Interval Predictor Models for Robust System Identification

Luis G. Crespo, Sean P. Kenny, Brendon K. Colbert, Tanner Slagel

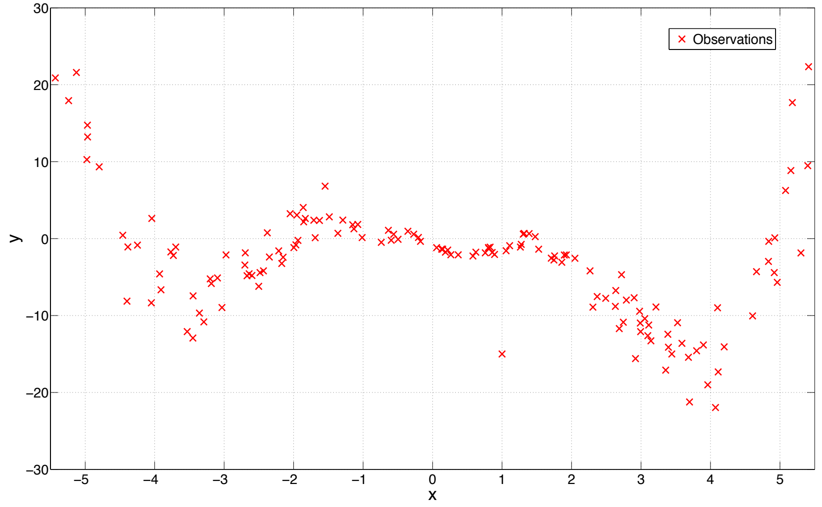
NASA Langley Research Center

60th IEEE CDC

Problem Statement

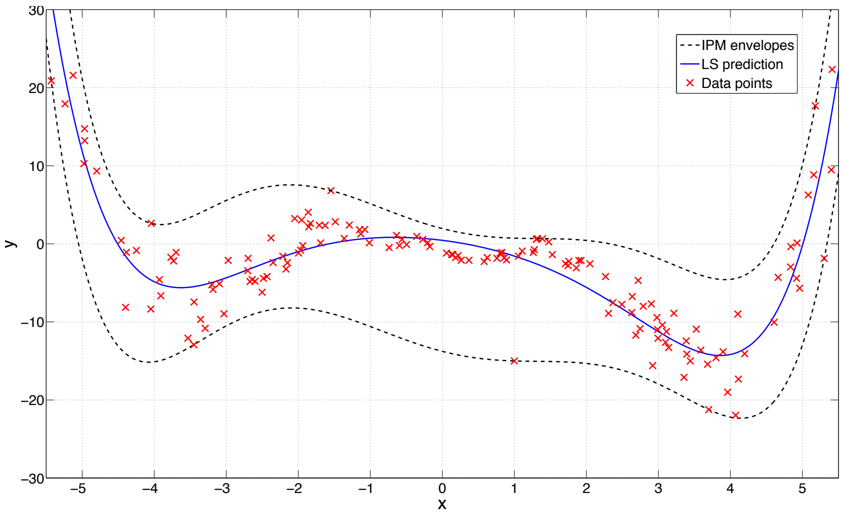
- Given a cloud of multivariate input-output data and a parametric model, make an informative prediction of the output at any input
- Informative: characterize the range of the data, and the probability of an unseen datum falling outside such a range, i.e., reliability

Data: $\mathcal{D} \triangleq \{\delta^{(i)}\}_{i=1}^{n},\ \delta^{(i)} = (x^{(i)}, y^{(i)})$



$$\boxed{y(x,p)}$$

$$I_y(y; x, \mathbb{P}) = \left[\,\underline{y}(x,\mathbb{P}),\ \overline{y}(x,\mathbb{P})\,\right]$$



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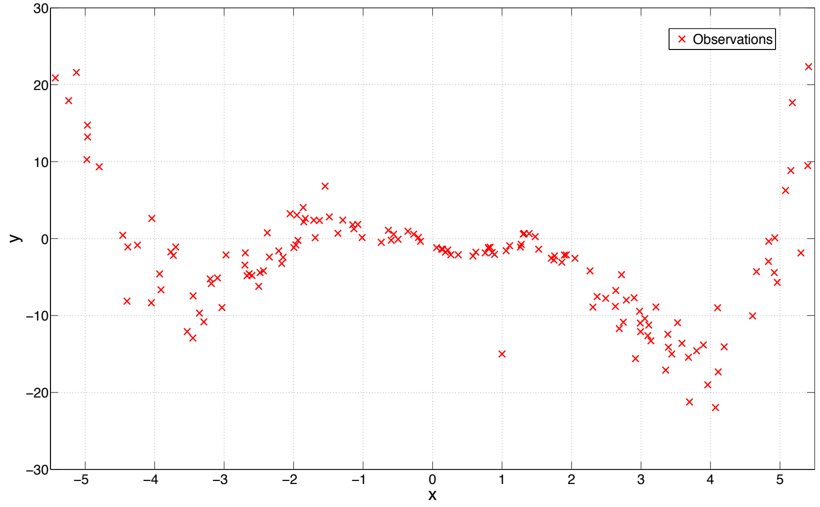

IID data, with noise effects filtered out

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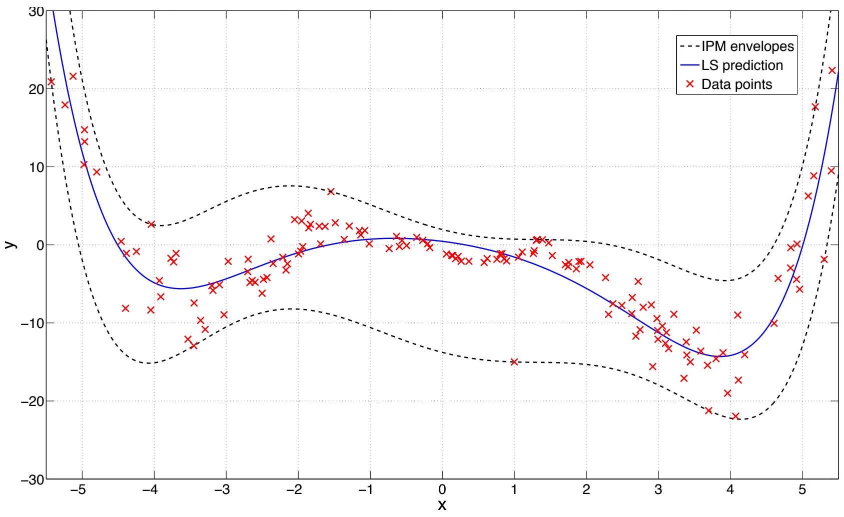

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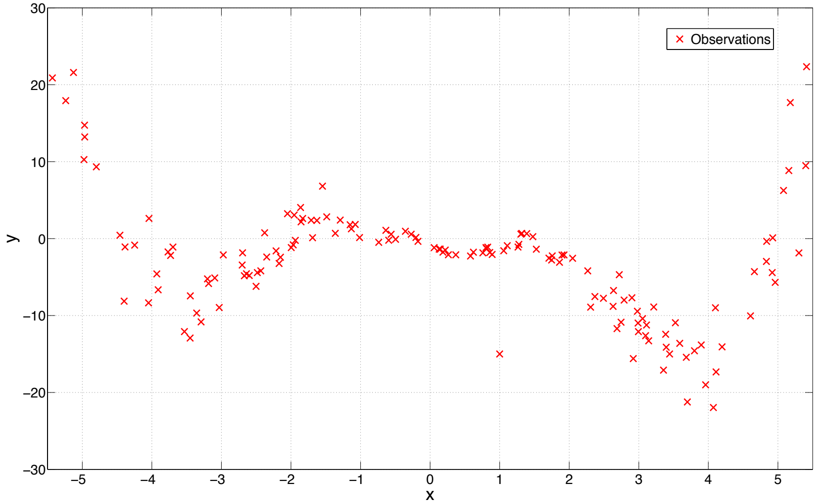

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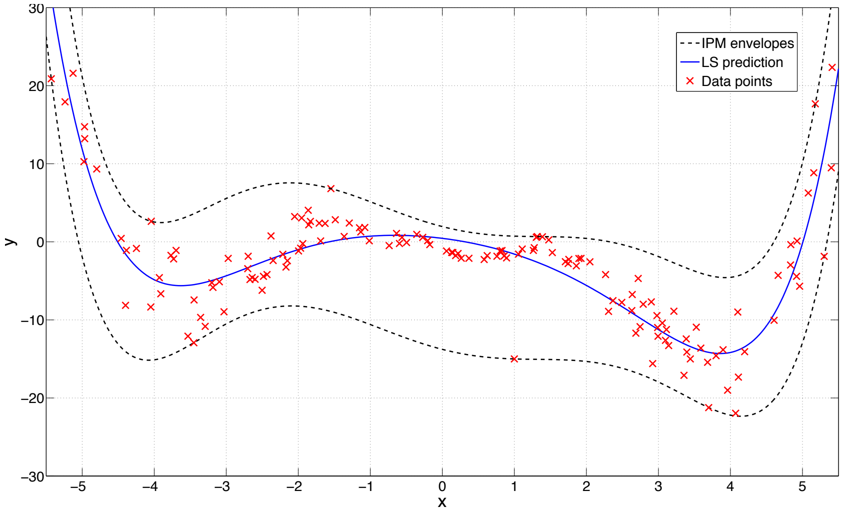

Multiple scenarios to account for variability

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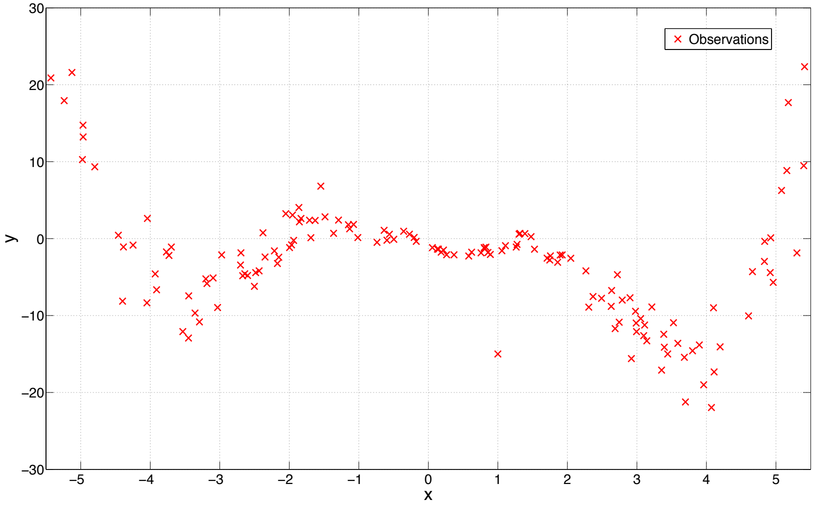

Set of continuous functions (possibly implicit and nonlinear)

$$\boxed{y(x,p)}$$

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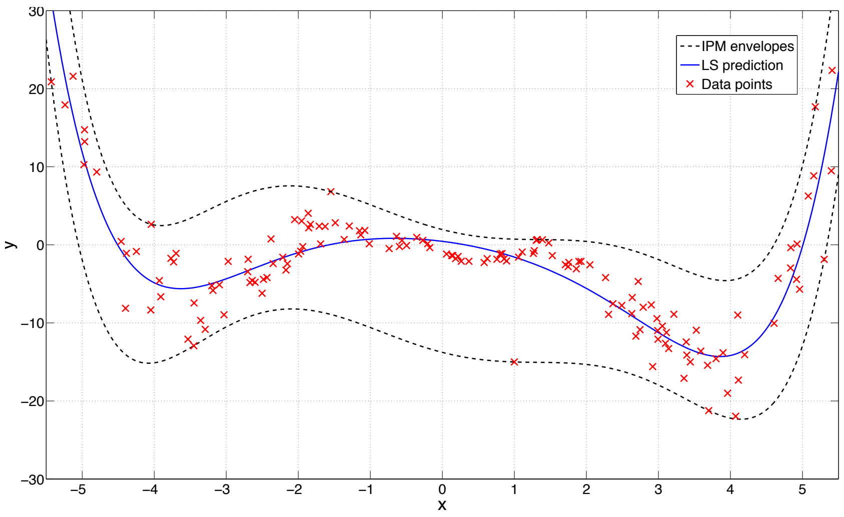

Interval Predictor Model (IPM)

- An IPM prescribes the parameters of a model as a path-connected set thereby making the output an interval-valued function of the input
- The formulation proposed seeks the set of a given class leading to the tightest enclosure of the input-output data
- The chosen classes: semi-algebraic, captures parameter dependencies

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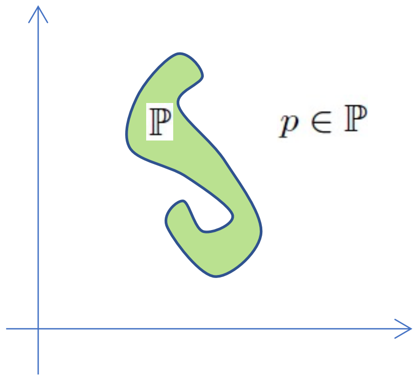

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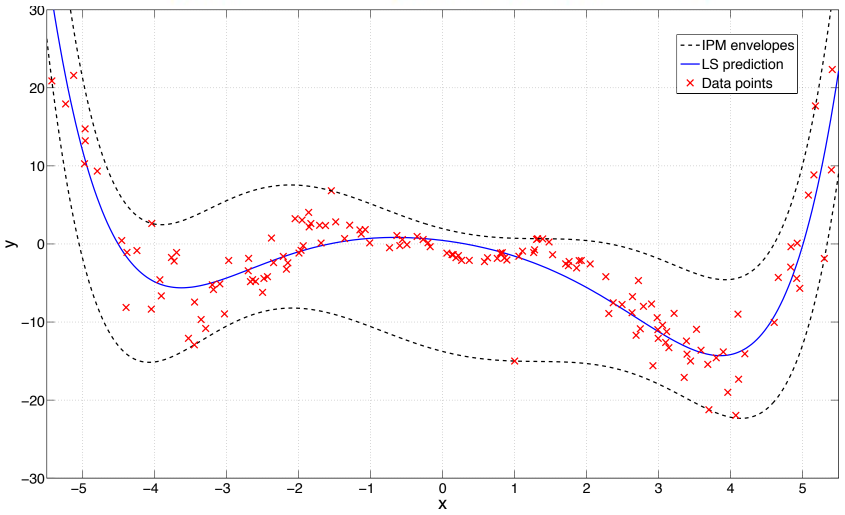

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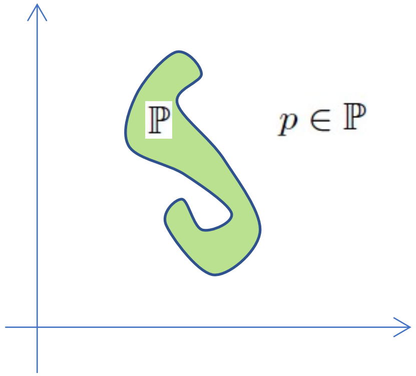
The multivariate IPMs are interdependent because they share the same $\mathbb{P}$

$$\boxed{y(x, p)}$$



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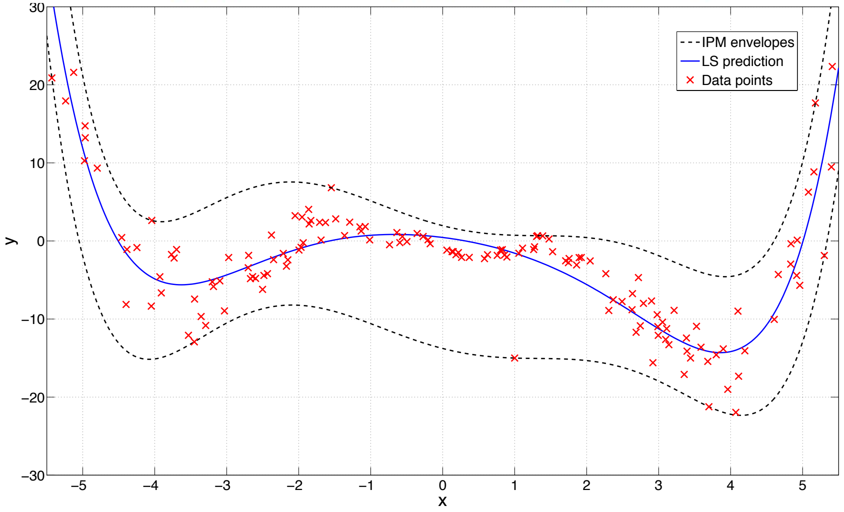

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All uncertainty realizations are feasible, e.g., FEM example

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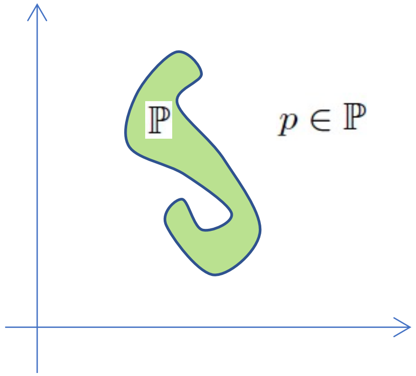

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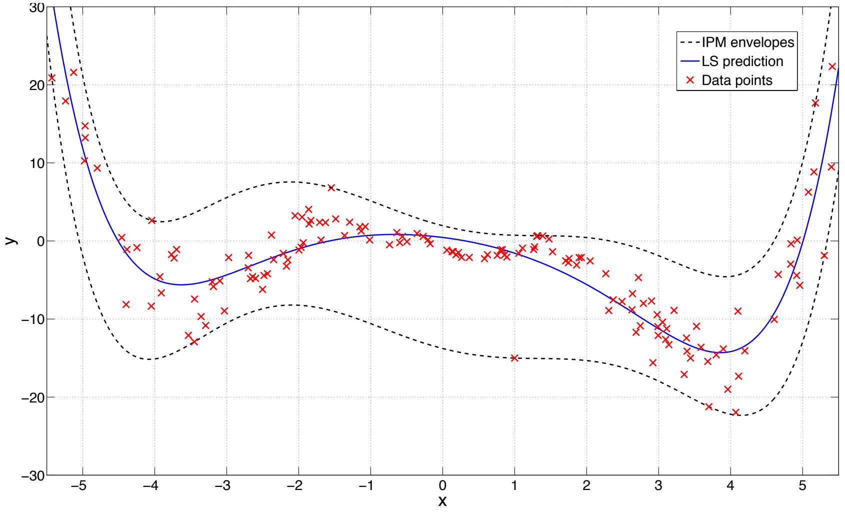

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Robustness analysis and robust control techniques based on polynomial optimization are applicable

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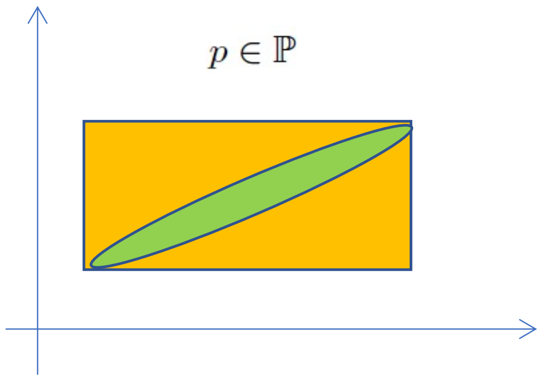
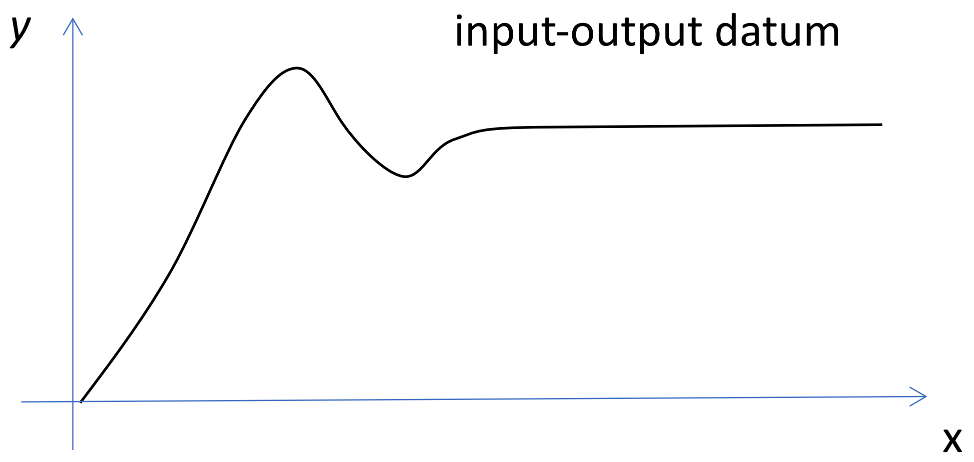
Eliminates unnecessary conservatism



Interval Predictor Model (IPM)

The set $\mathbb{P}$ is computed in a two-step procedure:

1. Data mapping: find the parameter sequence for which the multi-point IPM has minimal spread while containing the datum



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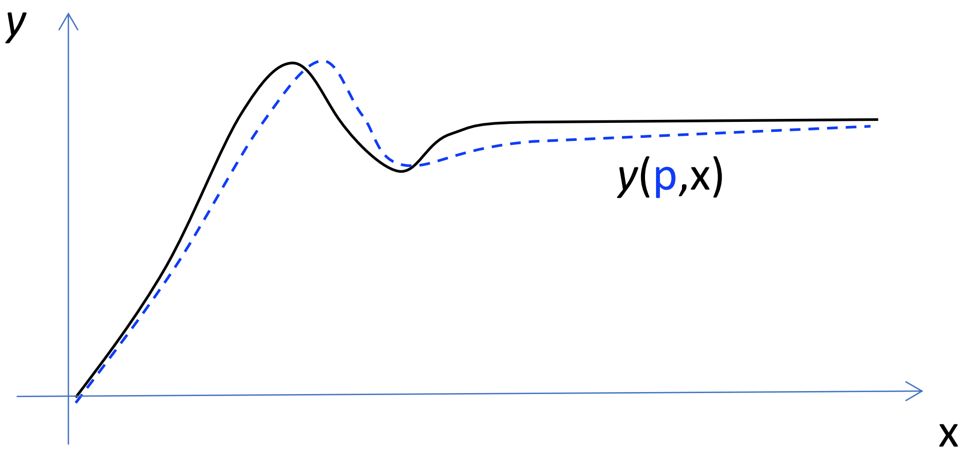

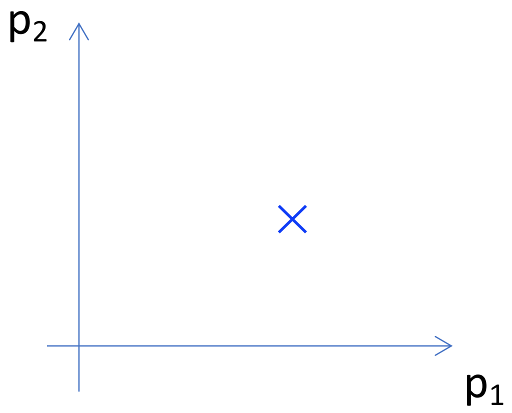

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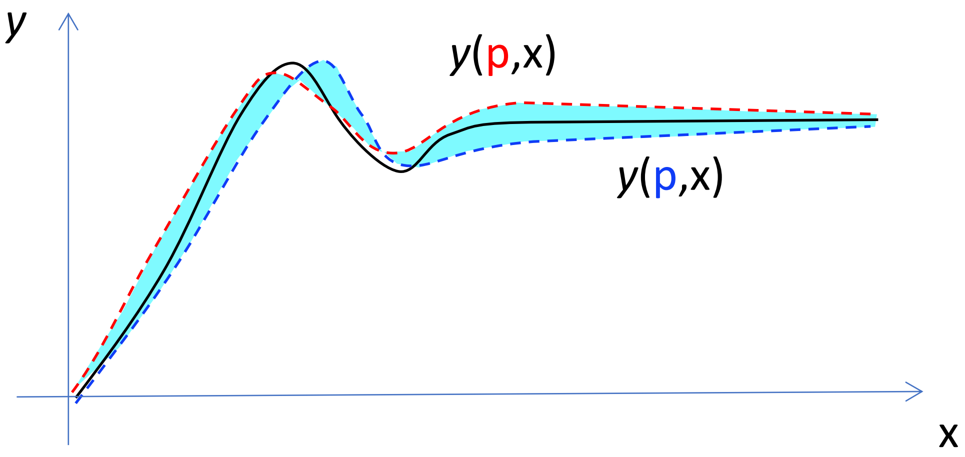

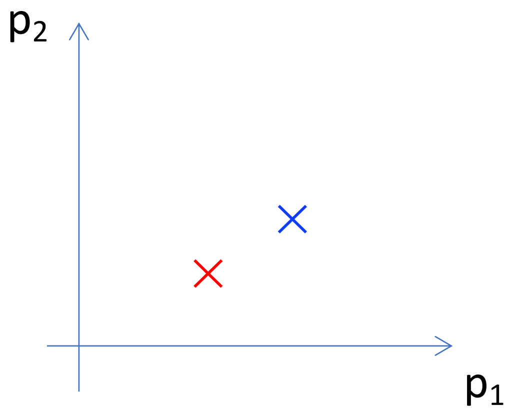

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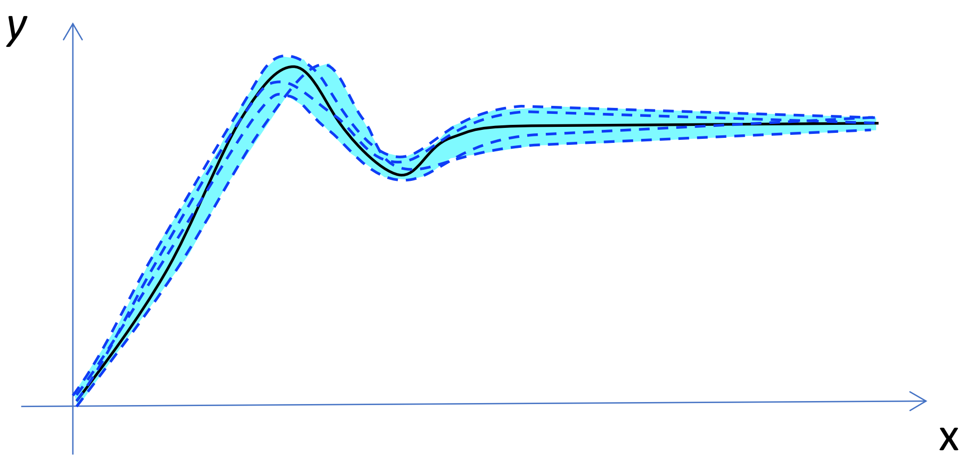

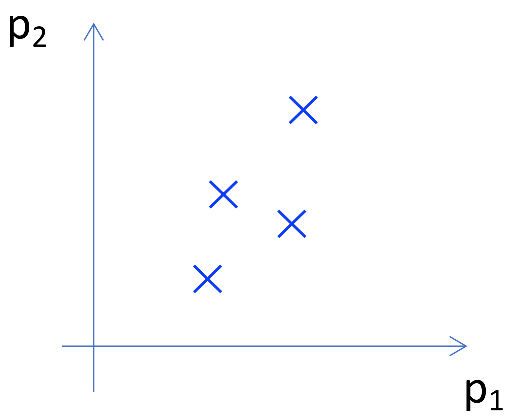

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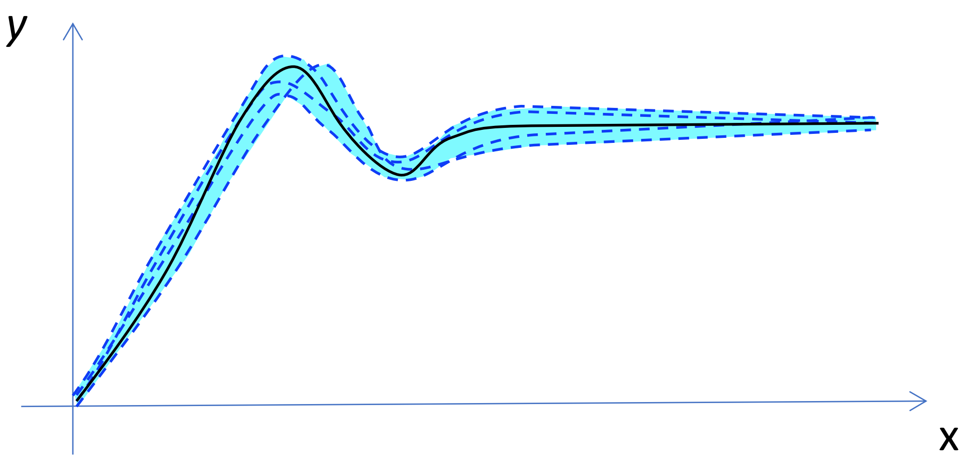
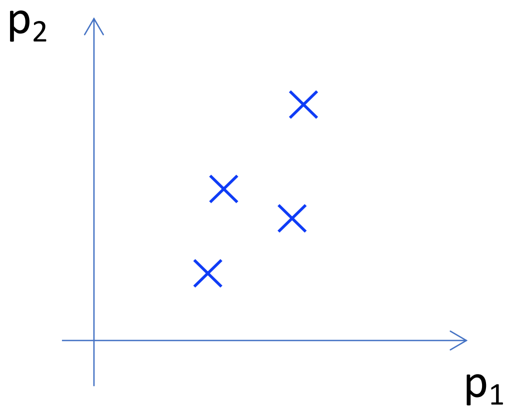
Input-output datum/scenario $\delta^{(i)} \rightarrow$ "surrogate" data set $Q^{(i)}$





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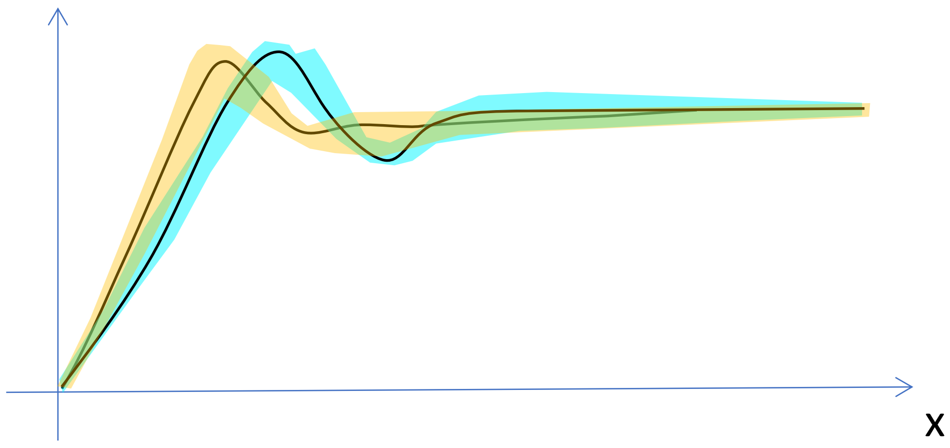

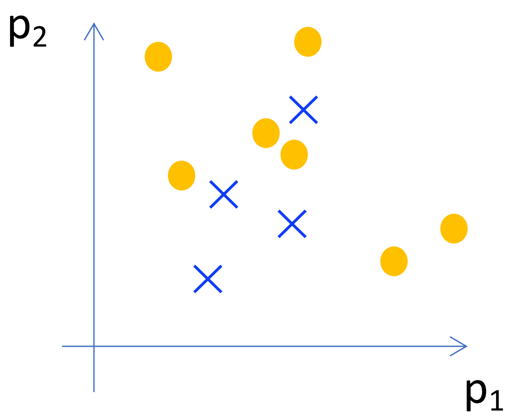

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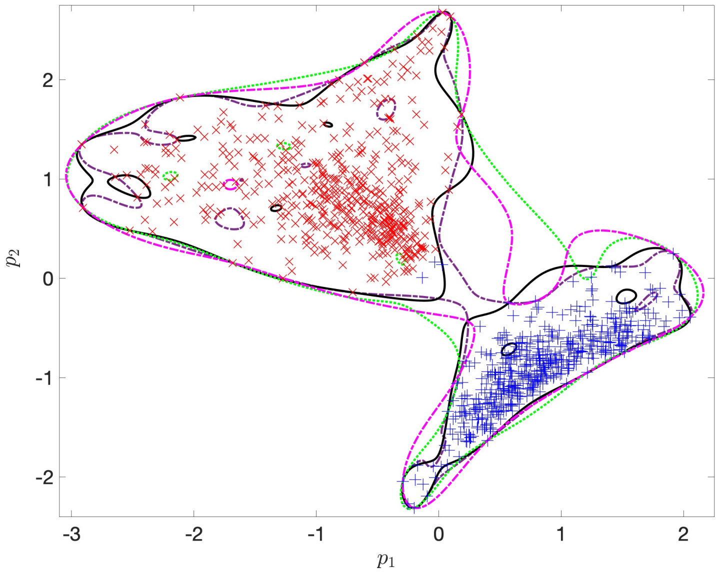
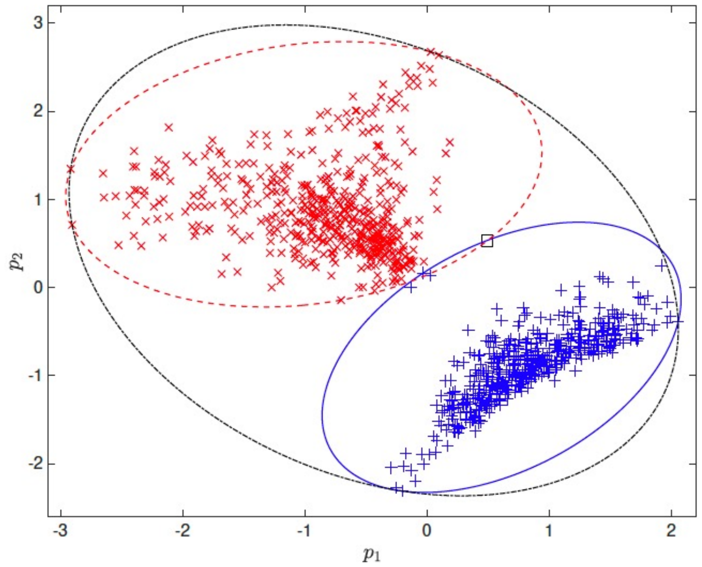
Single polynomial set

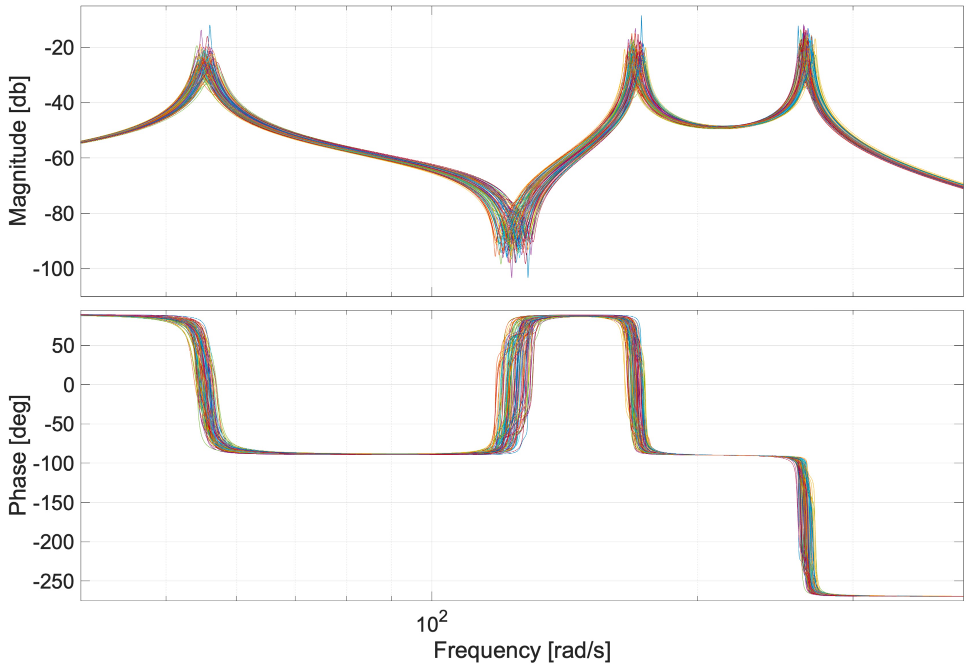


Union of intersecting ellipsoids



Example

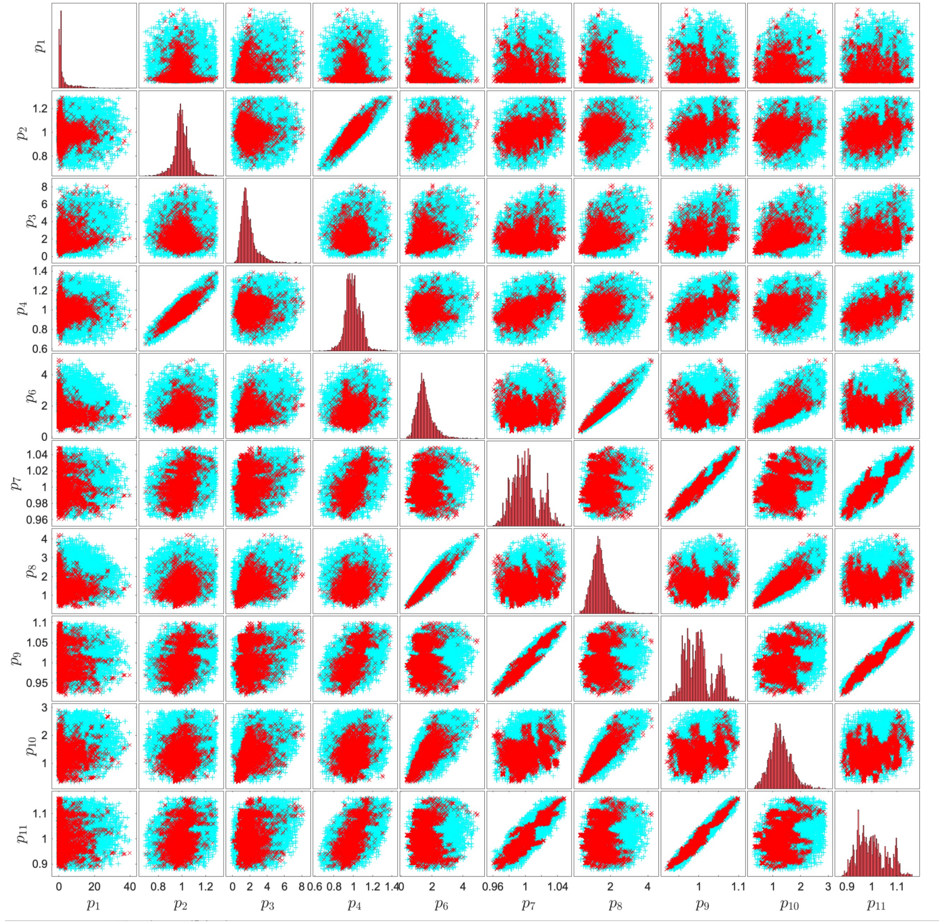
- SISO weakly nonlinear system with a non-collocated sensor-actuator pair
- 3DOF spring-mass-damper system
- Input/outputs: 2500 frequency points/corresponding mag and phase
- Data set has n=100 elements



$$H(s,p) = \frac{p_1 s^4 + p_2 s^3 + p_3 s^2 + p_4 s + p_5}{s^6 + p_6 s^5 + p_7 s^4 + p_8 s^3 + p_9 s^2 + p_{10} s + p_{11}}$$

Example

- The surrogate data set contains the 4135 parameter points shown in red
- The data-enclosure step leads a set containing the points shown in cyan

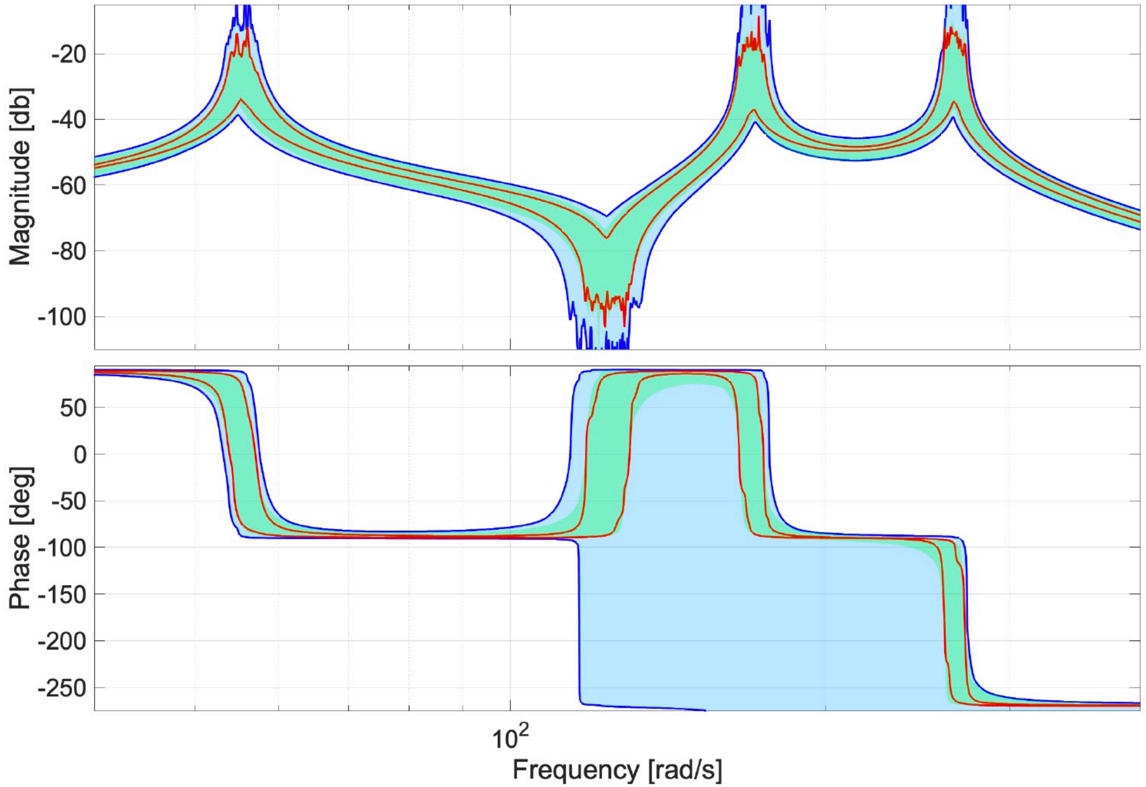


The data-enclosing set should capture parameter dependencies

The set $\mathbb{P}$ is chosen as the intersection of the asymptotically stable plants with the tightest data-enclosing ellipsoid and the tightest data-enclosing box

Example

- The envelopes of the input-output data are shown in red
- The IPM corresponding to the surrogate data is shown in green: tight
- The IPM corresponding to $\mathbb{P}$ is shown in blue: phase is loose



Performance and Reliability Analysis

- The smaller the spread of the IPM the more informative the prediction (the better the performance), but the lower the probability of future data falling within its boundaries (the worse its reliability)
- Performance metric: IPMs spread

$$S_k(\mathcal{Q}) \triangleq \mathbb{E}_x \left[\overline{y}_k(x, \mathcal{Q}) - \underline{y}_k(x, \mathcal{Q}) \right]$$

$$S(\mathcal{Q}) = \sum_{k=1}^{n_y} \omega_k S_k(\mathcal{Q})$$

Reliability Analysis

- Scenario optimization yields an upper bound on the probability of future data falling inside the IPM: non-asymptotic, tight, and distribution-free

$$V(\theta) \triangleq \mathsf{P}\left[\delta \in \Delta \mid \theta \notin h_\delta\right] \qquad \mathsf{P}^n\left[V(\theta^\star) \leq \epsilon\right] \geq 1-\beta$$

- The reliability of the optimum is an observable depending on its complexity
- The means to evaluate this bound depend on how the optimum was obtained. In regard to the proposed IPMs

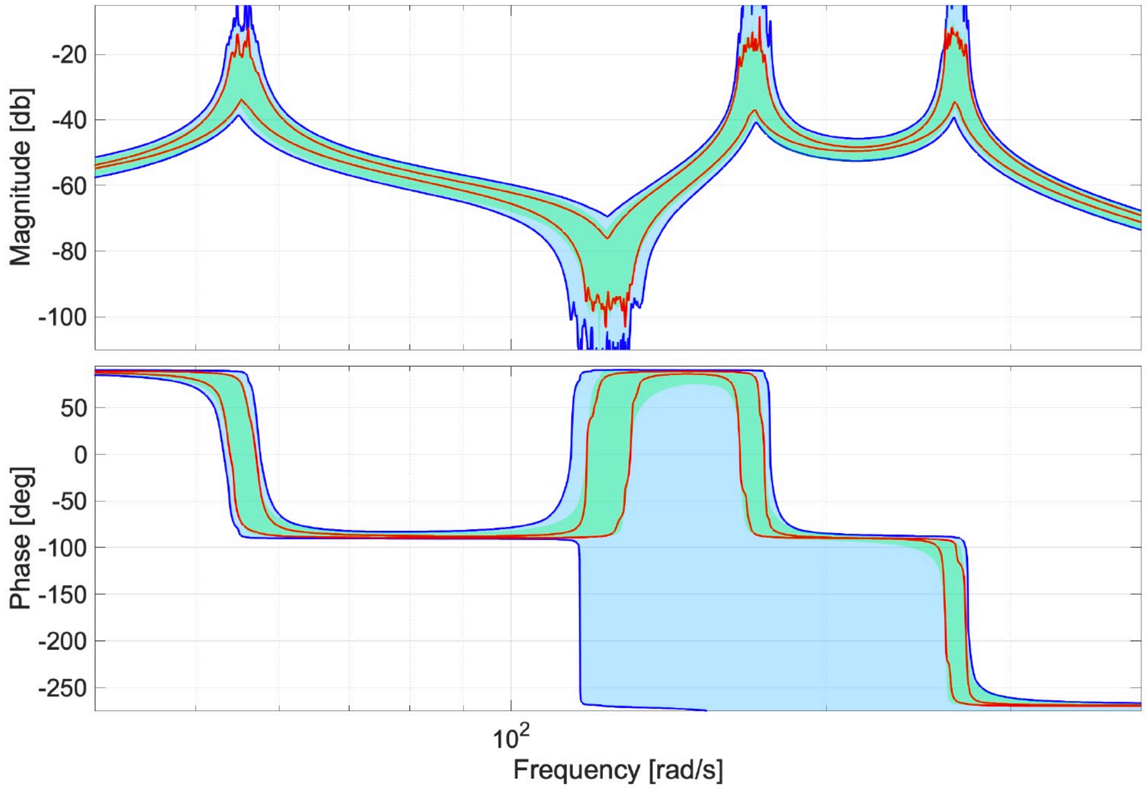
1. Data mapping step: immaterial from a reliability standpoint

$$\mathcal{Q}^{(n+1)} \in \mathbb{P} \text{ implies } y^{(n+1)}(x^{(n+1)}) \in I_y(y; x, \mathbb{P})$$

2. Data-enclosure step: the convexity of the optimization program used to compute $\mathbb{P}$ determines the applicable reliability framework

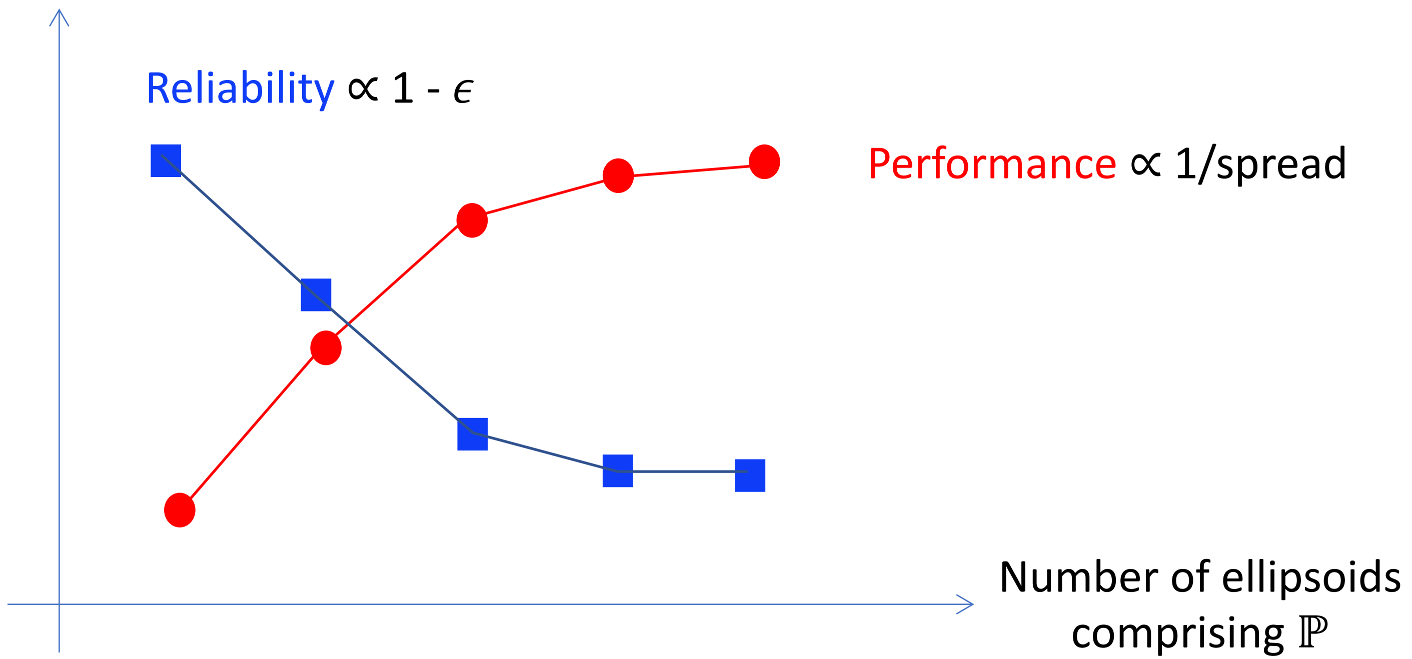
Example

- The probability of new datum falling outside the IPM is no greater than 0.401 with confidence 1-β=0.999 (convex scenario theory with n=100, β=1e3 and a number of support scenarios=21 yields ϵ=0.401)



Performance and Reliability Tradeoff

- The above metrics enable to formally tradeoff the performance and reliability of the prediction until the desired balance is attained, e.g.,



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