NASA/TM–20210019810

Analogy Between the Collatz Conjecture and Sliding Mode Control

Jared A. Grauer
Langley Research Center, Hampton, Virginia

September 2021

NASA STI Program Report Series

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA scientific and technical information (STI) program plays a key part in helping NASA maintain this important role.

The NASA STI program operates under the auspices of the Agency Chief Information Officer. It collects, organizes, provides for archiving, and disseminates NASA's STI. The NASA STI program provides access to the NTRS Registered and its public interface, the NASA Technical Reports Server, thus providing one of the largest collections of aeronautical and space science STI in the world. Results are published in both non-NASA channels and by NASA in the NASA STI Report Series, which includes the following report types:

- TECHNICAL PUBLICATION. Reports of completed research or a major significant phase of research that present the results of NASA Programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counterpart of peer-reviewed formal professional papers but has less stringent limitations on manuscript length and extent of graphic presentations.

- TECHNICAL MEMORANDUM. Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- CONTRACTOR REPORT. Scientific and technical findings by NASA-sponsored contractors and grantees.

- CONFERENCE PUBLICATION. Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.

- SPECIAL PUBLICATION. Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- TECHNICAL TRANSLATION. English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services also include organizing and publishing research results, distributing specialized research announcements and feeds, providing information desk and personal search support, and enabling data exchange services.

For more information about the NASA STI program, see the following:

- Access the NASA STI program home page at http://www.sti.nasa.gov

- Help desk contact information:

https://www.sti.nasa.gov/sti-contact-form/
and select the "General" help request type.

NASA/TM–20210019810

Analogy Between the Collatz Conjecture and Sliding Mode Control

Jared A. Grauer
Langley Research Center, Hampton, Virginia

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

September 2021

The use of trademarks or names of manufacturers in this report is for accurate reporting and does not constitute an official endorsement, either expressed or implied, of such products or manufacturers by the National Aeronautics and Space Administration.

Available from:

NASA STI Program / Mail Stop 148
NASA Langley Research Center
Hampton, VA 23681-2199
Fax: 757-864-6500

Abstract

The Collatz Conjecture is a famous mathematics problem that is simple to state and understand but remains unsolved. In this paper, the problem is recast as a discrete-time nonlinear system and viewed in a new light from the perspective of nonlinear systems and feedback control theory. In particular, connections are made between the Collatz sequence of numbers and the behavior of closed-loop dynamical systems designed using a feedback control method called sliding mode control. Trajectories of such systems are characterized by a reaching phase and a sliding phase, the latter of which exhibits exponential convergence. As sliding mode control design is rooted in Lyapunov stability theory, the analogy suggests a new direction for proving or disproving the Collatz Conjecture. Although several possible approaches are discussed, no formal proof is given in this paper.

Contents

Nomenclature

Roman

I	moment of inertia, slug-ft^2
k	sample index
N	number of samples
s	Laplace variable
sgn(.)	signum function
u	input
V	Lyapunov function
x	state variable
z^{-1}	discrete-time unit-shift operator

Greek

Γ	control input matrix
θ	attitude pitch angle, rad
σ	sliding variable
τ	applied moment, ft-lbf
Φ	state transition matrix

Other

$\dot{}$	time derivative

1 Introduction

The Collatz Conjecture is a notorious and unsolved problem whose introduction is generally credited to the German mathematician Lothar Collatz around the year 1937 [1]. The conjecture describes a sequence of numbers formed from the following process. Start by picking any positive integer. If the number is even, divide it by two; if the number is odd, multiply it by three and add one. Using the result, repeat this step many times. For example, the sequence starting from the initial value 3 is

$$3, \quad 10, \quad 5, \quad 16, \quad 8, \quad 4, \quad 2, \quad 1, \quad \ldots$$

The Collatz Conjecture states number sequences constructed in this manner eventually reach the value 1 from any initial value. The problem at hand, which remains to be solved, is to prove the conjecture true or false.

Although the problem seems to have little practical utility, it has remained alluring perhaps because of the simplicity in stating the problem, difficulty in finding its solution, and relevance to many branches of mathematics. Despite offering a reward for its solution, Paul Erdős said the problem was "Hopeless. Absolutely hopeless." [1]. The problem has been called "dangerous" and likened to a "siren's song" [2]. It was even stipulated that its circulation was intended to stifle technological progress in the United States [3]. Furthermore, the conjecture has inspired many forms of artwork [4] and has been the punchline of several jokes [5].

Undeterred by the difficulty, some progress on the problem has been made. In 2010, Jeffrey Lagarias published a collection of papers, including an annotated bibliography, reporting the current state of the art on the problem from a variety of perspectives including number theory, ergodic theory, probability, and others. Recently, Terrance Tao presented new results based on analogy to partial differential equations [3]. As of 2020, David Barina has verified the conjecture for trajectories starting from numbers up to about 10^{20} using direct computer simulations [6].

Upon reading the recent news of Ref. [3], the author likewise became curious about the problem. After some initial experimentation, similarities to the fields of nonlinear systems and feedback control theory became apparent, for which aspects of stability theory and nonlinear phenomena seemed relevant. In particular and most notably, it was observed that the Collatz system behaves similarly to a physical system operating under feedback designed using sliding mode control. These analogies and perspectives do not appear to have been discussed before in the open literature. Therefore, the purpose of this report is to point out those similarities and comparisons, thereby casting the problem in a new light and suggesting other directions for work towards proving or disproving the conjecture.

This paper is organized as follows. Section 2 views the number sequence introduced by Collatz as a nonlinear discrete-time dynamical system. Section 3 discusses concepts of linear and nonlinear stability in connection with the Collatz system, as well as limit cycle oscillations. Section 4 summarizes basic aspects of sliding mode control, illustrates its application with a simplified model of spacecraft attitude dynamics, and describes how the analogy could be used for proving or disproving the Collatz Conjecture. Section 5 concludes the report.

2 Dynamical System Model

The Collatz number sequence described in Section 1 can be written as

$$x_{k+1} = \begin{cases} x_k/2, & \text{for } x_k \text{ even} \quad (1a) \\ 3x_k + 1, & \text{for } x_k \text{ odd} \quad (1b) \end{cases}$$

which are equations of motion for a dynamical system represented in discrete time. The sequence is obtained from the scalar state variable x_k taken over the discrete samples $k = 1, 2, \ldots, N$.

The Collatz system in Eq. (1) can be classified in several ways. Due its discrete-time nature, the system is formed from difference equations, which are the counterparts of ordinary differential equations in continuous time. The system is single-input single-output (SISO), where the input is the 1 in Eq. (1b) and the output is the state. The system is called autonomous or time-invariant because k does not explicitly affect the dynamics. Although Eqs. (1a) and (1b) are both linear in terms of x_k and the input, the switching between the two functions based on the parity of x_k makes Eq. (1) a nonlinear system.

A block diagram representing the system dynamics is shown in Fig. 1. The next state value x_{k+1} is formed using a switch block based on the current state parity. When x_k is even, the value from Eq. (1a) is used, whereas when x_k is odd, the value from Eq.(1b) is used. The block z^{-1} is the discrete-time representation of the unit-shift operator, which delays x_{k+1} one sample to produce x_k. The arrows coming in to and out of the block diagram indicate the input and output of the system, respectively.

3 Stability Analyses

Broadly speaking, stability refers to behavior of a system after it is perturbed from some condition. Several types of stability analyses can be applied to the system in Eq. (1) in regard to the conjecture.

3.1 Linear Stability

As the Collatz system comprises two separate linear systems, a piecewise linear analysis [7] can be used to first analyze the linear stability of the system and provide some limited insight.

Linear, time-invariant, discrete-time, SISO systems, such as Eqs. (1a) and (1b), can be written in the canonical state-space form

$$x_{k+1} = \Phi x_k + \Gamma u_k \tag{2}$$

where Φ is the state transition matrix, Γ is the control matrix, and u_k is an applied input at sample k. The resulting state trajectory for Eq. (2) at an arbitrary sample is the convolution sum

$$x_k = \Phi^k x_1 + \sum_{i=1}^{k} \Phi^{k-i} \Gamma u_i \tag{3}$$

For a homogeneous linear system, the origin $x = 0$ is the only possible equilibrium solution. The stability of the system about this equilibrium point is characterized by eigenvalues of Φ [8,9]. Specifically for a scalar system, $|\Phi| < 1$ results in a convergence of the unforced part of the linear system to the origin, and $|\Phi| > 1$ results in a divergence from the origin. In both of these cases, the decay or growth of the homogenous solution is *exponential*, as seen by Eq. (3). A value $|\Phi| = 1$ is the intermediate marginally stable case, where there is no inherent stability to the origin. These characterizations of stability are *global* for a linear system because they are true for the entire state space and do not depend on the value of the state.

For the Collatz system, Eq. (1a) has $\Phi = 1/2$ and is therefore exponentially stable about the origin along even values of the state, whereas Eq. (1b) has $\Phi = 3$ and is unstable along odd values of the state. The linear stability properties of the system are piecewise and dependent upon the state value. The switching between these two linear systems is fundamentally what gives the Collatz system its nonlinear and interesting behavior. Despite this complexity, the origin $x = 0$ is the only equilibrium point of the Collatz system, although it is never reached, as described next.

3.2 Limit cycle oscillations

Although the piecewise linear analysis is helpful in understanding how the Collatz system behaves while the x_k remains even or odd, its usefulness in describing the system behavior over multiple samples is limited because of nonlinearities due to the switching between the two functions.

Once the solution has reached the value 1, Eq. (1) follows the sequence with 1, 4, 2, 1, ... which is a never-ending loop. This closed curve in the state trajectory, called a *limit cycle oscillation* [10], arises from the switching between the stable and unstable linear equations, and is a phenomenon found in some nonlinear systems. This limit cycle is stable in that nearby points (e.g., starting from the value 8) converge to it and remain indefinitely on the cycle. Other nearby points (e.g., starting from the values 3 or 5) can initially depart from the cycle, but eventually return to it. Furthermore, the limit cycle can be classified as an *invariant set* [10] because the solution never departs the limit cycle once it is reached.

In this light, the solution to the Collatz Conjecture could be reframed by asking the following different but equivalent questions:

1. Is the origin $x = 0$ globally stable?
2. Is the limit cycle 1, 4, 2, 1, ... globally stable?
3. Do other limit cycles that could entrain trajectories exist?

Establishing global stability of the origin or the limit cycle, perhaps under some qualifying conditions, would mean that solutions from all starting points converge to it, thereby proving the conjecture. Finding another limit cycle that could entrain the solution trajectory would, on the other hand, provide a counter-example and disprove the conjecture.

Those questions are also difficult and unanswered. Although some classical results are known for second-order systems [10], there is in general no systematic method for determining the existence of limit cycles; rather, laborious computer search and simulation is the standard approach. The region of attraction for the known limit cycle has been verified up to about 10^{20} using computer simulations and, currently, no other limit cycles have been found [6]. This fact provides an estimate of the region of attraction for the system, but does not constitute a formal proof for the conjecture.

3.3 Nonlinear Stability

The standard approach for determining the stability of a nonlinear system is the direct method of Lyapunov [7,10,11]. Inspired by physical systems, the approach involves forming a scalar "energy-like" function $V(x)$ called a *Lyapunov function.* By examining changes in $V(x)$ over time, an analyst can attempt to prove different characteristics of stability for the system.

The advantage of this method is that it is general and widely applicable to nonlinear systems, yet a powerful technique. Broad statements about the stability and other behaviors of the system can be inferred from the system dynamics without potentially expensive computer simulation or experimentation.

The generality of this method is also its drawback in that valid and useful Lyapunov functions are not unique, may be difficult to find, or may not exist. Although energy functions are readily available for physical systems (e.g., kinetic and potential energies of a pendulum) and some simple procedures exist for sometimes constructing useful Lyapunov functions [10], the same intuition cannot always be applied for purely mathematical constructs such as the Collatz system. As such, a successful stability analysis often requires experience, cleverness, and deeply-rooted intuition about the system.

The basic requirement of a Lyapunov function candidate is that it is a positive definite function: positive for all states except at the equilibrium, where it is zero. Because of this property, a Lyapunov function can be thought of as a generalized distance measure.

For example, one possible approach for solving the Collatz Conjecture is to select a Lyapunov function, such as

$$V(x_k) = \frac{1}{2}x_k^2 \tag{4}$$

and prove, perhaps under some conditions, that its value and its derivative, approximated by the forward difference

$$\dot{V}(x_k) = V(x_{k+1}) - V(x_k) \tag{5}$$

are such that the state trajectory eventually tends to the origin from any starting value. As energy functions are additive for physical systems, this could perhaps be done using the summation of two Lyapunov functions: one for Eq. (1a) and one for Eq. (1b). Rather, a discontinuous Lyapunov function could perhaps be of use, such as has been useful for other discontinuous systems. Another related approach is to define a Lyapunov function instead as a generalized distance from the limit cycle oscillation, and show convergence of that function to zero, which corresponds to the trajectory approaching the limit cycle. Another approach would be to estimate the region of ultimate boundedness and show it contains only the one known limit cycle.

Similar to other approaches taken towards solving the conjecture, those are also difficult routes without obvious solutions. First, there is little physical intuition with the system for refining Eq. (4). Second, it is difficult to analytically deal with the switching between Eqs. (1a) and (1b). Furthermore, state trajectories (e.g., the progression starting from the initial value 3 shown in Section 1) clearly move toward and away from the limit cycle instead of asymptotically converging towards it.

4 Analogy to Sliding Mode Control

In simulating the Collatz system from multiple initial conditions, it was observed that state trajectories have two distinct phases. The first phase consists of random-looking vacillations where the state increases and decreases in value. The second phase occurs when the state reaches a power of two (i.e., 2^n for some positive integer n). Note that all powers of two are even numbers, and subsequent iterations using Eq. (1a) are also even numbers. Therefore, once a power of two is attained, the solution remains even and decays exponentially to the value 1 and enters the limit cycle $1, 4, 2, 1$. For reasons elaborated on below, these portions of the state trajectory will be referred to as the *reaching* and *sliding* phases, respectively.

For example, consider the three trajectories resulting from initial conditions x_1 of 6, 9, and 10. These are shown in Fig. 2 using different colors and markers. Because these trajectories ended in the limit cycle, the horizontal axis was reversed to show the number of steps until the value $x_k = 1$ was reached. The three trajectories coalesced six steps from the limit cycle at the value of 10. The reaching phase consisted of the iterations until a power of two is obtained, which for these trajectories was $2^4 = 16$. After this value was reached, the solution followed the sliding phase and exhibited exponential decay to the value 1.

In the field of feedback control theory, improvements in the overall behavior of a system are sought by appropriately manipulating the system inputs using measurements of its output responses [12]. Examples of feedback control for physical systems include household thermostats, active smart structures, and aircraft flight control systems. Some feedback control systems are designed using a method called *sliding mode control* (SMC) [7, 10, 13] which, as will be discussed, manifests a switching behavior similar to the Collatz system. In addition to these synthetic sliding modes that have been designed into physical systems by means of adding feedback, naturally-occurring sliding modes have also been observed in biological systems [14].

In basic SMC, a sliding surface, on which the system will traverse the state space, is first designed. Using the direct method of Lyapunov, a control law is then formulated to force the solution to

1. Reach the sliding surface in finite time, and
2. Slide along the surface with exponential decay to the origin.

For context, a SMC design will be illustrated in the next section using a relatively simple second-order regulator system.

4.1 An Example of Sliding Mode Control

An idealized model for the single-axis attitude dynamics of a spacecraft, simplified from Euler's equations, is the equation of motion

$$I\ddot{\theta}(t) = \tau(t) \tag{6a}$$
$$= \tau_0 \operatorname{sgn}[u(t)] \tag{6b}$$

where

$$\operatorname{sgn}[u(t)] = \begin{cases} 1, & \text{for } u(t) > 0 \\ 0, & \text{for } u(t) = 0 \\ -1, & \text{for } u(t) < 0 \end{cases}$$

is the signum function. The states are the pitch angle θ and the pitch rate $\dot{\theta}$. The moment of inertia I is approximated as a constant for short time durations. The torque τ is applied by reaction control system (RCS) jet thrusters. The jets are assumed to be perfect actuators that can instantaneously switch between firing in the positive or negative direction with maximum torque τ_0, or turn off, according to the sign of the control signal u. For simplicity, these dynamics are represented in continuous time, rather than in discrete time.

For this example, take $I = 16,922$ kg-m^2 and $\tau_0 = 2,990$ N-m (four 445 N jets installed 1.68 m from the spacecraft center of mass), which roughly corresponds to the pitching dynamics of the Apollo Lunar Module (LM) during the descent stage to the surface of the moon [15, 16]. The goal of this example is to design u using SMC to stabilize the origin $\theta = \dot{\theta} = 0$.

A block diagram of the closed-loop system model is shown in Fig. 3. The control law block, to be designed, uses differences between the desired and measured values of θ and $\dot{\theta}$ to manipulate u. The spacecraft dynamics, given by Eq. (6), are represented by the series of blocks in the dashed rectangle. Division by the variable s represents the frequency-domain implementation of an integration.

To employ SMC in designing the control law, first establish a sliding surface defined by

$$\sigma(\theta, \dot{\theta}) = \lambda\theta + \dot{\theta} \tag{8}$$

where λ is a constant to select. Through the two-dimensional state space (θ–$\dot{\theta}$ plane), the level sets of $\sigma = c$ are parallel lines with c as the y-intercept. In particular, the level curve $\sigma = 0$ is the line

$$\dot{\theta} = -\lambda\theta \tag{9}$$

passing through the origin. Note that the surface is of lower order (one dimension) than the state space (two dimensions), and that Eq. (9) is a first-order linear differential equation of a single variable.

Next, form a Lyapunov function with which to derive a stabilizing control law. Instead of working in terms of the state variables θ and $\dot{\theta}$ directly, choose the Lyapunov function as a measure of the distance from the sliding surface as

$$V(\sigma) = \frac{1}{2}\sigma^2 \tag{10}$$

The particular value of this $V(\sigma)$ represents the vertical distance between the current state value and the line $\sigma = 0$ in the state space.

The energy flow of the system is ascertained from the time derivative of this Lyapunov function,

$$\dot{V}(\sigma) = \sigma\dot{\sigma} \tag{11}$$

For Eq. (11) to be negative definite, so the state trajectory is stable with respect to the line $\sigma = 0$, σ and $\dot{\sigma}$ need to have opposite signs. One control law that accomplishes this is

$$u = -\sigma \tag{12a}$$
$$= -\lambda\theta - \dot{\theta} \tag{12b}$$

With this control law, $\dot{V}(\sigma)$ always decreases and the trajectory converges towards the sliding surface for any attitude and for angular rates bounded by $|\dot{\theta}| < \tau_0/\lambda I$. (When this angular rate is exceeded, the system remains stable but RCS jets do not have enough control authority to follow the designed trajectory.) By these arguments, the control law in Eq. (12) therefore provides global asymptotic stability of the sliding surface.

Once the sliding surface has been reached, which occurs in a finite amount of time, the closed-loop dynamics are thereafter given by Eq. (9), which is a first-order linear system with exponential decay to the origin specified by λ. The initial conditions for the sliding phase are given by the values of θ and $\dot{\theta}$ at the moment $\sigma = 0$ is reached. The constant λ therefore defines both the sliding surface and the dynamics along it.

A simulation from an initial $\theta(0) = 10$ deg and $\dot{\theta}(0) = 20$ deg/s is shown as a phase portrait in Fig. 4 and as time histories in Fig. 5. The simulation was performed with a 50 Hz sampling rate ($\Delta t = 0.02$ s) and using a fourth-order Runge-Kutta numerical integration method. In Fig. 4, lines of constant σ are shown in blue, and arrows annotating the reaching and sliding phases are shown in black.

Starting from the initial condition, the control law fired the RCS jets in the negative direction. The ensuing reaching phase was a counter-clockwise parabola in the state space. The sliding surface was reached just after 3 s. From that time, the RCS jets reversed direction and the trajectory moved along the line with exponential convergence to the origin. Chattering between positive and negative RCS jet firings were needed to follow this trajectory during the sliding phase, which is characteristics of switching control systems.

As shown in Fig. 6, the variable σ decreases monotonically to zero. Because σ and $\dot{\sigma}$ have different signs, the Lyapunov function is always decreasing due to its derivative being negative definite.

4.2 Suggested Approach for a Proof

For all initial conditions so far documented, the Collatz system exhibits dynamics similar to control systems designed using SMC, such as in the spacecraft dynamics example. As discussed previously in this section, there is a reaching phase where the state increases and decreases in a random-looking manner. At some point, the state reaches a power of two, which acts as a sliding surface, and the solution then exhibits exponential decay to the limit cycle near the origin. Figure 2 provides examples of this behavior from three initial conditions.

The sliding surface $\sigma = 0$ for the Collatz system includes the set of points where x_k coincides with 2^n. Therefore, the distance from the state to the sliding surface can be represented as

$$\sigma(x_k) \triangleq \sigma_k = x_k - 2^n \tag{13}$$

where n is the value that gives 2^n closest to x_k and the smallest magnitude of σ. For example, $\sigma(5) = 1$ with $n = 2$, $\sigma(6) = 2$ with $n = 2$, and $\sigma(7) = -1$ with $n = 3$. This is analogous to the spacecraft example, where the function σ produced a generalized distance from the two-dimensional sliding surface through the origin.

A few points about Eq. (13) warrant further discussion. First, note that that the sliding surface is explicitly defined by the system dynamics in Eq. (1) — it is a property of the system. This is in contrast to sliding surfaces for feedback control systems, such as by Eq. (8) or (9) in the spacecraft example, which can be designed freely.

Second, analyzing the stability of a system relative to a sliding surface usually reduces the dynamics of a higher-order system to those of a first-order system. In the spacecraft example, there were two states but these were transformed into the single sliding variable σ for analysis. For the Collatz system, the state can take on the set of natural numbers, but the sliding surface in Eq. (13) can only take on a smaller subset of those numbers, namely powers of two. Furthermore, Eq. (13) represents a discontinuous set of points in the one-dimensional state space, whereas sliding surfaces used in SMC are typically continuous surfaces in multiple-dimension state spaces.

Last, note that the sliding surface can be said to be both a place and a dynamic. In the spacecraft example, $\sigma = 0$ in Eq. (9) represented a line through the state space — a place. It also represented a one-dimensional differential equation describing the evolution of σ — a dynamic. Similarly, $\sigma = 0$ represents a place for the Collatz system, which is the powers of two. Once on the surface, that state and all subsequent states are even, which restricts the dynamics of the motion to Eq. (1a).

Despite this analogy, it is difficult to use the same analysis from the spacecraft example to show stability of the Collatz system to its sliding surface. For example, again consider the trajectory starting from the initial condition $x_1 = 9$, which is shown as the blue points in Fig. 2. Following the strategy used for the spacecraft example, we could use the sliding surface defined in Eq. (13) and attempt the Lyapunov function

$$V(\sigma_k) = \frac{1}{2}\sigma_k^2 \tag{14}$$

with its time derivative approximated by the forward difference

$$\dot{V}(\sigma_k) = V(\sigma_{k+1}) - V(\sigma_k) \tag{15}$$

The ensuing time histories of the trajectory are shown in Fig. 7, including sliding and Lyapunov variables. Although the sliding phase is reached after 16 steps at the value 16, analysis of the Lyapunov function and its derivatives in Eqs. (14) and (15) offer little value in showing convergence to the limit cycle. This is because there does not seem to be an inherent stability of the Collatz system to the sliding surface, which is in contrast to the artificial stability intentionally designed into feedback systems using SMC. Again, perhaps other Lyapunov functions, maybe discontinuous or as summations of multiple functions, could provide more insight.

More likely, a successful approach for proving the conjecture could be the following. First note that unlike the powers of two, which create the sliding surface towards the origin, there appears to be no counterpart that takes the state to infinity along the set of odd numbers. This is because when an odd number is encountered, an even number follows, and then the subsequent number could be even or odd. There does not appear to be a sequence of numbers that are subsequently odd the way that powers of two are subsequently even given Eq. (1). The system therefore appears to be bounded from above by some potentially large but finite number. This bound is dependent upon the initial condition, and is potentially a non-uniform bound. The system is also bound from below by the limit cycle near the origin. Then confined to this region, and if there are no additional limit cycles in the state space, the state increases and decreases per Eq. (1) until a power of two is reached, from which the trajectory remains on the sliding surface and exhibits exponential decay to the limit cycle near the origin.

5 Conclusions

The Collatz Conjecture was viewed from the perspective of nonlinear system and feedback control theory. Within this framework, the number sequence was cast as a nonlinear difference equation, composed of two linear equations based on the instantaneous state value. Stability theory was discussed and a closed-loop behavior called a limit cycle was observed. It was shown that the system exhibits reaching and sliding phases with respect to a sliding surface, which behaves similarly to a feedback control approach called sliding mode control. Based on this analogy and an illustrative example, an approach for proving the conjecture was suggested.

No formal proof of the conjecture was given. Rather, an approach was offered for a potential proof. Depending on the initial value, the Collatz sequence appears to have a large but finite upper limit. In the examples considered, the trajectories increased and decreased in value, bounded below by the limit cycle and bounded from above by the upper bound, until a power of two was reached. Once this happened, the sliding surface was reached and the equations of motion resulted in subsequent numbers remaining powers of two as the sequence decayed exponentially to the limit cycle.

6 Acknowledgements

This research was supported by NASA under the Human Landing System (HLS) ground-based simulation task for the Artemis program. Helpful discussions with Joseph Galante and Robert Sanner are acknowledged and appreciated. This paper is dedicated to the memory of William Grauer Jr. and Joyce Spittel.

References

1. Lagarias, J.: *The Ultimate Challenge: The* $3x + 1$ *Problem*. American Mathematical Society, 2010.

2. Hartnett, K.: "Mathematician Proves Huge Result on 'Dangerous' Problem". Quanta Magazine, Dec. 2019.

3. Tau, T.: "Almost All Orbits of the Collatz Map Attain Almost Bounded Values". arXiv eprint 1909.03562, Sep. 2019.

4. Bellos, A.: "Collatz Conjecture in Color". `https://youtu.be/LqKpkdRRLZw`, accessed Aug. 2021.

5. Munroe, R.: "XKCD". `https://xkcd.com/710`, accessed Aug. 2021.

6. Barina, D.: "Convergence Verification of the Collatz Problem". *The Journal of Supercomputing*, 2020.

7. Khalil, H.: *Nonlinear Systems*. Prentice-Hall, 3rd ed., 2002.

8. Zadeh, L.; and Desoer, C.: *Linear System Theory: The State-Space Approach*. McGraw-Hill, 1963.

9. Ogata, K.: *Modern Control Engineering*. Prentice-Hall, 1970.

10. Slotine, J.; and Li, W.: *Applied Nonlinear Control*. Prentice-Hall, 1991.

11. Salle, J. L.; and Lefschetz, S.: *Stability by Liapunov's Direct Method*. Academic Press, 1961.

12. Åström, K.; and Murray, R.: *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton University Press, 2008.

13. Utkin, V.: *Sliding Modes and Their Application in Variable Structure Systems*. Mir Publishers, 1978.

14. Abbadi, M.; and Spurgeon, S.: "Growth Dynamics of Algal-bacterial Cocultures: A Control Engineering Perspective". *European Control Conference*, 2018.

15. Anon.: CSM / LM Spacecraft Operational Data Book: Volume III Mass Properties. NASA SNA-8-D-027, Aug. 1969.

16. Stengel, R.: "Manual Attitude Control of the Lunar Module". *Journal of Spacecraft and Rockets*, vol. 7, no. 8, Aug. 1970, pp. 941–948.

Figures

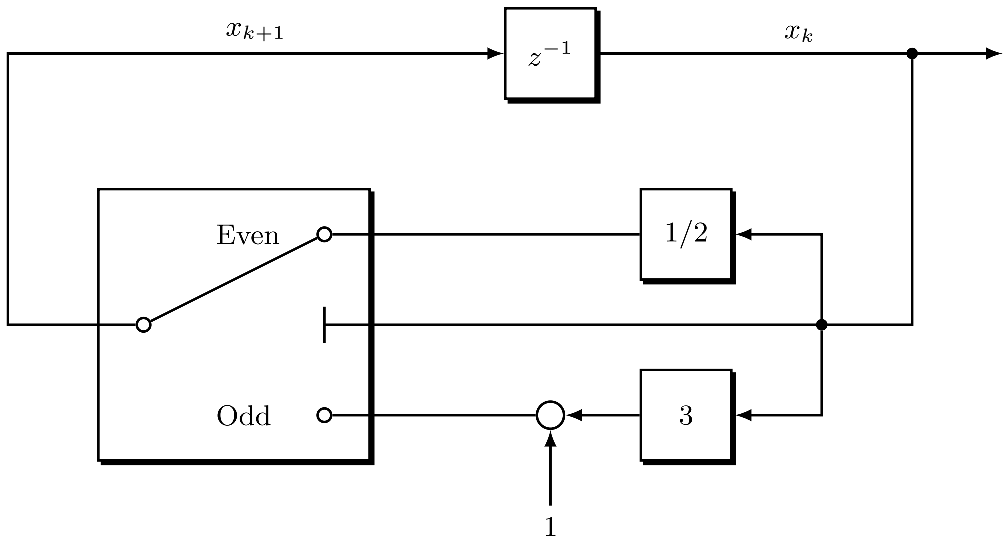

Figure 1: Block diagram representation of the Collatz system.

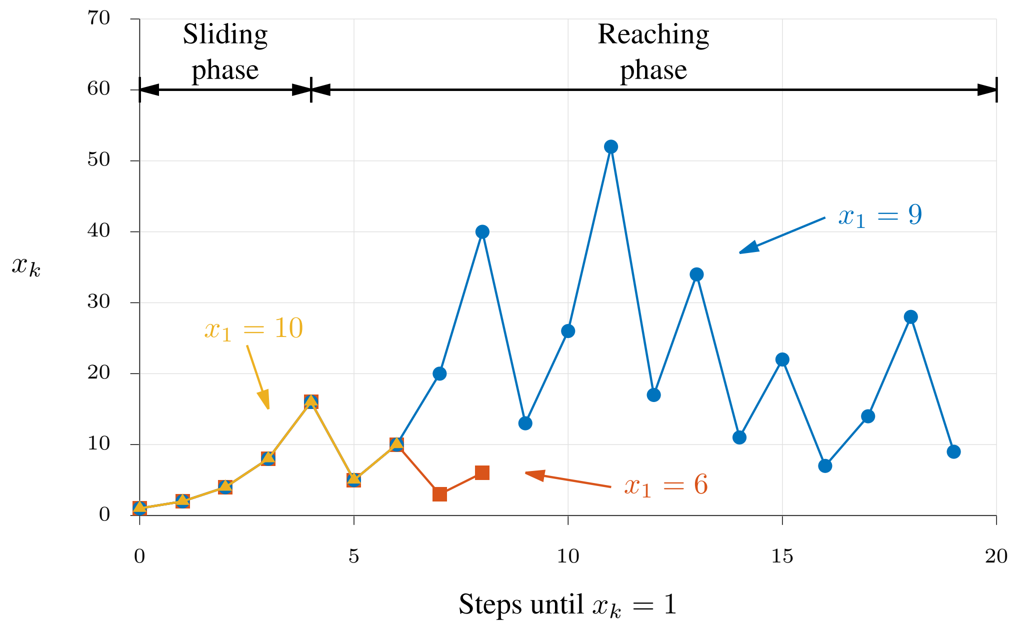

Figure 2: Solution trajectories of the Collatz system from three initial conditions.

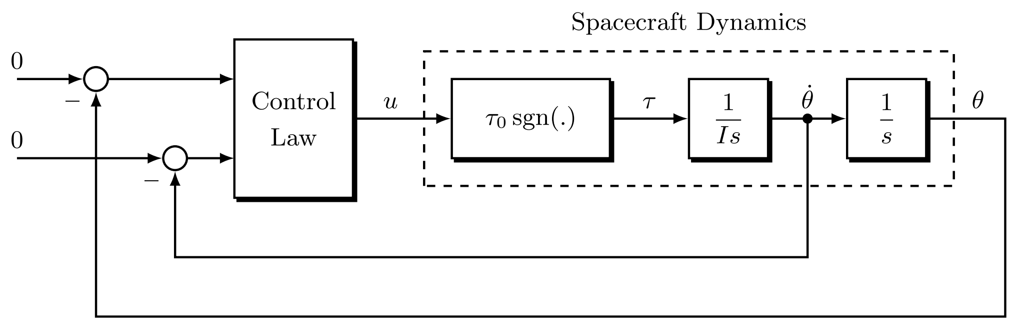

Figure 3: Block diagram of the closed-loop spacecraft attitude dynamics.

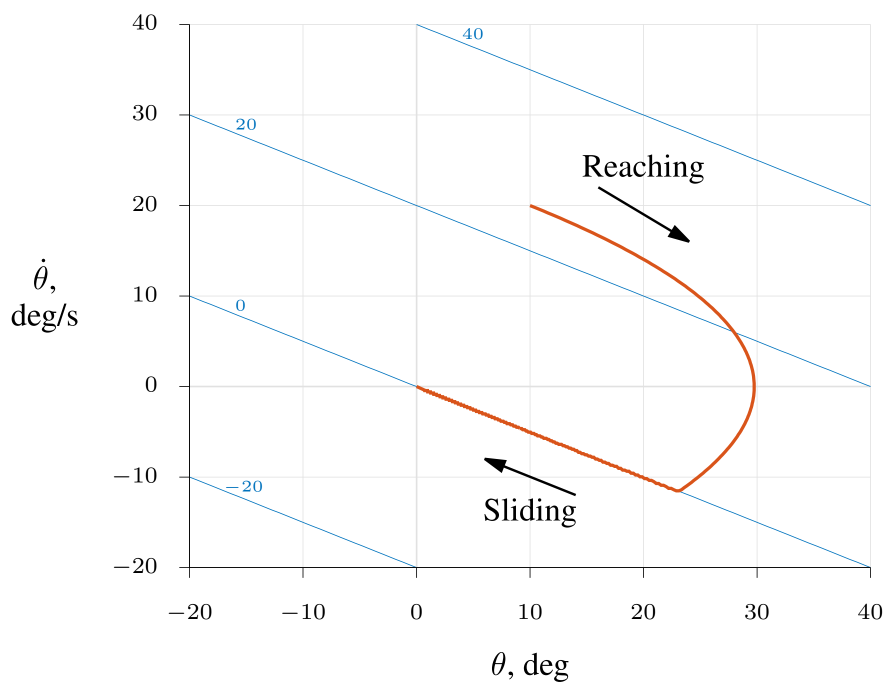

Figure 4: Phase portrait showing a state trajectory (red) and lines of constant σ (blue).

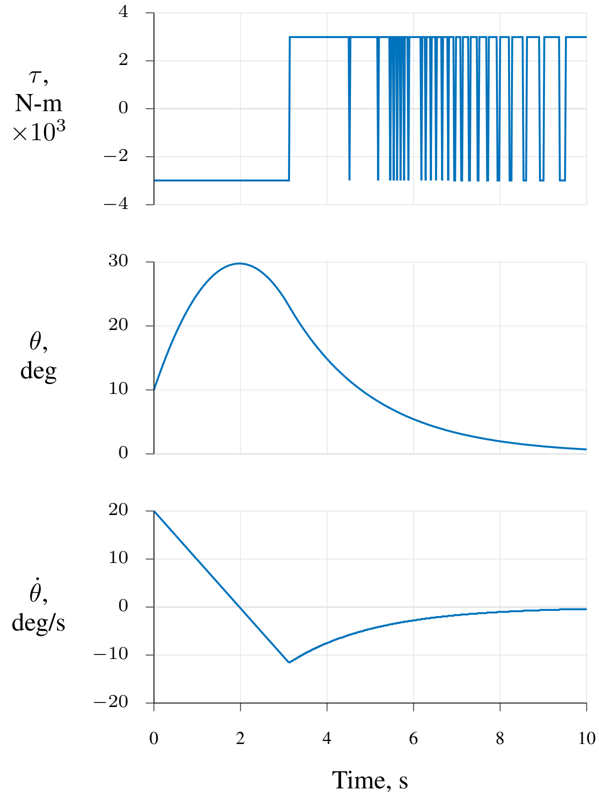

Figure 5: Physical variables.

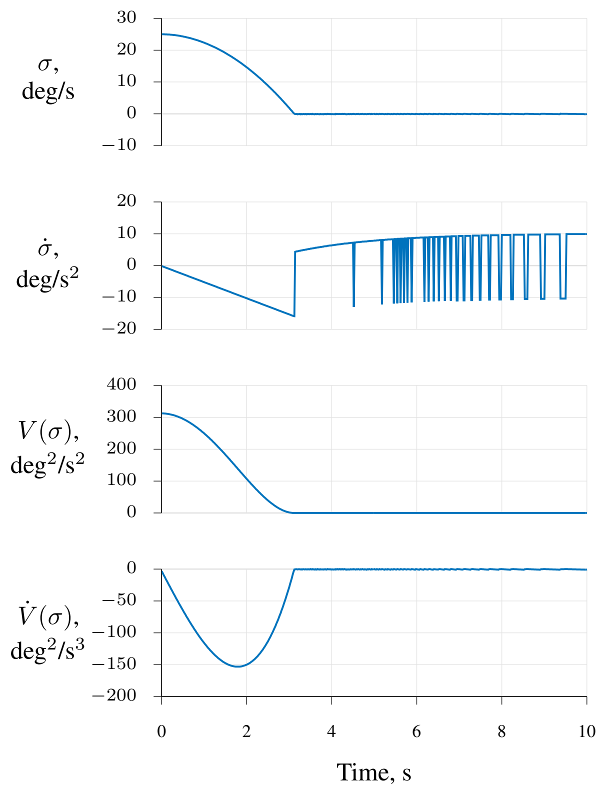

Figure 6: Simulated time histories from $\theta(0) = 10$ deg and $\dot{\theta}(0) = 20$ deg/s.

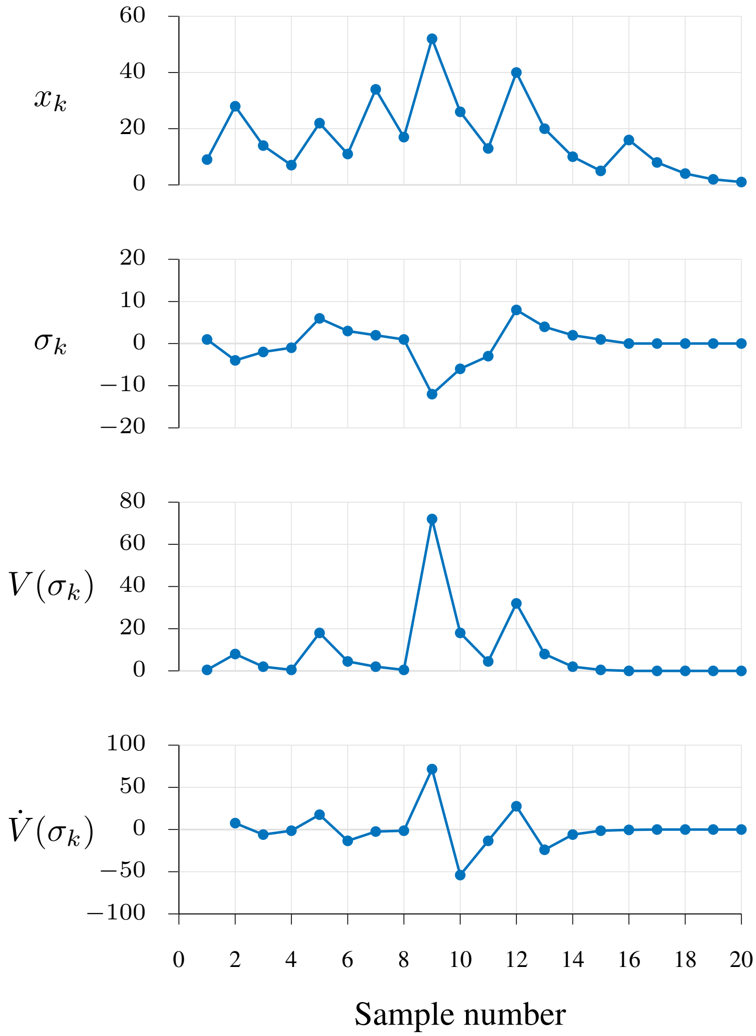

Figure 7: Trajectory of Collatz system from $x_1 = 9$ including sliding and Lyapunov variables.

REPORT DOCUMENTATION PAGE	*Form Approved* *OMB No. 0704–0188*

The public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Department of Defense, Washington Headquarters Services, Directorate for Information Operations and Reports (0704-0188), 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to any penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.
PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
01-09-2021	Technical Memorandum	

4. TITLE AND SUBTITLE	**5a. CONTRACT NUMBER**
Analogy Between the Collatz Conjecture and Sliding Mode Control	**5b. GRANT NUMBER**
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	**5d. PROJECT NUMBER**
Jared A. Grauer Langley Research Center, Hampton, Virginia	**5e. TASK NUMBER**
	5f. WORK UNIT NUMBER 954879.14.23.01

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	**8. PERFORMING ORGANIZATION REPORT NUMBER**
NASA Langley Research Center Hampton, Virginia 23681-2199	L–?
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	**10. SPONSOR/MONITOR'S ACRONYM(S)**
National Aeronautics and Space Administration Washington, DC 20546-0001	NASA
	11. SPONSOR/MONITOR'S REPORT NUMBER(S) NASA/TM–20210019810

12. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified-Unlimited
Subject Category 08
Availability: NASA STI Program (757) 864-9658

13. SUPPLEMENTARY NOTES

14. ABSTRACT

The Collatz Conjecture is a famous mathematics problem that is simple to state and understand but remains unsolved. In this paper, the problem is recast as a discrete-time nonlinear system and viewed in a new light from the perspective of nonlinear systems and feedback control theory. In particular, connections are made between the Collatz sequence of numbers and the behavior of closed-loop dynamical systems designed using a feedback control method called sliding mode control. Trajectories of such systems are characterized by a reaching phase and a sliding phase, the latter of which exhibits exponential convergence. As sliding mode control design is rooted in Lyapunov stability theory, the analogy suggests a new direction for proving or disproving the Collatz Conjecture. Although several possible approaches are discussed, no formal proof is given in this paper.

15. SUBJECT TERMS

Frequency response, aircraft system identification, maximum likelihood estimation, output error parameter estimation, multisine inputs

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			STI Help Desk (email: help@sti.nasa.gov)
U	U	U	UU	22	**19b. TELEPHONE NUMBER** ***(Include area code)*** (757) 864-9658