



Boundary-Layer Instability and Transition Prediction

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AIAA Short Course on
Hypersonic Propulsion Concepts: Design, Control, Operation
and Testing

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Notes

- This lecture is based on previous research work collaborated with many colleagues over the past decades
 - I'd like to acknowledge Mujeeb Malik, Fei Li, Meelan Choudhari, Heather Kline, and Pedro Paredes
- In many places, the focus is on the theory or flow physics without mentioning origins/authors of the research, interested readers can trace them with references provided in the slides
- Theoretical formulation follows NASA's LSTRAC software manual (NASA TM 2004-213233)

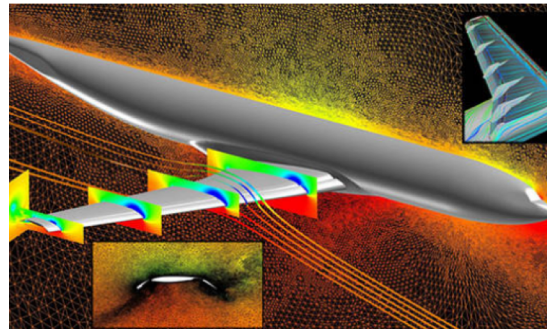
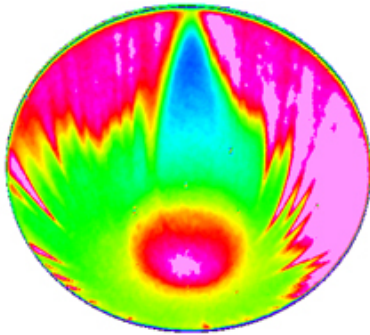


Outline of the Lecture

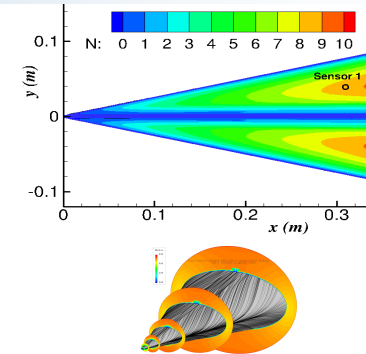
1. Introduction
2. Boundary-layer stability theory and computations
 - Linear stability theory
 - Linear parabolized stability equations
3. Transition prediction using N-factor method
 - Transition mechanisms and modal instability
 - Nonmodal, transient growth
4. Absolute amplitude methods and DNS
 1. Nonlinear PSE
 2. Biglobal stability (2D eigenvalue problems)
5. 3D boundary layer
6. Chemical/thermal nonequilibrium flows
7. Summary



Transition Prediction in CFD Computations



<https://www.nasa.gov/aeroresearch/programs/tacp/ttt/cfd-vision-2030-study/>



- NASA's CFD Vision 2030 study and the Revolutionary Computational Aerosciences (RCA) subproject under T³ identify *physics-based transition prediction and modeling* as one of the technical challenges and areas for further development
- Transition modeling in CFD solvers
 - RANS-based approach
 - Solving additional transport equations for modeled quantities to predict transition onset and turn on turbulent eddy viscosity accordingly
 - Popular due to ease of integration to existing CFD codes
 - **Physics-based approach (non-RANS)**
 - Using either Linear Stability Theory (LST) or parabolized stability equations (PSE) approach
 - Solving a separate set of perturbation equations on a body-fitted coordinate system and perform N factor integration
 - NASA's LSTRAC transition prediction software
 - Stringent meanflow grid requirement and higher-level of expertise required
 - More compliant to transition flow physics but harder to integrate with existing CFD solvers
 - On the way to be integrated into structured or unstructured mesh CFD solvers
- Transition prediction/control important for hypersonic propulsion

Laminar-Turbulent Transition (I)

Morkovin 1994

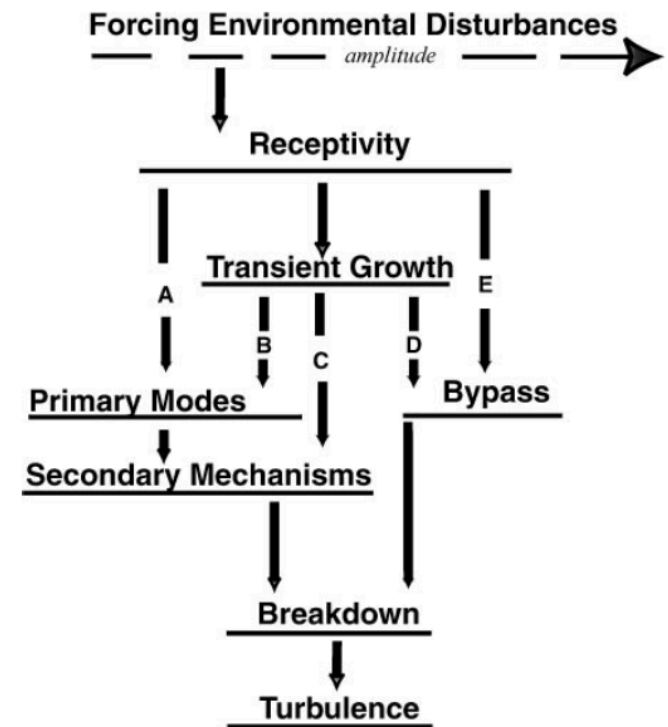
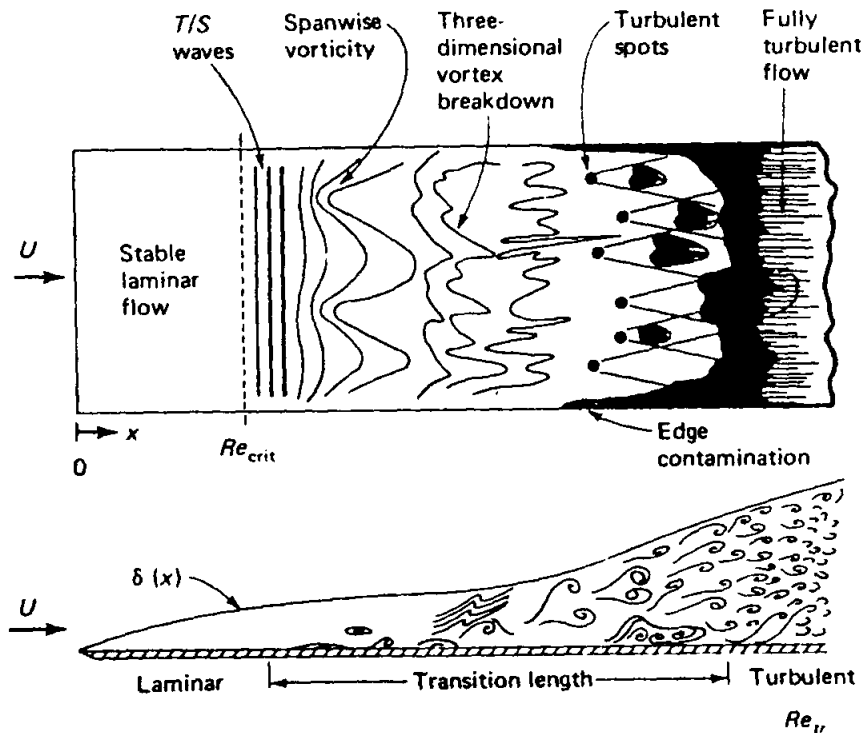


FIGURE 4.33 Transition to turbulence over a flat plate. (From White, 1993.)

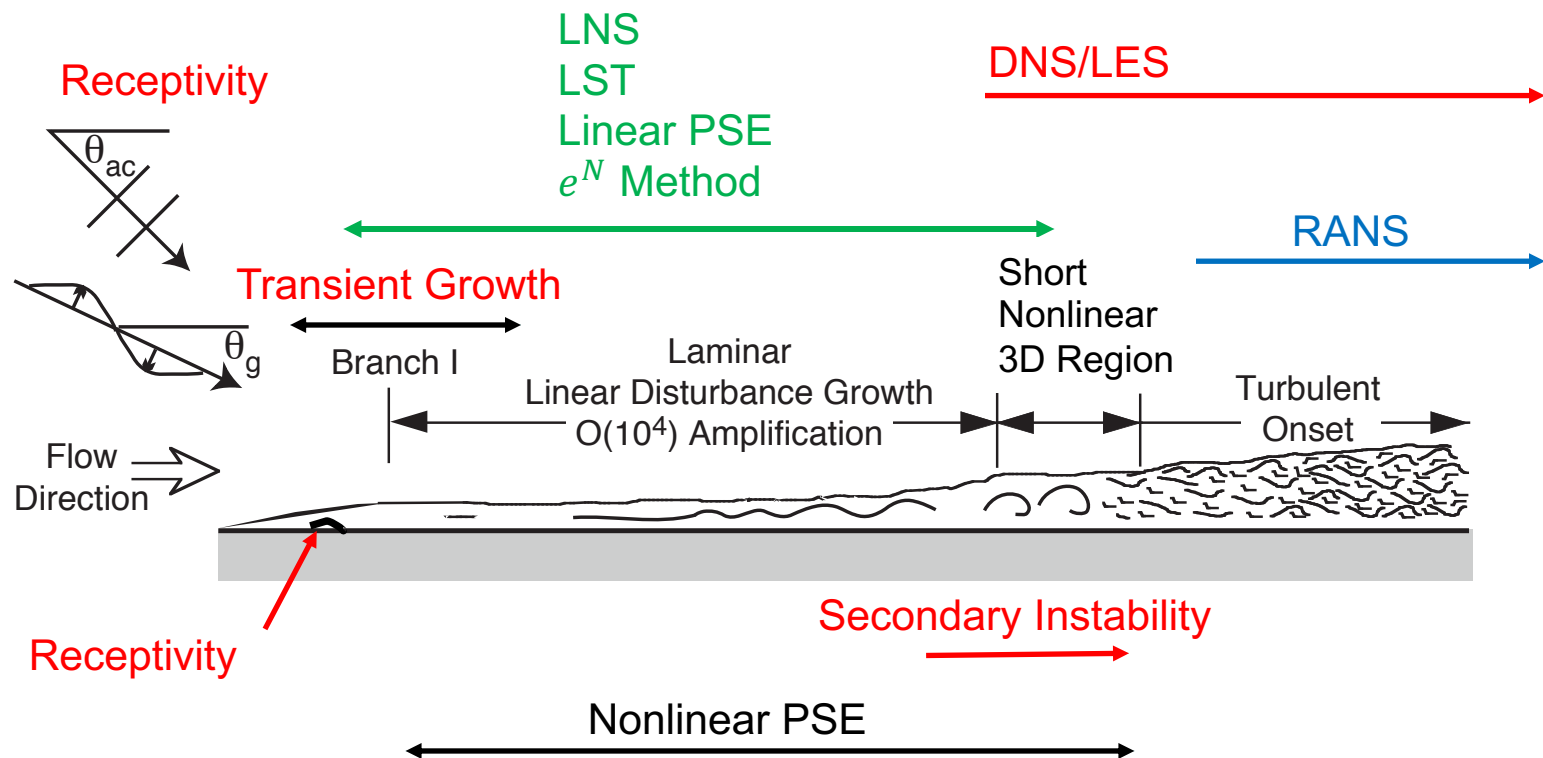
https://commons.wikimedia.org/wiki/File:Path_to_transition.png

- Under very noisy environment like in turbomachines, bypass transition more likely to happen
- Receptivity process leads to the internalization of environmental disturbances into the boundary layer, will not be discussed in this lecture due to time limitation



Laminar to Turbulent Transition (II)

- Usual route to transition and suggested analysis tools



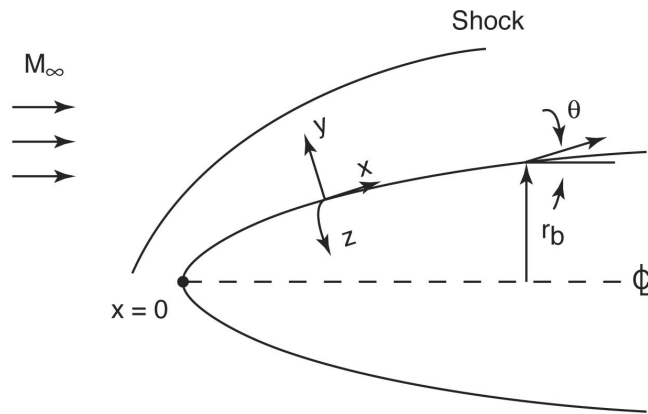


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Navier-Stokes Equations in 2D Body-fitted Curvilinear Coordinates



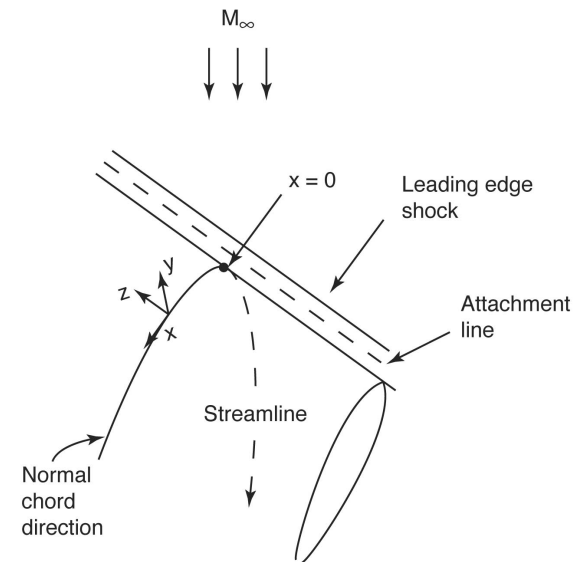
$$dl = \sqrt{(h_1 dx)^2 + dy^2 + (h_3 dz)^2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right] = - \nabla p + \frac{1}{R} (\nabla [\lambda (\nabla \cdot \vec{V})] + \nabla \cdot [\mu (\nabla \vec{V} + \nabla \vec{V}^T)])$$

$$\rho C_p \left[\frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T \right] = \nabla \cdot (k \nabla T) + \frac{\partial p}{\partial t} + (\vec{V} \cdot \nabla) p + \frac{\Phi}{R}$$

$$\Phi = \lambda (\nabla \cdot \vec{V})^2 + \frac{\mu}{2} [\nabla \vec{V} + \nabla \vec{V}^T]^2$$



$$p = \rho \mathcal{R} T$$

s = Stokes parameter

R = Reynolds Number



Navier-Stokes Perturbation Equations

- Assume steady-state mean flow exists, dependent variables expressed as

$$u = \bar{u} + u', v = \bar{v} + v', w = \bar{w} + w'$$

$$p = \bar{p} + p', \rho = \bar{\rho} + \rho', T = \bar{T} + T'$$

$$\mu = \bar{\mu} + \mu', \lambda = \bar{\lambda} + \lambda', k = \bar{k} + k'$$

- Navier-Stokes equations in perturbation form:

$$\phi = (p', u', v', w', T')^T$$

$$\begin{aligned} \Gamma^l \frac{\partial \phi}{\partial t} + \frac{A^l}{h_1} \frac{\partial \phi}{\partial x} + B^l \frac{\partial \phi}{\partial y} + \frac{C^l}{h_3} \frac{\partial \phi}{\partial z} + D^l \phi = \frac{1}{R_0} \left(\frac{V_{xx}^l}{h_1^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{V_{xy}^l}{h_1} \frac{\partial^2 \phi}{\partial x \partial y} + V_{yy}^l \frac{\partial^2 \phi}{\partial y^2} \right. \\ \left. + \frac{V_{xz}^l}{h_3} \frac{\partial^2 \phi}{\partial x \partial z} + \frac{V_{yz}^l}{h_3} \frac{\partial^2 \phi}{\partial y \partial z} + \frac{V_{zz}^l}{h_3^2} \frac{\partial^2 \phi}{\partial z^2} \right) \end{aligned}$$

- Direct numerical simulations (DNS) equations in perturbation form
- Better accuracy for small perturbations as compared to total flow variables in conservative form in most CFD codes
- All coefficient matrices are functions of both mean and perturbed quantities

$$A = A^l + A^n, \quad A^l = A^l(\bar{p}, \bar{u}, \bar{v}, \bar{w}, \bar{T}, \dots), \quad A^n = A^n(\bar{p}, \bar{u}, \dots, p', u', \dots)$$



Linearized Navier-Stokes (LNS) Equations

- Assume small perturbations, $u' \ll \bar{u}$, $T' \ll \bar{T}$, etc.
- Neglect all nonlinear terms (such as $u'T'$ or $\rho'u'v'$)

$$\Gamma^l \frac{\partial \phi}{\partial t} + \frac{A^l}{h_1} \frac{\partial \phi}{\partial x} + B^l \frac{\partial \phi}{\partial y} + \frac{C^l}{h_3} \frac{\partial \phi}{\partial z} + D^l \phi = \frac{1}{R_0} \left(\frac{V_{xx}^l}{h_1^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{V_{xy}^l}{h_1} \frac{\partial^2 \phi}{\partial x \partial y} + V_{yy}^l \frac{\partial^2 \phi}{\partial y^2} + \frac{V_{xz}^l}{h_3} \frac{\partial^2 \phi}{\partial x \partial z} + \frac{V_{yz}^l}{h_3} \frac{\partial^2 \phi}{\partial y \partial z} + \frac{V_{zz}^l}{h_3^2} \frac{\partial^2 \phi}{\partial z^2} \right) \quad (\text{LNS1})$$

- Solution requires accurate integration in time until a quasi steady-state is reached
- Instability waves asymptote to a temporal-spatial wave solution at large time
- Consider large-time wave solution for a single disturbance mode with a distinct frequency in time and wave number in at least one (spanwise, or z-) direction

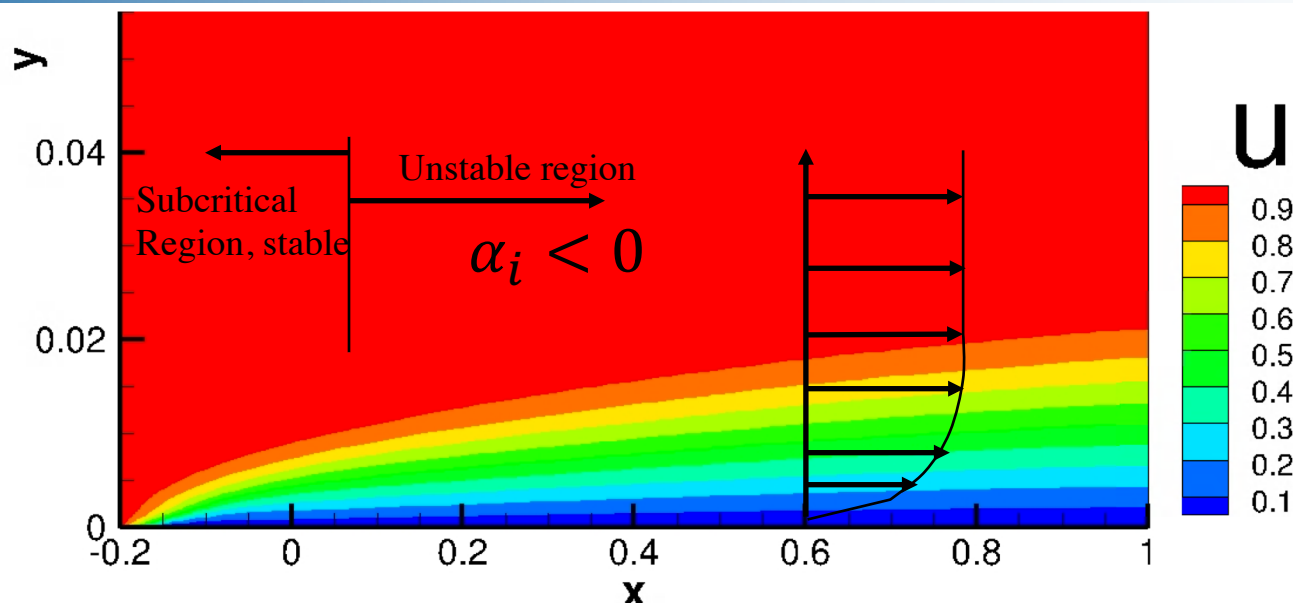
$$\phi = \chi(x, y) e^{i(\beta z - \omega t)}$$

$$\left(\frac{A^l}{h_1} - \frac{i\beta V_{xz}^l}{h_3 R_0} \right) \frac{\partial \chi}{\partial x} + \left(B^l + \frac{i\beta V_{yz}^l}{h_3 R_0} \right) \frac{\partial \chi}{\partial y} + \left(D^l - i\omega \Gamma^l + \frac{i\beta C^l}{h_3} + \frac{\beta^2 V_{zz}^l}{h_3^2 R_0} \right) \chi = \frac{1}{R_0} \left(\frac{V_{xx}^l}{h_1^2} \frac{\partial^2 \chi}{\partial x^2} + \frac{V_{xy}^l}{h_1} \frac{\partial^2 \chi}{\partial x \partial y} + V_{yy}^l \frac{\partial^2 \chi}{\partial y^2} \right) \quad (\text{LNS2})$$

- Steady-state equations, no time iteration required
- Direct solution possible but requires solution of a large sparse matrix linear equation $L \chi = f$
- Iterative solution possible if matrix L is diagonally dominant
- Governs all disturbances, acoustic, entropy, or vorticity waves



Linear Stability Theory (LST)



- Quasi-parallel assumption *Approximation through mode shape, not omitting terms in LNS

$$\chi(x, y) = \psi(y)e^{i\alpha x} \quad \text{Substitute into (LNS2)}$$

$$\phi(x, y, z, t) = \psi(y)e^{i(\alpha x + \beta z - \omega t)}$$

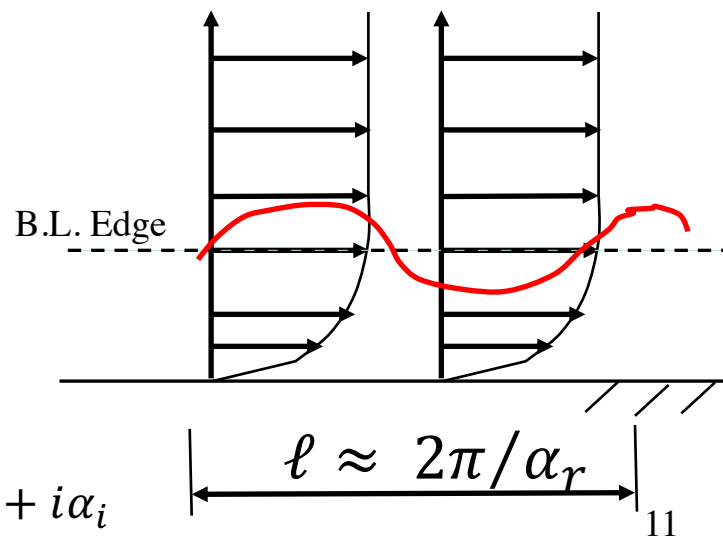
$$\left(B^l + \frac{i\beta V_{yz}^l}{h_3 R_0} - \frac{i\alpha V_{xy}^l}{h_1 R_0} \right) \frac{d\psi}{dy} + \left(D^l - i\omega \Gamma^l + \frac{i\alpha A^l}{h_1} + \frac{\alpha^2 V_{xx}^l}{h_1^2 R_0} - \frac{i\beta V_{xz}^l}{h_3 R_0} \right. \\ \left. + \frac{i\beta C^l}{h_3} + \frac{\beta^2 V_{zz}^l}{h_3^2 R_0} \right) \psi = \frac{V_{yy}^l}{R_0} \frac{d^2 \psi}{dy^2}$$

$$y = \hat{u} = \hat{v} = \hat{w} = \hat{T} = 0$$

$$\hat{u} = \hat{v} = \hat{w} = \hat{T} = 0, \quad y \rightarrow \infty$$

$$L\psi = 0 \quad (\text{ODE})$$

- Spatial eigenvalue problems $\alpha = \alpha(\omega_r, \beta) = \alpha_r + i\alpha_i$





Discretized LST Solutions

$$\frac{\partial \psi}{\partial \eta} = \frac{-\psi_{i,j+2} + 8\psi_{i,j+1} - 8\psi_{i,j-1} + \psi_{i,j-2}}{12\Delta\eta}$$

$$\frac{\partial^2 \psi}{\partial \eta^2} = \frac{-\psi_{i,j+2} + 16\psi_{i,j+1} - 30\psi_{i,j} + 16\psi_{i,j-1} - \psi_{i,j-2}}{12\Delta\eta^2}$$

- Use 4th-order central differencing, form a banded matrix operator

$$L\psi = 0$$

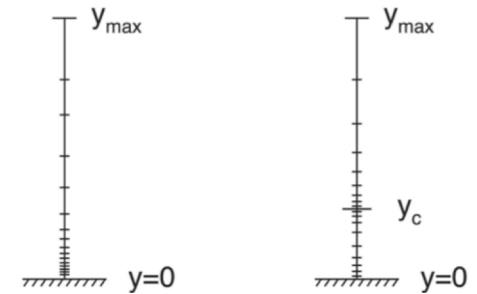
$$(L_0 + \alpha L_1 + \alpha^2 L_2) \psi = \omega_r L_3 \psi$$

- Nonlinear in α , approximated global eigenvalue search

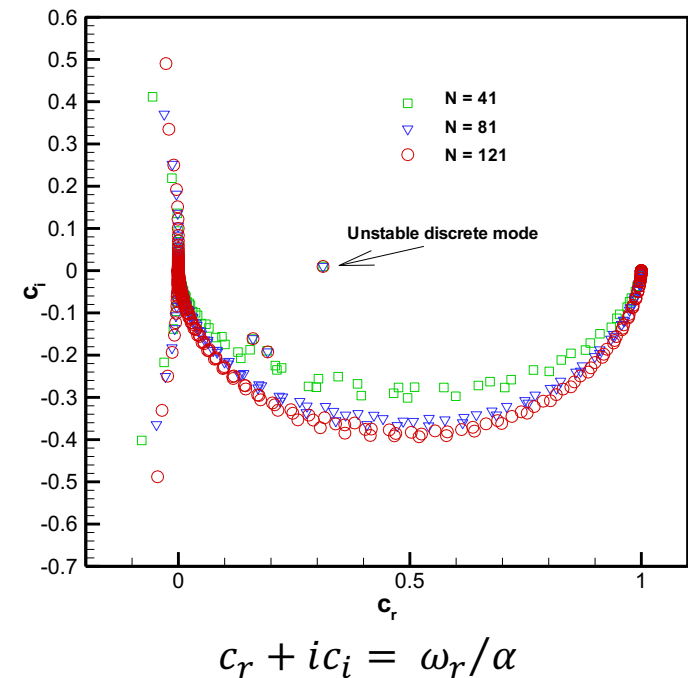
$$(L_0 + \alpha L_1) \psi = \omega_r L_3 \psi$$

- Solve by QR or LR algorithm, pick discrete modes α_0
- Refine eigenvalues by Newton's method
 - Pick a known quantity, τ (e.g. $\hat{u} = 0$)
 - Iterate until that quantity is converged via

$$\alpha^{n+1} = \alpha^n - \tau^n / \left(\frac{\partial \tau}{\partial \alpha} \right)^n$$

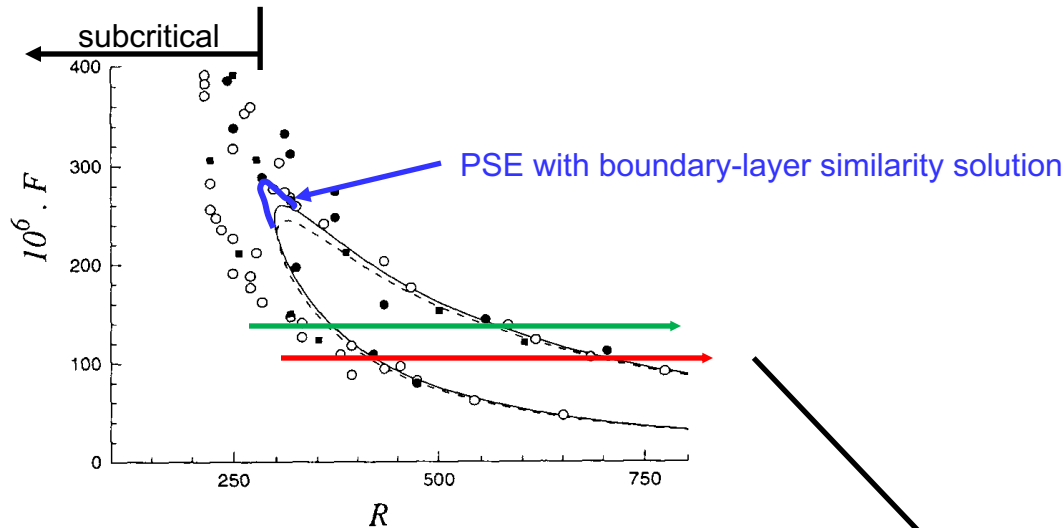


Total N points





Tollmien-Schlichting (TS) Wave over Incompressible Flat Plate



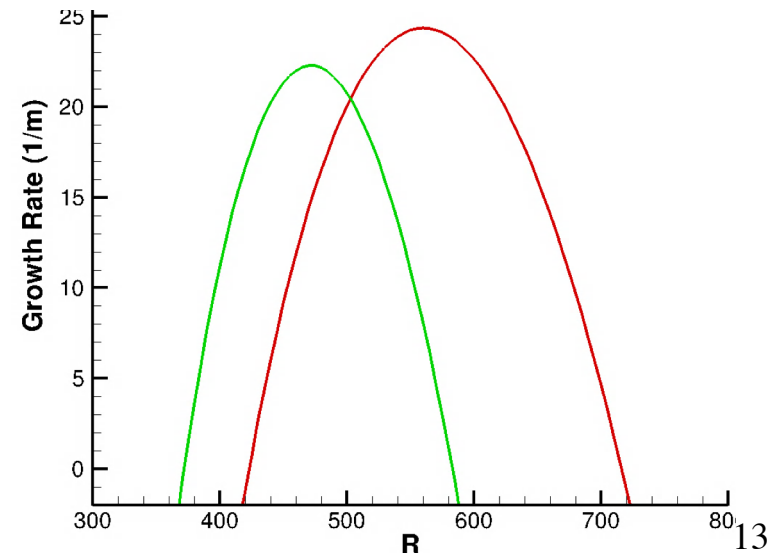
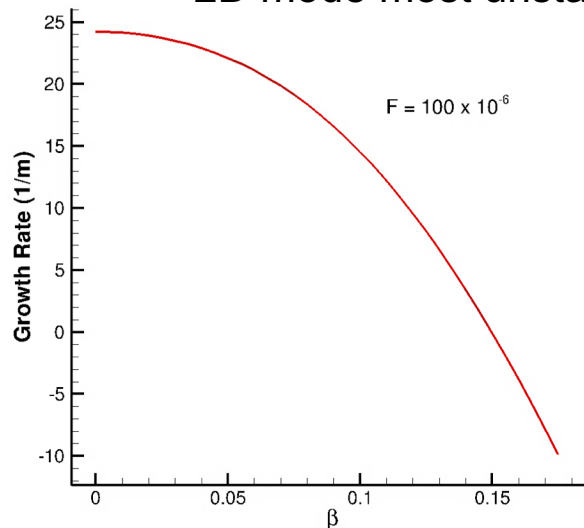
$$\ell = \sqrt{\frac{\nu_e x}{U_e}}$$

$$R = U_e \ell / \nu_e$$

$$F = \frac{2\pi \nu_e f}{U_e^2}$$

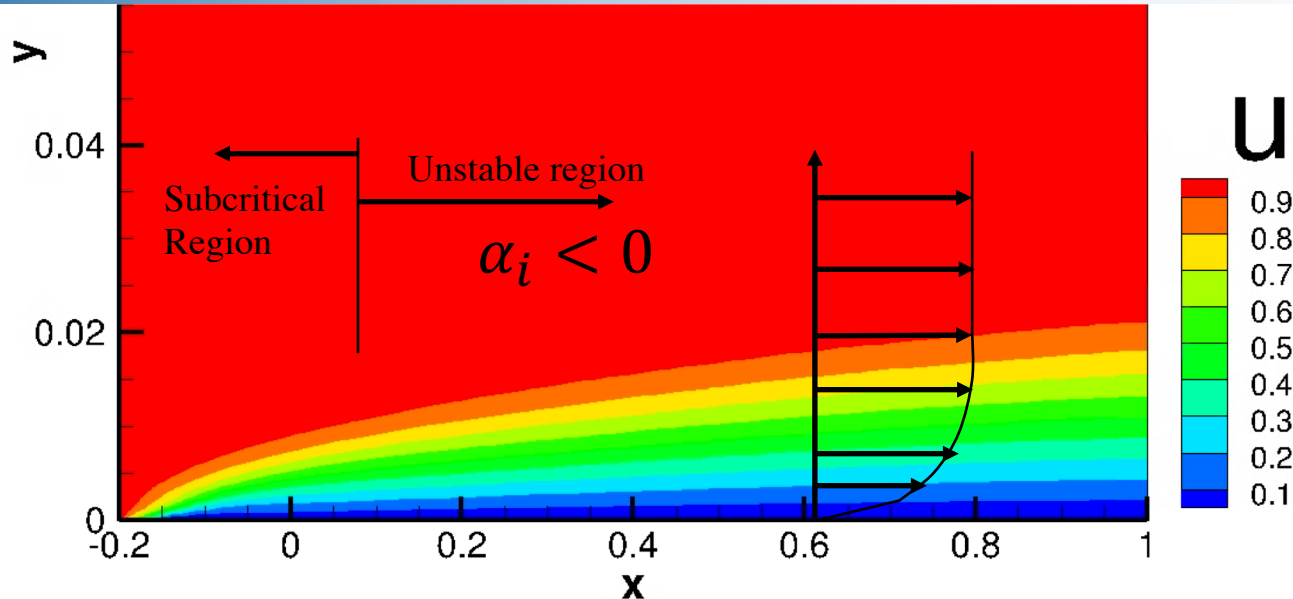
FIGURE 4.6 Neutral curve for flat-plate boundary layer: — — — quasi-parallel, — non-parallel; experiment: ○ Ross *et al.* (1970), ● Schubauer and Skramstad (1947), ■ Kachanov *et al.* (1977).

2D mode most unstable





Parabolized Stability Equations (PSE)



- Boundary layer is growing, $e^{i\alpha x}$ does not represent the instability wave

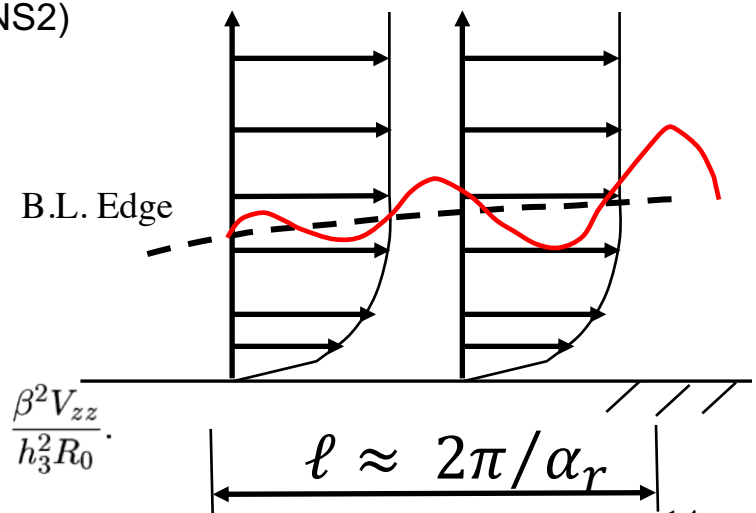
$$\chi(x, y) = \psi(x, y) e^{i \int_0^x \alpha(\xi) d\xi} \quad \text{Substitute into (LNS2)}$$

$$\hat{A} \frac{\partial \psi}{\partial x} + \hat{B} \frac{\partial \psi}{\partial y} + \hat{D} \psi = \frac{1}{R_0} \left(\frac{V_{xx}^l}{h_1^2} \frac{\partial^2 \psi}{\partial x^2} + \frac{V_{xy}^l}{h_1} \frac{\partial^2 \psi}{\partial x \partial y} + V_{yy}^l \frac{\partial^2 \psi}{\partial y^2} \right)$$

$$\hat{A} = \frac{A^l}{h_1} - \frac{2i\alpha V_{xx}^l}{h_1^2 R_0} - \frac{i\beta V_{xz}^l}{h_3 R_0}$$

$$\hat{B} = B^l - \frac{i\alpha V_{xy}^l}{h_1 R_0} - \frac{i\beta V_{xz}^l}{h_3 R_0}$$

$$\hat{D} = D^l - i\omega \Gamma^l + \frac{i\alpha A^l}{h_1} + \frac{i\beta C^l}{h_3} - \left(i \frac{d\alpha}{dx} - \alpha^2 \right) \frac{V_{xx}^2}{h_1^2 R_0} + \frac{\alpha \beta V_{xz}^l}{h_1 h_3 R_0} + \frac{\beta^2 V_{zz}^2}{h_3^2 R_0}$$



*Mode shape accounts for growing BL, no approximation



Parabolized Stability Equations (Cont'd)

$$\chi(x, y) = \psi(x, y) e^{i \int_0^x \alpha(\xi) d\xi}$$

$$\hat{A} \frac{\partial \psi}{\partial x} + \hat{B} \frac{\partial \psi}{\partial y} + \hat{D} \psi = \frac{1}{R_0} \left(\frac{V_{xx}^l}{h_1^2} \frac{\partial^2 \psi}{\partial x^2} + \frac{V_{xy}^l}{h_1} \frac{\partial^2 \psi}{\partial x \partial y} + V_{yy}^l \frac{\partial^2 \psi}{\partial y^2} \right)$$

- Discretized solutions still require solving large sparse matrix equations
- For convective instabilities, disturbances only propagate downstream, a marching solution is much more efficient than LNS solvers
- To this end, first drop second derivatives of shapefunctions (only) in x

$$\hat{A} \frac{\partial \psi}{\partial x} + \hat{B} \frac{\partial \psi}{\partial y} + \hat{D} \psi = \frac{V_{yy}^l}{R_0} \frac{\partial^2 \psi}{\partial y^2}$$

*second derivative of the wave part retained

- Still non-parabolic because all inviscid terms are retained
 - For locally supersonic flows, equation is parabolic, can march downstream
 - For locally subsonic flows, there is a minimal step size to maintain numerical stability
- Parabolize the inviscid terms for locally subsonic flows by modifying $\hat{A}$ such that

*dropping pressure shapefunction x gradient

$$\frac{\partial p'}{\partial x} = (i\alpha \hat{p} + \Omega \frac{\partial \hat{p}}{\partial x}) e^{i(\int_{x_0}^x \alpha d\xi + \beta z - \omega t)}$$

$$\Omega = \sigma_p \text{Min}(1, M_x^2)$$

- Resulting PSE have numerically stable marching (non-upstream mode) solutions for any streamwise step sizes with a given initial solution at $x = x_0$



Discretized PSE Solutions

- Wall normal grid similar to LST, but scale with local boundary-layer thickness, streamwise grid is either constant increment or scale with local Reynolds number
- Streamwise derivatives by first- or second-order backward difference, wall-normal by 4th-order finite difference

- Determination of α

- As much in α as possible
- Local iteration at each x via

$$\chi(x, y) = \psi(x, y) e^{i \int_0^x \alpha(\xi) d\xi}$$

$$\psi_{KE} = \int_0^{y_{max}} (\hat{u}^* \hat{u} + \hat{v}^* \hat{v} + \hat{w}^* \hat{w}) dy$$

$$\alpha_e = \alpha_n - \frac{1}{\psi_{KE}} \int_0^{y_{max}} \left(\hat{u}^* \frac{\partial \hat{u}}{\partial x} + \hat{v}^* \frac{\partial \hat{v}}{\partial x} + \hat{w}^* \frac{\partial \hat{w}}{\partial x} \right) dy$$

- When converged, α corresponds to wave number measured by total kinetic energy
- Non-eigenvalue problems, must provide initial conditions by
 - Using LST eigenvalue solutions, but results in transient behavior
 - Modal solution from other solvers or from Fourier decomposed solutions
 - Local nonparallel eigenvalue approximations formed by consecutive discretized PSE
 - Details in LSTRAC manual



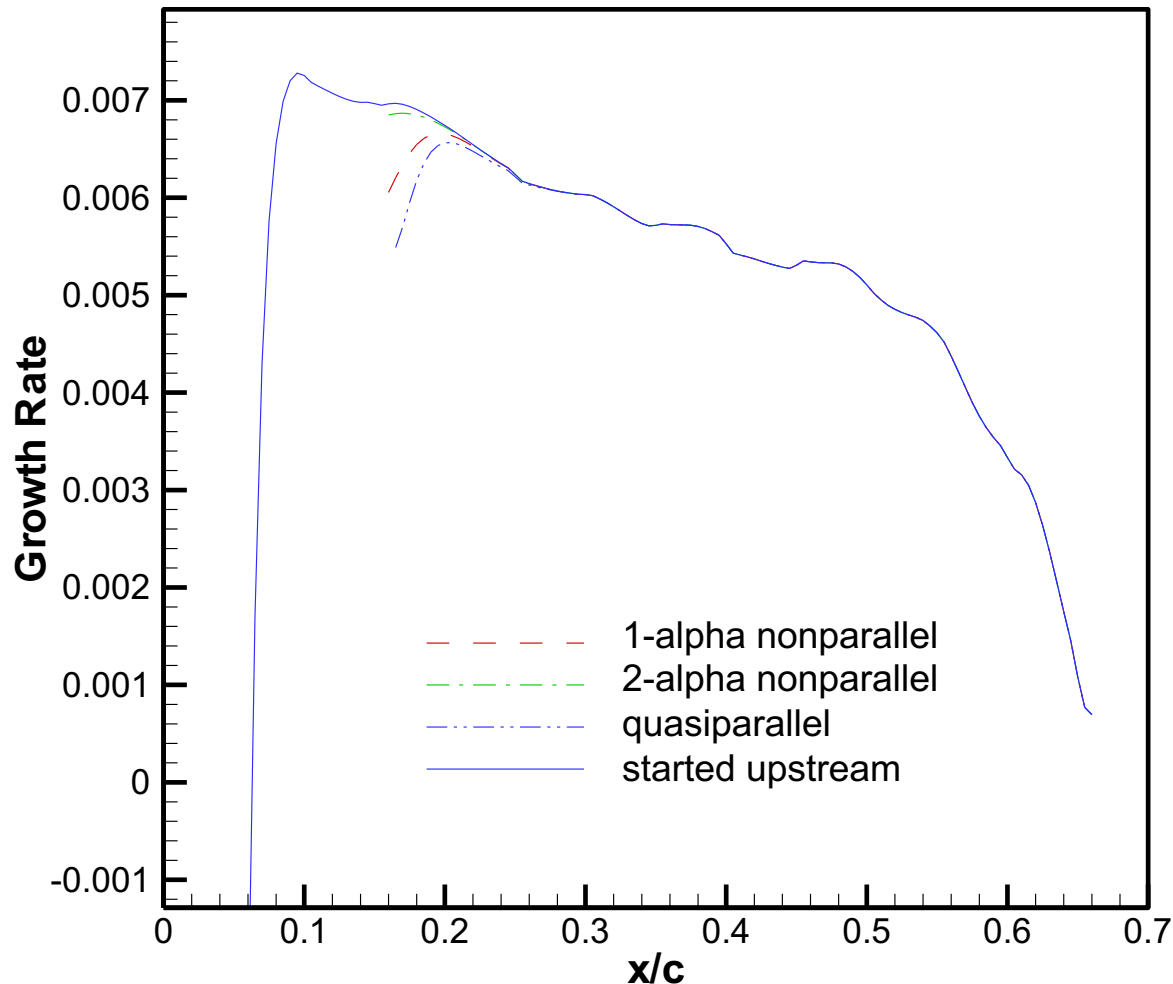
Common Myths about PSE

- PSE has a minimum step size restriction
 - Not so, with parabolization by dropping pressure shapefunction gradient, there is none
- PSE always traces the most unstable mode
 - No, when initializing PSE with a stable eigenmode, it remains stable all the way through
 - "Locking" on to most unstable mode only takes place if the initial disturbance structure warrants such transition physically
 - Agreement of LPSE and DNS for supersonic modes (shown later) proves that
- PSE cannot calculate acoustic modes
 - See the supersonic modes case shown later
- PSE cannot be used for transient growth (results shown later)
 - Nonlinear PSE tracks streamwise vortices all the time
- PSE cannot track upstream propagating modes or dual modal structure at the same frequency and wavenumber due to a single α used
 - Such situations unlikely to happen for realistic problems



Initiation of PSE Marching (Transient Effect)

- Local nonparallel eigenmode initiation
 - One-alpha for both i and $i+1$ stations
 - Two-alpha (α_i, α_{i+1})



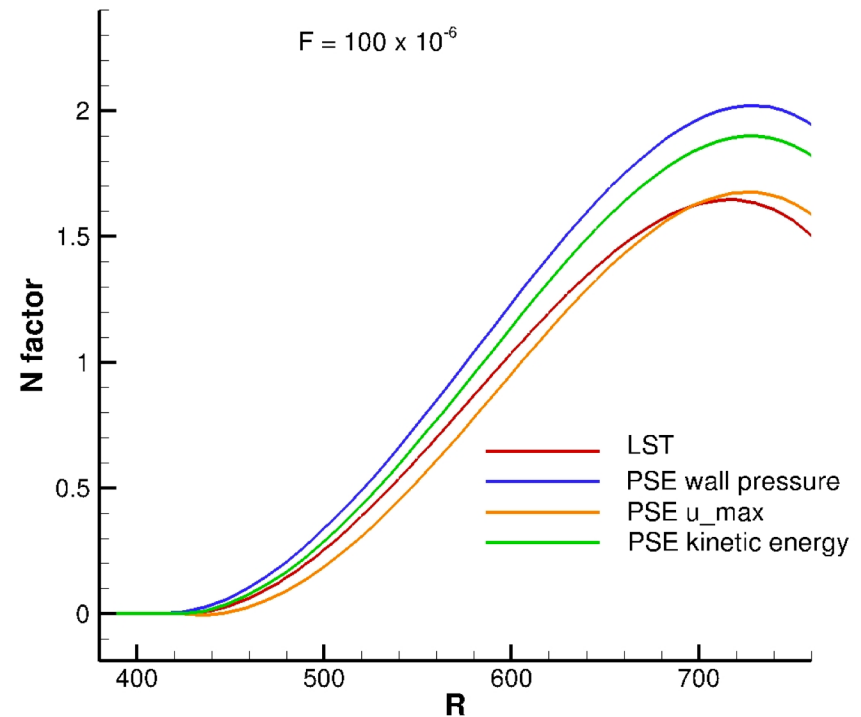
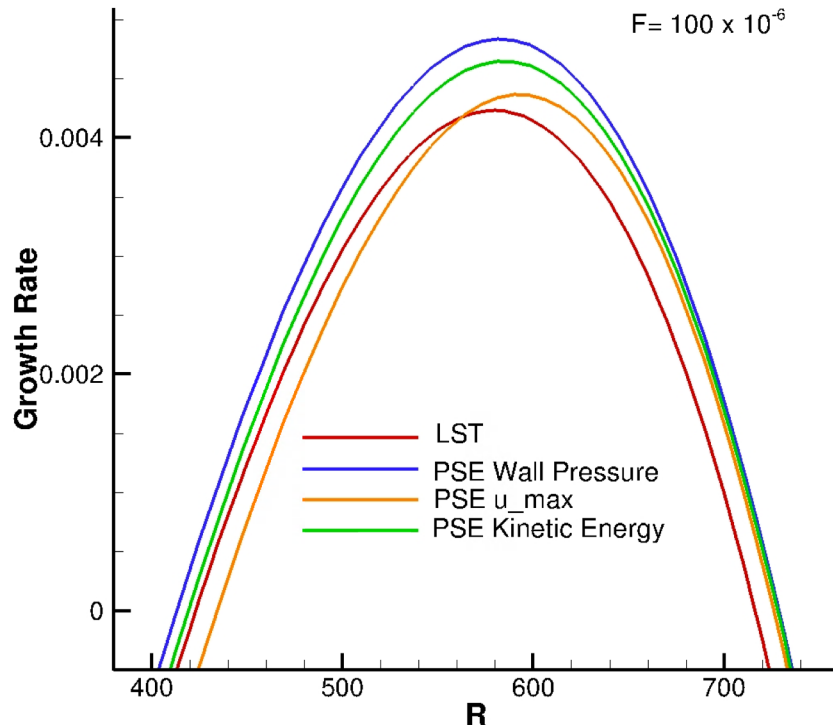


Nonparallel Effects on TS Waves

- Effective streamwise wave number for a nonparallel boundary layer
 - For a given flow quantity ψ in the shapefunction

$$\alpha_{eff} = \alpha - i \frac{1}{\psi} \frac{\partial \psi}{\partial x}$$

- Disturbance growth rate depends on what is used to measure it





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Instability Wave and Transition Prediction

- Small disturbances inside the boundary layer begins to grow beyond the neutral point x_0

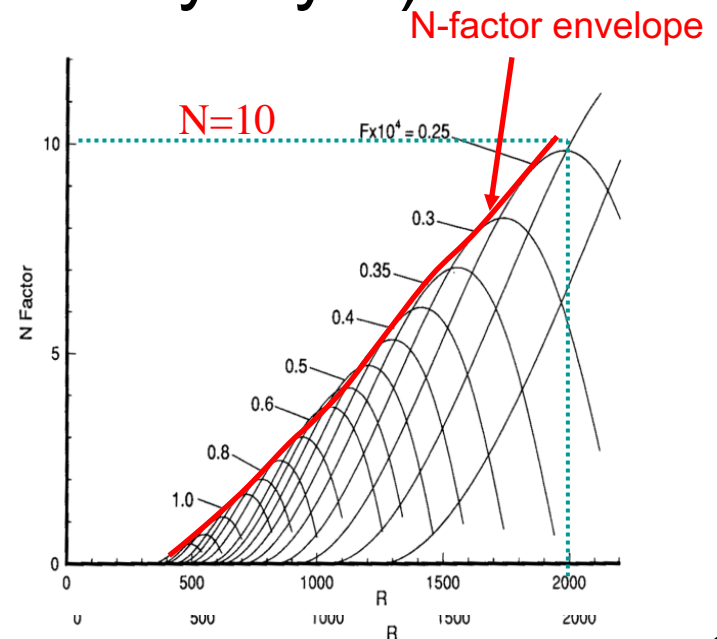
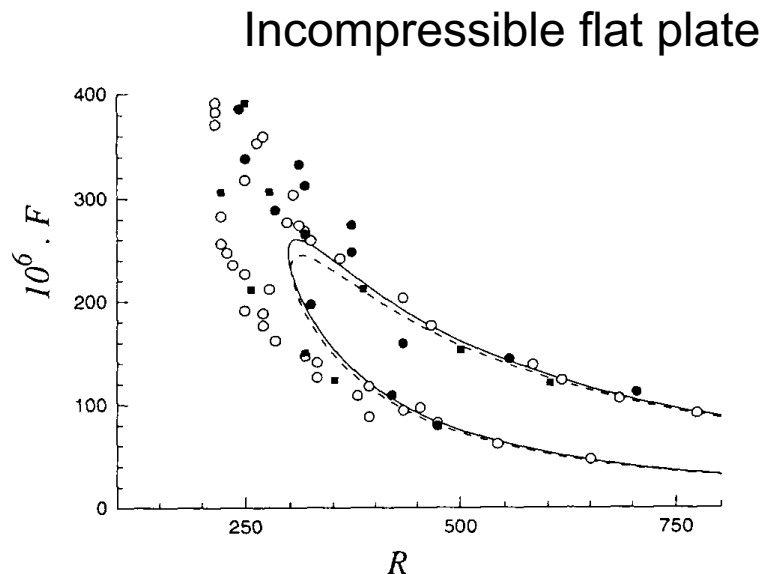
$$\phi'(x, y, z, t) = \hat{\phi}(x, y) e^{i(\beta z - \omega t + \int_{x_0}^x \alpha d\xi)}$$
$$N(x) = - \int_{x_0}^x \alpha_i d\xi$$

- Overall strength of any instability wave measured by the N factor, larger N value means larger overall amplification
- When the overall amplification reaches a preset threshold value (say $N = 10$), flow is transitional – the e^N method
 - Works for boundary layers with a significant linear growth region
 - Both LST and PSE are commonly used for N factor correlation
- N-factor methods are empirical but also predictive
 - Instability mechanism known
 - Unstable range in frequency or spatial extent also predicted
 - A “physics based” empirical method
 - In the absence of receptivity and disturbance environment information, N-factor method is the only feasible way to get transition location by far



Relevant Transition Mechanisms

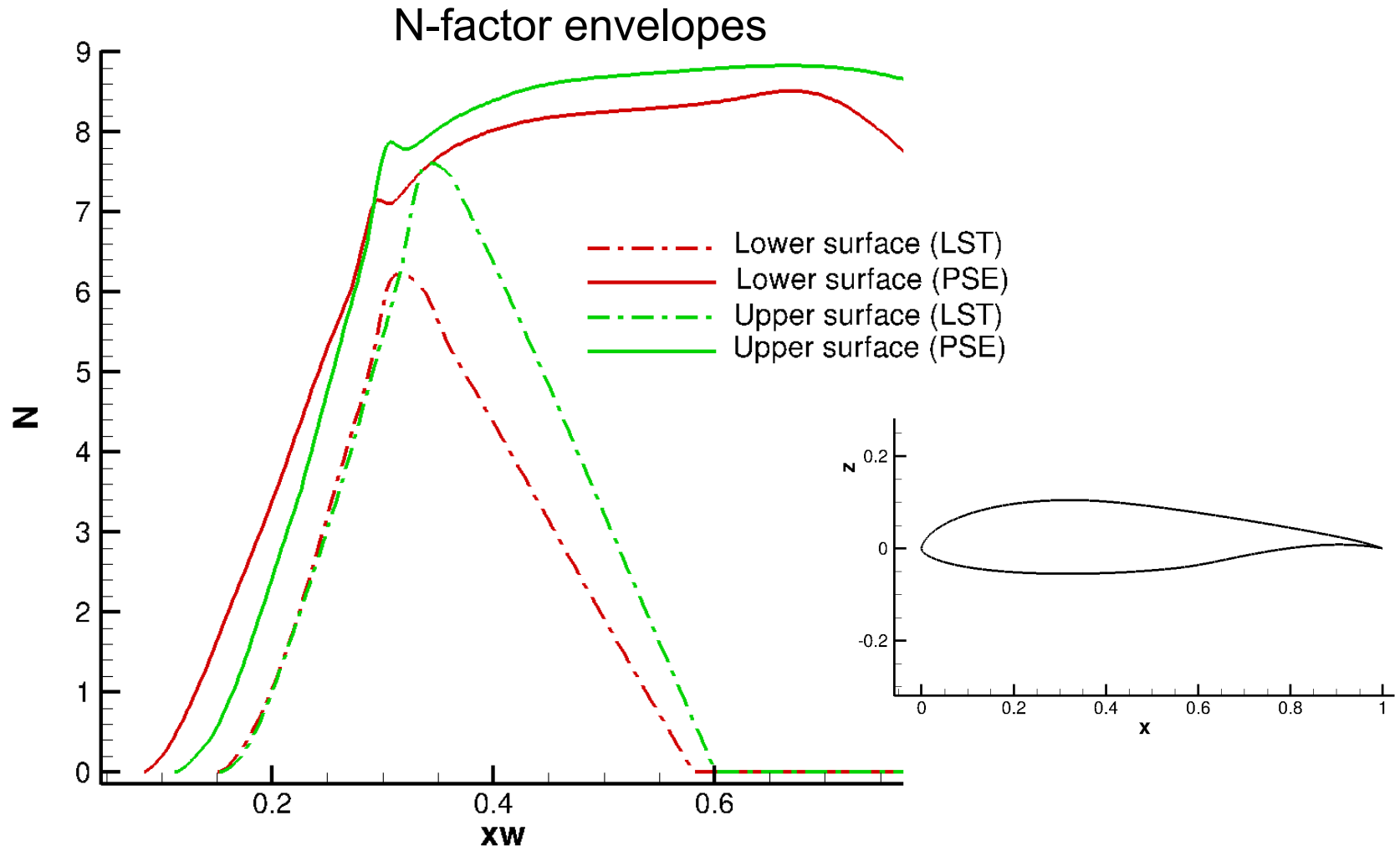
- TS wave (subsonic)
- First-mode instability (low supersonic)
- Mack's second-mode instability ($M > 4$ or so)
- Görtler instability over concave wall (all speed)
- Crossflow instability (3D boundary layer)





TS Waves over 2D Airfoils

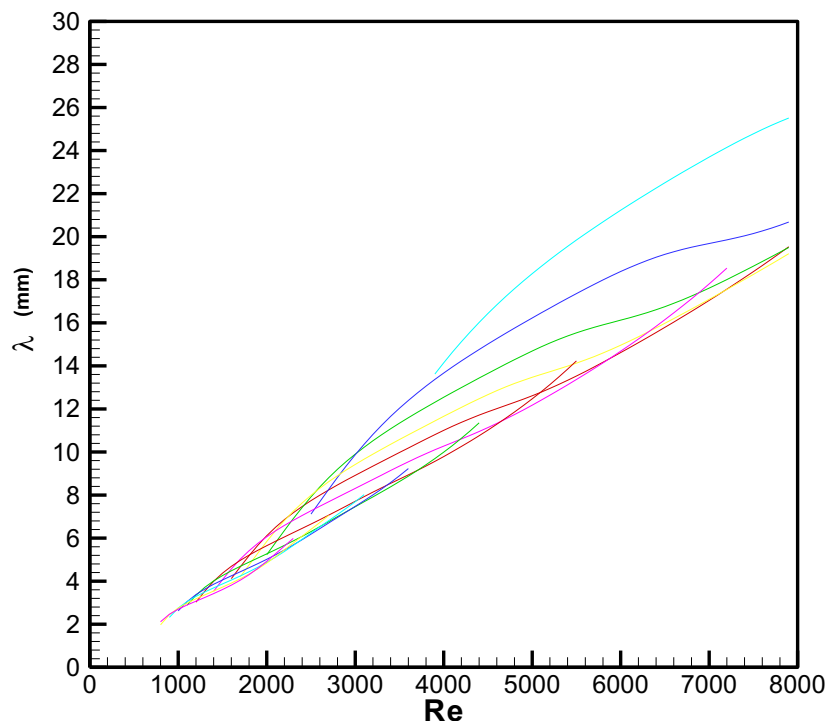
- Wall cooling stabilizes and negative pressure gradient destabilizes TS waves



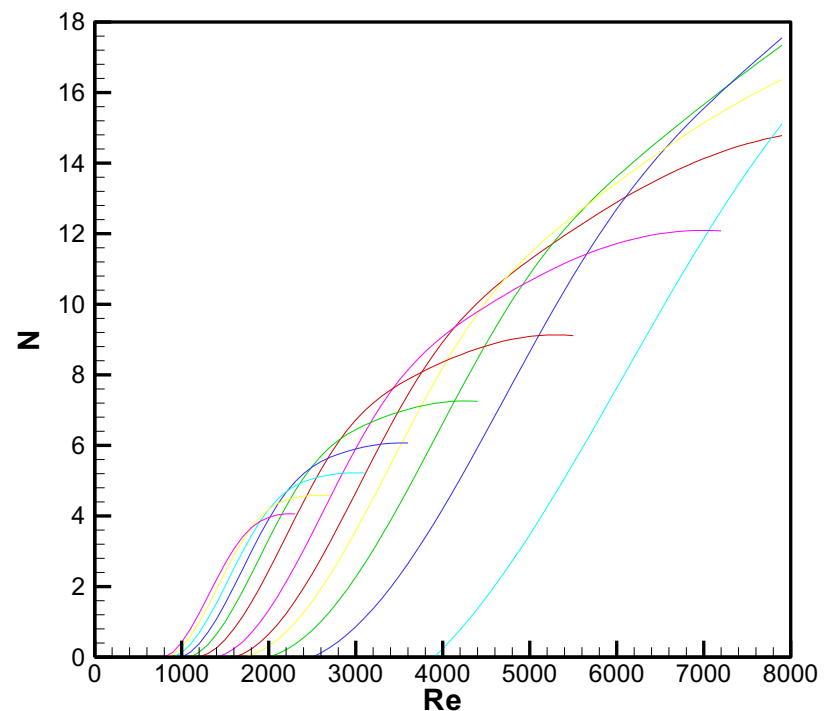


First-Mode Dominant Transition

- Most unstable when oblique with about $50^\circ - 70^\circ$ wave angle
- Cooling stabilizes first-mode waves
- Mach 2 flat plate
 - Oblique waves with different spanwise wavelengths



Wave length

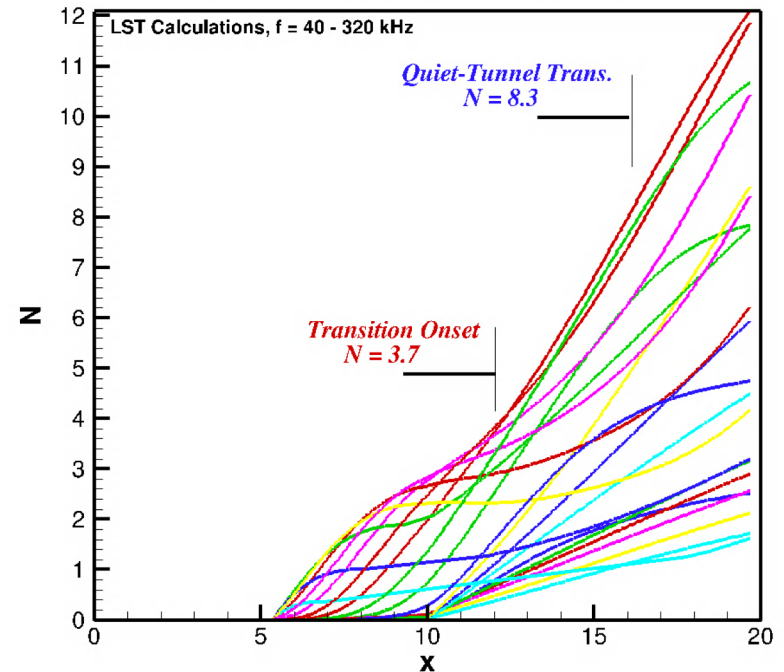
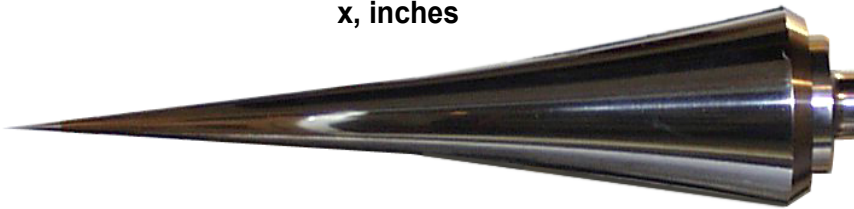
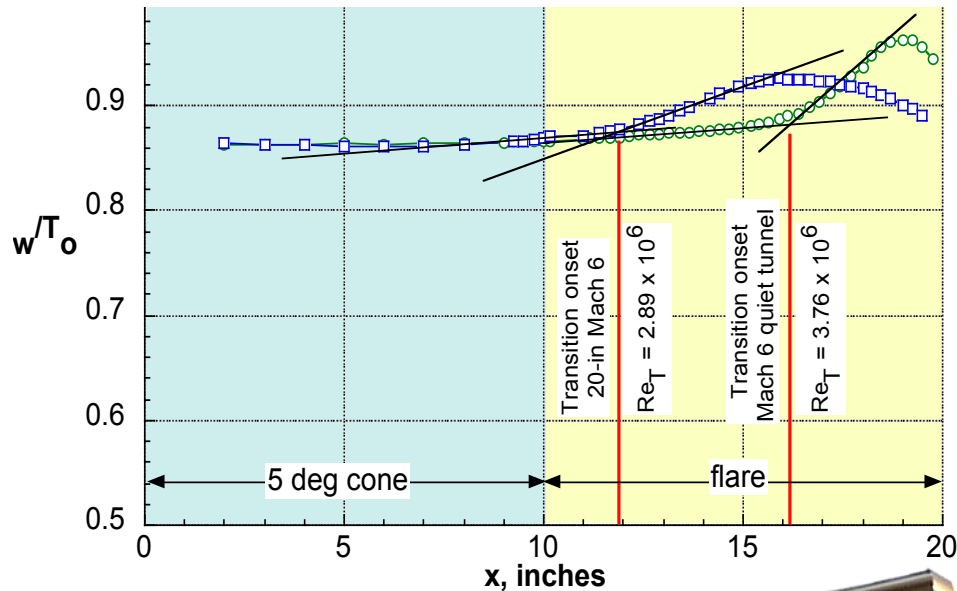


N factors



Transition in a 2D/axisymmetric Boundary Layer

- Second mode dominant transition
 - Wall cooling destabilizing

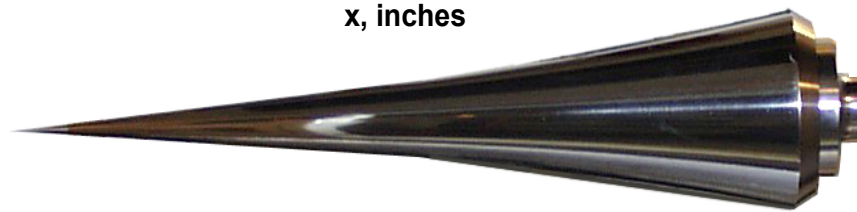
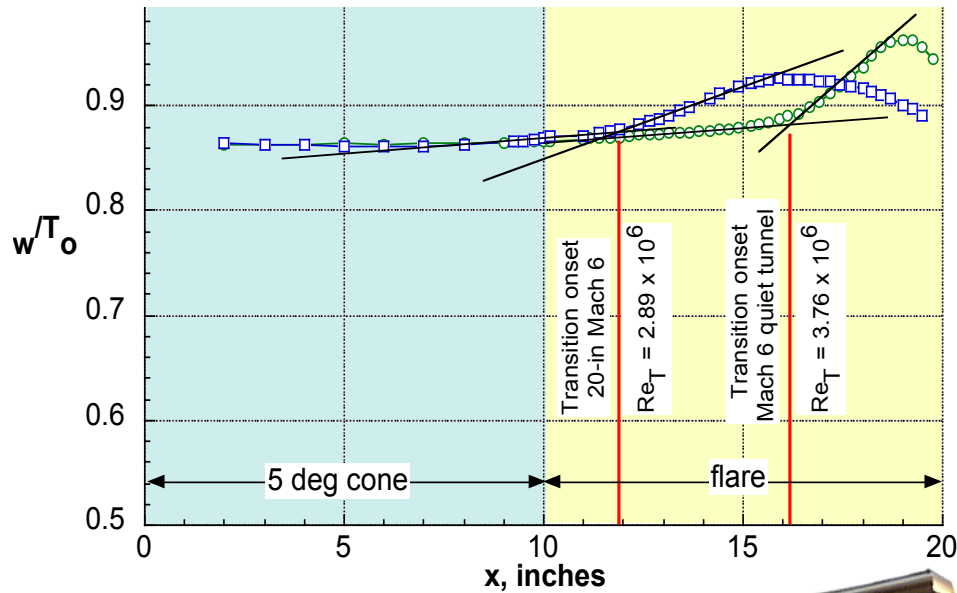


Mach 6 Flow over a Flared Cone

From Horvath et al., AIAA Paper 2002-2743.



Mach 6 Flared Cone Görtler Instability



From Horvath et al., AIAA Paper 2002-2743.

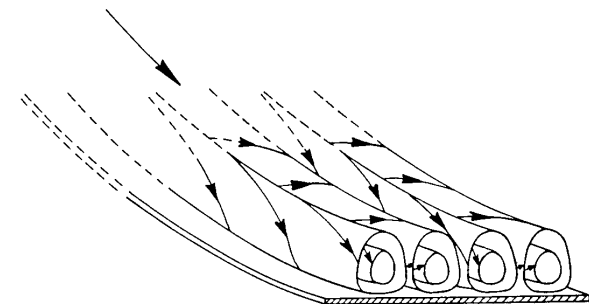
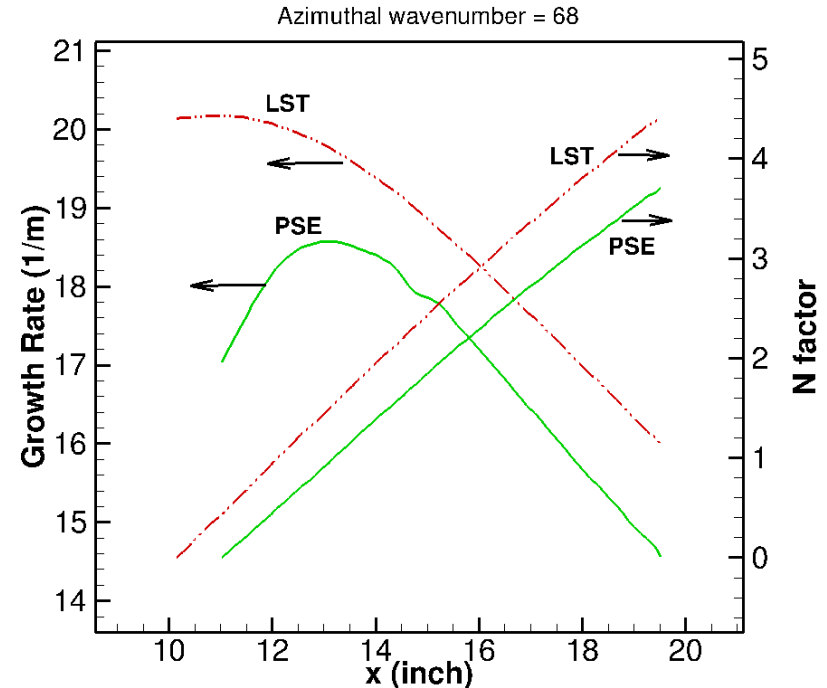
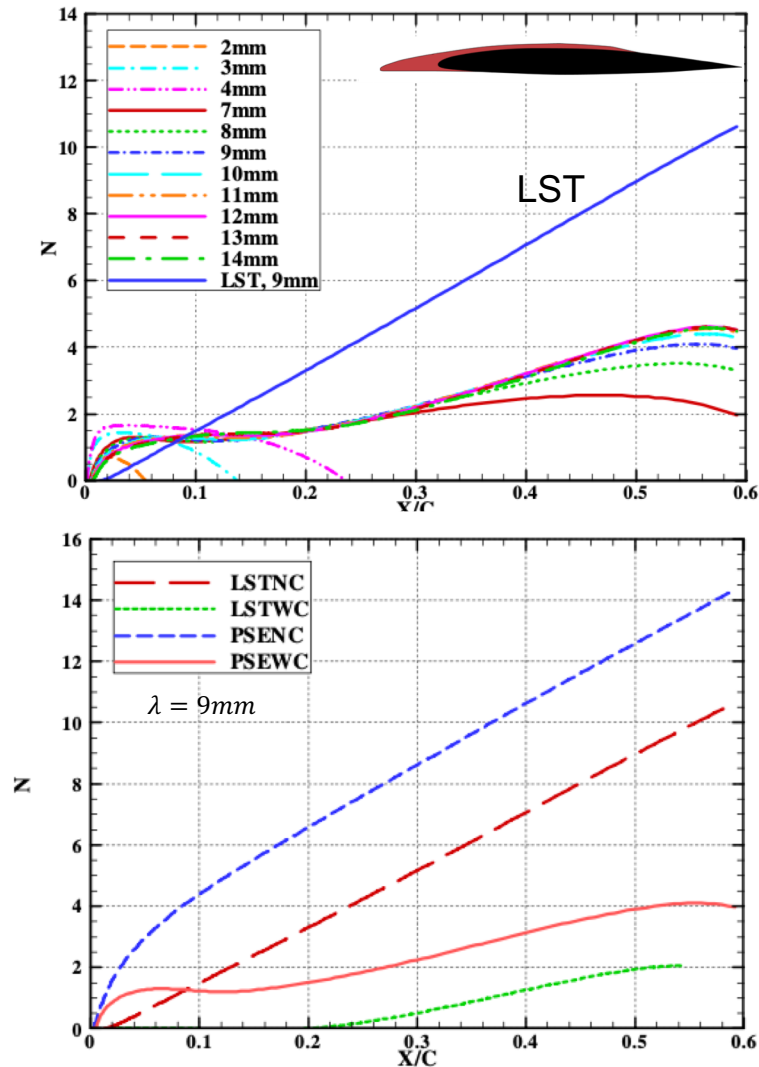


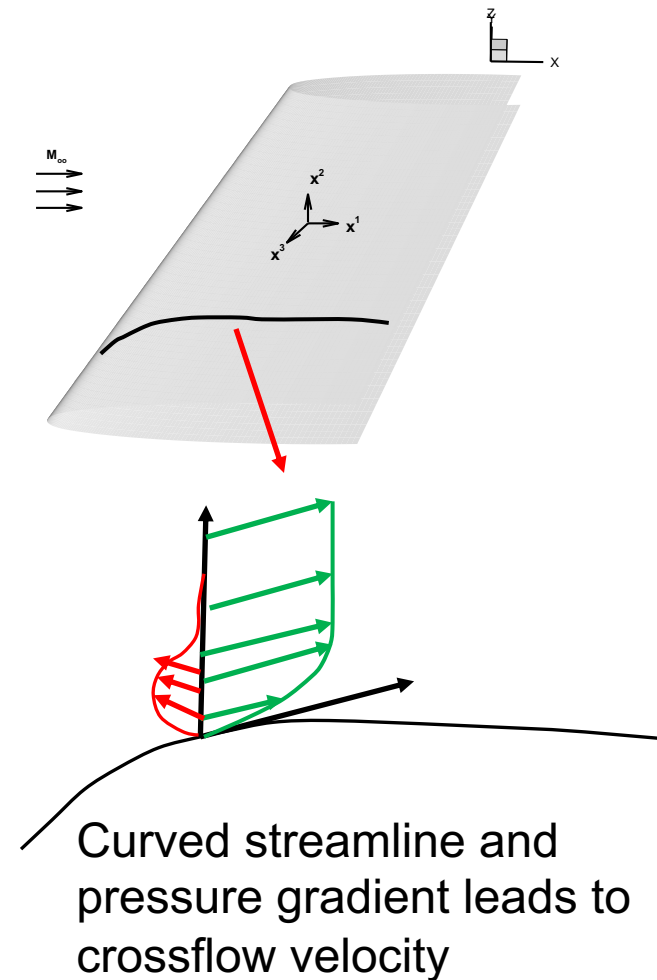
FIGURE 4.25 Vortex disturbances in the flow of a fluid on a concave wall, axes of vortices parallel to principal flow direction. (From Görtler, 1954.)

Crossflow Instability

Gulfstream-III Wing glove



Malik et al. AIAA 2011-3525

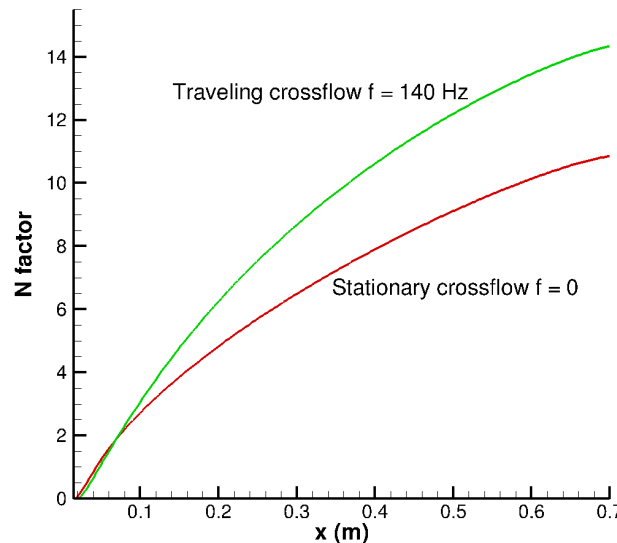
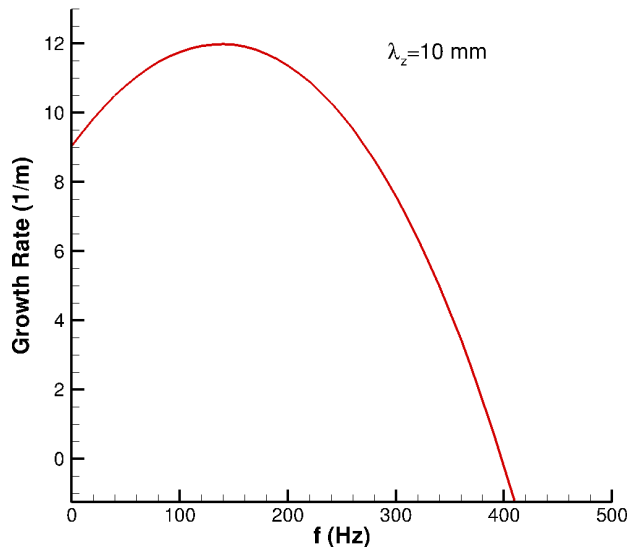
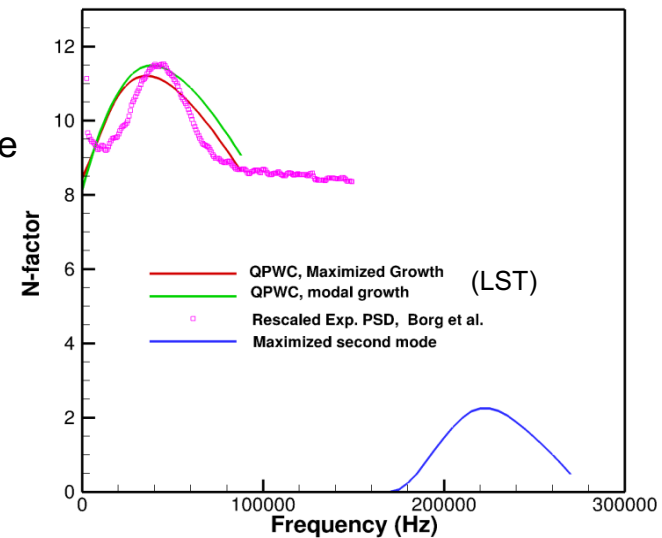


***Curvature & nonparallel effects equally important**



Stationary vs. Traveling Crossflow Instability

- Traveling crossflow more unstable
- More attention to stationary crossflow instability due to the assumption of quiet environments in flight/quiet tunnels and surface roughness could induce stationary crossflow instability naturally
 - Traveling crossflow observed in quiet tunnel experiments
 - Should consider both for transition prediction
 - More experiments needed



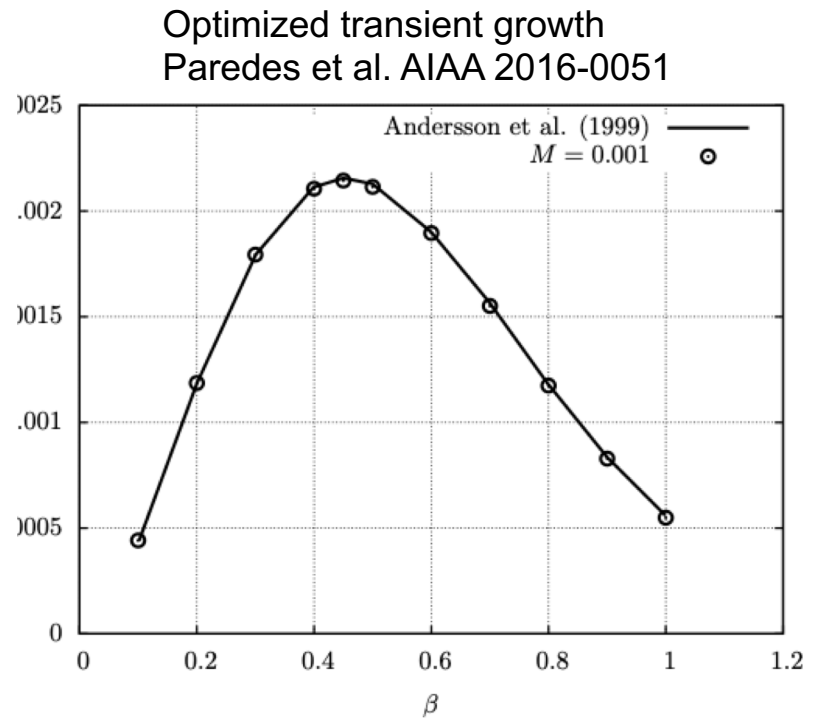
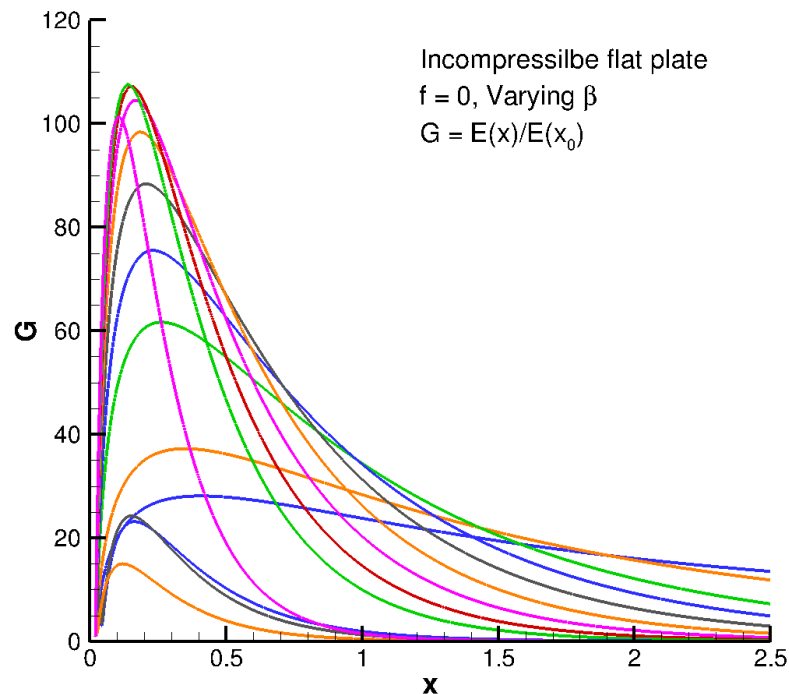
Mach 6 elliptic Cone
Purdue quiet tunnel exp.

Crossflow instability in subsonic swept-wing boundary layers



Nonmodal Transient Growth

- Disturbances consist of linear combinations of stable eigenmodes can grow algebraically at subcritical Reynolds numbers and even in the absence of unstable eigenmodes
- Streamwise vortices due to transient growth can be predicted by optimizing disturbances with the PSE approach
 - Shown to grow to large amplitudes that cause strong secondary instability
 - Could be major actor under a noisy environment



PSE computed streamwise vortices in incompressible flat plate



Outline of the Lecture

1. Introduction
2. Boundary-layer stability theory and computations
 - Linear stability theory
 - Linear parabolized stability equations
3. Transition prediction using N-factor method
 - Transition mechanisms and modal instability
 - Nonmodal, transient growth
4. **Absolute amplitude methods and DNS**
 1. Nonlinear PSE
 2. Biglobal stability (2D eigenvalue problems)
5. 3D boundary layer
6. Chemical/thermal nonequilibrium flows
7. Summary



Nonlinear PSE

- Disturbance initial amplitudes known from either measurements or receptivity computations

$$\Gamma \frac{\partial \phi}{\partial t} + \frac{A}{h_1} \frac{\partial \phi}{\partial x} + B \frac{\partial \phi}{\partial y} + \frac{C}{h_3} \frac{\partial \phi}{\partial z} + D\phi = \frac{1}{R_0} \left(\frac{V_{xx}}{h_1^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{V_{xy}}{h_1} \frac{\partial^2 \phi}{\partial x \partial y} + V_{yy} \frac{\partial^2 \phi}{\partial y^2} + \frac{V_{xz}}{h_3} \frac{\partial^2 \phi}{\partial x \partial z} + \frac{V_{yz}}{h_3} \frac{\partial^2 \phi}{\partial y \partial z} + \frac{V_{zz}}{h_3^2} \frac{\partial^2 \phi}{\partial z^2} \right)$$

- Fourier decomposition in time and spanwise coordinate
 - Assume periodic in z

$$\phi(x, y, z, t) = \sum_{m=-M}^M \sum_{n=-N}^N \chi_{mn}(x, y) e^{i(n\beta z - m\omega t)}$$

$$\chi_{mn} = \psi_{mn}(x, y) A_{mn}(x)$$

$$A_{mn}(x) = e^{i \int_{x_0}^x \alpha_{mn}(\xi) d\xi}$$

Amplitude



$$\hat{A}_{mn} \frac{\partial \psi_{mn}}{\partial x} + \hat{B}_{mn} \frac{\partial \psi_{mn}}{\partial y} + \hat{D}_{mn} \psi_{mn} - \frac{V_{yy}^l}{R_0} \frac{\partial^2 \psi_{mn}}{\partial y^2} = \frac{F_{mn}}{A_{mn}}$$

NPSE for Mode (m, n)

$$F^n = -\Gamma^n \frac{\partial \phi}{\partial t} - \frac{A^n}{h_1} \frac{\partial \phi}{\partial x} - B^n \frac{\partial \phi}{\partial y} - \frac{C^n}{h_3} \frac{\partial \phi}{\partial z} - D^n \phi + \frac{1}{R_0} \left(\frac{V_{xx}^n}{h_1^2} \frac{\partial^2 \phi}{\partial x^2} + \frac{V_{xy}^n}{h_1} \frac{\partial^2 \phi}{\partial x \partial y} + V_{yy}^n \frac{\partial^2 \phi}{\partial y^2} + \frac{V_{xz}^n}{h_1 h_3} \frac{\partial^2 \phi}{\partial x \partial z} + \frac{V_{yz}^n}{h_3} \frac{\partial^2 \phi}{\partial y \partial z} + \frac{V_{zz}^n}{h_3^2} \frac{\partial^2 \phi}{\partial z^2} \right)$$

All modes interact through F^n

$$F^n(x, y, z, t) = \sum_{m=-M}^M \sum_{n=-N}^N F_{mn}(x, y) e^{i(n\beta z - m\omega t)}$$



Nonlinear PSE (cont'd)

- Initialize all known Fourier modes with mode shapes and amplitudes at x_0
- Solve PSE for each Fourier mode $(m\omega_0, n\beta_0)$ at a given x location for fundamental frequency ω_0 and fundamental spanwise wavenumber β_0 (domain size)
- Iterate until convergence of all Fourier modes, then march on to next x

$$\hat{A}_{mn} \frac{\partial \psi_{mn}}{\partial x} + \hat{B}_{mn} \frac{\partial \psi_{mn}}{\partial y} + \hat{D}_{mn} \psi_{mn} - \frac{V_{yy}^l}{R_0} \frac{\partial^2 \psi_{mn}}{\partial y^2} = \frac{F_{mn}}{A_{mn}}$$

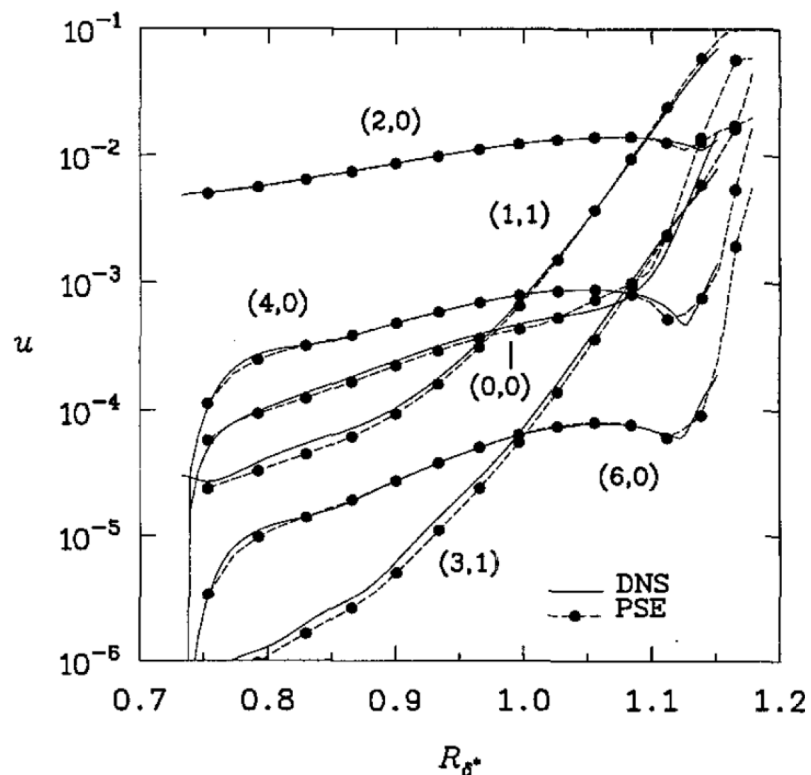
$$\hat{A}_{mn} = \frac{A^l}{h_1} - \frac{2i\alpha_{mn}V_{xx}^l}{h_1^2 R_0} - \frac{in\beta V_{xz}^l}{h_3 R_0}$$

$$\hat{B}_{mn} = B^l - \frac{i\alpha_{mn}V_{xy}}{h_1 R_0} - \frac{in\beta V_{xz}^l}{h_3 R_0}$$

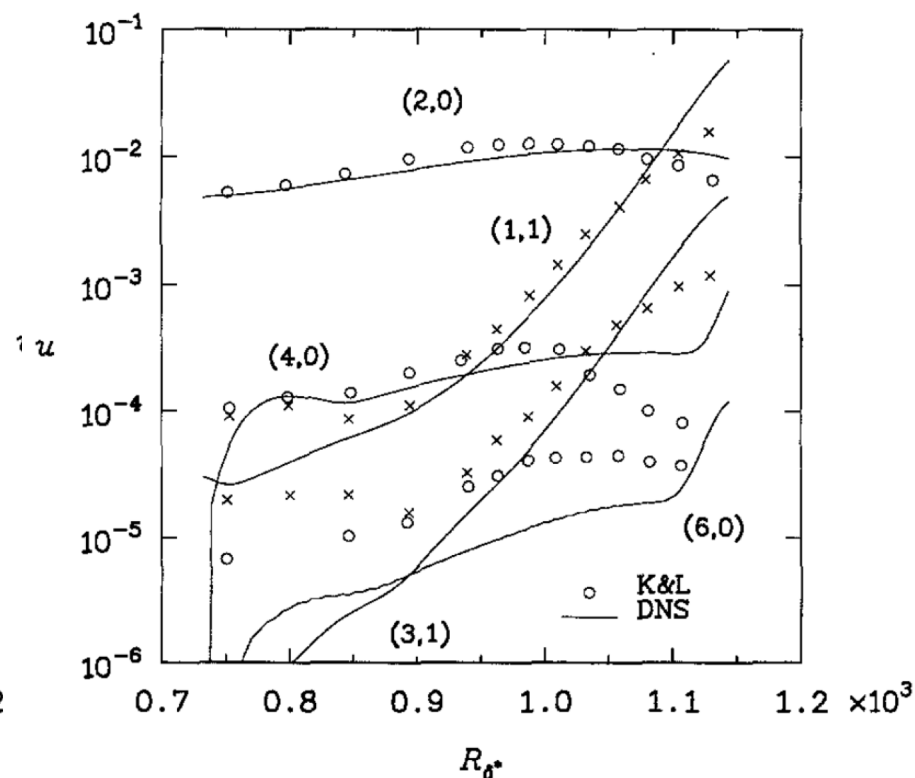
$$\hat{D}_{mn} = D^l - im\omega\Gamma^l + \frac{i\alpha_{mn}A^l}{h_1} + \frac{in\beta C^l}{h_3} - \left(i\frac{d\alpha_{mn}}{dx} - \alpha_{mn}^2\right) \frac{V_{xx}^2}{h_1^2 R_0} + \frac{n\alpha\beta V_{xz}^l}{h_1 h_3 R_0} + \frac{n^2\beta^2 V_{zz}}{h_3^2 R_0}$$

Subharmonic Breakdown to Turbulence

- Experiment done by Kachanov & Levchenko (K&L) (1984)
- Primary mode (2, 0) is a TS wave, and subharmonic (1, 1) is an oblique wave that is linearly stable with given amplitudes at an upstream location



Comparison with DNS

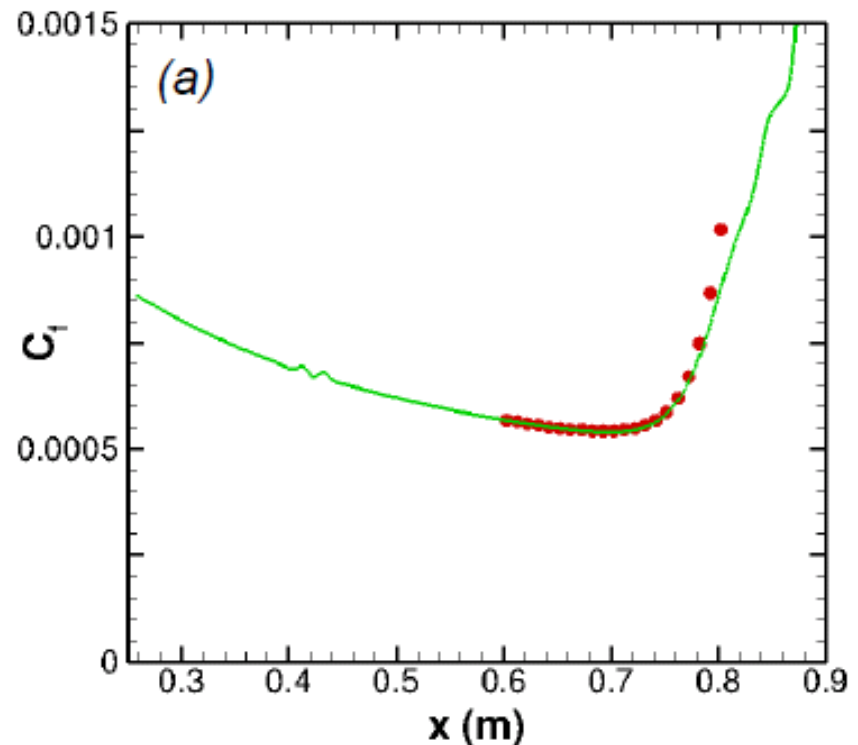
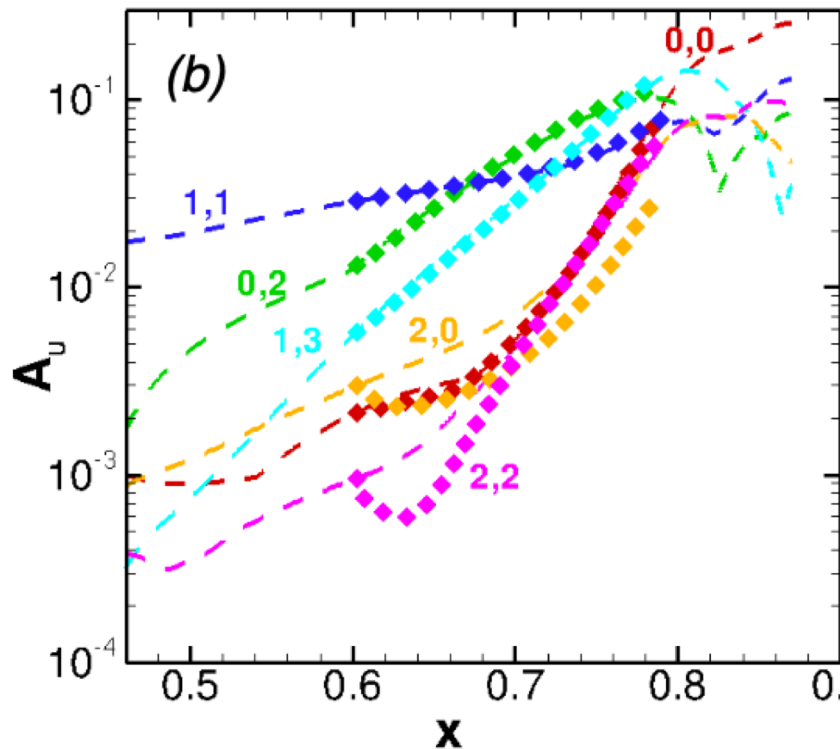


Comparison with experiments



Oblique-Mode Breakdown in Supersonic Boundary Layer

- A pair of unstable oblique first-mode waves ($1, \pm 1$) with a finite amplitude can nonlinearly interact to form a growing streamwise vortex mode ($0, 2$); in turn, the triad interaction eventually leads to transition
 - Likely to happen at low supersonic Mach numbers (less than 4)

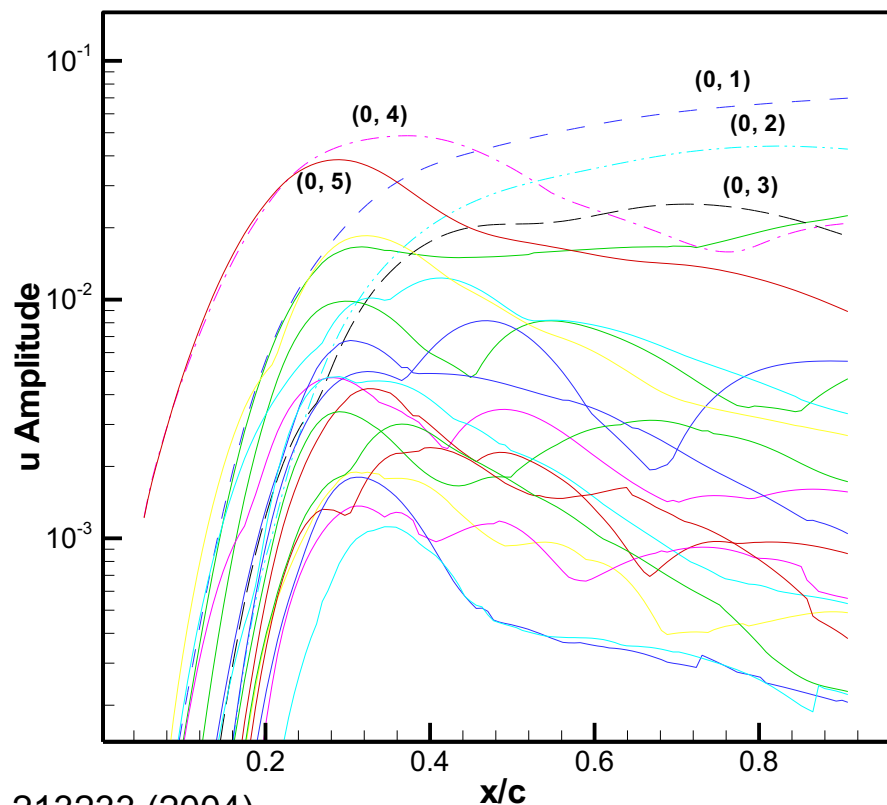


Symbols:NPSE, lines:DNS (U. Arizona)



Nonlinear Development of Stationary Crossflow Instability

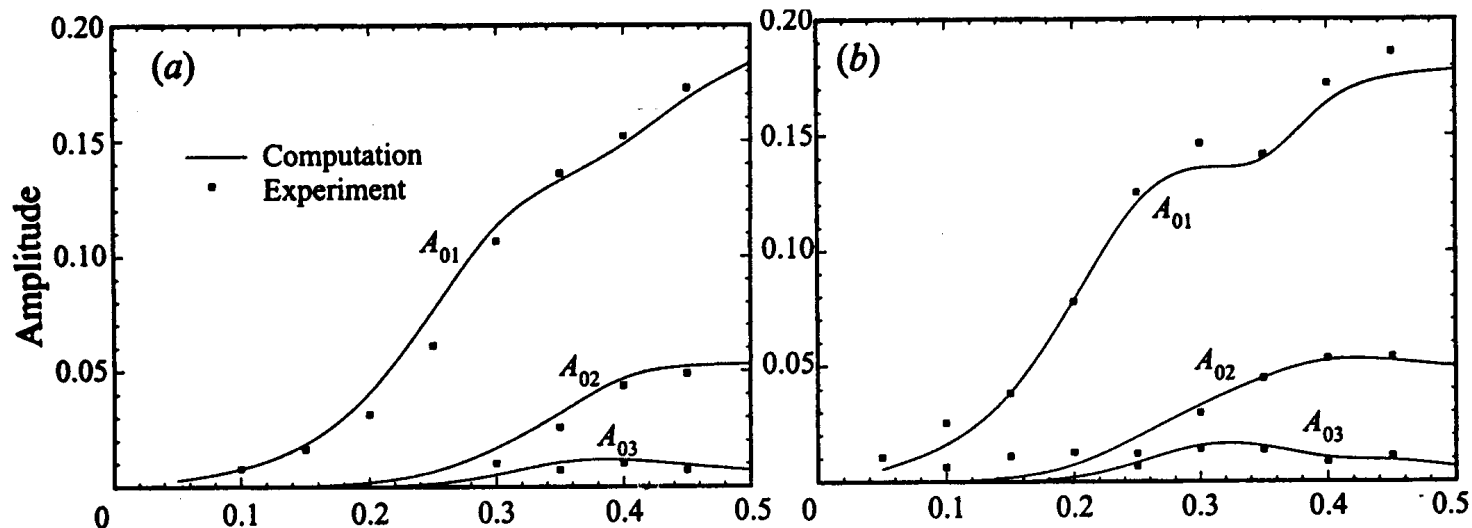
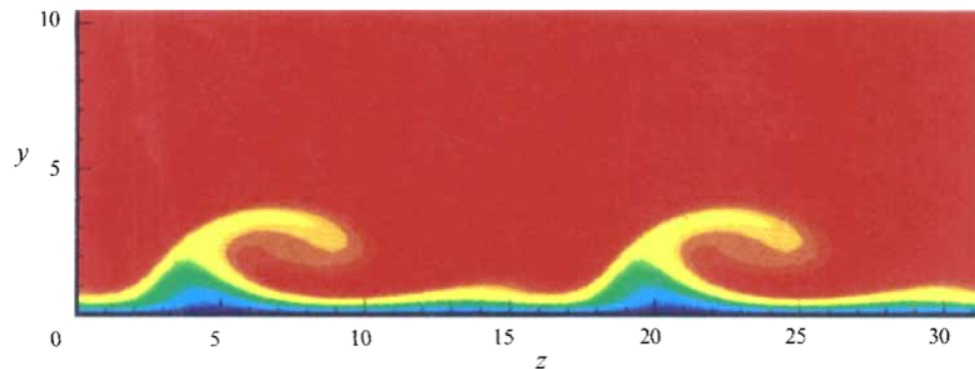
- One or more unstable crossflow modes with finite amplitudes can interact and grow to a large-amplitude, saturated state with many energetic harmonics
- Fundamental crossflow vortex mode with $\beta_0 = 2\pi/40 \text{ mm}$
 - Two equal-amplitude modes with wavelength of 8 mm ((0, 5) mode) and 10 mm ((0, 4) mode)
 - Lead to very unstable (0, 1), (0, 2), and (0, 3) modes with wavelength of 40 mm, 20 mm, and 40/3 mm in addition to meanflow distortion (0, 0)





Secondary Instability of ASU Swept-Wing Boundary Layer

- Incompressible ASU experiments (1996)
- Comparison with nonlinear PSE computations for different initial amplitudes
- When saturation occurs, vortex is rolled-up, resulting in strong variation on the y - z plane, causing potential secondary instability





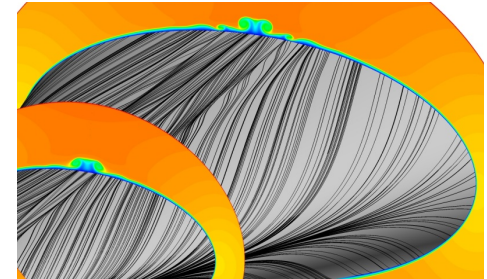
2D Eigenvalue or Biglobal Stability Analysis

- Disturbance structure is strongly two-dimensional on the cross-plane to the instability wave evolution direction
- Must use a different disturbance wave form

$$\phi(x, y, z, t) = \chi(x, y, z)e^{-i\omega t}$$

- Quasiparallel assumption (dropping all superscripts l)

$$\chi(x, y, z) = \psi(y, z)e^{i\alpha x} \quad \text{Substitute into (LNS1)}$$



$$\begin{aligned} \left(B - \frac{i\alpha V_{xy}}{h_1 R_0}\right) \frac{\partial \psi}{\partial y} + \left(C - \frac{i\alpha V_{xz}}{h_1 h_3 R_0}\right) \frac{\partial \psi}{\partial z} + \left(D - i\omega \Gamma + \frac{i\alpha A}{h_1} + \frac{\alpha^2 V_{xx}}{h_1^2 R_0}\right) \psi \\ = \frac{V_{yy}}{R_0} \frac{\partial^2 \psi}{\partial y^2} + \frac{V_{yz}}{h_3 R_0} \frac{\partial^2 \psi}{\partial y \partial z} + \frac{V_{zz}}{h_3^2 R_0} \frac{\partial^2 \psi}{\partial z^2} \end{aligned}$$

- Apply 4th-order finite difference in both y and z directions and homogeneous boundary conditions
- 2D eigenvalue problem in (ω, α)

$$(L_0 + \alpha L_1 + \alpha^2 L_2) \psi = \omega L_3 \psi$$

- Temporal: α real and ω complex
- Spatial: ω real and α complex
- Large block pentadiagonal matrix due to 2D discretized points



2D Eigenvalue Solution Procedure

- In general, solve as a temporal eigenvalue problem, i.e., given α , compute ω using inverse Rayleigh method

$$L_\alpha \psi = \omega L_3 \psi$$

- L_3 is a diagonal matrix, L_α block penta-diagonal $5N_z$ by $5N_z$ block size and a length of N_y
- Intel MKL library quite efficient for solving such banded matrices
- QR method generates all eigenvalues but work increases as $(5N_z N_y)^3$ for a full matrix
- Given initial guess of ω_0 and ψ^0

$$(L_\alpha - \omega_k L_3) \psi^{k+1} = L_3 \psi^k$$

$$(L_\alpha - \omega_k L_3)^{tr} \varphi^{k+1} = L_3^{tr} \varphi^k$$

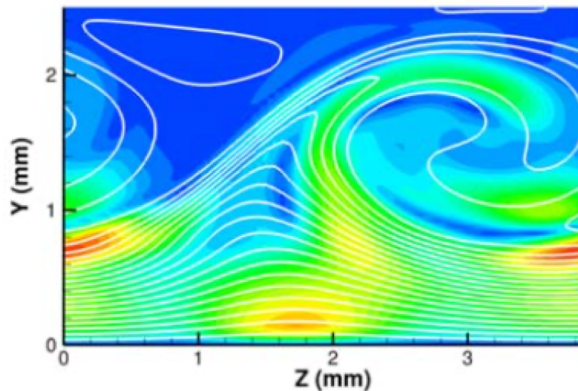
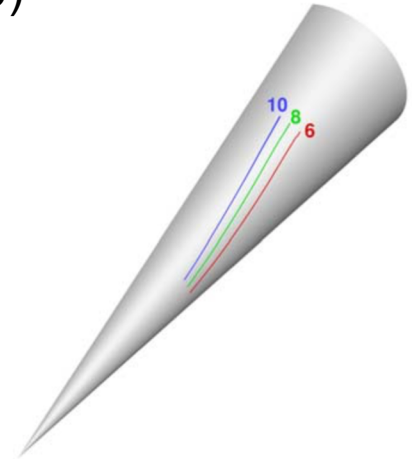
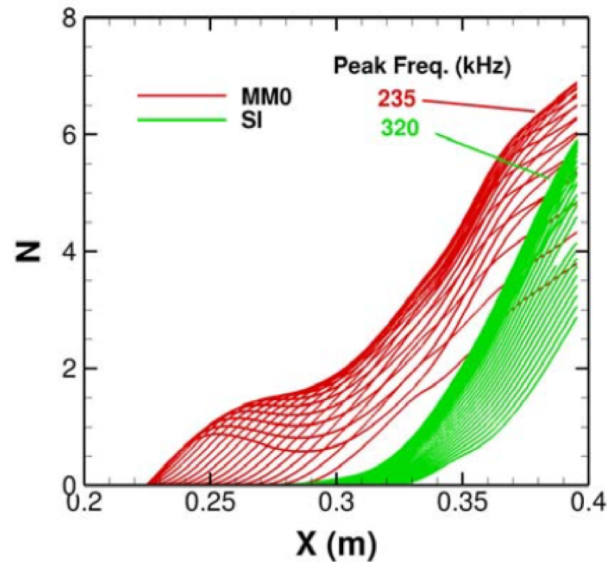
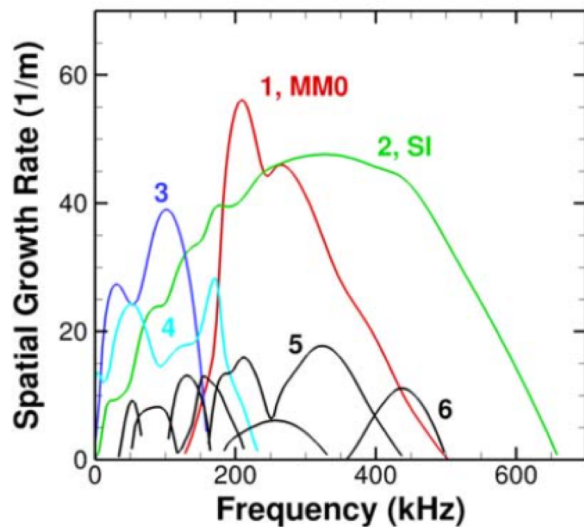
$$\omega_{k+1} = \frac{\langle \varphi^k, L_\alpha \psi^{k+1} \rangle}{\langle \varphi^{k+1}, L_3 \psi^{k+1} \rangle}$$

- Iterate until convergence
- Use Arnoldi's method to accelerate eigenvalue search for larger dimensions
- Spatial stability problem can be solved by iterations until $\omega_i = 0$ while setting α complex according to $\alpha_i = -\omega_i / c_r$ ($c_r = \omega_r / \alpha_r$)

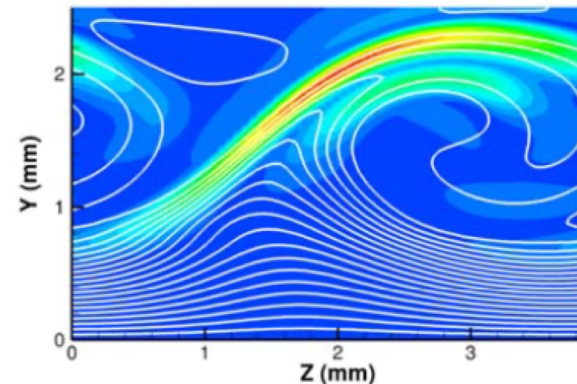


Secondary Instability of Stationary Crossflow Vortices

- Mach 6 flow over a sharp cone at 6° angle of attack (vortex 10)



(h) Mode MM0, $X = 0.375$ m

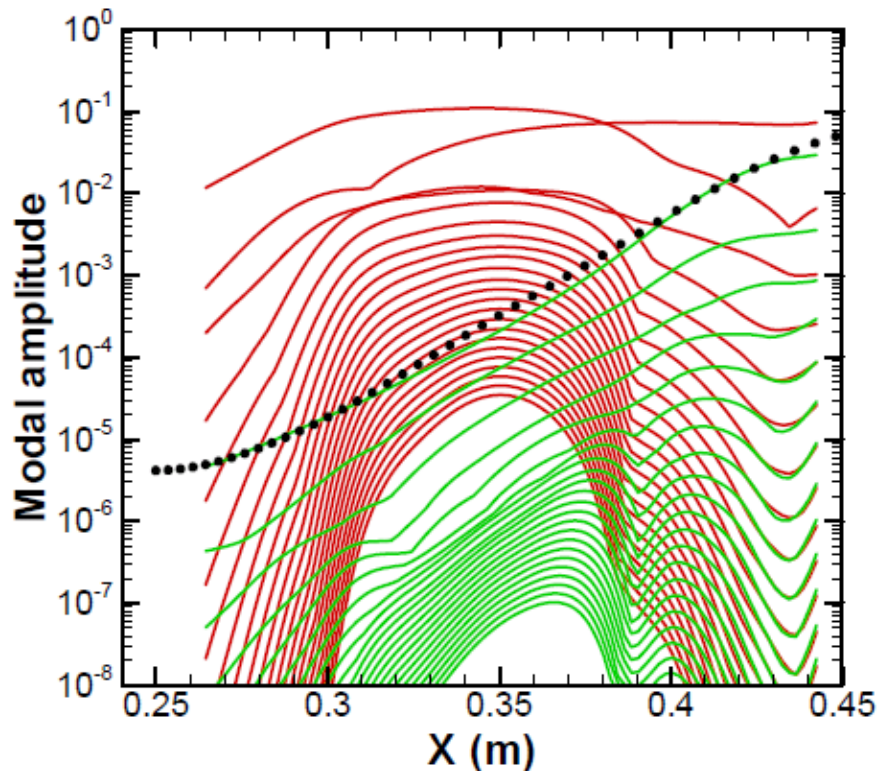
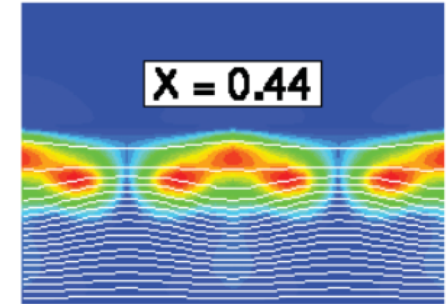


(i) Mode SI, $X = 0.375$ m

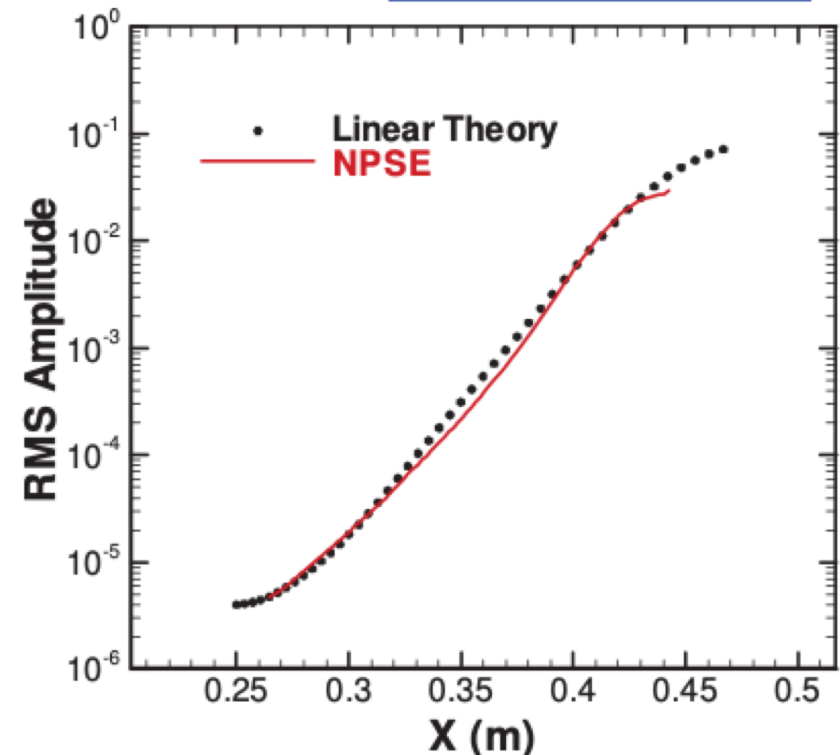


Secondary Instability of Second-Mode Waves over HIFiRE-1 Cone

- Under Purdue Mach 6 Quiet Tunnel conditions
- Subharmonic instability computed by nonlinear PSE
- Excellent agreement with linear biglobal theory



Lines: NPSE, symbols: biglobal theory



Comparison with nonlinear PSE prediction



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Extension of Biglobal Stability Theory to Plane-Marching PSE

- To fully account for boundary-layer growth in the streamwise direction
- Can also handle finite-amplitude disturbances if nonlinear terms kept

$$\phi(x, y, z, t) = \chi(x, y, z)e^{-i\omega t}$$

$$\chi(x, y, z) = \psi(x, y, z)e^{i \int_{x_0}^x \alpha(\xi) d\xi} \quad \text{Substitute into (LNS1)}$$

$$\hat{A} \frac{\partial \psi}{\partial x} + \hat{B} \frac{\partial \psi}{\partial y} + \hat{C} \frac{\partial \psi}{\partial z} + \hat{D} \psi = \frac{1}{R_0} \left(\frac{V_{xy}}{h_1} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{V_{xz}}{h_1 h_3} \frac{\partial^2 \psi}{\partial x \partial z} + V_{yy} \frac{\partial^2 \psi}{\partial y^2} + \frac{V_{yz}}{h_3} \frac{\partial^2 \psi}{\partial y \partial z} + \frac{V_{zz}}{h_3^2} \frac{\partial^2 \psi}{\partial z^2} \right)$$

$$\hat{A} = \frac{A}{h_1} - \frac{2i\alpha V_{xx}}{h_1^2 R_0} \quad \hat{B} = B - \frac{i\alpha V_{xy}}{h_1 R_0} \quad \hat{C} = C - \frac{i\alpha V_{xz}}{h_1 h_3 R_0}$$

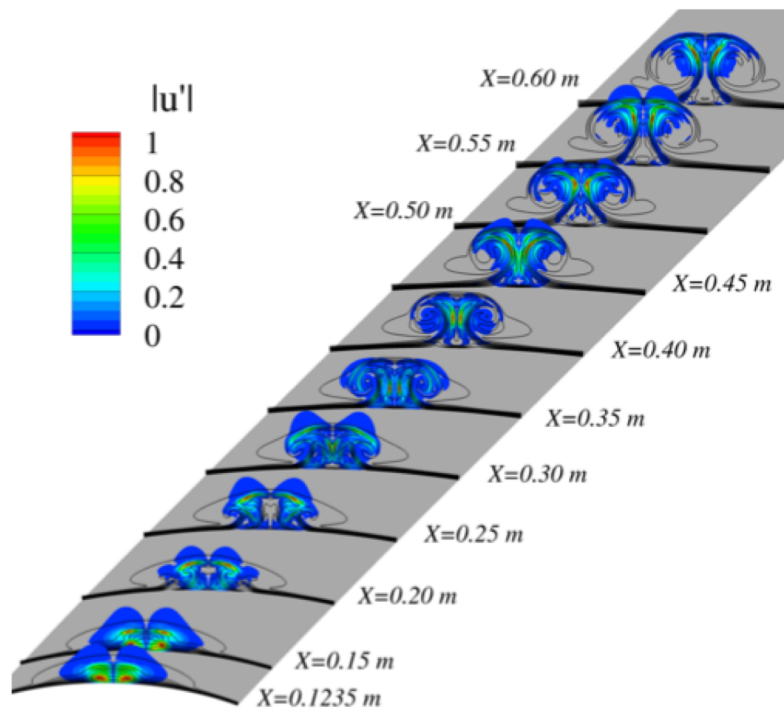
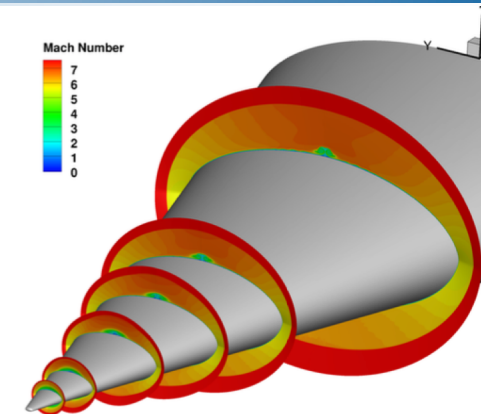
$$\hat{D} = D - i\omega\Gamma + \frac{i\alpha A}{h_1} - \left(i \frac{d\alpha}{dx} - \alpha^2 \right) \frac{V_{xx}}{h_1^2 R_0}$$

- Given initial disturbance field on the initial y-z plane $x = x_0$ and corresponding α , discretize the plane using 4th-order finite difference
- Compute ψ at new x by solving the banded matrix discretized equations and iterate until the norm in α is converged
- Only one α is assumed, must focus on a window of distinct instability mechanism such as crossflow vortex, etc. to ensure convergence



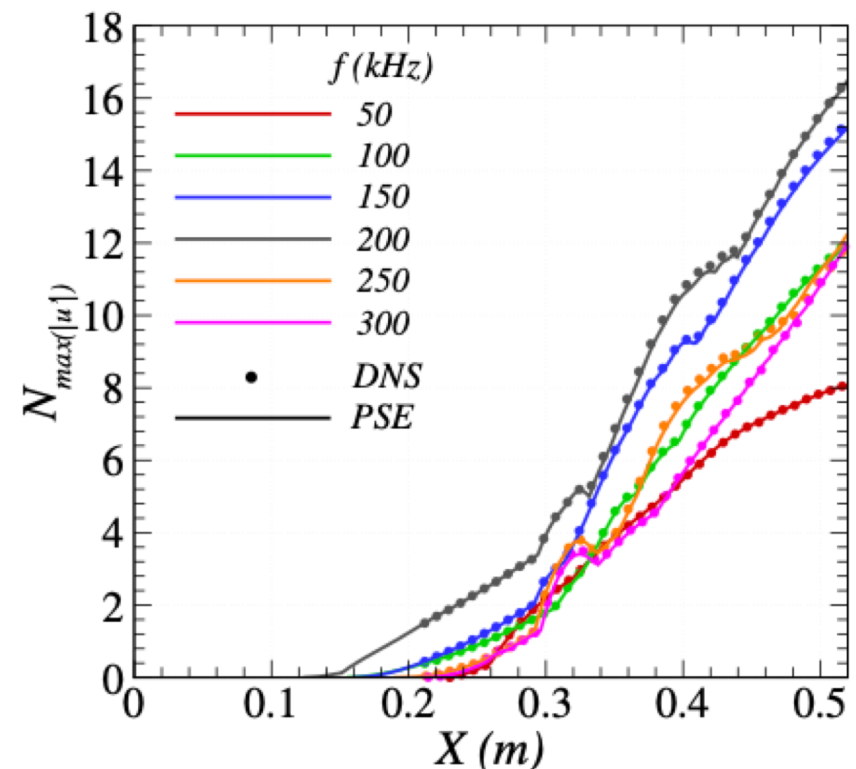
Plane-Marching PSE for Hypersonic Streak Instability

- Mach 7.8 flow over HIFiRE-5b elliptic cone under flight conditions



(a) Antisymmetric mode (PSE)

Unstable mode at 200 kHz



Comparison with DNS

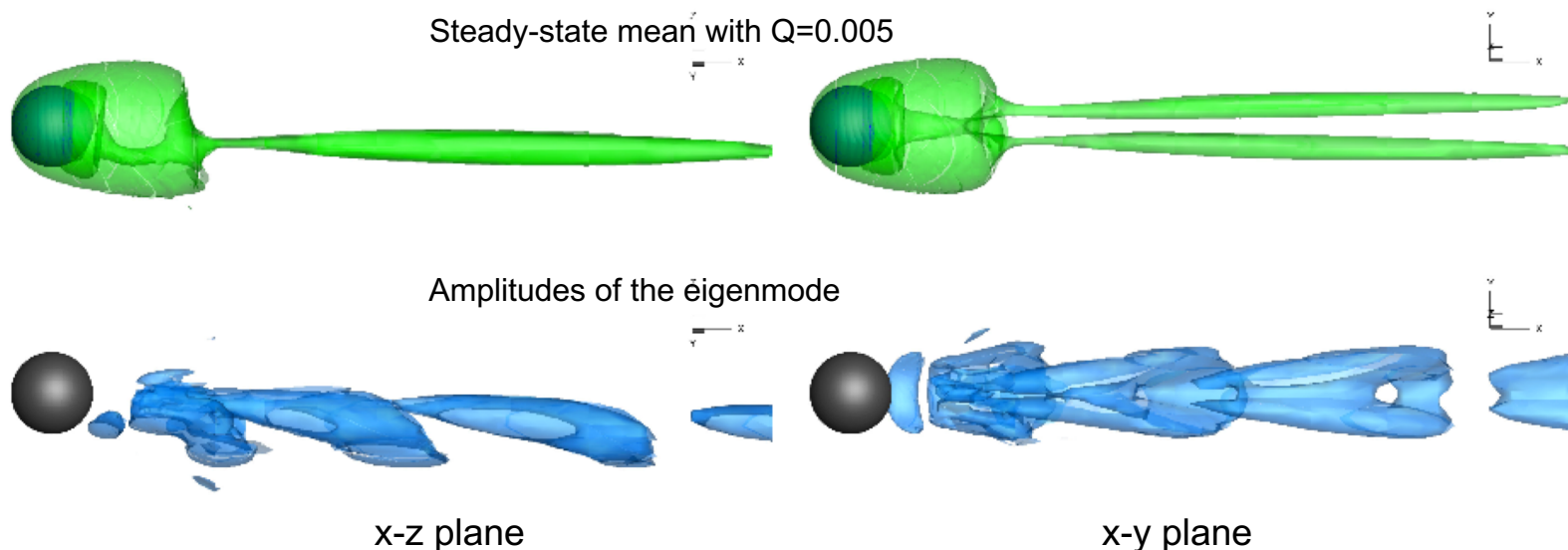


Tri-global Stability Analysis

- Disturbance form (and no further assumptions other than linear perturbations)
$$\phi(x, y, z, t) = \chi(x, y, z)e^{-i\omega t}$$
- Discretization along all three directions over the entire domain results in a very large system of equations in the eigenvalue problem

$$L\chi = \omega\chi$$

- Usually solved by Arnoldi's method or a variant of it to save time
- Example: incompressible flow over a sphere at $Re=300$, showing most unstable mode





Efficient Alternatives for 3D Boundary Layers

- Plane-marching PSE can become costly for complex disturbance structures or when multiple instability modes are present over the entire configuration
 - Lower fidelity methods are more suitable for engineering type of applications
 - Physics-based transition prediction models often require multiple iterations between CFD solvers and transition prediction tools
- Two groups of methods have been used
 - Local line marching along a physics-compliant, pre-determined path
 - Surface marching
 - Similar to 3D boundary-layer solvers
 - Marching solution over the 3D surface at each surface vertex consecutively based on the concept of domain of influence in boundary-layer theory
 - Wave numbers α and β along x and z directions, respectively
 - Governing equations similar to plane-marching PSE

$$\chi(x, y, z) = \psi(x, y, z) e^{i(\int_{x_0}^x \alpha(\xi) d\xi + \int_{z_0}^z \beta(\eta) d\eta)}$$

$$\hat{A} \frac{\partial \psi}{\partial x} + \hat{B} \frac{\partial \psi}{\partial y} + \hat{C} \frac{\partial \psi}{\partial z} + \hat{D} \psi = \frac{1}{R_0} \left(\frac{V_{xy}}{h_1} \frac{\partial^2 \psi}{\partial x \partial y} + \frac{V_{xz}}{h_1 h_3} \frac{\partial^2 \psi}{\partial x \partial z} + V_{yy} \frac{\partial^2 \psi}{\partial y^2} + \frac{V_{yz}}{h_3} \frac{\partial^2 \psi}{\partial y \partial z} + \frac{V_{zz}}{h_3^2} \frac{\partial^2 \psi}{\partial z^2} \right)$$

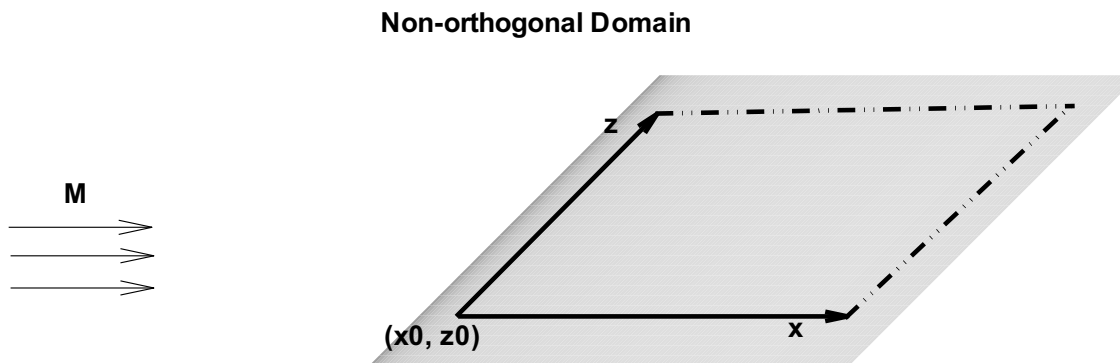


Surface-Marching PSE

- How to determine α and β dynamically?
 - Irrotational condition suggested in 1990s

$$\frac{\partial \alpha}{\partial z} = \frac{\partial \beta}{\partial x} \quad \text{or} \quad \frac{\partial \alpha_r}{\partial z} = \frac{\partial \beta_r}{\partial x}$$

- Derived by assuming wave fronts follow irrotational condition, which is only valid if the boundary layer is truly parallel in both x and z
- Often diverge during PSE marching
- Only working relation found is to assume β real and apply PSE normalization (based on kinetic energy) along x and z separately to update α and β
- Initial conditions must be known at both $x = x_0$ and $z = z_0$



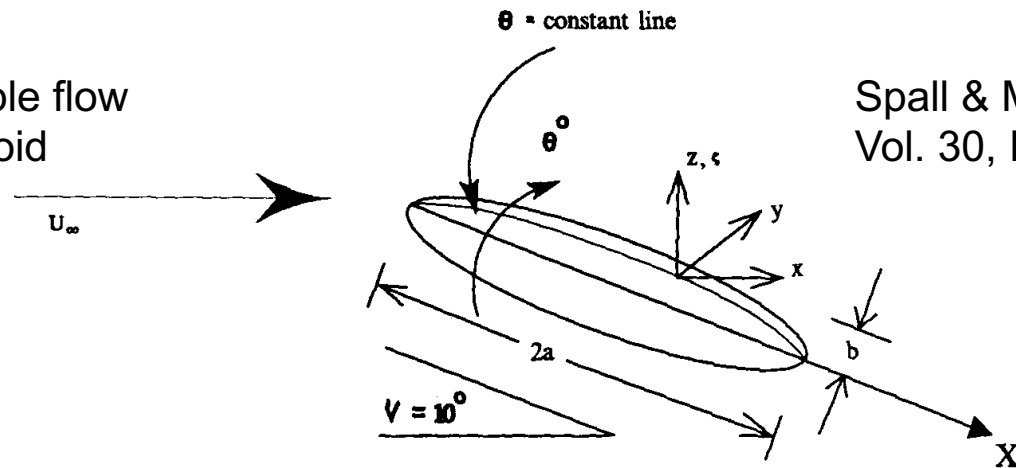


Transition in 3D Boundary Layers

How to integrate N factors?

SPALL AND MALIK: LINEAR STABILITY OF BOUNDARY LAYERS

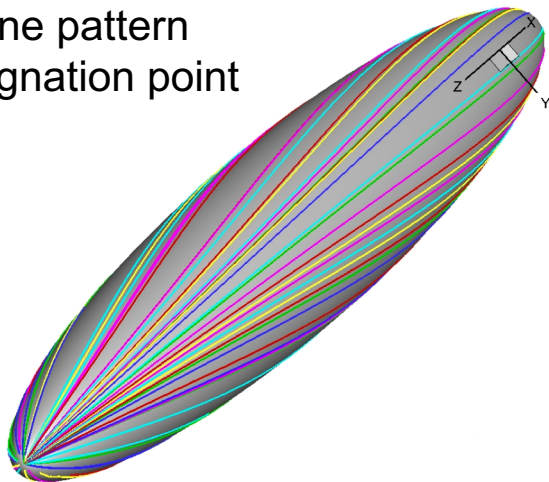
Incompressible flow
over a spheroid



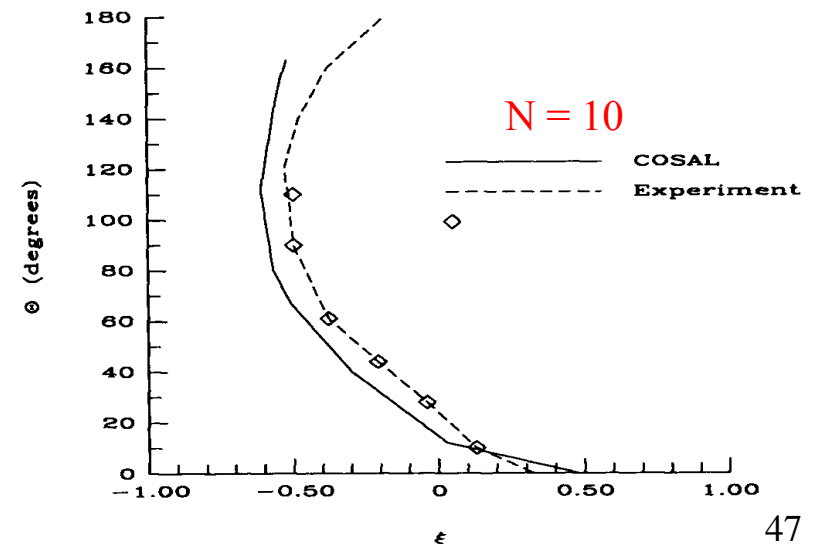
Spall & Malik *AIAA J.*
Vol. 30, No. 4, 1991

Fig. 1 Geometry and coordinate system for prolate spheroid.

Streamline pattern
from stagnation point



LST along group velocity lines





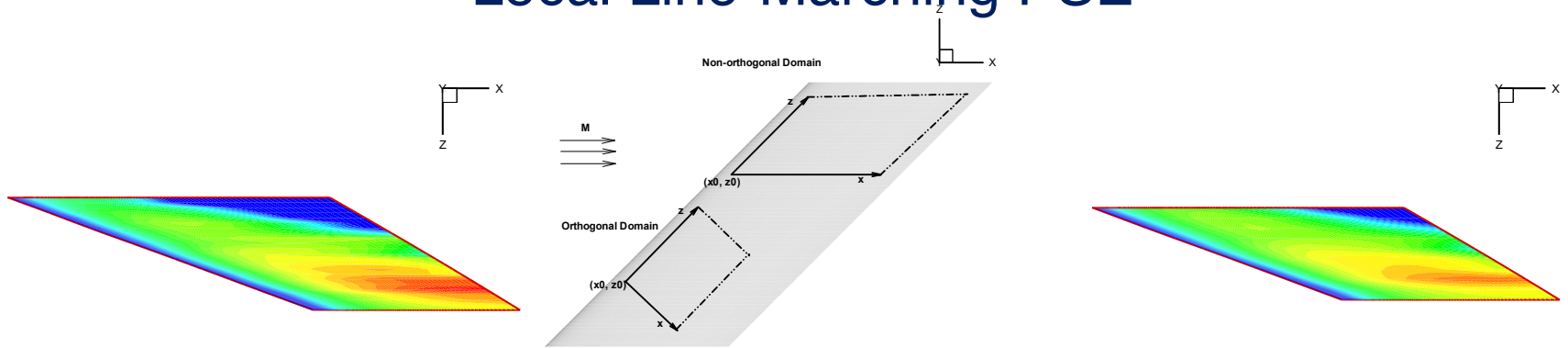
N-factor Integration Paths over 3D Boundary Layers

- Needed for any *local* line marching LST/PSE transition predictions
- Commonly-used options for LST computations
 - Group velocity lines (maximized growth rates, saddle point method)
 - Streamlines (single wave length or maximized growth)
 - Generalized inflection-point paths (for stationary modes only)
 - Vortex Lines (for both traveling and stationary crossflow modes)
- For PSE computations
 - Must follow individual modes with a fixed spanwise wave number (no growth-rate maximization)
 - Streamlines or generalized inflection-point lines
 - Predetermined (from LST or other met) paths including group velocity or vortex lines
 - Grid lines for wing-like configurations



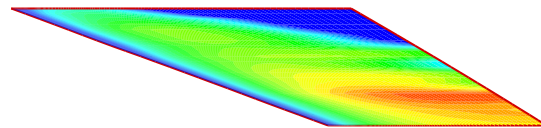
Local & Nonlocal Line Marching PSE Methods

Comparison of Surface-Marching and Local Line-Marching PSE



Surface-Marching (by solving PDE over all surface points sequentially)

Line-Marching with Spanwise Correction

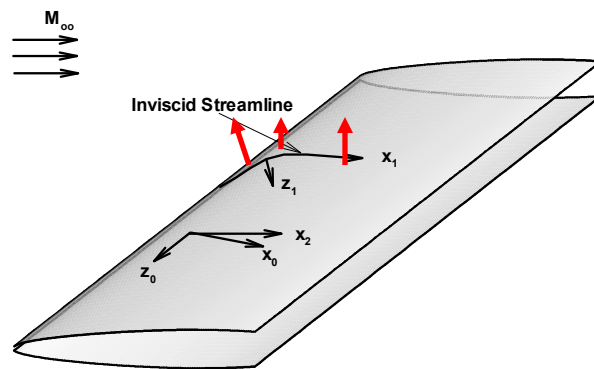


Most widely used $\Rightarrow$ Line-Marching without Spanwise Correction

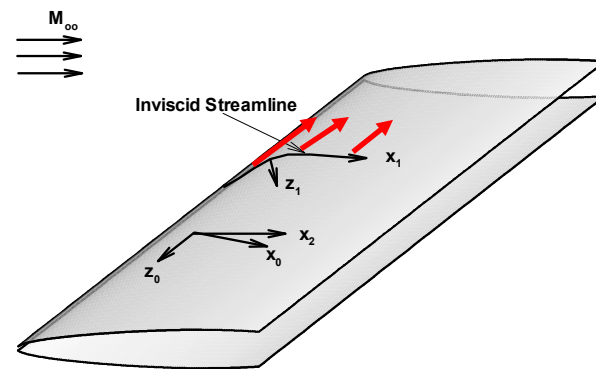


Local Line-Marching PSE

- Must follow individual modes at each point along the marching path
 - Past experiences show better correlation than maximized growth-rate options
 - Each mode normally has a unique spanwise wave number (real, or integer for a conical body)
 - N-factor integration to comply with physics requires a non-orthogonal spanwise axis that varies along the path



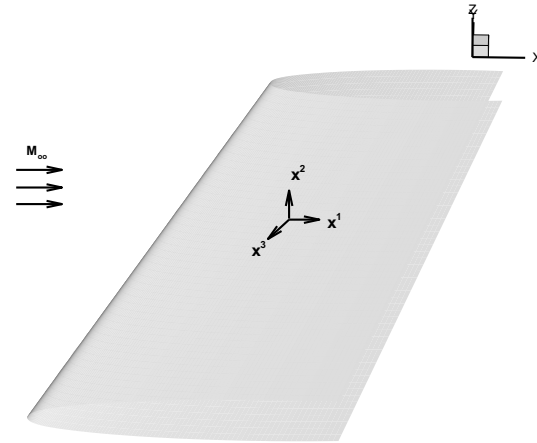
Orthogonal spanwise axis



Non-orthogonal spanwise axis

Navier-Stokes Equations in Non-orthogonal Coordinate System

$$(x^1, x^2, x^3)$$



$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0$$

$$\rho \left(\frac{\partial v^i}{\partial t} + v^j v^i_{,j} \right) = -g^{ij} p_{,j} + g^{ij} (\lambda v^k_{,k})_{,j} + \left[\mu (g^{jm} v^i_{,m} + g^{in} v^j_{,n}) \right]_{,j}$$

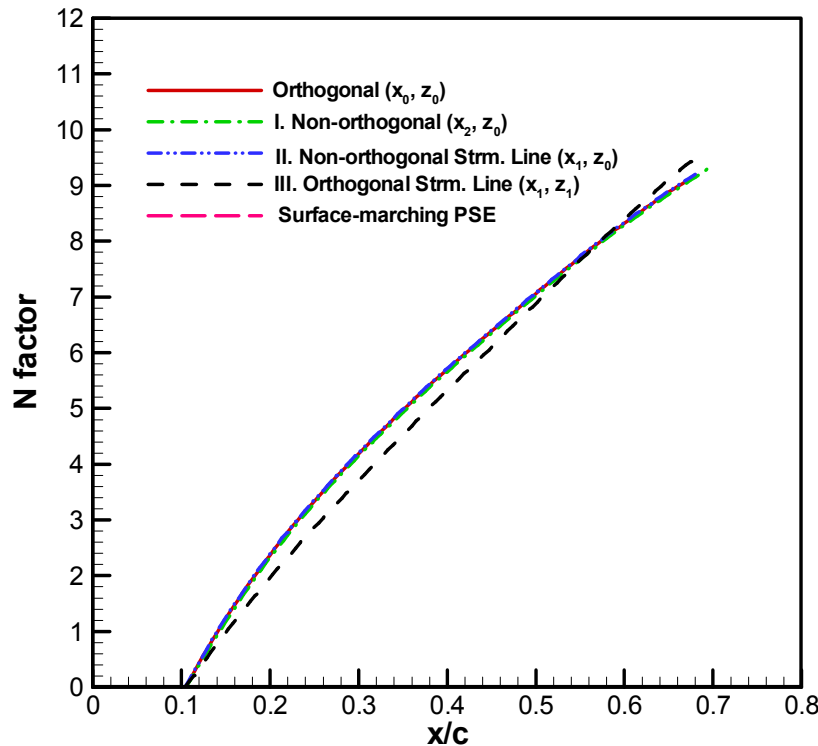
$$\rho C_p \left(\frac{\partial T}{\partial t} + v^j T_{,j} \right) = (\kappa g^{jk} T_{,k})_{,j} + \frac{\partial p}{\partial t} + v^j p_{,j} + \Phi$$

Governing PDEs can be cast in the same form as before

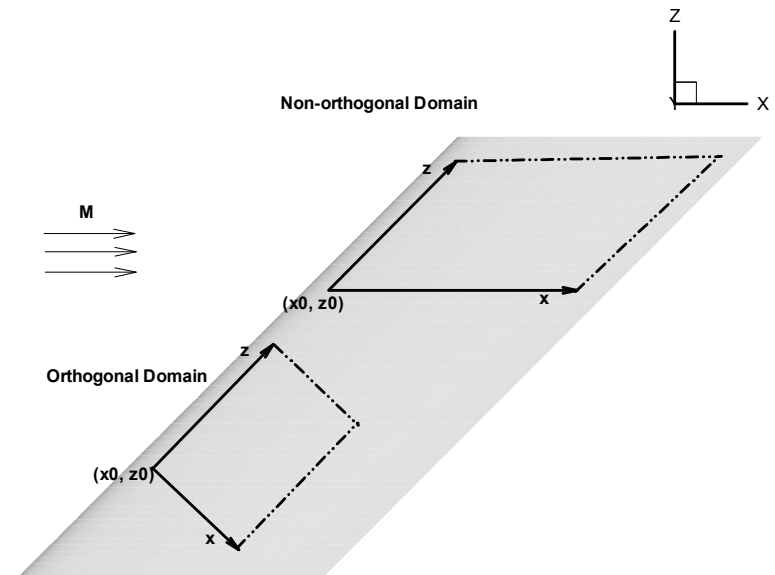


Validation with an Infinite Swept Wing

- Infinite swept wing solved as a 3D boundary layer



PSE predictions with various methods

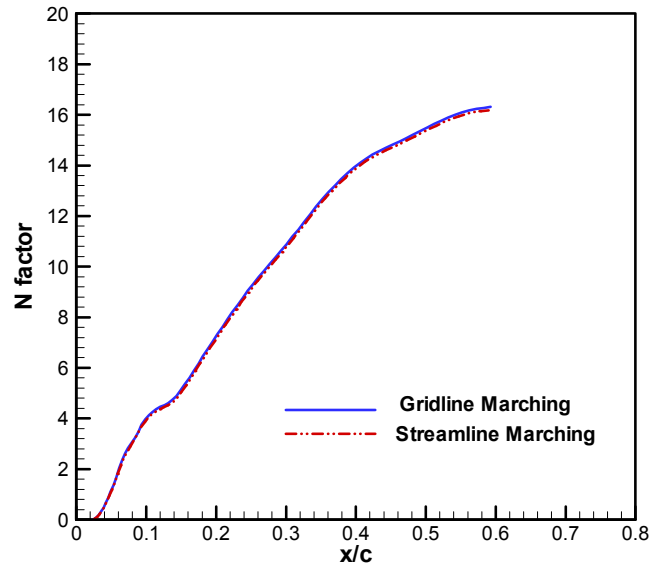
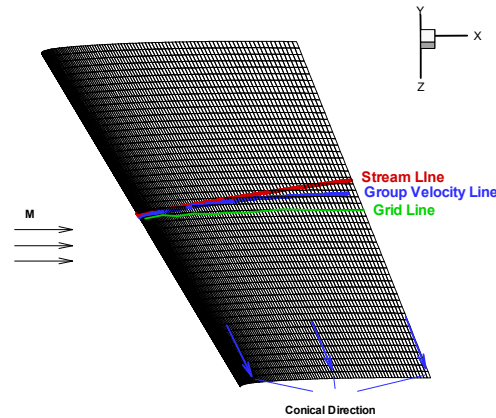
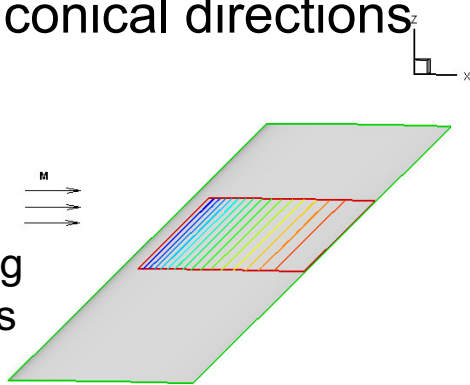




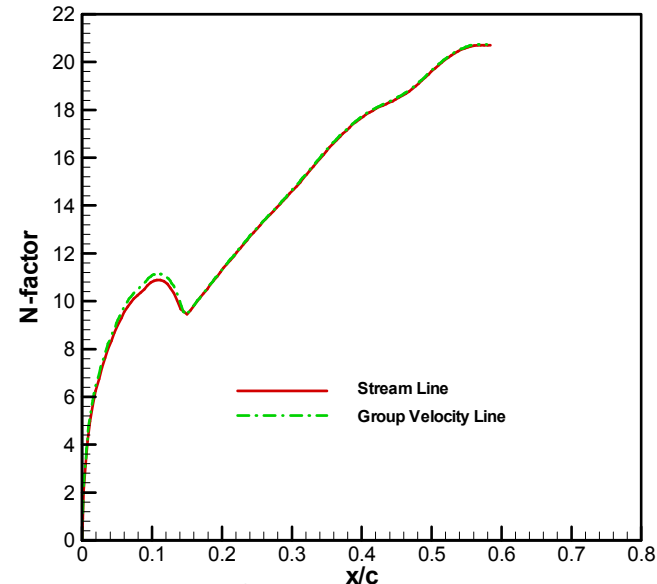
Transonic Tapered Wing (Mach 0.8, $f=1.5$ kHz, $\lambda = 12$ mm)

- Leading and trailing edge sweep angles are 28° and 14° , respectively
- Spanwise conical directions

Surface Marching
N-factor contours



PSE

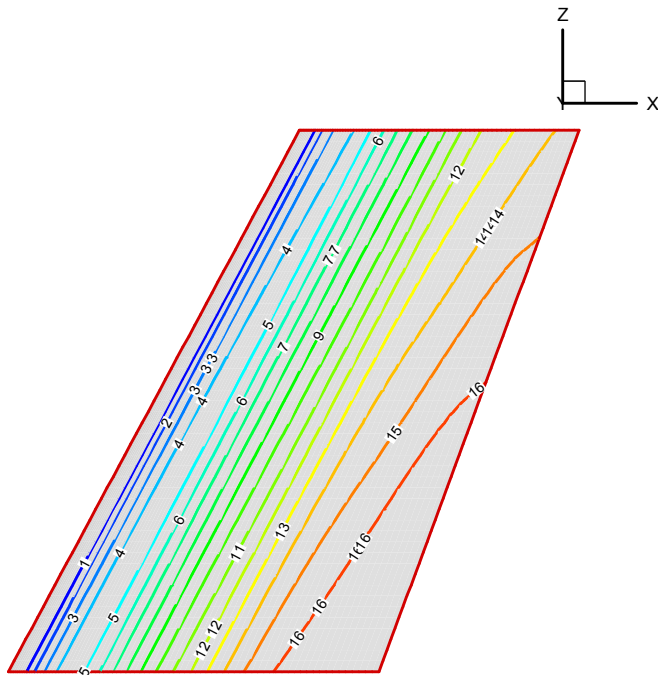


LST

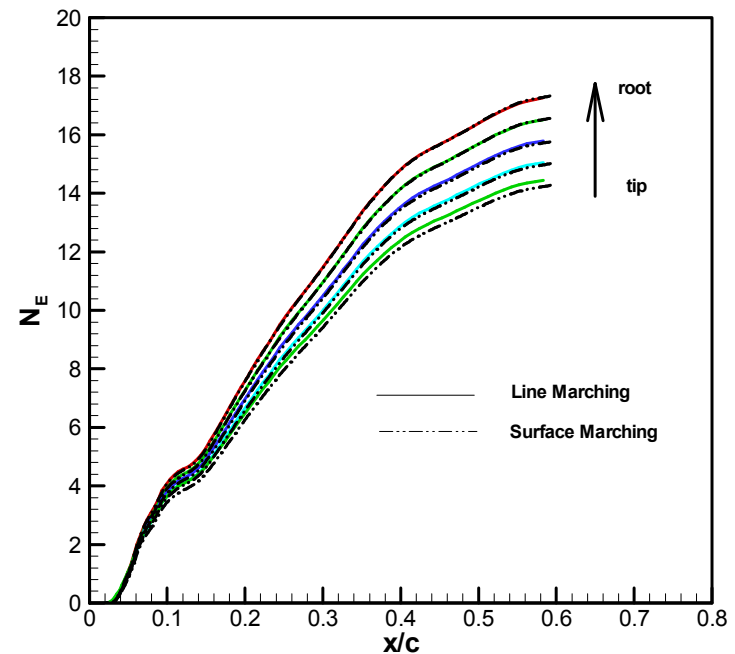


Transonic Tapered Wing (Mach 0.8, $f=1.5$ kHz, $\lambda = 12$ mm)

- Comparison of local-line and surface marching PSE



Surface marching N-factor contours

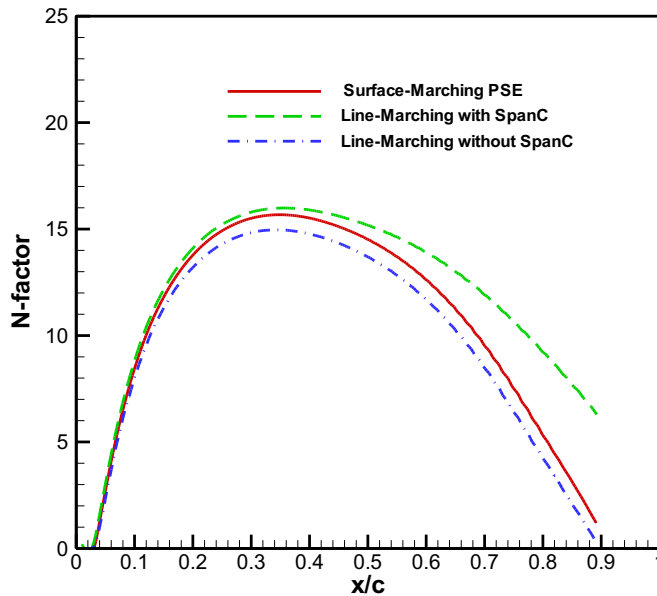
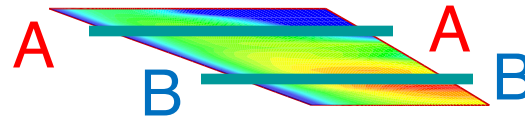


Comparison with local-line marching

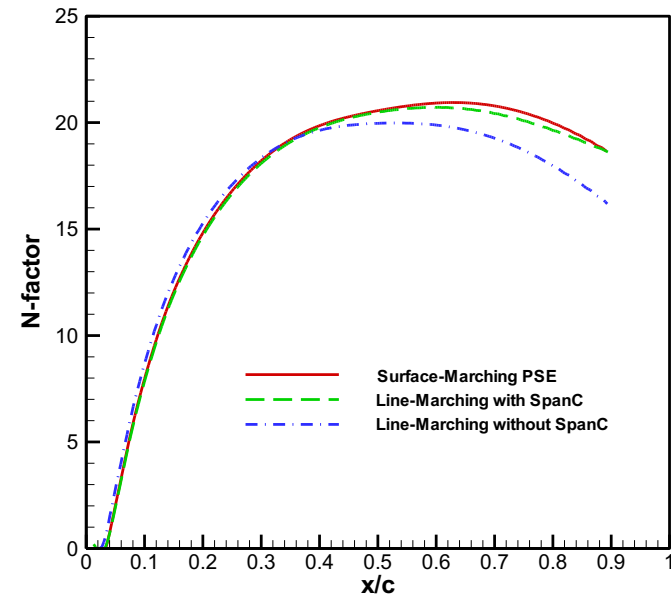


Pegasus Supersonic Finite Swept Wing

Comparison of Surface-Marching and Line-Marching



A-A

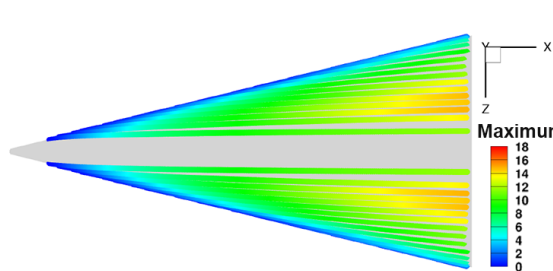


B-B

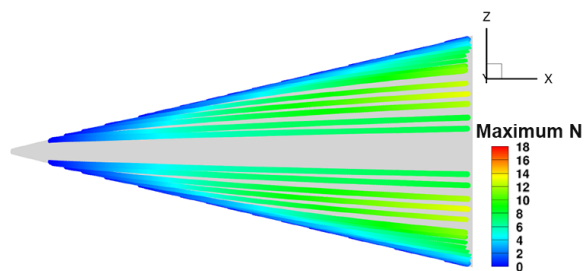


HiFiRE5b Fight Test by Local Line Marchig

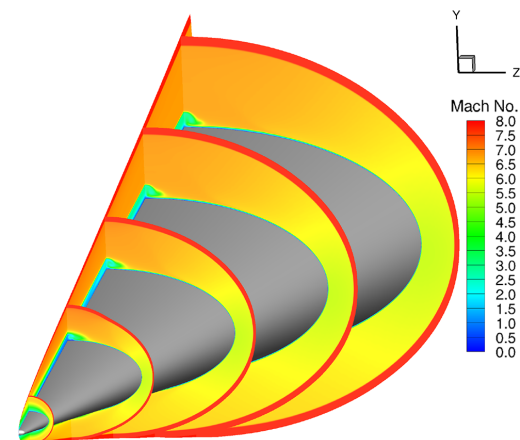
- Tufts et al. *J. Spacecraft & Rockets* (Vol. 55, No. 6, 2018)
- All stability (PSE) calculations by LASTRAC



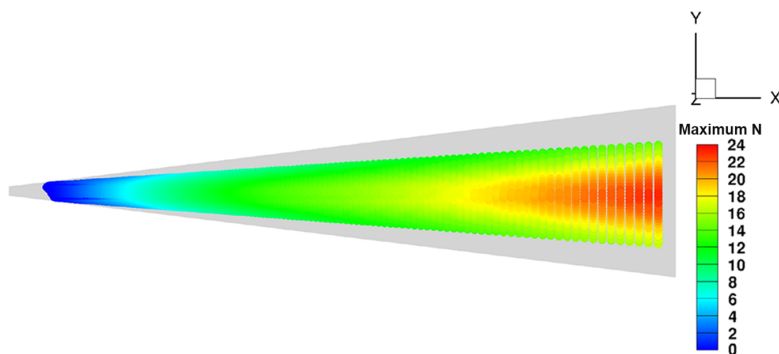
a) Leeward



b) Windward



Stationary crossflow



Second mode

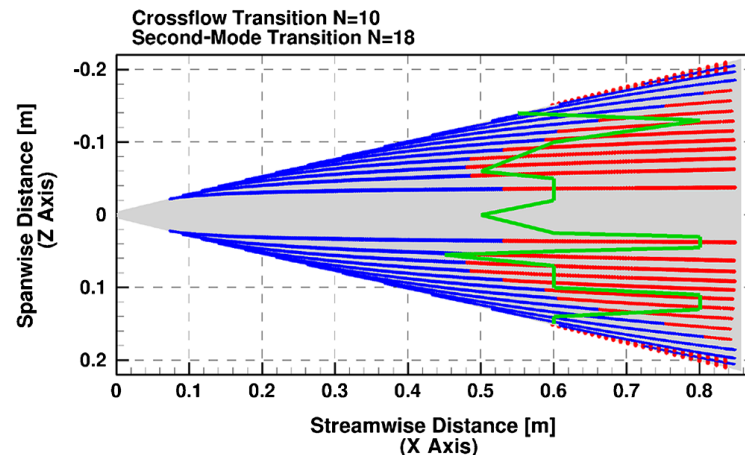


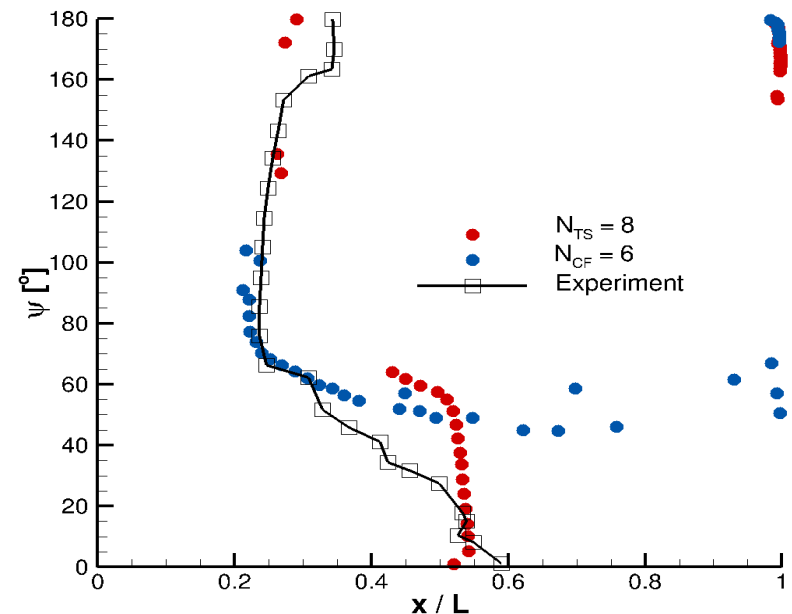
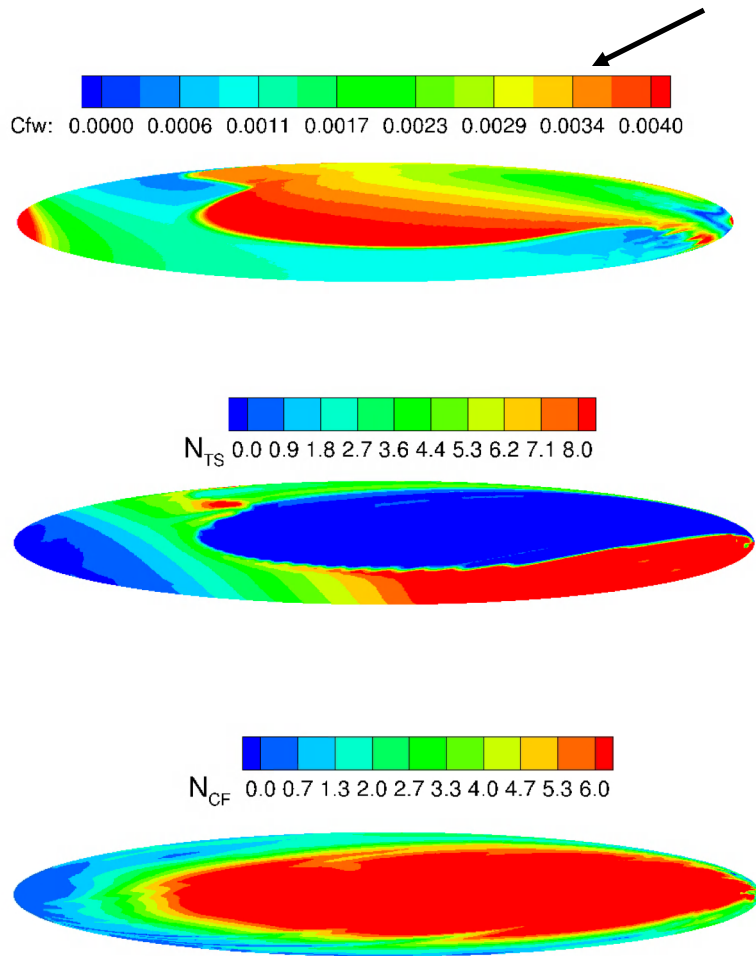
Fig. 18 Flight conditions: $\alpha = -1.2$ deg, $\beta = 0$ deg, and preliminary estimated leeward transition locations, with experimental in green.

Comparison with flight



Incompressible Flow over a Prolate Sphere at Angle of Attack (Linear PSE)

Meanflow computed by OVERFLOW using SST-2003-LM2015 transition model



Experiment extracted from Krimmelbein et al. 2018.

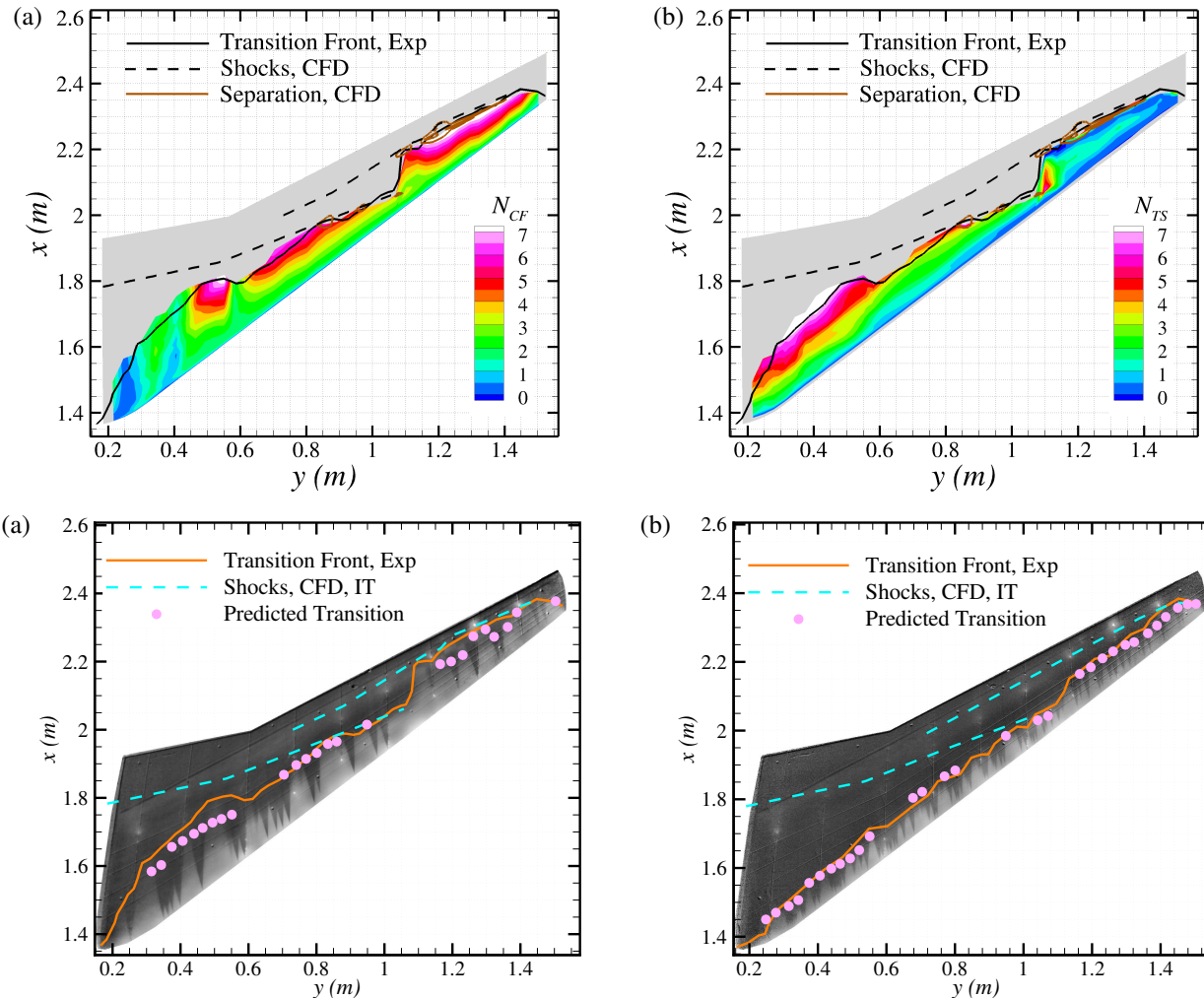
LSTRAC predictions using converted unstructured quad surface mesh



CRM-NLF Transonic Wing

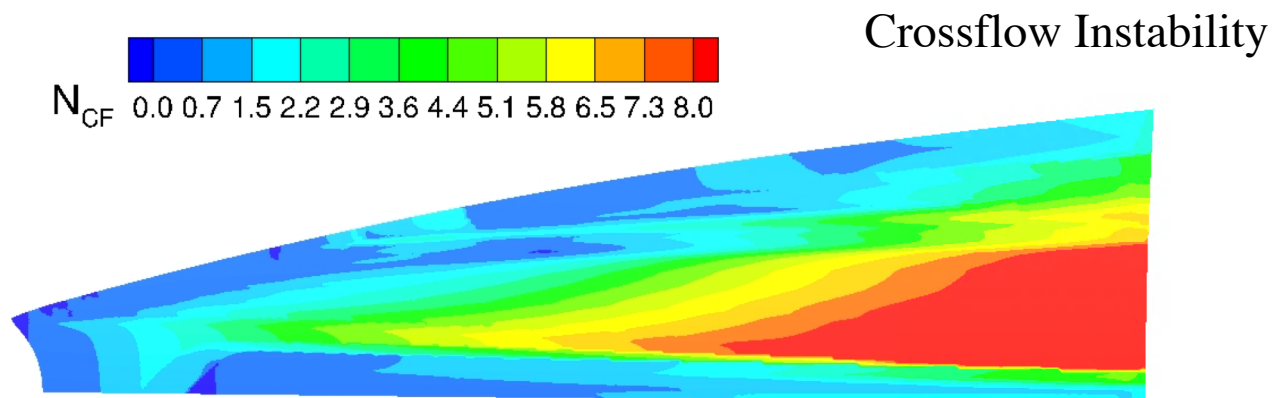
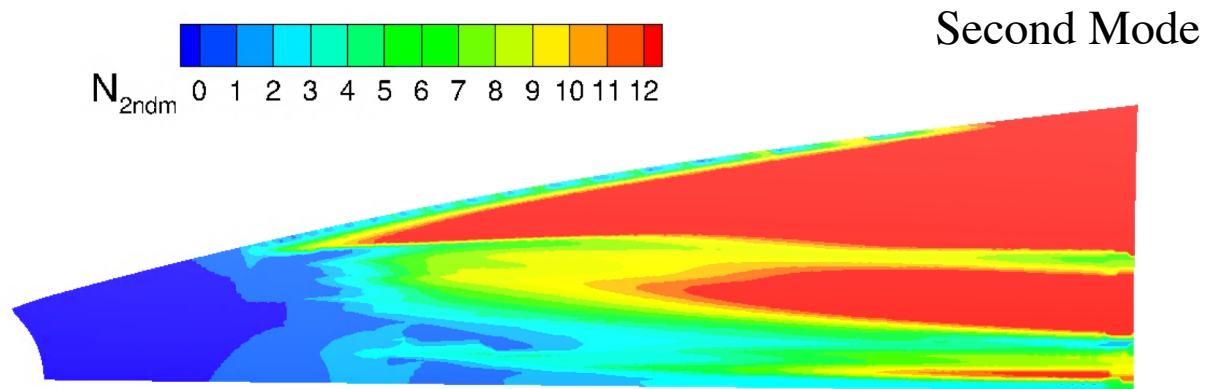
- NTF tunnel measurements (Mach 0.86)
- PSE calculations with $N_{TS} - N_{CF}$ correlations

From Paredes et al., AIAA Paper 2021-1431



Linear PSE Transition Prediction of BOLT-1 Flight Experiment

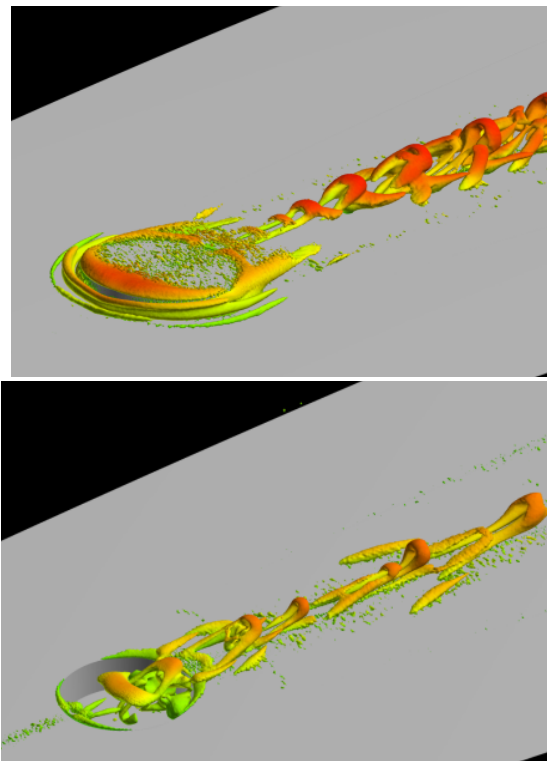
- BOLT-1 flight conditions at descent phase 488.0 s
- Mach number around 7.8



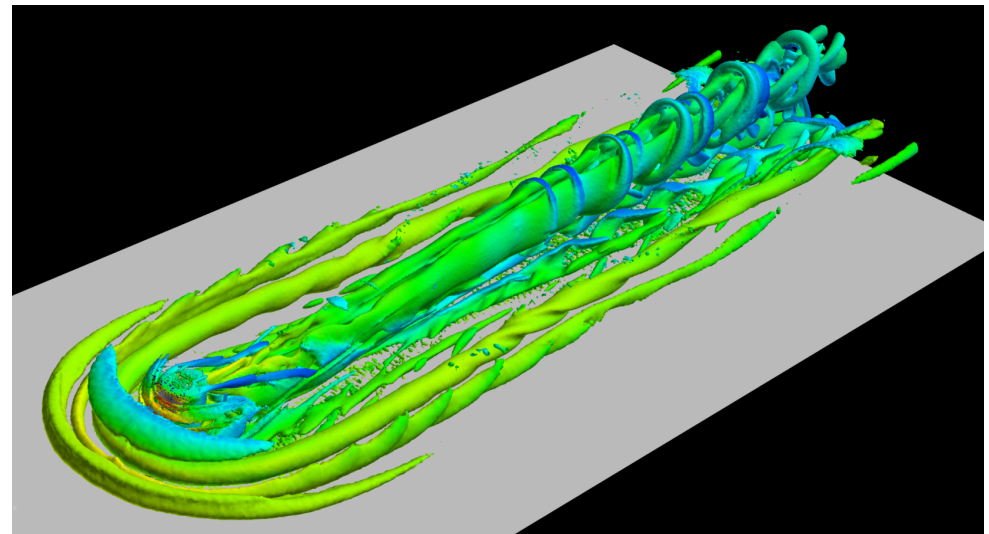


Roughness Induced Transition in 3D Boundary Layers

- Roughness heights h and corresponding best analysis tools, let δ be boundary-layer thickness
 - $h/\delta \ll 1$: receptivity + LST/LPSE + NPSE/DNS
 - $h/\delta < 0.5$: meanflow resolved + biglobal LST/plane-marching PSE in the wake
 - $h/\delta > 0.5$: DNS starting from upstream of the roughness



Subsonic flow over protuberance/cavity



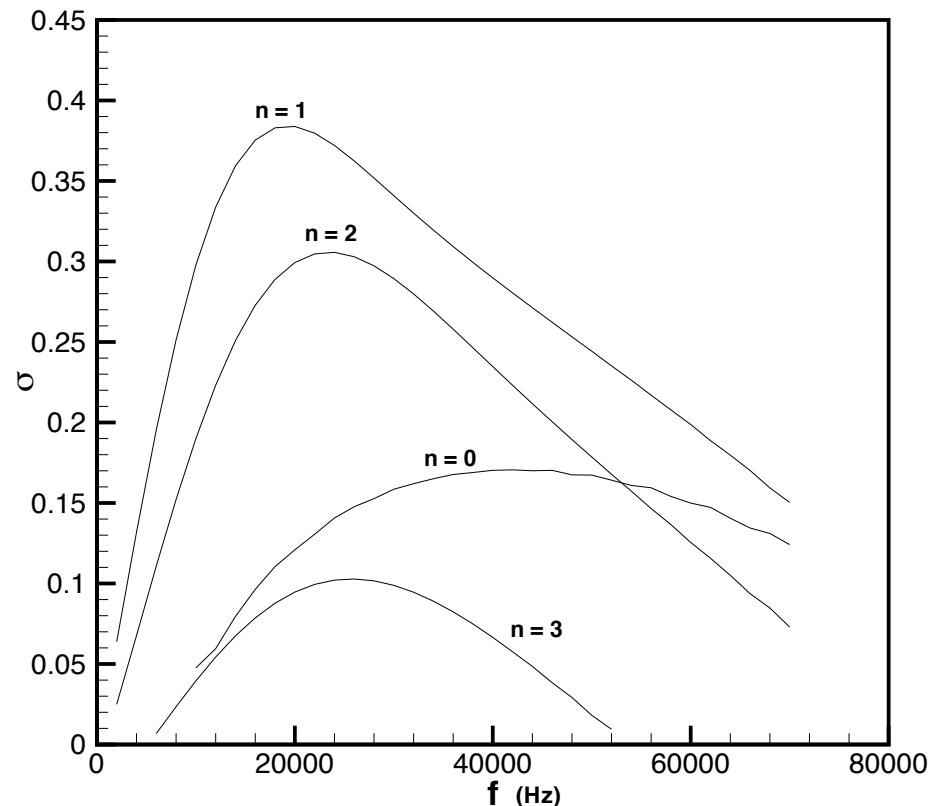
Supersonic flow over a tall protuberance

Chang et al. AIAA 2011-3245



Jets and Shear Layers

- Predicting jet and shear-layer instability is relevant in hypersonic propulsion
- Not a focus in the transition research community
- Mach 2.5 round jet



Disturbance growth rates vs. frequency for various azimuthal wave numbers computed by LASTRAC (from NASA TM 2004-213233)



Outline of the Lecture

1. Introduction
2. Boundary-layer stability theory and computations
 - Linear stability theory
 - Linear parabolized stability equations
3. Transition prediction using N-factor method
 - Transition mechanisms and modal instability
 - Nonmodal, transient growth
4. Absolute amplitude methods and DNS
 1. Nonlinear PSE
 2. Biglobal stability (2D eigenvalue problems)
5. 3D boundary layer
6. Chemical/thermal nonequilibrium flows
7. Summary



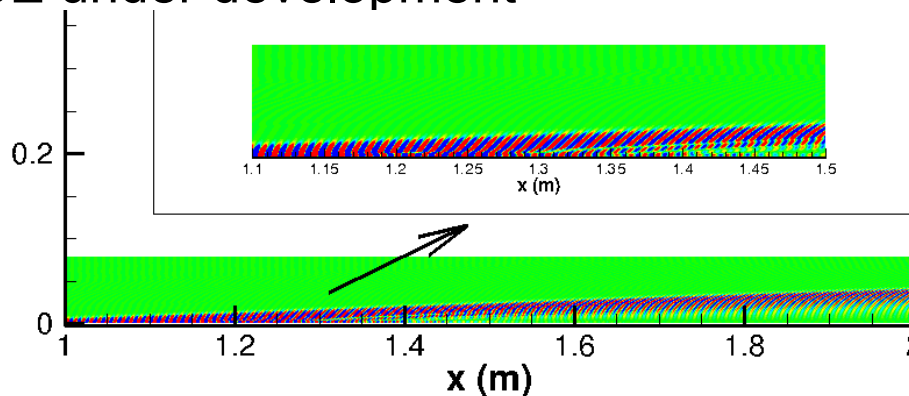
Chemically Reacting Hypersonic Boundary Layer

- Chemically reacting boundary-layer flows take place in hypersonic configurations either on vehicle surfaces or at engine inlets when high temperatures cause gas molecules to dissociate
- Instability waves and transition analyses of such high-enthalpy flows must account for full effects of the nonequilibrium gases
- Implemented nonequilibrium chemistry modules in the popular NASA transition prediction software LSTRAC to allow transition prediction of hypersonic high-enthalpy flows over general 3D configurations using the suite of LST/LPSE/NPSE tools to enable/enhance transition prediction capabilities in hypersonic vehicles



LASTRAC 3.x

- Extends publicly released LASTRAC software for transition predictions of multicomponent chemically reacting hypersonic flows over general 3D boundary layers
- New features
 - Hypersonic reacting flow (equilibrium or nonequilibrium) over *general 3D bodies* with practically *all transition mechanisms*
 - Mack modes, crossflow, Görtler, and shear-layer/jet instability
 - User input chemistry models (e.g., can handle Mars or any other combination of gases)
 - Nonlinear, absolute amplitude transition prediction using nonlinear PSE under development



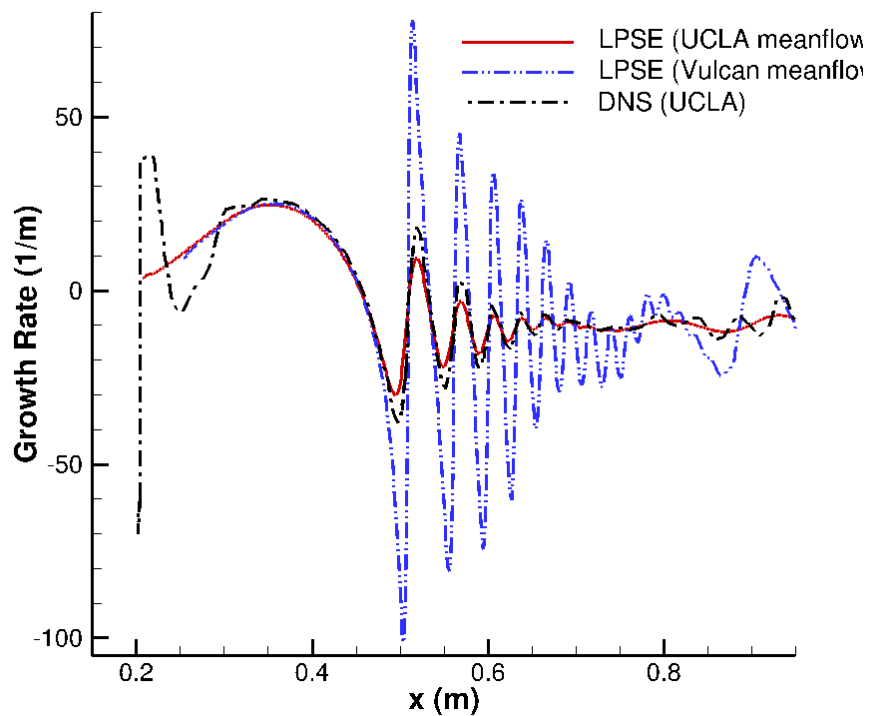
Mach 10 supersonic modes over a cone



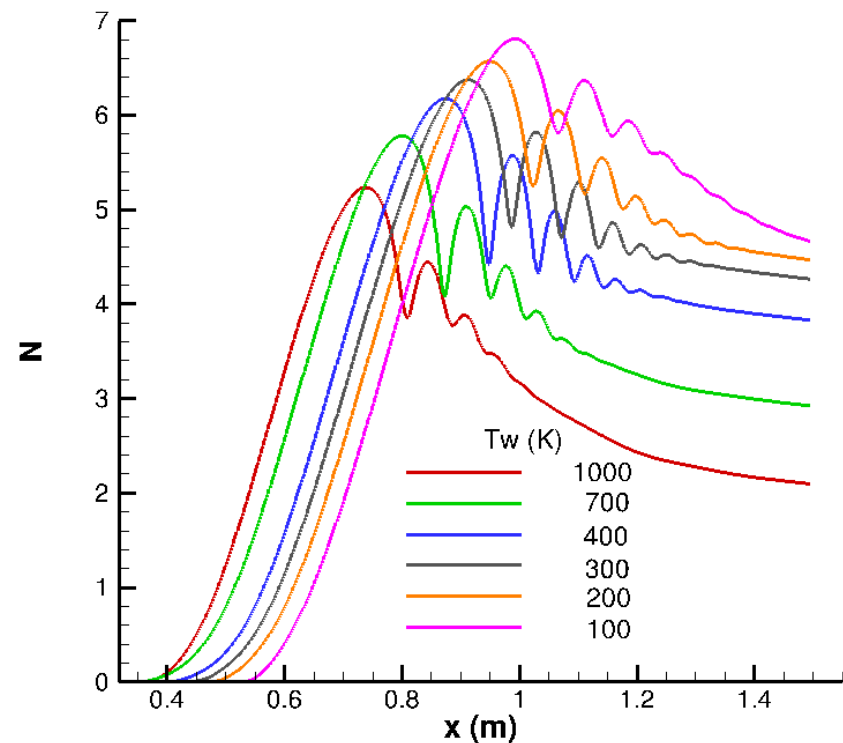
Supersonic Modes in High-Enthalpy Hypersonic Boundary Layers

- Mach 10 flow over a 7° cone under high-enthalpy conditions
- Chemical and thermal nonequilibrium chemistry models used
- Supersonic modes are present

Chang et al. *AIAA J.* online
<https://doi.org/10.2514/1.J060161>



Comparison of PSE and DNS predictions using two different meanflows



Effects of wall cooling on supersonic modes overall growth



Summary

- We have reached a stage of “maturing” prediction tools at various fidelity levels
 - All tools compliment each other at different regions of applications based on flow physics
 - High fidelity tools need to be more efficient for parametric studies
 - Overall, PSE predictions are very accurate and should be used in lieu of DNS until near the onset of transition
- Areas that need further development
 - Receptivity models and disturbance environment characterizations
 - DNS tools that can handle interactions of strong shocks with disturbances or even the presence of shocklets for many hypersonic applications
 - Improvements of various line-marching methods in 3D boundary layers with the help of validations against experimental transition front measurements
 - Coupling physics-based N-factor transition prediction tools/models to CFD solvers for more accurate laminar-transitional-turbulent CFD predictions
 - Nonequilibrium chemistry models need to be further developed along with high-enthalpy transition experiments either via wind tunnel or flight tests