6$, all levels are relative to $\max |\tilde{w}_{||}|$) corresponding to the labelled modes in Fig. 3, e.g., panels (b1) through (b4) correspond to the modes labelled 1 through 4 in Fig. 3(b). (a) Orthogonal approach ($\theta = 0^\circ$) and excluding the $\partial\bar{Q}/\partial x$ -terms, (b) nonorthogonal approach ($\theta = 30^\circ$) and excluding the $\partial\bar{Q}/\partial x$ -terms, and (c) nonorthogonal approach ($\theta = 30^\circ$) and including the $\partial\bar{Q}/\partial x$ -terms.

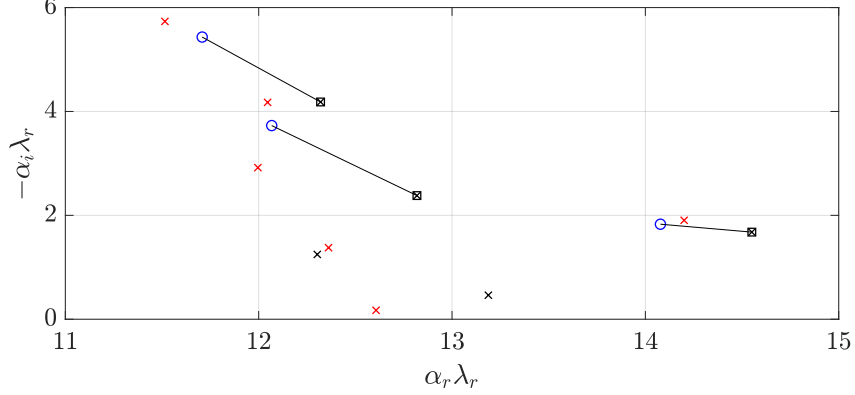


Fig. 5: Eigenvalues as computed with different approaches at $f = 1275$ Hz and $x = 4$ mm. Direct eigenvalue problem excluding/including the $\partial \bar{Q}/\partial x$ -terms (black/red crosses), adjoint eigenvalue problem excluding the $\partial \bar{Q}/\partial x$ -terms (black squares), and eigenvalue correction using Eq. (23) (blue circles).

B. Inclusion of the $\partial \bar{Q}/\partial x$ -terms

The growth rates computed while including the $\partial \bar{Q}/\partial x$ -terms in the eigenvalue problem (see the orange terms in Eqs. (14a-f)) and using the nonorthogonal coordinate system ($\theta = 30^\circ$), are shown in Fig. 3(c). Clearly, another 2 units are added to the largest growth rate, which is now achieved at $f = 2.125$ kHz. This effect is not homogeneous across the different modes. Indeed, the modes that are labelled 1, 2, and 4 in Fig. 3(b, $\partial \bar{Q}/\partial x$ -terms excluded) each experience a destabilization of about +2 units in $-\alpha_i \lambda_r$. Mode 3 in Fig. 3(b), however, is destabilized to a much smaller extent, so that it is the fourth-most unstable mode in Fig. 3(c, $\partial \bar{Q}/\partial x$ -terms included). This is possibly related to the fact that mode 3 is the one mode for which the $\tilde{w}_{||}$ -eigenfunction is found to have a larger magnitude than the $\tilde{u}_{\perp}$ -component. The match-up of the modes across Figs. 3(b) and (c) is supported by the eigenfunctions shown in Figs. 4(b1-b4) and (c1-c4), respectively. Note that, despite the fact that slightly different frequencies have been selected, the eigenfunctions change only slightly in having included the $\partial \bar{Q}/\partial x$ -terms. The relative magnitudes of $\tilde{u}_{\perp}$ and $\tilde{w}_{||}$ are qualitatively the same for the case where the $\partial \bar{Q}/\partial x$ -terms were dropped.

In relation to the argument that the $\partial \bar{Q}/\partial x$ -terms are assumed to be small, the large destabilizing contribution may come as a surprise and therefore deserves a careful justification. To this end, the eigenvalues are recomputed in a different way, where the effect of the $\partial \bar{Q}/\partial x$ -terms is explicitly assumed to be infinitesimal. This is done using the approach outlined in Appendix B.B, which makes use of the adjoint eigenfunctions for the case where the $\partial \bar{Q}/\partial x$ -terms are excluded from the eigenvalue problem. As a verification of the adjoint eigenfunctions, the bi-orthogonality product (21) was evaluated (while excluding the premultiplying factor $\alpha_k - \alpha_j$). Upon providing eigenfunctions of $O(1)$, this relationship evaluated to complex values with magnitudes smaller than $O(10^{-14})$ when using noncorresponding direct and adjoint eigenfunctions (i.e., $P_{jk} = \Phi_j^H(B + (\alpha_j + \alpha_k)C)\Xi_k$ for $j \neq k$).[‡]

When using the corresponding direct and adjoint eigenfunctions (i.e., P_{jk} for $j = k$), the bi-orthogonality product evaluates to values with magnitudes of $O(10^{-3})$.[§] The fact that this product occurs in the denominator of the expression for the correction for α , Eq. (23), explains the large change of the eigenvalues as a consequence of including the small $\partial \bar{Q}/\partial x$ -terms.

The correction to α is computed with Eq. (23) for the 3 most unstable modes as governed by the eigenvalue problem in which the $\partial \bar{Q}/\partial x$ -terms are excluded. The results for $x = 4$ mm and $f = 1275$ Hz are shown as the blue circles in Fig. 5; they should match up with the red crosses. Indeed, despite a noticeable difference, the correction bridges the majority of the difference between the solutions satisfying the eigenvalue problems including and excluding the $\partial \bar{Q}/\partial x$ -terms. The blue circles and red crosses do not match up exactly, because 1) Eq. (23) is (exactly) valid only in the limit of (infinitesimally) small operator changes and 2) Eq. (23) does not account for the change in the other terms of the system (black terms in Eqs. (14a-f)) as a consequence of including the $\partial \bar{Q}/\partial x$ -terms. In solving the eigenvalue problem while including the $\partial \bar{Q}/\partial x$ -terms, these terms do commit a change to the other terms in the system, as they are accounted for while a new internal balance is struck. Lastly, the change in the eigenfunctions, $\delta \Xi$, is small, as qualitatively observed in

[‡]To achieve this, it was found especially necessary to use the appropriate eigenvalues in evaluating $\Phi_j^H(B + (\alpha_j + \alpha_k)C)\Xi_k$. For example, the quantity $\Phi_1^H(B + (\alpha_1 + \alpha_2)C)\Xi_2$ is of $O(10^{-14})$, while $\Phi_1^H(B + 2\alpha_1 C)\Xi_2$ is much larger.

[§] $P_{jj} \times 10^3 = 0.158 - 1.237i, 2.428 + 1.312i, \text{ and } 0.023 + 0.125i$ for $j = 1, 2, \text{ and } 3$, corresponding to the labels in Fig. 3(b).

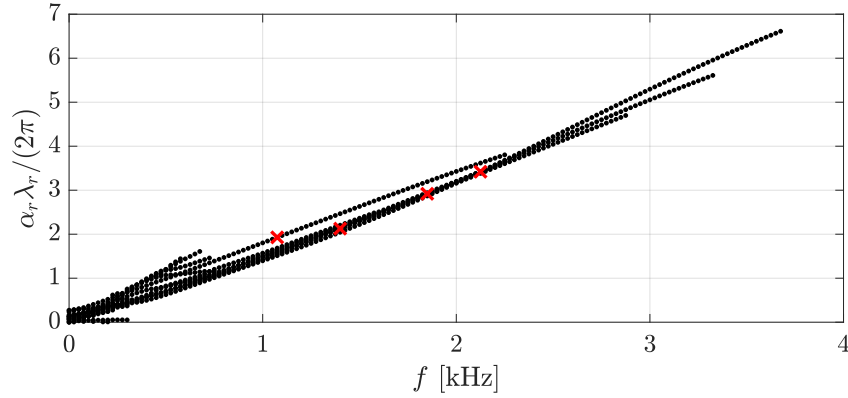


Fig. 6: Wavenumber in the freestream (x -)direction versus frequency, obtained with stability analysis at $x = 4$ mm, while using the nonorthogonal coordinate system (with $\theta = 30^\circ$), and including the $\partial \bar{Q} / \partial x$ -terms. Crosses indicate the solutions that have the largest growth rate versus the frequency, see Fig. 3(c).

Figs. 4(b1-b4) and (c1-c4). Therefore, the unaccounted higher-order terms in Eq. (22b) involving the change in the eigenfunctions do not have a large contribution.

These results support the conclusion that, although the growth rates undergo a large associated change, the effect of the $\partial \bar{Q} / \partial x$ -terms is appropriately considered to be small in terms of the slow-evolution assumption, because it can be recovered to a very reasonable extent in an approach that assumes the contribution to be infinitesimal.

C. Potential of parabolization

As mentioned in §III.B, a method with a higher fidelity for the present problem is a plane-marching approach, which relies upon parabolizing the general, linear perturbation equations in the direction in which the flow evolves slowly. It has already been demonstrated that the $\partial \bar{Q} / \partial x$ -terms can be treated as small terms from this perspective, which was the second criterion referred to in §III.B.

The first criterion required showing that the presently considered solutions represent convective instability mechanisms. To this end, the wavenumber is shown versus the frequency in Fig. 6, at $x = 4$ mm, while using the nonorthogonal coordinate system (with $\theta = 30^\circ$), and including the $\partial \bar{Q} / \partial x$ -terms. This clearly shows that the real part of $c_g = \partial \omega / \partial \alpha$, which is approximately equal to $\partial \omega / \partial \alpha_r$, is positive. A lower bound for the real part of c_g for the solutions of interest is $0.2 \bar{U}_{s,e}$. Despite the presence of the reverse-flow region, these solutions are convective and will propagate downstream in the x -direction.

All proposed preliminary criteria required for parabolization are satisfied. Based on these qualitative arguments and their satisfaction for the other considered x -stations (not presented here), there is no reason to believe that parabolization in x will not work.

D. Improved comparison with SPOD modes

Upon using the nonorthogonal coordinate system and including the $\partial \bar{Q} / \partial x$ -terms, the likeness of the BiGlobal eigenfunctions and the SPOD shape functions is reexamined. To obtain the SPOD modes, 10,000 snapshots were obtained with an acquisition rate of 8 kHz while focusing the field of view on approximately one crossflow-vortex wavelength including the range $z \in [0, \lambda_r]$, see Eppink [5] for more details. Spectral Proper Orthogonal Decomposition (SPOD) is a modal decomposition technique that is similar to the snapshot POD method, but provides the added benefit of including frequency information. The approach used in this paper is the one described by Schmidt and Colonius [18], in which POD is applied to the fast-Fourier transform of the data. The SPOD results were computed using a block size of 200 frames, with 50% overlap. All three velocity components were included in the computation. The results from this specific SPOD analysis consisted of 99 modes for each of the 101 frequency bands, with a frequency resolution of $\Delta f = 40$ Hz. Because of the presence of the instabilities in the flow downstream of the step, the data are generally found to be low-rank, which means that the data can be represented by a single or a small number of basis vectors. This can be inferred from the SPOD eigenvalues, which are also returned for each mode as part of the SPOD analysis. Generally, it is found that, immediately downstream of the step, the first mode contains significantly more energy than the rest of the modes.

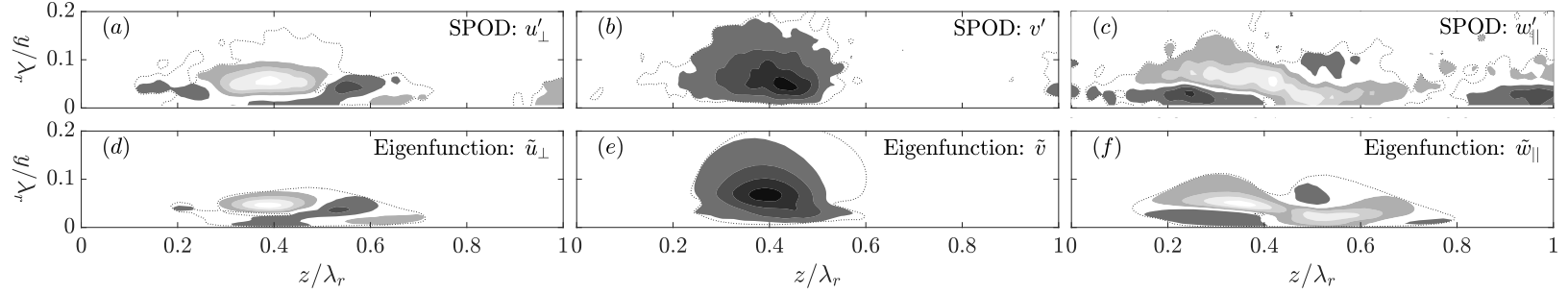


Fig. 7: Comparison of the measured perturbation content (a-c, at $x = 10$ mm) with the stability analysis (d-f, at $x = 4$ mm) for the frequency $f = 1400$ Hz. (a) $u'_{\perp}$, (b) v' , (c) $w'_{\parallel}$ -shape functions corresponding to the most energetic SPOD mode. (d) $\tilde{u}_{\perp}$, (e) $\tilde{v}$, (f) $\tilde{w}_{\parallel}$ -eigenfunctions corresponding to the most unstable eigenmode. Functions' phase is set such that the real part of the wall-normal perturbation velocity component attains the maximum absolute value in the plane (real part: filled contours, levels: $\pm[1, 3, 5, 7]/8$; absolute value: dotted contours, level: $1/4$, scaling the maximum absolute value to 1)

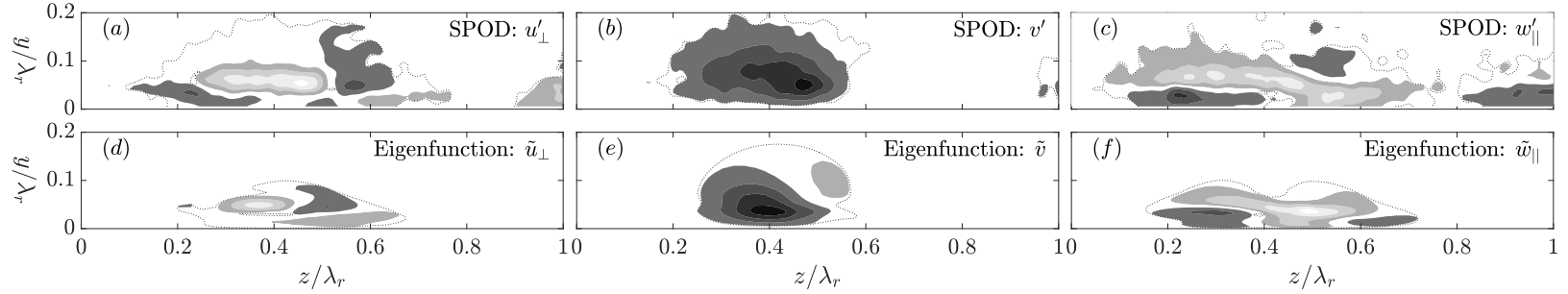


Fig. 8: Comparison of the measured perturbation content (a-c, at $x = 10$ mm) with the stability analysis (d-f, at $x = 4$ mm) for the frequency $f = 1800$ Hz. (a) $u'_{\perp}$, (b) v' , (c) $w'_{\parallel}$ -shape functions corresponding to the most energetic SPOD mode. (d) $\tilde{u}_{\perp}$, (e) $\tilde{v}$, (f) $\tilde{w}_{\parallel}$ -eigenfunctions corresponding to the most unstable eigenmode. Functions' phase is set such that the real part of the wall-normal perturbation velocity component attains the maximum absolute value in the plane (real part: filled contours, levels: $\pm[1, 3, 5, 7]/8$; absolute value: dotted contours, level: $1/4$, scaling the maximum absolute value to 1)

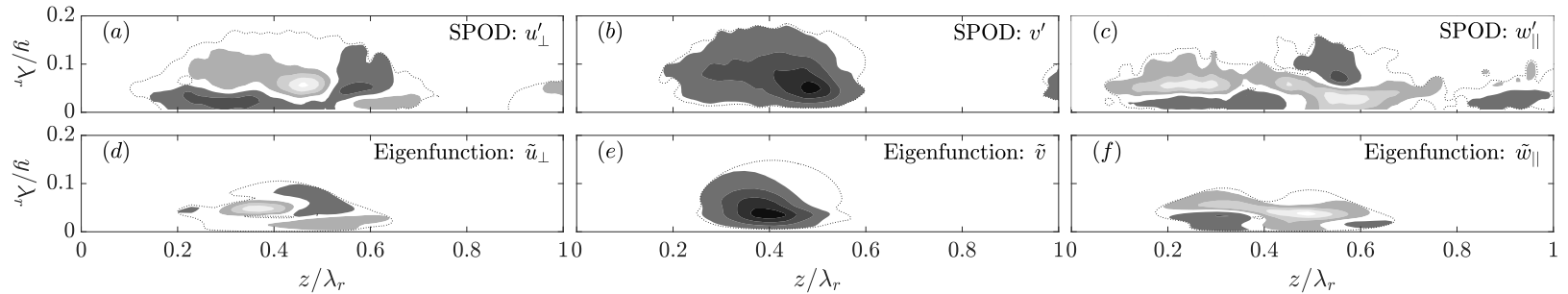


Fig. 9: Comparison of the measured perturbation content (a-c, at $x = 10$ mm) with the stability analysis (d-f, at $x = 4$ mm) for the frequency $f = 2200$ Hz. (a) $u'_{\perp}$, (b) v' , (c) $w'_{\parallel}$ -shape functions corresponding to the most energetic SPOD mode. (d) $\tilde{u}_{\perp}$, (e) $\tilde{v}$, (f) $\tilde{w}_{\parallel}$ -eigenfunctions corresponding to the most unstable eigenmode. Functions' phase is set such that the real part of the wall-normal perturbation velocity component attains the maximum absolute value in the plane (real part: filled contours, levels: $\pm[1, 3, 5, 7]/8$; absolute value: dotted contours, level: $1/4$, scaling the maximum absolute value to 1)

The SPOD modes are obtained at $x = 10$ mm. The BiGlobal results for that location, including all effects, were found to be locally stable, which renders the process of selecting modes non-trivial. Furthermore, time-resolved PIV data are not available at $x = 4$ mm. Nevertheless, the eigenfunctions are found to be nearly invariant over the x -range for which the relevant frequencies are unstable. Hence, for the purposes of the present paper, we will perform the comparison with the BiGlobal results at $x = 4$ mm.

The comparison between the computed growth rates and the SPOD eigenvalues are shown in Figs. 3(c) and (d), respectively. Upon having formulated the stability problem in the nonorthogonal coordinate system and including the $\partial\bar{Q}/\partial x$ -terms, the overlap of the retrieved frequency bands is highly improved. In order to compare the spatial organization of the perturbation field, a comparison is performed at the frequencies 1.4, 1.8, and 2.2 kHz, which is shown in Figs. 7 to 9, respectively. Tracing these frequencies back to Figs. 3(c) and (d), they are shown to closely correspond to the frequency yielding the largest eigengrowth (2.2 kHz) and two lower frequencies for which the SPOD eigenvalues are still relatively large.

For the lowest frequency, 1.4 kHz corresponding to Fig. 7, the match between the eigenfunctions and SPOD shape functions is excellent. This applies for all perturbation velocity components, and up to contours representing a relatively small perturbation amplitude. There is a noticeable difference in the wall-normal extent of the structures, which is likely due to the mismatch in the considered streamwise distance from the step.

For the higher frequencies, 1.8 and 2.2 kHz in Figs. 8 and 9, respectively, the functions can still be compared well, but differences emerge. This is expected to be caused by the difference in the streamwise distance at which the data are evaluated. To close the gap between the present stability and measurement data, it is necessary to integrate the stability results in the downstream direction in order to 1) compare results at the same location and 2) obtain amplification factors besides growth rates. This will be the subject of future investigations.

V. Conclusion

This paper presents results in an on-going investigation of the perturbation dynamics of the flow in the direct downstream vicinity of a forward-facing step subject to crossflow vortices. The sweep angle is $\Lambda = 30^\circ$, the (supercritical, i.e., transition-at-the-step-inducing) step height is $h = 1.7$ mm, the wavelength of the incident crossflow vortices equals $\lambda_r = 11$ mm, and the resultant edge velocity and angle of the inviscid streamline are $\bar{U}_{s,e} = 24.3$ m/s and 35.3° , respectively, in the direct downstream vicinity of the step. The high-resolution, stereographic Particle Image Velocimetry (stereo-PIV) measurement dataset for this flowfield provides all velocity components in a plane that is aligned with the wall-normal and step-parallel directions. This represents a complete description of the base flow for the execution of a BiGlobal stability analysis in that plane. By using this approach, it was demonstrated in previous work [9] that the flow supports several families of unstable eigenmodes.

In the present paper, the stability approach is improved with respect to the previous work in regard to two aspects: 1) the stability problem is formulated in a nonorthogonal coordinate system and 2) the contribution of the out-of-plane base-flow derivatives, i.e., the $\partial\bar{Q}/\partial x$ -terms, is locally accounted for. Both aspects reduce the modeling error by decreasing the size and number of terms that are dropped in the process to simplify the three-dimensional, linear perturbation equations. Upon having included the improvements in the stability approach, much larger growth rates are retrieved and the unstable frequency range is extended. Furthermore, the most unstable eigenfunctions are now found to be located on the top of the regions supporting reverse flow.

The new stability results are compared against SPOD modes obtained from time-resolved measurement data. This is done here at mismatching distances from the location of the step (the stability results are evaluated at $x = 4$ mm and the SPOD at $x = 10$ mm), because the new eigensolutions are locally stable at $x = 10$ mm and time-resolved PIV data are not available at $x = 4$ mm. The match is deemed appropriate nevertheless, because unstable eigenfunctions vary slowly for the considered stations in the x -direction. The match with SPOD data is significantly improved with respect to previous results [9] by having included the changes in the formulation of the stability problem. This manifests itself in a significant improvement in the overlap between the unstable frequency range and the frequency range for which the dominant SPOD eigenvalue is large. Furthermore, despite the difference in the considered streamwise stations for the analyses, an excellent match is obtained in terms of the eigenfunctions and SPOD shape functions at a relatively low frequency. Although a qualitative match can be achieved for frequencies closer to the most unstable one, the mismatch in the considered distance from the location of the step is expected to be responsible for noticeable differences in those cases.

The large destabilizing contribution by the $\partial\bar{Q}/\partial x$ -terms deserves a careful justification in relation to the argument that the $\partial\bar{Q}/\partial x$ -terms are assumed to be small. To this end, the eigenvalues were computed with two approaches: 1) by

including the $\partial \bar{Q}/\partial x$ -terms in the eigenvalue problem, and 2) by using a correction formula for the eigenvalue that assumes the $\partial \bar{Q}/\partial x$ -terms to be infinitesimal and invokes the adjoint eigenfunctions. It is shown that the correction formula accounts for the majority of the destabilizing effect by the $\partial \bar{Q}/\partial x$ -terms. Furthermore, the terms in the formula point out that the magnitude of the $\partial \bar{Q}/\partial x$ -terms is actually small; it is the large sensitivity of the eigensolution that causes the effect of the terms to be amplified by a factor of 10^3 . This is quantified with the bi-orthogonality product with the adjoint eigenfunctions.

On the path toward applying a plane marching technique to the present problem, it is first established qualitatively whether or not the perturbation problem can be parabolized or not. Two qualitative criteria are evaluated to make sure parabolization is not impossible on an a priori basis. The first criterion is that the terms that are omitted upon considering a local approach are small. The terms for which this can be checked in the present context are the $\partial \bar{Q}/\partial x$ -terms. These terms are indeed found to be small as per the arguments posed in the previous paragraph. The second criterion is that the perturbation mechanisms of interest should be convective. This is not a trivial matter for the present case, due to the presence of reverse-flow regions. The perturbations are demonstrated to be convective by assessing the relationship between the out-of-plane wavenumber and the frequency, which demonstrates that the group speed of the most unstable eigensolutions does not approach zero.

Appendix A. BiGlobal equations in a nonorthogonal coordinate system

The perturbation problem is closest to being periodic in the direction parallel to the step, $z_{||}$, while the distortion of the crossflow vortices causes the variation of the base flow to be least in the direction of the crossflow vortices' axes. The latter direction is close to the freestream direction, x , which differs from the direction orthogonal to the step, $x_{\perp}$, by the sweep angle. This implies that the stability problem ought to be formulated in a nonorthogonal coordinate system in order to minimize the model error. A similar approach has been used by Li & Choudhari [19].

Using a nonorthogonal coordinate system is uncommon, hence we elaborate on the formulation of the governing equations in this appendix. See Fig. 10 and Eqs. (4) and (5) for the relation between the various coordinates that are used. In what follows, the angle θ will be assumed to be constant.

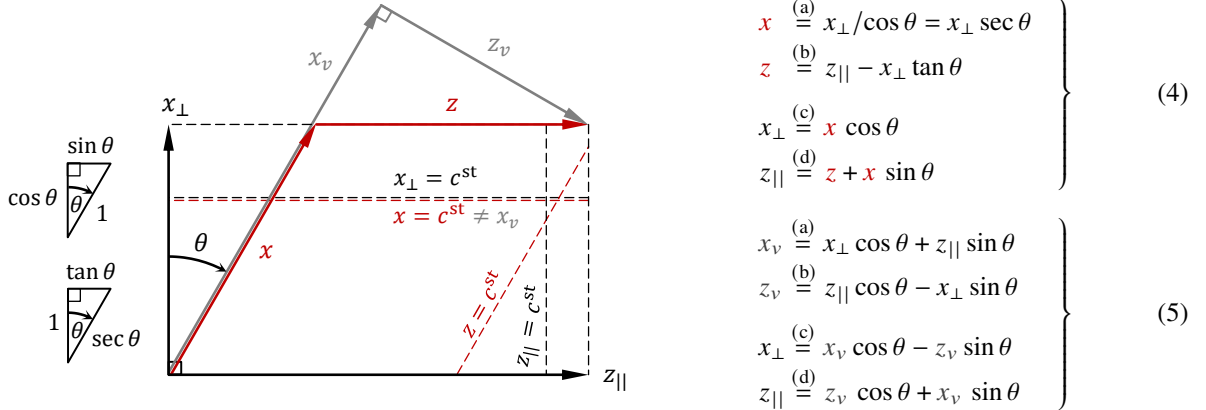


Fig. 10: Coordinate systems and transformations.

To reexpress the equations, the first step is to realize that the same function is expressed in different ways:

$$f_{\perp}(x_{\perp}, z_{||}) = f(x, z). \quad (6)$$

However, the expression for the function changes upon changing the coordinate system, hence $f_{\perp}(x, z) \neq f(x, z)$. The original equations are given in terms of the orthogonal $x_{\perp}$ - and $z_{||}$ -coordinates. Therefore, we want to rewrite the derivatives with respect to those coordinates, which is done as follows:

$$\left. \frac{\partial f_{\perp}}{\partial x_{\perp}} \right|_{z_{||}} = \left. \frac{\partial f}{\partial x_{\perp}} \right|_{z_{||}} = \left. \frac{\partial f}{\partial x} \right|_z \left. \frac{\partial x}{\partial x_{\perp}} \right|_{z_{||}} + \left. \frac{\partial f}{\partial z} \right|_x \left. \frac{\partial z}{\partial x_{\perp}} \right|_{z_{||}} = \left. \frac{\partial f}{\partial x} \right|_z \sec \theta - \left. \frac{\partial f}{\partial z} \right|_x \tan \theta; \quad (7a)$$

$$\left. \frac{\partial f_{\perp}}{\partial z_{||}} \right|_{x_{\perp}} = \left. \frac{\partial f}{\partial z_{||}} \right|_{x_{\perp}} = \left. \frac{\partial f}{\partial x} \right|_z \left. \frac{\partial x}{\partial z_{||}} \right|_{x_{\perp}} + \left. \frac{\partial f}{\partial z} \right|_x \left. \frac{\partial z}{\partial z_{||}} \right|_{x_{\perp}} = 0 + \left. \frac{\partial f}{\partial z} \right|_x, \quad (7b)$$

where Eq. (6) is used in the equations on the far left and Eqs. (4a,b) were used to evaluate the partial derivatives in letting $f = f(x(x_\perp), z(x_\perp, z_{||}))$. The inverse relations are obtained by going through the above derivation reversely; they are written as follows:

$$\left. \frac{\partial f}{\partial x} \right|_z \stackrel{(a)}{=} \left. \frac{\partial f_\perp}{\partial x_\perp} \right|_{z_{||}} \cos \theta + \left. \frac{\partial f_\perp}{\partial z_{||}} \right|_{x_\perp} \sin \theta; \quad \left. \frac{\partial f}{\partial z} \right|_x \stackrel{(b)}{=} \left. \frac{\partial f_\perp}{\partial z_{||}} \right|_{x_\perp}. \quad (8)$$

This leads to the following, respectively covariant and contravariant, metric tensors:

$$g_{ij} = \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \stackrel{\text{expressing}}{\text{as a matrix}} \begin{bmatrix} 1 & 0 & \sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & 1 \end{bmatrix}; \quad g^{ij} = \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^k} \stackrel{\text{expressing}}{\text{as a matrix}} \begin{bmatrix} \sec^2 \theta & 0 & -\sec \theta \tan \theta \\ 0 & 1 & 0 \\ -\sec \theta \tan \theta & 0 & \sec^2 \theta \end{bmatrix}, \quad (9)$$

where summation over the index k is implied and x^i and y^i represent the nonorthogonal and Cartesian coordinates, respectively. Note that $g_{ij}g^{jk}$ and $g^{kj}g_{ji}$ are equal to the Kronecker delta, δ_i^k .

Upon reexpressing the Laplacian, the second derivatives are required:

$$\frac{\partial^2 f_\perp}{\partial x_\perp^2} \stackrel{(a)}{=} \frac{\partial^2 f}{\partial x^2} \sec^2 \theta - 2 \frac{\partial^2 f}{\partial x \partial z} \sec \theta \tan \theta + \frac{\partial^2 f}{\partial z^2} \tan^2 \theta; \quad \frac{\partial^2 f_\perp}{\partial z_{||}^2} \stackrel{(b)}{=} \frac{\partial^2 f}{\partial z^2}, \quad (10)$$

where it is useful to note that, upon reexpressing $\partial^2/\partial x_\perp^2 + \partial^2/\partial z_{||}^2$, the coefficients of $\partial^2/\partial z^2$ above combine into: $1 + \tan^2 \theta = \sec^2 \theta$.

In deriving the stability equations, the instantaneous flow variables, q , are decomposed as:

$$q(x, y, z, t) = \bar{Q}(x, y, z) + q'(x, y, z, t), \quad (11)$$

where $\bar{Q}$ is the steady base flow, which evolves slowly in x , and q' is the perturbation, which is further decomposed as follows:

$$q' = \tilde{q}(x, y, z) e^{i(\int^x \alpha(\bar{x}) d\bar{x} - \omega t)} + c.c., \quad (12)$$

where $\tilde{q}$ is the perturbation's shape function, α the streamwise wavenumber, and ω the angular frequency. Furthermore, $\bar{x}$ represents the dummy integral variable corresponding to x and $c.c.$ denotes the complex conjugate. Advantage is taken of the slow development of the base-flow variables in x by specifying a slow development of the shape function and wavenumber in x . By differentiating Eq. (12), one obtains:

$$\frac{\partial q'}{\partial x} = \left(\frac{\partial \tilde{q}}{\partial x} + i\alpha \tilde{q} \right) e^{\dots} + c.c.; \quad \frac{\partial^2 q'}{\partial x^2} = \left(\frac{\partial^2 \tilde{q}}{\partial x^2} + 2i\alpha \frac{\partial \tilde{q}}{\partial x} - \alpha^2 \tilde{q} + i \frac{d\alpha}{dx} \tilde{q} \right) e^{\dots} + c.c. \quad (13)$$

By using ansatz (12) in the linearized Navier-Stokes equations for incompressible flow, one obtains:

$$\frac{\tilde{D} \tilde{u}_\perp}{\tilde{D} t} + \tilde{u}_\perp \sec \theta \frac{\partial \bar{U}_\perp}{\partial x} + \tilde{v} \frac{\partial \bar{U}_\perp}{\partial y} + (\tilde{w}_{||} - \tilde{u}_\perp \tan \theta) \frac{\partial \bar{U}_\perp}{\partial z} = - \left(\frac{\partial \tilde{p}}{\partial x} + i\alpha \tilde{p} \right) \sec \theta + \frac{\partial \tilde{p}}{\partial z} \tan \theta + \frac{\tilde{L} \tilde{u}_\perp}{Re}; \quad (14a)$$

$$\frac{\tilde{D} \tilde{v}}{\tilde{D} t} + \tilde{u}_\perp \sec \theta \frac{\partial \bar{V}}{\partial x} + \tilde{v} \frac{\partial \bar{V}}{\partial y} + (\tilde{w}_{||} - \tilde{u}_\perp \tan \theta) \frac{\partial \bar{V}}{\partial z} = - \frac{\partial \tilde{p}}{\partial y} + \frac{\tilde{L} \tilde{v}}{Re}; \quad (14b)$$

$$\frac{\tilde{D} \tilde{w}_{||}}{\tilde{D} t} + \tilde{u}_\perp \sec \theta \frac{\partial \bar{W}_{||}}{\partial x} + \tilde{v} \frac{\partial \bar{W}_{||}}{\partial y} + (\tilde{w}_{||} - \tilde{u}_\perp \tan \theta) \frac{\partial \bar{W}_{||}}{\partial z} = - \frac{\partial \tilde{p}}{\partial z} + \frac{\tilde{L} \tilde{w}_{||}}{Re}; \quad (14c)$$

$$\sec \theta \left(\frac{\partial \tilde{u}_\perp}{\partial x} + i\alpha \tilde{u}_\perp \right) + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}_{||}}{\partial z} - \tan \theta \frac{\partial \tilde{u}_\perp}{\partial z} = 0, \quad (14d)$$

where $\tilde{D}/\tilde{D}t$ and $\tilde{L}$ are the material-derivative operator and transformed Laplacian, respectively:

$$\frac{\tilde{D}}{\tilde{D} t} = -i\omega + \bar{U}_\perp \sec \theta \left(\frac{\partial}{\partial x} + i\alpha \right) + \bar{V} \frac{\partial}{\partial y} + (\bar{W}_{||} - \bar{U}_\perp \tan \theta) \frac{\partial}{\partial z}; \quad (14e)$$

$$\tilde{L} = \sec^2 \theta \left(\frac{\partial^2}{\partial x^2} + 2i\alpha \frac{\partial}{\partial x} - \alpha^2 + i \frac{d\alpha}{dx} \right) - 2 \sec \theta \tan \theta \left(\frac{\partial^2}{\partial x \partial z} + i\alpha \frac{\partial}{\partial z} \right) + \frac{\partial^2}{\partial y^2} + \sec^2 \theta \frac{\partial^2}{\partial z^2}. \quad (14f)$$

Small terms, first-order x -derivatives, are printed in green and orange and negligibly small terms, second-order x -derivatives, in blue. The negligibly small terms are neglected in both the local and marching approach. The small terms are neglected in a consistent local approach, while they are included in a marching approach. The orange terms are small terms that still ‘fit’ in the eigenvalue problem formulation, because they do not involve x -derivatives of the unknown variables.

Note that $\{\bar{U}_\perp, \bar{V}, \bar{W}_\parallel\}$ and $\{\tilde{u}_\perp, \tilde{v}, \tilde{w}_\parallel\}$ correspond to the base-flow and perturbation velocity components in the *orthogonal* $x_\perp$ -, y -, and $z_\parallel$ -directions. Note that the same applies to the Eqs. (14a) through (14c); while expressed in the nonorthogonal (x -, y -, z -)coordinates, they still represent the equations for the perturbation momenta in the $x_\perp$ -, y -, and $z_\parallel$ -directions. There are several evident occurrences in which it could be convenient to combine the following terms:

$$\bar{U} = \bar{U}_\perp \sec \theta; \quad \tilde{u} = \tilde{u}_\perp \sec \theta; \quad (15a)$$

$$\bar{W} = \bar{W}_\parallel - \bar{U}_\perp \tan \theta; \quad \tilde{w} = \tilde{w}_\parallel - \tilde{u}_\perp \tan \theta, \quad (15b)$$

which correspond to the velocity components pointing along the directions of the coordinates in the nonorthogonal system. This follows from the Eqs. (4a,b) for the associated coordinates. Lastly, note that upon setting $\theta = 0$, the system in Cartesian coordinates is recovered [20, Appendix C.3].

Appendix B. Adjoint eigenfunctions and their use

Adjoint (or left) eigenfunctions enable projecting a generic signal onto the eigenbasis, i.e., they are used to compute the amplitudes (or Fourier coefficients) in the eigenfunction expansion of a generic signal. As a consequence, adjoint eigenfunctions play an important role in determining the region of influence for a given eigenmode [21] and they allow computing derivatives of the solutions with respect to parameters of the problem. In this appendix, the problem for the adjoint eigenfunctions for the quadratic eigenvalue problem is defined, the bi-orthogonality relationship is derived, and the approach to compute the change in the eigenvalue based on parameter variations is outlined.

A. Adjoint eigenvalue problem and bi-orthogonality relation

The direct/primal/right eigenvalue problem for α is quadratic and can be written in the following forms:

$$A\Xi \stackrel{(a)}{=} \alpha B\Xi + \alpha^2 C\Xi, \quad \begin{bmatrix} A & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} \Xi \\ \alpha\Xi \end{bmatrix} \stackrel{(b)}{=} \alpha \begin{bmatrix} 0 & C \\ I & 0 \end{bmatrix} \begin{bmatrix} \Xi \\ \alpha\Xi \end{bmatrix}, \quad (16)$$

where Eq. (16b) represents the companion matrix form [22] and $\Xi = [\tilde{u}_\perp \ \tilde{v} \ \tilde{w}_\parallel \ \tilde{p}]^T$ in the present context, i.e., Ξ is the column vector consisting of all (direct/primal/right) eigenfunctions; the T represents transposition. Besides the discretized version of the BiGlobal partial-differential equations, the matrices A , B , and C also contain constraints like the boundary conditions. Note that solving Eq. (16a) is equivalent to finding the roots of $\det(A - \alpha B - \alpha^2 C)$.

The adjoint/left eigenvalue problem is defined as:

$$\Phi^H A = \alpha^\dagger \Phi^H B + (\alpha^\dagger)^2 \Phi^H C, \quad (17)$$

where $\alpha^\dagger$ is the ‘‘adjoint eigenvalue’’, $\Phi = [\tilde{u}_\perp^\dagger \ \tilde{v}^\dagger \ \tilde{w}_\parallel^\dagger \ \tilde{p}^\dagger]^T$ is the column vector of adjoint/left eigenfunctions and H represents taking the Hermitian transpose (= regular tranposition + taking the complex conjugate). Evaluating the Hermitian transpose of Eq. (17) yields:

$$A^H \Phi \stackrel{(a)}{=} (\alpha^\dagger)^* B^H \Phi + ((\alpha^\dagger)^*)^2 C^H \Phi, \quad \begin{bmatrix} A^H & 0 \\ -B^H & I \end{bmatrix} \begin{bmatrix} \Phi \\ A^H \Phi / (\alpha^\dagger)^* \end{bmatrix} \stackrel{(b)}{=} (\alpha^\dagger)^* \begin{bmatrix} 0 & I \\ C^H & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ A^H \Phi / (\alpha^\dagger)^* \end{bmatrix}, \quad (18)$$

where the star indicates complex conjugation. Equation (18b) is found by recasting Eq. (18a) such that the resulting matrices are the Hermitian-transposed versions of the companion matrices in Eq. (16b). Solving Eq. (18a) is equivalent to finding the roots of:

$$\det\left(A^H - (\alpha^\dagger)^* B^H - ((\alpha^\dagger)^*)^2 C^H\right) = \det\left(A^T - \alpha^\dagger B^T - (\alpha^\dagger)^2 C^T\right)^* = \det\left(A - \alpha^\dagger B - (\alpha^\dagger)^2 C\right)^*, \quad (19)$$

where the second equation follows from the invariance of determinants with respect to transposition. Note, however, that the determinant on the far right-hand side of Eq. (19) (upon undoing the complex conjugation) represents the same polynomial as $\det(A - \alpha B - \alpha^2 C)$, i.e., the polynomial corresponding to the direct eigenvalue problem. Hence:

$$\alpha^\dagger = \alpha; \quad (20)$$

the direct and adjoint eigenvalue problem have the same spectrum of eigenvalues. Therefore, the dagger will from now on be dropped from α upon considering the adjoint eigenvalue problem.

In Eq. (18b), the adjoint eigenvalue problem is written in a form that can be solved with a typical eigensolver. Note that, in doing so, the solver will return $(\alpha^\dagger)^* = \alpha^*$, i.e., the complex conjugate of the eigenvalues corresponding to the direct problem. The top half of the returned eigenvector will contain the vector of adjoint eigenfunctions: Φ . In this work, the discretized adjoint eigenvalue problem is obtained by: discretizing the direct differential equations and boundary conditions to form the matrices A , B , and C *first*; and forming the matrices in Eq. (18b) by performing Hermitian transposition of the matrices in Eq. (16b) *second*. This is referred to as the discrete-adjoint approach.

A pivotal relationship can be derived as follows. Let j and k denote indices corresponding to specific, not necessarily distinct eigensolutions of the system, then:

$$\begin{aligned} 0 &= \Phi_j^H (A \Xi_k) - (\Phi_j^H A) \Xi_k \\ &= \Phi_j^H \left(\alpha_k B \Xi_k + \alpha_k^2 C \Xi_k \right) - \left(\alpha_j \Phi_j^H B + \alpha_j^2 \Phi_j^H C \right) \Xi_k \\ &= (\alpha_k - \alpha_j) \Phi_j^H B \Xi_k + (\alpha_k^2 - \alpha_j^2) \Phi_j^H C \Xi_k \\ &= (\alpha_k - \alpha_j) \Phi_j^H B \Xi_k + (\alpha_k - \alpha_j) (\alpha_k + \alpha_j) \Phi_j^H C \Xi_k \\ &= (\alpha_k - \alpha_j) \Phi_j^H (B + (\alpha_k + \alpha_j) C) \Xi_k. \end{aligned} \quad (21)$$

The second equation follows by substituting Eqs. (16a) and (17a); the definitions of the direct and adjoint eigenvalue problems. Equation (21) shows that, whenever $\alpha_j \neq \alpha_k$, then $\Phi_j^H (B + (\alpha_k + \alpha_j) C) \Xi_k$ must be equal to zero. In other words, the vectors Φ_j and Ξ_k are orthogonal *under the weight* $B + (\alpha_k + \alpha_j) C$, if they correspond to different eigenvalues. This demonstrates the mutual orthogonality of the bases formed by the direct and adjoint eigenvectors. For this reason, this equation is also known as the bi-orthogonality relationship.

B. Eigenvalue corrections

The bi-orthogonality property is highly useful for many purposes. One particular such application is determining corrections to the eigenvalues for perturbations of the operator matrices A , B , and C . Here, we will restrict ourselves to just perturbing A , because the terms in the equations that will be perturbed never include the factor α or α^2 ; see the orange $\partial Q / \partial x$ -terms in Eqs. (14a-f). If B and C are to be perturbed as well, the approach generalizes without additional complications.

Assume α , Ξ , and Φ solve Eqs. (16a) and (17a). In adding δA to A , the eigenvalue and -functions will respond by changing by the amounts $\delta \alpha$ and $\delta \Xi$, respectively, such that:

$$(A + \delta A) (\Xi + \delta \Xi) = (\alpha + \delta \alpha) B (\Xi + \delta \Xi) + (\alpha + \delta \alpha)^2 C (\Xi + \delta \Xi), \quad (22a)$$

which expands into:

$$\begin{aligned} (A - \alpha B - \alpha^2 C) \Xi + (A - \alpha B - \alpha^2 C) \delta \Xi &= -\delta A \Xi + \delta \alpha (B + 2\alpha C) \Xi \\ &\quad - \delta A \delta \Xi + \delta \alpha (B + 2\alpha C) \delta \Xi + \delta \alpha^2 C (\Xi + \delta \Xi). \end{aligned} \quad (22b)$$

The first term on the left-hand side drops, because it represents the direct eigenvalue problem, Eq. (16a). Upon left-multiplying Eq. (22b) with Φ^H , the second term on the left-hand side (and the fourth $(\delta \alpha \dots \delta \Xi)$ term on the right-hand side) will cancel due to the satisfaction of the adjoint eigenvalue problem, Eq. (17a). Upon assuming the perturbation δA is sufficiently small, $\delta \alpha$ and $\delta \Xi$ are small as well, such that all terms of quadratic and higher order can be dropped from the right-hand side. In dropping these terms, all occurrences of $\delta \Xi$ are eliminated and the resulting expression can be solved for $\delta \alpha$:

$$\delta \alpha = \frac{\Phi^H \delta A \Xi}{\Phi^H (B + 2\alpha C) \Xi}. \quad (23)$$

The denominator, $\Phi^H(B + 2\alpha C)\Xi$, corresponds to the bi-orthogonality product in Eq. (21) upon setting $\alpha_j = \alpha_k = \alpha$. Note that Eq. (23) breaks down if δA attains a large magnitude. Strictly speaking, Eq. (23) is exact if δA is restricted to be infinitesimal. Writing $\delta A = \delta p (\partial A / \partial p)$, where p is a scalar parameter to be varied, Eq. (23) yields the exact expression for $\partial \alpha / \partial p$. That is, the derivatives of an eigenvalue with respect to any parameter can be calculated without having to explicitly vary the parameters. This comes at the cost of having to compute the adjoint eigenfunctions corresponding (only) to that eigenvalue.

In the specific case of providing the correction to the eigenvalue with respect to the $\partial \bar{Q} / \partial x$ -terms in Eqs. (14a-f), the product $\Phi^H \delta A \Xi$ evaluates to:

$$\Phi^H \delta A \Xi = \begin{bmatrix} \tilde{u}_\perp^\dagger \\ \tilde{v}^\dagger \\ \tilde{w}_\parallel^\dagger \\ \tilde{p}^\dagger \end{bmatrix}^H \begin{bmatrix} \sec \theta \partial \bar{U}_\perp / \partial x & 0 & 0 & 0 \\ \sec \theta \partial \bar{V} / \partial x & 0 & 0 & 0 \\ \sec \theta \partial \bar{W}_\parallel / \partial x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_\perp \\ \tilde{v} \\ \tilde{w}_\parallel \\ \tilde{p} \end{bmatrix} = \sec \theta \left((\tilde{u}_\perp^\dagger)^H \frac{\partial \bar{U}_\perp}{\partial x} \tilde{u}_\perp + (\tilde{v}^\dagger)^H \frac{\partial \bar{V}}{\partial x} \tilde{v} + (\tilde{w}_\parallel^\dagger)^H \frac{\partial \bar{W}_\parallel}{\partial x} \tilde{w}_\parallel \right). \quad (24)$$

The inner products in this equation are executed with the column-stacked versions of the discretized functions on the computational domain [14, Eq. (3.15)]. Hence the inner products represent (otherwise unweighted[¶]) summations over all collocation points. In terms of the primitive variables, the matrices B and C are expressed as:

$$B = i \sec \theta \begin{bmatrix} \mathcal{D} & 0 & 0 & -I \\ 0 & \mathcal{D} & 0 & 0 \\ 0 & 0 & \mathcal{D} & 0 \\ -I & 0 & 0 & 0 \end{bmatrix}, \quad C = -\frac{\sec^2 \theta}{Re} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{where: } \mathcal{D} = -\bar{U}_\perp + \frac{2}{Re} \left(\sec \theta \frac{\partial}{\partial x} - \tan \theta \frac{\partial}{\partial z} \right), \quad (25)$$

while A contains all other terms in Eqs. (14a-f), i.e., those that do not contain a factor α .

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[¶]An interpretation is that the adjoint eigenfunctions that result from the discrete-adjoint approach are implicitly weighted such that no further weight is needed when evaluating the summation over all the points in the domain. In the continuous-adjoint approach, instead, the bi-orthogonality relationship is expressed in terms of integrals over the domain; the continuous-adjoint eigenfunctions will therefore differ from the discrete-adjoint eigenfunctions by a scaling with the local dydz-area elements corresponding to the grid for the stability problem. If the dydz-area at a given position is much smaller than the unit area in the non-dimensional coordinates, the discrete-adjoint eigenfunctions are much smaller than the continuous-adjoint eigenfunctions at the same position.

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