Numerical Fluxes and Shocks

Abraham R. Flores
Mentor : Dr. Alireza Mazaheri

NASA Langley Research Center
July 30th 2021

Partial Differential Equations

Motivation & Background

- Equations involving rates of change
- How we describe the world around us
- Vast array of equations and applications
- We want fast, accurate and robust solvers

General Form of First Order PDE

$$F(x_1, x_2, ..., x_n, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, ..., \frac{\partial y}{\partial x_n}) = 0$$

1D Advection Equation

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} = 0$$

Flux f

Scalar Advection

$$f = aU$$

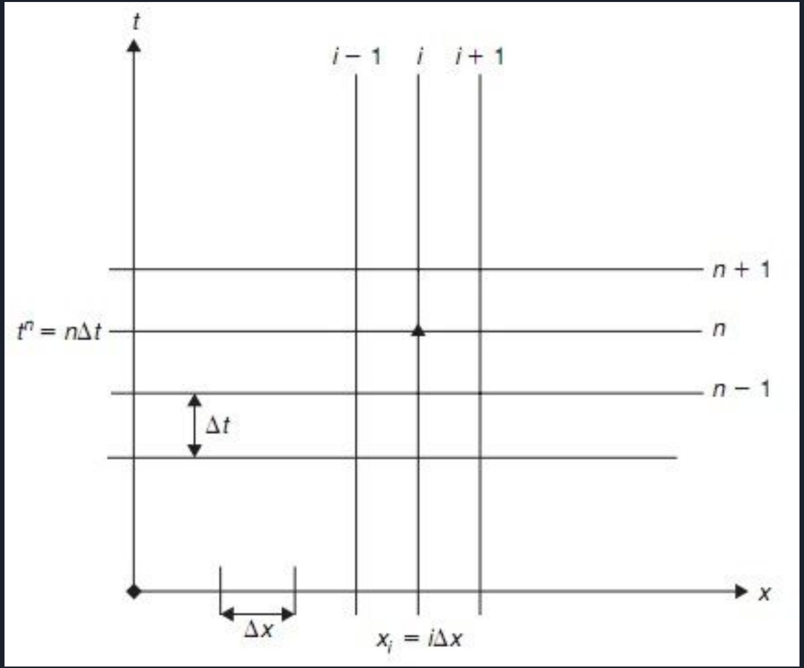
Advective Burger's Equation

$$f = \frac{1}{2}U^2$$

Discretization (Grids)

- How we slice up our domain
- Transition from continuous to discrete domain
- How we inform computers to evaluate our solutions.

Δi Refers to the spacing between to grid points



Discretization of 1D Advection Equation

Finite Differences

- Application of the definition of the derivative
- Fast method but can require fine grid sizes for accurate solutions
- Trivally programmable

Forward

$$\frac{\partial f}{\partial i} = \frac{f(i+\Delta i) - f(i)}{\Delta i} + \mathcal{O}(\Delta i)$$

Backward

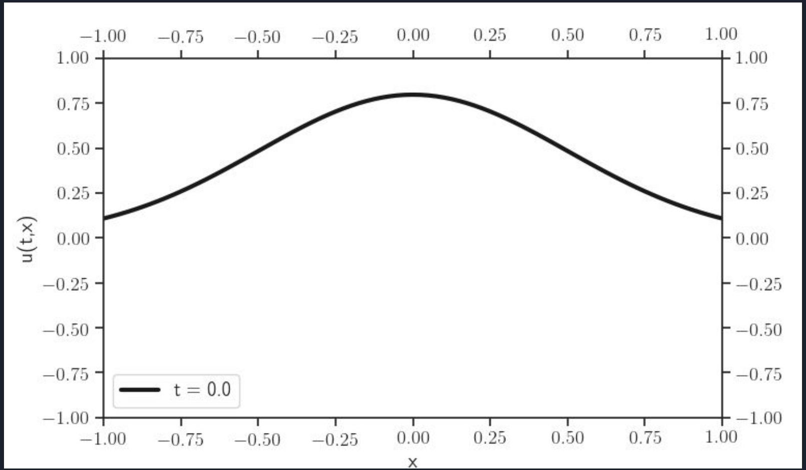
$$\frac{\partial f}{\partial i} = \frac{f(i) - f(i-\Delta i)}{\Delta i} + \mathcal{O}(\Delta i)$$

Central

$$\frac{\partial f}{\partial i} = \frac{f(i+\Delta i) - f(i-\Delta i)}{2\Delta i} + \mathcal{O}(\Delta i^2)$$

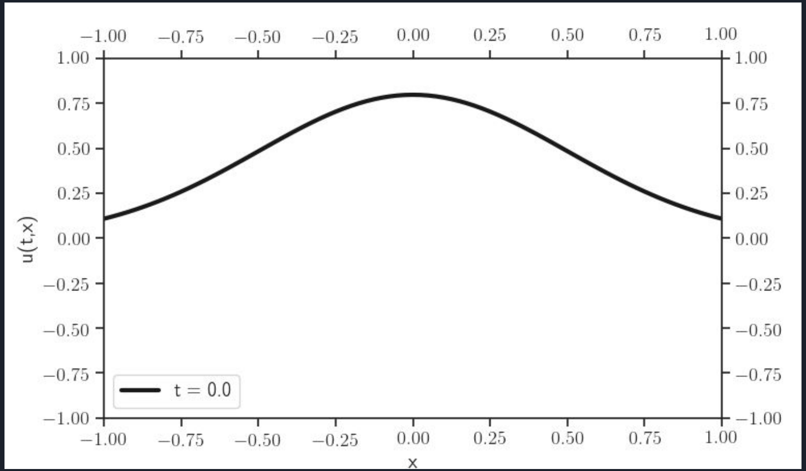
An example in 1D

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$



Forward Time Backward Space

(Stable)



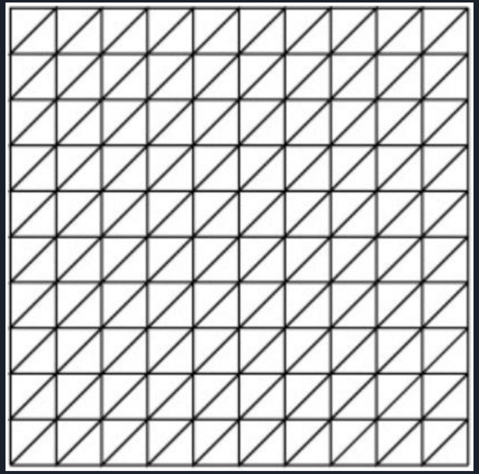
Forward Time Central Space

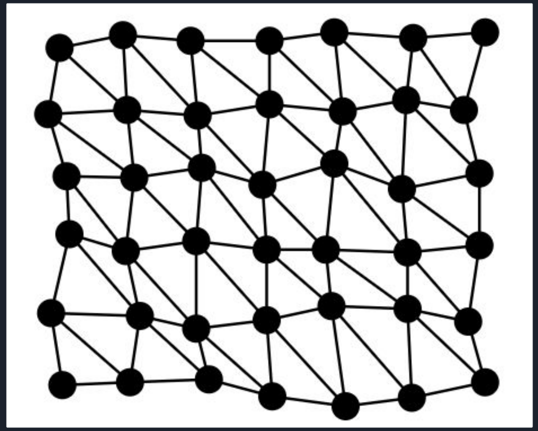
(Unstable)

Meshes

- Spatial representation of discrete cells
- Does not have to be uniform
- Various types of cells
  - 2D
    - Triangle
    - Quadrilateral
  - 3D
    - Tetrahedron
    - Pyramid
    - Triangular Prism
    - Hexahedron
    - ... Polyhedron

- Classification of Meshes
  - Structured : Pattern forms grid layout
  - Unstructured : Stored connectivity graph





Finite Volume Method

- Cell averages as computational nodes
- Flux through cells are conserative
- Straight forward implementation
- Can require fine grids for improved accuracy

Finite Volume 1D Advection Equation

$$\frac{\partial \hat{U}}{\partial t} + \frac{1}{\Delta x}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) = 0$$

Numerical Fluxes

Define : $f_{i\pm\frac{1}{2}}$

- Lax-Friedrichs

$$f_{i+\frac{1}{2}} = \frac{1}{2}(f_i^n + f_{i+1}^n) - \frac{1}{2}\frac{\Delta x}{\Delta t}(u_{i+1}^n - u_i^n)$$

- Lax-Wendroff

$$f_{i+\frac{1}{2}} = f(u_{i+\frac{1}{2}}^{lw})$$

$$u_{i+\frac{1}{2}}^{lw} = \frac{1}{2}(u_i^n + u_{i+1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}(f_{i+1}^n - f_i^n)$$

- Godunov centred flux

$$f_{i+\frac{1}{2}} = f(u_{i+\frac{1}{2}}^{gc})$$

$$u_{i+\frac{1}{2}}^{gc} = \frac{1}{2}(u_i^n + u_{i+1}^n) - \frac{\Delta t}{\Delta x}(f_{i+1}^n - f_i^n)$$

Shocks (discontinuities)

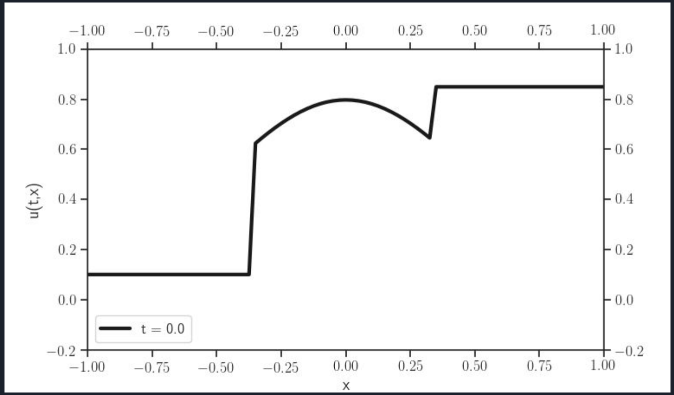
Riemann Problem

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0$$

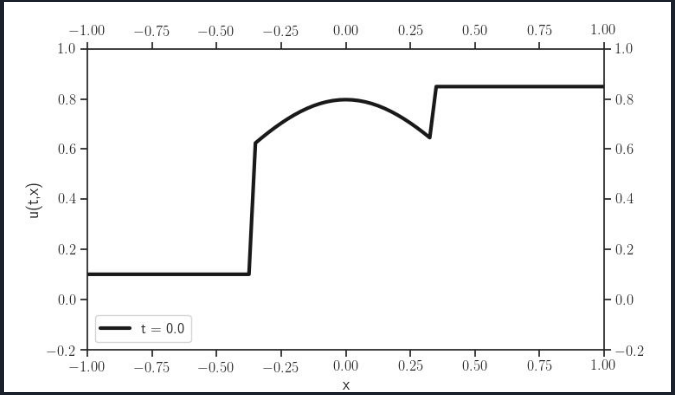
$$u(x,0) = u^- \qquad u(x,0) = u^+$$

$$x < 0 \qquad x > 0$$

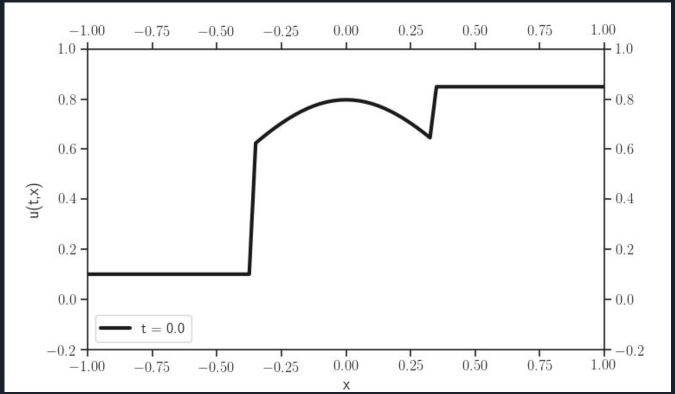
Shocks



Lax-Friedrichs



Lax-Wendroff



Godunov

Compressible Euler Equations

1D Fluid Mechanics

Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

Momentum

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + p) = 0$$

Energy

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}(Eu + pu) = 0$$

u : Velocity

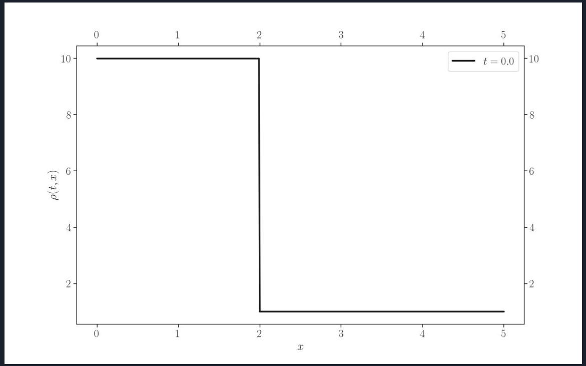
ρ : Density

p : Pressure

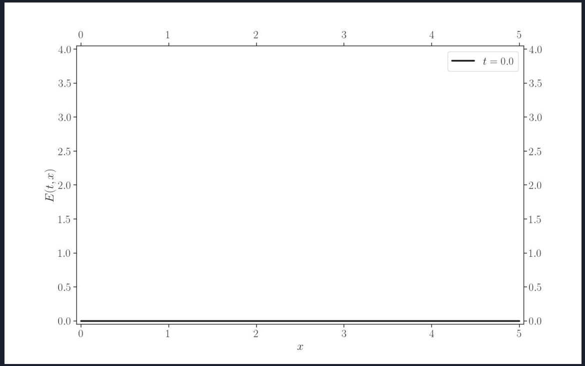
E : Energy

Euler Equations and Exact Solution

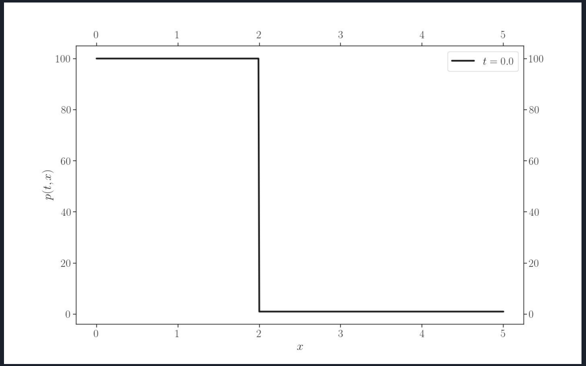
1D Fluid Mechanics



Density



Energy



Pressure

Euler Equations and Finite Volume Method

1D Fluid Mechanics

Thanks for listening!

EXTRA SLIDES

Flux Limiters

1D Linear Advection Example

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$u_i^{n+1} = u_i^n - a\frac{\Delta t}{\Delta x}(u_i - u_{i-1}) - (\psi_i f_{i+\frac{1}{2}} - \psi_{i-1} f_{i-\frac{1}{2}})$$

Flux Limiter

$$\psi(R_i)$$

Forward-Backward Ratio

$$R_i = \frac{u_{i+1} - u_i}{u_i - u_{i-1}}$$

- superbee $\min[\max[1, R], 2, 2R]$
- Barth-Jesperson $\frac{1}{2}(R+1)\min[\min[1, \frac{4R}{R+1}], \min[1, \frac{4}{R+1}]]$
- van Leer $\frac{2R}{R+1}$
- van Albada $\frac{R^2+R}{R^2+1}$
- min $\min[1, R]$

Finite Element Method

- Divides the PDE on to the mesh in order to solve within each cell (finite element)
- Can be made accurate with high order methods
- Complex methods become extremely complex programming problems

Approximate solution with a basis, evaluated on the grid

$$u \approx \sum u_i \psi_i$$

Multiply PDE by test function (ψ) and integrate over domain

$$\int_\Omega \left[u\frac{\partial \psi}{\partial t} + f\frac{\partial \psi}{\partial x} \right] = 0$$