# Towards Formalization of Advanced Linear Algebra with Applications to Dynamical Systems using PVS

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# Why Linear Algebra?

1. Dynamical Systems and Stability
   Let $\mathbf{x}_k \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Given a safety-critical dynamical system with initial condition $\alpha$:
   i) discrete:
   $$\begin{cases} \mathbf{x}_{k+1} = f(\mathbf{x}_k) \\ \mathbf{x}_0 = \alpha \end{cases}$$
   ii) continuous:
   $$\begin{cases} \mathbf{x}'(t) = f(\mathbf{x}(t)) \\ \mathbf{x}(0) = \alpha \end{cases}$$
   Reasoning about dynamics (either discrete or continuous) often reduces to reasoning about linear algebra.
2. Eigenvalues & Eigenvectors
   Stability analysis of such a dynamical system often reduces to an eigenvalue problem. That is, finding a non-zero $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.


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# Motivating Example

Consider an example of a discrete dynamical system from safety "vehicle" control[1]. The dynamics of such a system are modeled in the following animation (show animation):

[1] Mahyar R. Malekpour: *Achieving Equilibrium for Dense, Integrated Vehicle Navigation*, AIAA SciTech 2021, Virtual, pp. 8, January 2021.



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# Motivating Example

The example considered on the last slide can be formulated as a discrete-time dynamical system. We want to be able to find (approximately) the eigenvalues and eigenvectors to study stability:

$$\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k \\ \mathbf{x}_0 = \alpha \end{cases}$$

In this example, we define:

- $\mathbf{x} \in \mathbb{R}^{n+1}$ gives position of vehicles, where $n$ is number of vehicles
- $p = \dfrac{1}{m}$ is the $\%$ of distance traveled to the front vehicle, where $m \geq 1$
- $\mathbf{b} = (1-p)\mathbf{e}_1 + p\mathbf{e}_n$ and $B$ is the circulant matrix specified by $\mathbf{b}$
- A (i.e. the dynamics matrix) has structure:

$$\left(\begin{array}{ccc|c} & & & 1 \\ & B & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right)$$


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# Motivating Example

$$\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k \\ \mathbf{x}_0 = \alpha \end{cases}$$

Challenges presented in this example:

- $A$ has complex eigenvalues/eigenvectors $\implies$ need multivariate complex reasoning in PVS
- $A$ is arbitrarily large $\implies$ challenging to find roots of characteristic polynomial
- $A$ is **not** diagonalizable $\implies$ not easy to reason about dynamics or repeated multiplication by A

$$A = \left(\begin{array}{ccc|c} & & & 1 \\ & B & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 & 1 \end{array}\right)$$


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Goal: Develop multivariate complex library to reason about eigenvalues and eigenvectors, and provide rigorous approximations in PVS


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# Outline

- Elements of complex 2 x 2 matrices
- Stability in higher dimensions
- Power Method for approximating eigenvalues and eigenvectors
- Future directions and further work


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# Formalization of complex 2 x 2 matrices

Formalized in PVS:

- Basic operations on complex 2 x 2 complex matrices
- Scalar invariants (i.e. trace, determinant)
- Characteristic polynomial in terms of trace and determinant
- Matrix inverse
- Inverse using Cayley-Hamilton theorem (specified, not verified)
- Eigenvalues and eigenvectors
- Stability criterion for discrete-time dynamical systems


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- **Characteristic polynomial in terms of trace and determinant**
- Matrix inverse
- Inverse using Cayley-Hamilton theorem (to be proven)
- **Eigenvalues and eigenvectors**
- **Stability criterion for discrete-time dynamical systems**


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# A simple example of a discrete dynamical system

## Fibonacci's rabbit population

A population of rabbits can be modeled by the second order difference equation $x_{n+1} = x_n + x_{n-1}$ with initial values $x_0 = 0$, $x_1 = 1$ and $y_n = x_{n-1}$. It can be rewritten as:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$


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In PVS, it is possible to:

- Find the scalar invariants
- Find the characteristic polynomial
- Find the eigenvalues and eigenvectors
- Determine stability of equilibrium points


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- $\text{tr}(A) = 1 + 0 = 1$ and $\det(A) = 0(-1) - 1(1) = -1$


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- $p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - \lambda - 1$



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- $\lambda^2 - \lambda - 1 = 0 \implies \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$


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In PVS, it is possible to:

- Find the scalar invariants
- Find the characteristic polynomial
- Find the eigenvalues and eigenvectors
- **Determine stability of equilibrium points**

- Equilibrium point is the origin, and is a saddle point (unstable). That is, whenever one of the eigenvalues lies in the complex unit disk, i.e. $|\lambda_1|^2 < 1$ and the other outside, i.e. $|\lambda_2|^2 > 1$


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- Elements of complex 2 x 2 matrices
- Stability in higher dimensions
- Power Method for approximating eigenvalues and eigenvectors
- Future directions and further work


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# Challenges of determining stability in higher dimensions

**Stability of fixed points in n-dimensions**

If $\mathbf{x}_* \in \mathbb{R}^n$ is a fixed point of $f$, then $\mathbf{x}_*$ is stable whenever all of the eigenvalues of $Df(\mathbf{x}_*)$ (matrix of partial derivatives) all lie inside the unit disk, i.e. $|\lambda_i|^2 < 1$ for all $i = 1, \cdots, n$

- In practice, it is quite difficult to find eigenvalues of systems which are 5-dimensional or higher. Why?


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- This is because the characteristic polynomial will be of degree 5 or higher. Polynomial equations of degree 5 or higher are unsolvable (leads to Galois theory).


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- Modeling dynamics is often equivalent to repeated matrix-vector multiplication, which is in essence the power method.


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- This is because the characteristic polynomial will be of degree 5 or higher. Polynomial equations of degree 5 or higher are unsolvable (leads to Galois theory).
- Modeling dynamics is often equivalent to repeated matrix-vector multiplication, which is in essence the power method.
- We shall use the power method to rigorously approximate eigenvalues in PVS.



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# Power Method for approximating eigenvalues and eigenvectors

## The Power Method

Assume $A$ is a complex $n \times n$ matrix with its eigenvalues ordered as $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ and that $A$ has a complete set of eigenvectors. Let $\mathbf{z}_0$ be an initial guess for the eigenvector. An approximation for the (dominant) eigenpair is given by:

$$(\mathbf{z}_k, \lambda_k) = \left(\frac{A\mathbf{z}_{k-1}}{\|A\mathbf{z}_{k-1}\|}, \frac{\bar{\mathbf{z}_k}^T A\mathbf{z}_k}{\bar{\mathbf{z}_k}^T \mathbf{z}_k}\right)$$

- Note that this method also applies to real matrices with a full set of eigenvectors.
- By dominant, we mean the eigenvalue $\lambda_1$, which is largest in modulus.
- In the approximation of the eigenvector, any norm will do, due to norm equivalency.


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Goal: Specify in PVS a fully executable power method and prove results about convergence


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# Power Method for approximating eigenvalues and eigenvectors

To achieve our goal, we propose the following step-by-step approach:

1. Specify an executable power method in PVS for real, diagonalizable $n \times n$ matrices.
2. Prove that the real power method is equivalent to the complex power method.
3. Convergence of the real power method will follow from convergence of the complex version, by previous step.

Formalized in PVS (for complex $n \times n$ matrices):

- Basic properties of eigenvalues/eigenvectors
- Matrix integer powers and their eigenvalues
- Complex norms (e.g. $l_1, l_2, l_\infty$)
- Real power method using $l_\infty$ norm for eigenvector approximation
- "Lift" real power method to complex matrices
- Result stating equivalence between complex and real power methods
- Example of using the real or complex power method


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- Example of using the real or complex power method
- Definition of diagonalizability
- Circulant matrices

Question: Why do we want to use the $l_\infty$ norm to specify the real power method?

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Question: Why do we want to use the $l_\infty$ norm to specify the real power method?

Answer: We want to have an **executable norm** in PVS, without any square roots! Moreover, none of the complex norms are executable (because of modulus)-hence why we insist on the real power method.


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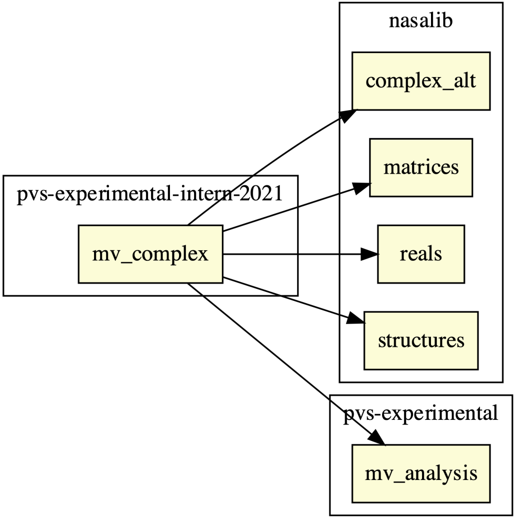
# Power Method for approximating eigenvalues and eigenvectors





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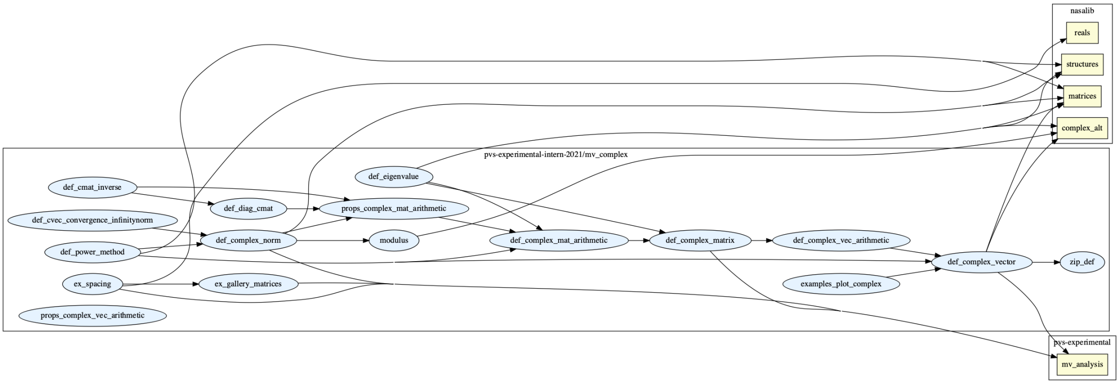
# Power Method for approximating eigenvalues and eigenvectors





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# Power Method for approximating eigenvalues and eigenvectors

Recall our motivational example. Can we apply the power method directly?

$$\begin{cases} \mathbf{x}_{k+1} = A\mathbf{x}_k := A^{k+1}\mathbf{x}_0 \\ \mathbf{x}_0 = \alpha \end{cases}$$

with:

$$A = \left(\begin{array}{ccc|c} & B & & \begin{matrix} 1 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & \dots & 0 & 1 \end{array}\right), B \text{ is a circulant matrix}$$

- $A$ is **not** diagonalizable, so the power method does not guarantee convergence to an eigenvector
- However, there is still hope. Instead of considering position $\mathbf{x}_k$, we consider velocities $\mathbf{v}_k := \mathbf{x}_{k+1} - \mathbf{x}_k$.
- Indeed, $\mathbf{v}_k := \mathbf{x}_{k+1} - \mathbf{x}_k =: A^{k+1}\mathbf{x}_0 - A^k\mathbf{x}_0 = A^k(A - I)\mathbf{x}_0 = A^k\mathbf{v}_0$
- To try: Apply our executable power method in PVS to velocities, with initial guess $\mathbf{v}_0 = (A - I)\mathbf{x}_0$


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# Future directions and further work

1. Proof of convergence of the power method in PVS
2. Specify matrix exponential $e^{At}$ and properties
3. Apply above to solve simple ODEs of the form $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}(t)$
4. Extend definition of trace, determinant, characteristic polynomial for $n \times n$ complex matrices
5. Specify the inverse power method (for finding smallest eigenvalue in modulus)
6. Jordan canonical forms
7. Block power method using QR factorization for finding multiple eigenvalues


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Summary:

1. Made contributions to the complex vectors and matrices libraries in PVS
2. Had fun while doing it!
3. Total Proofs: $275$, Lines of Spec: $2{,}001$, Number of Spec files: $17$


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