

Anisotropic Goal-Based Mesh Adaptation Metric Clarification and Development

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Adaptive unstructured mesh techniques have a limited, but growing impact on production analysis workflows to control discretization error for reliable simulation results. Multiple independent implementations of flow solvers, anisotropic metric construction methods, and anisotropic mesh adaptation mechanics have matured. Goal-based metrics target estimated error in output functions, such as lift and drag, through the guidance of an adjoint solution. A unification of goal-based anisotropic metrics is presented for steady viscous flows, which is an active area of research. These goal-based metrics drive robust and efficient anisotropic mesh adaptation for the calculation of output functions. The super-convergent functional output error behavior of stabilized finite-element methods is exploited without a formal proof, and evidence of super-convergence is shown in numerical experiments. Mesh adapted drag and lift outputs for two simple bodies in compressible viscous flow show convergence of error to less than a single drag count. Asymptotic behavior established for relatively coarse meshes shows the efficiency of this goal-based metric when compared to solution interpolation error control and expert-guided meshing. Anisotropic mesh adaptation techniques are applied to a transport aircraft in a high-lift configuration where variation between approaches decreases with mesh refinement, but asymptotic behavior is not observed with available resources.

I. Introduction

Reynolds-averaged Navier–Stokes (RANS) solutions have become a critical tool for the design of aerospace vehicles. However, the RANS-based Computational Fluid Dynamics (CFD) analysis and design process has not reached the level of automation desired by practitioners as described in the CFD Vision 2030 Study by Slotnick et al. [1] with a five year update by Cary et al. [2]. The Vision Study provides a number of case studies to illustrate the current state of CFD capability and capacity. The Study identifies mesh generation and adaptivity as continuing bottlenecks in the CFD workflow. Unstructured mesh adaptation addresses this bottleneck, e.g., Kleb et al. [3] shows the potential breadth and impact of automated viscous meshing. Alauzet and Loseille [4] documented the dramatic progress made in anisotropic

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solution-adaptive methods to automatically resolve shocks and boundary layers. They also identified where continued investment is necessary to improve this automated process for complex simulations. Park et al. [5] documented the state of solution-based anisotropic mesh adaptation and motivated further development with the impacts that improved capability would have on aerospace analysis and design. Both of these reviews provide descriptions of the investments necessary to accomplish the goals established by the CFD Vision 2030 Study. One of these investments, goal-based^a error estimation for anisotropic metric construction, is developed. A review of goal-based error estimation is provided by Fidkowski and Darmofal [6].

The components of unstructured mesh adaptation are shown in Fig. 1. Starting with an initial mesh, a primal flow solution is computed, followed by an adjoint that corresponds to a particular output quantity. The information from the flow and adjoint solutions are used to estimate error and specify a new mesh resolution and orientation request via a Riemannian metric field. The current mesh system is modified by mesh mechanics to adhere to the metric. Once the adapted mesh is available, the previous flow solution is interpolated on the new mesh to provide an initial condition that approximates the steady-state solution on that adapted mesh. The process is repeated until exit criteria are met (e.g., accuracy requirement, resource limit).

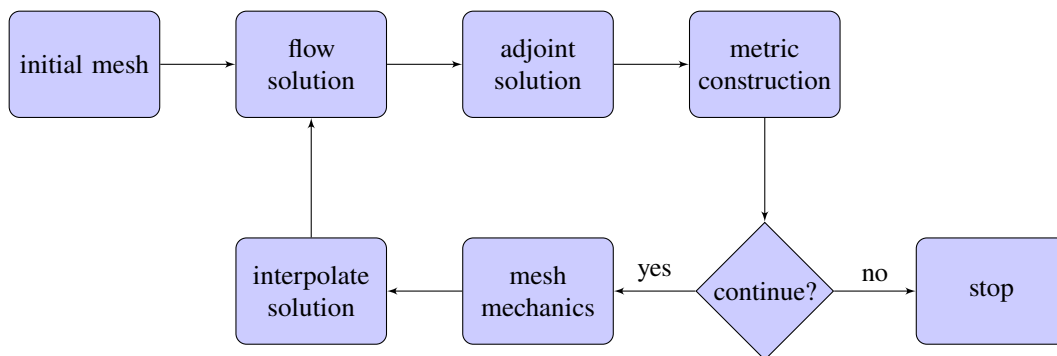


Fig. 1 Solution-based mesh adaptation process.

Adaptive unstructured mesh techniques have a limited, but growing impact on production analysis workflows where the control of discretization error is critical to obtaining reliable simulation results. Mesh adaptation methods develop the mesh in concert with the solution, so verification exercises must include the entire mesh adaptation process (Fig. 1) to establish the confidence required to impact production simulation workflows.

Anisotropic metric construction to control a viscous goal-based error estimate is described by Alauzet and Frazza [7]. This viscous metric was developed through a series of publications. Michal et al. [8] introduced a continuous-mesh metric for RANS. Belme, Alauzet, and Dervieux [9] and Belme [10] developed a continuous-mesh metric for elliptic equations. A linearized form of the inviscid error estimate of Loseille, Dervieux, and Alauzet [11] is used in the viscous metric. These approaches are based on the continuous-mesh framework described by Loseille and Alauzet [12, 13].

The main focus of this paper is establishing a unified context for the explicit anisotropic error estimates first proposed in Ref. [8] and generalized in Refs. [7] and [14]. These unified anisotropic error estimates are presented in a framework of general continuous finite-element discretizations, which reduces the generality adopted in Ref. [7]. The primary reason for this specialization is pursuing super-convergent functional output error behavior, which is exhibited by many stabilized finite-element discretizations. While a formal proof of Streamlined Upwind Petrov-Galerkin (SUPG) [15] adjoint consistency is not available, section IV presents numerical evidence of super-convergence. The error indicators proposed in Refs. [7] and [14] are generalized to optionally combine adjoint solution anisotropy with primal solution anisotropy. Finally, an approach is proposed to eliminate the heuristic aspects of the original variant of the error estimate in Ref. [8].

^aGoal-based is a synonym for output-based or adjoint-based in literature. Here we use goal-based.

II. Goal-based Error Indicator

Consider the system of RANS equations in the following general form:^b

$$U_{,t} + (F_i[U])_{,x_i} - (G_i[U, \nabla U])_{,x_i} - S[U, \nabla U] = 0 \quad (1)$$

in a d -dimensional domain Ω with well-posed boundary conditions on $\partial\Omega$. A subscript in $f_{,z}$ denotes the partial derivative of a function $f[\dots, z, \dots]$ with respect to the independent variable z , which will usually be a spatial coordinate or a state vector, interpreted as a derivative with respect to that vector. In the equation above, $F_i[U]$ are inviscid fluxes, $G_i[U, \nabla U]$ are viscous fluxes, and $S[U, \nabla U]$ are source terms in the RANS model; $\nabla U = (U_{,x_1}, \dots, U_{,x_d})$. The spatial coordinates are x_i , $i = \overline{1, d}$ with d the dimension of physical space. The vector of state conservation variables U , fluxes F_i and G_i , and RANS model source term S are M -column vectors with M depending on d and the particular choice of the RANS model. Hereafter, summation convention over repeated indices is adopted.

Viscous fluxes can be written in the following quasilinear form:

$$G_i = K_{ij}[U]U_{,x_j}, \quad i, j = \overline{1, d} \quad (2)$$

and the inviscid flux Jacobian matrices $A_i[U] = F_{i,U}$ are introduced. Both A_i and K_{ij} are $M \times M$ matrices. The steady state equations $U_{,t} = 0$ are assumed with a general projectional form to approximate Eq. (1):

$$b^{(h)}(W^{(h)}; U^{(h)}) = 0, \quad \forall W^{(h)} \text{ in a test space } \mathcal{V}^{(h)}. \quad (3)$$

$U^{(h)}$, an approximation to U and solution of Eq. (1), is sought for a trial space, $\mathcal{U}^{(h)}$. Both $\mathcal{U}^{(h)}$ and $\mathcal{V}^{(h)}$ will usually consist of piecewise polynomials on some finite partition $\mathcal{T}_h$ of the domain $\Omega = \cup K$, $K \in \mathcal{T}_h$. The form $b^{(h)}(\cdot; \cdot)$ is considered linear with respect to its first functional argument and nonlinear with respect to the second. In general, $b^{(h)}(\cdot; \cdot)$ will have stabilization terms to assure stability of the discretization. Optionally, other desirable properties can be included like adjoint consistency. The approximated boundary conditions are included in the definition of $b^{(h)}(\cdot; \cdot)$ or (generally nonlinear) constraints imposed on trial and test space functions.

The anisotropic error estimate derivation follows the framework of continuous, stabilized finite-element discretizations of the RANS equations, where SUPG is a common example. The following simplifying assumptions are made. The error estimate and corresponding local error indicators are based on the unstabilized, pure Galerkin part of $b^{(h)}(\cdot; \cdot)$. The Galerkin contribution is treated as the leading term in the full output error expansion, which is a reasonable assumption in many cases, at least, asymptotically. Boundary terms are ignored in the error estimate.

The Galerkin part of the RANS discretization in Eq. (3) has the form:

$$b(W; U^{(h)}) \equiv \int_{\Omega} W_{,x_i}^T \left(-F_i[U^{(h)}] + G_i[U^{(h)}, \nabla U^{(h)}] \right) dx - \int_{\Omega} W^T S[U^{(h)}, \nabla U^{(h)}] dx, \quad (4)$$

with test function $W \in \mathcal{V}^{(h)}$.

The dual-weighted residual (DWR) paradigm [16] and the functional output $\phi(U)$ create an error estimate for $\epsilon = \phi(U) - \phi(U^{(h)})$ coming from the Galerkin terms:^c

$$\begin{aligned} \epsilon &= b'(Z - Z^{(h)}, U - U^{(h)}) = b'(\tilde{U}^{(h)}; Z - Z^{(h)}, U - U^{(h)}) \\ &= \int_{\Omega} \left(Z - Z^{(h)} \right)_{,x_i}^T \left(-A_i[\tilde{U}^{(h)}] \left(U - U^{(h)} \right) + K_{ij}[\tilde{U}^{(h)}] \left(U_{,x_j} - U_{,x_j}^{(h)} \right) \right) dx \\ &\quad - \int_{\Omega} \left(Z - Z^{(h)} \right)^T \left(S_{,U}[\tilde{U}^{(h)}, \nabla \tilde{U}^{(h)}] \left(U - U^{(h)} \right) + S_{,\nabla U}[\tilde{U}^{(h)}, \nabla \tilde{U}^{(h)}] \left(U_{,x_i} - U_{,x_i}^{(h)} \right) \right) dx \end{aligned} \quad (5)$$

where $b'(\cdot; \cdot, \cdot)$ stands for the linearization of the Galerkin b form taken at some state $\tilde{U}^{(h)}$. Z and $Z^{(h)}$ are exact and approximate adjoint solutions corresponding to the chosen functional $\phi(U)$. More precisely, the adjoint system is the linearization of $b^{(h)}(\cdot; \cdot)$ and the functional at $\tilde{U}^{(h)}$, see Ref. [16] for more detail. Practical considerations require the

^bSquare brackets are used to denote functional dependencies, increase readability, and eliminate potential ambiguities.

^cThe terms corresponding to $G_{i,U}$ are omitted in this error representation to better expose certain symmetries in the primal and dual local error estimates introduced below.

use of computable states $\tilde{U}^{(h)}$ for b and ϕ linearizations, so most of the statements below should be considered accurate to $O(\|U - U^{(h)}\|^2)$. The notation $S_{,\nabla_i}U$ is adopted in place of $S_{,U,x_i}$ to ease readability.

The error representation in Eq. (5) relies on the Galerkin orthogonality property of the $b^{(h)}$ form. The estimate of Eq. (5) is driven by the convergence rate of both primal and adjoint solutions in a multiplicative manner. The actual convergence rates will depend on the consistency and stability properties of the *stabilized* discretization (both primal and dual), but the unstabilized part represented by b' , will (potentially) inherit superconvergent asymptotic behavior.

As a next step, we introduce regularizations $\hat{U}^{(h)}$ and $\hat{Z}^{(h)}$ of the discrete solutions $U^{(h)}$ and $Z^{(h)}$. The rationale for doing this is twofold. Regularizations can provide a better mean linearization state $\tilde{U}^{(h)}$ (in some sense, an intermediate state between $U^{(h)}$ and the unknown U). More importantly, we are relying on the regularization to implicitly account for the interface jump terms described below. These regularizations are obtained via some local reconstruction techniques described below and are used in the following expressions.

The spatial derivatives are stripped from the primal solutions by integrating Eq. (5) by parts:

$$\epsilon^{(1)} = \int_{\Omega} \left(\bar{\mathcal{R}}^* \left[Z - Z^{(h)} \right] \right)^T \left(U - U^{(h)} \right) dx. \quad (6)$$

In general, $\bar{\mathcal{R}}^*[W]$ will be a distribution in Ω due to the presence of the interface jump terms related to $Z^{(h)}$ involving normal viscous fluxes (and possibly source derivatives for RANS). These terms require careful attention and cannot be simply dismissed. To take these terms into account, reconstruction techniques substitute $U^{(h)}$ and $Z^{(h)}$ by their regularizations $\hat{U}^{(h)}$ and $\hat{Z}^{(h)}$, respectively, which don't have interface jumps in normal derivatives. Such reconstructions can be obtained^d by applying L_2 projections of discontinuous gradients onto a space of continuous elements $\mathcal{V}^{(h)}$ or $\mathcal{U}^{(h)}$ [17]. In this vein, the regular part of the $\bar{\mathcal{R}}^*$ residual is introduced with a slight abuse of notation due to using the same symbol:

$$\begin{aligned} \bar{\mathcal{R}}^*[W] &\equiv \bar{\mathcal{R}}^*[W; \hat{U}^{(h)}] \\ &= -A_i^T [\hat{U}^{(h)}] W_{,x_i} - \left(K_{ij}^T [\hat{U}^{(h)}] W_{,x_j} \right)_{,x_i} - S_{,U}^T [\hat{U}^{(h)}, \nabla \hat{U}^{(h)}] W + \left(S_{,\nabla_i U}^T [\hat{U}^{(h)}, \nabla \hat{U}^{(h)}] W \right)_{,x_i}. \end{aligned} \quad (7)$$

The following compact expression for the error is available by assuming all the spatial derivatives are taken in the interiors of the grid elements where the discrete adjoint solution is smooth (polynomial):

$$\epsilon^{(1)} = \int_{\Omega} \left(\bar{\mathcal{R}}^* \left[Z - \hat{Z}^{(h)}; \hat{U}^{(h)} \right] \right)^T \left(U - U^{(h)} \right) dx. \quad (8)$$

Incidentally, the reconstructions $\hat{Z}^{(h)}$ provide nontrivial second spatial derivatives for Eq. (7).

The error expression is transformed by taking into account that Z is actually the exact adjoint solution,

$$\epsilon^{(1)} = \int_{\Omega} \left(S^* - \bar{\mathcal{R}}^* \left[\hat{Z}^{(h)}; \hat{U}^{(h)} \right] \right)^T \left(U - U^{(h)} \right) dx \equiv \int_{\Omega} \left(\mathcal{R}^* \left[\hat{Z}^{(h)}; \hat{U}^{(h)} \right] \right)^T \left(U - U^{(h)} \right) dx. \quad (9)$$

The volume term S^* is nontrivial only if the output functional $\phi(\cdot)$ has a volume contribution. The dual error expression is further transformed by exploiting an alternative differentiation-by-parts scheme:

$$\epsilon^{(2)} = \int_{\Omega} \left(Z - Z^{(h)} \right)^T \mathcal{R}[\hat{U}^{(h)}] dx, \quad (10)$$

with the primal residual operator $\mathcal{R}(\cdot)$ given by the left hand side of Eq. (1) with $U_{,t} = 0$. Another option is to employ linearized primal residual at a reconstructed state $\hat{U}^{(h)}$.

The residual operators introduced as $\mathcal{R}$ and $\bar{\mathcal{R}}^*$ naturally arise as consistency terms in the context of the residual-stabilized finite-element methods: the former for SUPG, Galerkin Least-Squares (GLS), and Variational multiscale (VMS) when applied to the trial functions and the latter for VMS when applied to the test functions.

^dThere are subtle issues in the actual construction of the regularization, which are discussed in the implementation description in a following section.

The localization for Eq. (6) starts with an additive representation of the global error followed by the estimate:

$$|\epsilon^{(1)}| \leq \sum_K \left| \int_K \left(\mathcal{R}^* \left[\hat{Z}^{(h)} \right] \right)^T (U - U^{(h)}) dx \right| = \sum_K \epsilon_K^{(1)}; \quad \epsilon_K^{(1)} \geq 0. \quad (11)$$

At this point, there are options for how to estimate each term in this elementwise sum. The original estimate proposed in section D of Ref. [8] relied on the discrete Cauchy-Schwarz inequality as follows:

$$\begin{aligned} \epsilon_K^{(1)} &\leq \int_K \left| \left(\mathcal{R}^* \left[\hat{Z}^{(h)} \right] \right)^T (U - U^{(h)}) \right| dx \\ &\leq \int_K \left| \mathcal{R}^* \left[\hat{Z}^{(h)} \right] \right| |U - U^{(h)}| dx \leq \| |\mathcal{R}^*| \|_{L_\infty(K)} \| |U - U^{(h)}| \|_{L_1(K)}, \end{aligned} \quad (12)$$

where $|\mathbf{x}| = |(x_1, \dots, x_M)|_{l_2, \mathbb{R}^M} = (x_1^2 + \dots + x_M^2)^{1/2}$. In Ref. [8], certain heuristics were suggested to make the discrete L_2 norms in the rightmost expression dimensionally consistent, which somewhat compromised generality of the suggested scheme. Below, we demonstrate how symmetrization variables for the Navier-Stokes system can be employed to avoid these heuristics. This type of estimate gives rise to a single metric field.

An alternative approach proposed by Alauzet and Frazza [7] and refined in [14] sharpens the above estimate:

$$\epsilon_K^{(1)} \leq \sum_{m=1}^M \int_K \left| \mathcal{R}_m^* (U_m - U_m^{(h)}) \right| dx \leq \sum_{m=1}^M \| \mathcal{R}_m^* \|_{L_\infty(K)} \| U_m - U_m^{(h)} \|_{L_1(K)}. \quad (13)$$

This estimate is componentwise where subscript m is the scalar component of a state or residual M -column vector. With this approach no additional heuristic-based modifications are required, and it is based on a sharper local estimate, but creates multiple metric fields, which can be combined into one as discussed below.

A similar argument applied to the $\epsilon^{(2)}$ as given in Eq. (10) leads to the dual estimate:

$$\epsilon_K^{(2)} \equiv \left| \int_K \left(\mathcal{R} \left[\hat{U}^{(h)} \right] \right)^T (Z - Z^{(h)}) dx \right| \leq \sum_{m=1}^M \| \mathcal{R}_m \|_{L_\infty(K)} \| Z_m - Z_m^{(h)} \|_{L_1(K)}. \quad (14)$$

by defining positive scalar weights $r_m^* = \| \mathcal{R}_m^* \|_{L_\infty(K)}$ and $r_m = \| \mathcal{R}_m \|_{L_\infty(K)}$. The use of L_∞ norms reflects the “pointwise” nature of r_m^* and r_m and allows the extraction of weights as isotropic scalar fields from the metrics below. The interpolation error norms $\| Z_m - Z_m^{(h)} \|_{L_1(K)}$ are replaced by their computable counterparts: $\| \hat{Z}_m^{(h)} - Z_m^{(h)} \|_{L_1(K)}$. The interpolation errors are never computed explicitly; they are represented as Hessians of the reconstructed solution $\hat{Z}_m^{(h)}$ and used to define metric fields below. $\hat{Z}_m^{(h)}$ should be at least quadratic inside $K \in \mathcal{T}_h$ to construct the Hessians. To construct the following metric fields, the L_1 norm is used for interpolation errors in both Eqs. (13) and (14), in accordance with theory in Ref. [12] for L_p -optimal meshes:

$$\mathcal{M}^* = \det(\mathcal{H}^*)^{-\frac{1}{2+d}} \mathcal{H}^* \quad \text{and} \quad \mathcal{M} = \det(\mathcal{H})^{-\frac{1}{2+d}} \mathcal{H}. \quad (15)$$

The Hessian matrices $\mathcal{H}^*$ and $\mathcal{H}$ arising from the interpolation errors in each of the m terms in Eqs. (13) and (14) must be combined into a single metric for mesh adaptation. The most straightforward (but not necessarily optimal) approach is combining the absolute values of the componentwise matrices as suggested in Ref. [7]:

$$\mathcal{H}^* = \sum_{m=1}^M r_m^* \left| \text{Hess}(\hat{U}_m^{(h)}) \right| \quad \text{and} \quad \mathcal{H} = \sum_{m=1}^M r_m \left| \text{Hess}(\hat{Z}_m^{(h)}) \right|, \quad (16)$$

which sums the weighted absolute value of the second derivative terms. The log-Euclidean formalism for metric fields [18] appears to be an attractive alternative. A numerical verification case in section IV.B compares these two approaches to combine the error estimates from multiple equations into a single metric. $\mathcal{H}^*$ captures anisotropy in the primal solution and $\mathcal{H}$ captures anisotropy in the adjoint. Combining the two (by taking the mean in the simplest case) captures both types of the anisotropy.

When the entropy variables V (e.g., Ref. [19]) are used instead of the conservation variables in the above primal formulation, the error estimates, local error indicators, and metric fields are modified. Entropy variables are a natural choice for the entropy output of the Navier-Stokes equations [20]. Equation (9) becomes,

$$\epsilon^{(1)} = \int_{\Omega} \left(\tilde{\mathcal{R}}^* \left[\hat{Z}^{(h)}; \hat{V}^{(h)} \right] \right)^T \left(V - V^{(h)} \right) dx, \quad (17)$$

with the symmetric form of the adjoint residual $\tilde{\mathcal{R}}^*[\cdot]$:

$$\begin{aligned} \tilde{\mathcal{R}}^*[W] &\equiv \tilde{\mathcal{R}}^*[W; \hat{V}^{(h)}] \\ &= S^* - \tilde{A}_i[\hat{V}^{(h)}]W_{,x_i} - \left(\tilde{K}_{ji}[\hat{U}^{(h)}]W_{,x_i} \right)_{,x_j} - S_{,V}^T W + \left(S_{,\nabla_i V}^T W \right)_{,x_i}. \end{aligned} \quad (18)$$

The inviscid and viscous flux Jacobians are $\tilde{A}_i = (F_i[U[V]])_{,V} = A_i[U]U_{,V}$ and $\tilde{K}_{ij} = K_{ij}U_{,V}$. The symmetry properties of $\tilde{A}_i$ and $\tilde{K}_{ij}$ (see Ref. [19]) are used in Eq. (18).

For the entropy adjoint ($Z = V$), the error estimate becomes:

$$\epsilon^{(1)} = \int_{\Omega} \left(\tilde{\mathcal{R}}^* \left[\hat{V}^{(h)}; \hat{V}^{(h)} \right] \right)^T \left(V - V^{(h)} \right) dx. \quad (19)$$

The formulas related to the $\epsilon^{(2)}$ error don't change (with the understanding that $U = U[V]$). For the case of the entropy adjoint, $\epsilon^{(1)}$ is a linearized version of $\epsilon^{(2)}$.

Entropy variables can be employed to make the error estimate in Eq. (12) dimensionally consistent. The matrix $\tilde{A}_0 = U_{,V}$ is symmetric and positive semidefinite due to the convexity of the entropy function. Therefore, employing generalized Cauchy-Schwarz inequality for the dot product $(\tilde{A}_0 \mathbf{x}, \mathbf{y})_{l_2}$ produces:^e

$$\begin{aligned} \epsilon_K^{(1)} &\leq \int_K \left| \left(\mathcal{R}^* \left[\hat{Z}^{(h)} \right] \right)^T \left(U - U^{(h)} \right) \right| dx \\ &\leq \int_K \left(\tilde{A}_0 \mathcal{R}^*, \mathcal{R}^* \right) \left(\tilde{A}_0^{-1} (U - U^{(h)}), U - U^{(h)} \right) dx \\ &\leq \left\| |\mathcal{R}^*|_{\tilde{A}_0} \right\|_{L_{\infty}(K)} \left\| U - U^{(h)} \right\|_{\tilde{A}_0^{-1}} \Big|_{L_1(K)}. \end{aligned} \quad (20)$$

The discrete norms in this estimate are dimensionally consistent. This form of the local error estimate suggests the quantity $V^T U$ as a scalar to construct the metric field for the Navier-Stokes equation. The latter quantity is proportional to the specific entropy per unit mass.

III. Implementation

Two implementations of goal-based metrics are described. The first closely follows the description in the previous section. The second closely follows the metric described in Alauzet and Frazza [7, 14].

A. Boeing General Geometry Navier-Stokes T1 and EPIC adaptation codes

T1 is the latest generation of the finite-element solvers developed at Boeing within the long term General Geometry Navier-Stokes effort [21] with the goal to provide robust solution-adaptive RANS technology to model flows of production complexity. The T1 solver is built around continuous stabilized finite-element discretization, SUPG, second order for the baseline practical applications and with optional high-order (quadratic and higher finite elements) capability. As it has become a standard for this sort of discretizations, the strong solver based on the Newton's method with pseudotransient continuation to steady state is used. The T1 solver relies on the framework provided by the PETSc library [22] customizing most of the elements of the algebraic solver, such as, line-search and drop tolerance-based preconditioner for the Krylov linear solver.

^eFor a symmetric positive-definite matrix S in $\mathbb{R}^M$ defining the dot product $(S \cdot, \cdot)$, $|\cdot|_S$ is the notation used for the corresponding norm.

The EPIC (Edge Primitive Insertion and Collapse) anisotropic mesh adaptation package developed at Boeing provides a modular framework for anisotropic mesh adaptation that can be linked with external flow solvers [23]. EPIC relies on repeated application of edge break, edge collapse, element reconnection and vertex movement operations to modify a mesh such that element shapes match a given anisotropic metric tensor field. The metric field on the adapted mesh is continuously interpolated from the initial metric field. Several methods are available to preprocess the initial metric field, helping to ensure that the resulting mesh is appropriate for the flow solver. These include limiting the global minimum and maximum local metric sizes, the maximum cell anisotropy, and the spatial variation of cell size. In addition, the metric distribution can be limited relative to the initial mesh (to slow down how fast the mesh changes from adaptive iteration to iteration) and/or to the local geometry surface curvature (to ensure that geometry features are retained at a minimum resolution). The geometry is defined by Initial Graphics Exchange Specification (IGES) import, an unstructured mesh geometry surrogate, or a combination of the two sources. The surface mesh is maintained on the geometry definition with geometric projections and a local remeshing procedure.

Implementation of the goal-based error indicators and construction of the related metrics in T1 closely follows the general framework presented in section II and includes two essential stages. The unstructured grids are assumed to be purely tetrahedral.

The first stage is a reconstruction of the second derivatives needed primarily for the Hessians and the high-order terms in the primal and adjoint PDE residuals $\mathcal{R}$ and $\mathcal{R}^*$ for the error estimates. The baseline linear finite elements do not provide meaningful information on the second derivatives of the solution (both primal and adjoint). Postprocessing constructs second derivatives, which are beyond the linear finite-element discretization. A patch-reconstruction technique is applied in T1 with the properties required by the assumptions in the previous section.

First, the nodal values of the solution gradients are reconstructed. For every node, the gradients are integrated over each cell in the grid having this node as a vertex (i.e. constant cell-based gradients with a cell-volume weight are assembled by summing up the individual cell contributions). This nodal sum is then divided by the overall volume of all cells sharing the node.

For each edge connecting two nodes, a cubic fit is constructed based on the nodal values of the solution and nodal reconstructed gradients obtained in the previous step. At the endpoints, the nodal gradients are projected onto the unit direction defined by the edge. On a given edge, the midpoint value of the cubic differs from the mean of the endpoint values by,

$$(\nabla_a Q - \nabla_b Q)|e_{ab}|/8, \quad (21)$$

where ∇_a and ∇_b denote the mentioned projected nodal gradients, $|e_{ab}|$ stands for the length of the edge with the endpoints a and b , and Q is the reconstructed solution for the variables as discussed below. Park [24, 25] applied this same edge reconstruction method to an embedded linear solution and showed it equivalent to a least-squares quadratic fit of edge endpoint solution and derivative.

After the edge midpoint reconstruction step, the nodal and midedge values of the solution provide a quadratic representation of the solution in the interior of each tetrahedral cell (e.g., standard finite-element hierarchical quadratic Lagrangian basis). This reconstruction is globally continuous. Nontrivial Hessian values are now available in the interior of each cell and they are constant due to the use of the exact quadratic reconstruction. T1 implements this kind of the Hessian reconstruction uniformly for all nodes including those lying on the viscous walls. Other schemes were suggested in the literature (e.g., Ref. [26] uses extrapolation of the Hessians into the boundary nodes from inside the domain). Boundary extrapolation appears more natural when metrics is assigned to the nodes of the grid, not to the cells as in this implementation.

Cell-based quadratic reconstruction is naturally suited to providing Hessian information. Unfortunately, this cell-based reconstruction is not guaranteed to result in the continuous gradients across the cell interfaces when directly differentiated (once) in the cell interiors. Continuous cell interface gradients are an essential assumption of the previous section. The solution is to directly use the gradients linearly reconstructed from the nodal values for the $\nabla \hat{U}^{(h)}$ and $\nabla \hat{Z}^{(h)}$ terms in formulas like Eqs. (6) and (10).

For the construction of weights r_m and r_m^* , the T1 implementation relies on a uniform application of symmetrization/entropy variables, as briefly described at the end of the previous section. From the implementation viewpoint, entropy variables are used regardless of the variables used in the discretization (i.e. which variables are actually represented as polynomials in the finite-element setting). For the error indicator reconstruction, conversion to the entropy variables is always done. (This step is trivial for T1; the baseline choice of discretization variables coincides with the entropy variables.) For the reconstruction procedure described above, nodal values of the solution are converted to entropy variables and the corresponding polynomial reconstructions are used for these variables in the cell interiors.

This conversion is only applied to the primal solution reconstructions; the adjoint solution is naturally tied to the entropy variables.

With entropy variables, it is possible to use the same viscous and inviscid coefficient matrices for both primal and adjoint (transposed) PDE residuals in the definitions of the isotropic weights r_m and r_m^* in the local error estimates $\epsilon^{(1)}$ and $\epsilon^{(2)}$. This entropy-variable approach is similar, but not exactly equivalent to, relying on the use of conservation variables to form detailed explicit expressions for the weights in Ref. [7]. In terms of the formalism developed in section II, the T1 implementation is based on the Eq. (19) representation of the goal-based error estimate, while the implementation in Refs. [7] and [14] is derived from Eq. (9).

For the multiscale interpolation error indicator, a similar nodal conversion procedure is done for the Mach number scalar field. Consequently, the multiscale error indicators are also defined inside cells in the approach presented here.

During the second stage of goal-based metric construction, either an adjoint or primal PDE is applied to the quadratic reconstruction of the corresponding solution depending on the type of the error estimate. The quadratic reconstruction allows the inclusion of viscous terms in the PDE-based residuals. In the T1 implementation in particular, the PDE is applied to the reconstructed solution at each quadrature point inside a cell. To provide some regularization to a potentially noisy $L_\infty(K)$ norm of the residual (as required in Eq. (13) and Eq. (14)), the $L_1(K)$ norm scaled by the cell volume $|K|$ is used. Explicit formulas for the $\tilde{A}_i$ and $\tilde{K}_{ij}$ matrices needed for this implementation can be found in Refs. [19] and [27]. With the explicit knowledge of the symmetrization variables, these matrices can be easily obtained from the more conventional conservation or primitive variable matrices.

Regardless whether the error estimate is multiscale or goal-based, the resultant $\mathcal{H}$ field is passed to EPIC and converted to a metric via Eq. (15). A global scale factor must then be applied to the metric field to result in a metric of the desired complexity (or its discrete analog: to produce a desired number of degrees of freedom). This scale factor is found iteratively. The scaled metric is interpolated to the mesh vertices using a volume-weighted log-Euclidean interpolation. If $\omega(v)$ is the set of elements surrounding vertex v , the interpolated vertex metric $\mathcal{M}_v$ is:

$$\mathcal{M}_v = \exp \left(\frac{1}{\sum_{K \in \omega(v)} |K|} \sum_{K \in \omega(v)} |K| \log(\mathcal{M}_K) \right) \quad (22)$$

where $\log(\cdot)$ is the matrix natural logarithm and $\exp(\cdot)$ is the matrix exponential. Next, any metric limits or modifications as requested by the user are applied to $\mathcal{M}_v$ and the metric field is integrated over the mesh to compute the metric complexity. Finally, the scale factor is updated using a line-search algorithm and the process repeated until a preset tolerance to the desired mesh size is reached.

B. FUN3D-SFE

FUN3D-SFE [28] is a continuous Stabilized Finite-Element (SFE) discretization within the FUN3D (Fully-Unstructured Navier-Stokes 3D) framework [29]. SUPG is used for stabilization of the continuous Galerkin discretization of FUN3D-SFE. The boundary conditions are enforced weakly and forces are computed directly from the residual routines within FUN3D-SFE. A linear nodal basis is used in this study, which is designed to be second-order accurate in space.

To advance the solution of the Navier-Stokes equations to a steady state, density, velocities, and temperature are updated with a Newton-type solver described by Anderson, Newman, and Karman [28]. Here, an initial update to the flow variables is computed using a locally varying time-step parameter that is later multiplied by the current Courant-Friedrichs-Lewy (CFL) number. The CFL number is automatically adjusted based on the L_2 norm of the residual. At each iteration, a linearized residual matrix is formed and solved using the generalized minimal residual (GMRES) [30] algorithm with a preconditioner based on an Incomplete Lower/Upper (ILU) preconditioner for primal and adjoint problems [31]. The implementation of the adjoint has been verified by comparing the adjoint values, obtained by solving the adjoint, with those obtained using a finite-difference method in Ref. [32]. The exact residual Jacobian is available. A line search is conducted to determine an appropriate relaxation factor such that the L_2 norm of the unsteady residual decreases.

The *refine* mesh adaptation package uses a combination of classic split, collapse, and element swap operations in a distributed-memory parallel environment [25], which have been extended via cavity operators [33] to allow more complex element connectivity changes that escape the element validity constraints of a sequential application of classic operators. Nonsmooth optimization based on Freitag and Ollivier-Gooch [34] relocates vertices for poorly shaped elements. Vertex relocation is performed to improve adjacent element shape with a convex combination of ideal vertex locations [35] for elements with medium shape measures. As elements approach an ideal shape in the metric, pliant

smoothing of Bossen and Heckbert [36] is used to drive edge length distributions toward unity. Geometry is accessed by *refine* through the Engineering Geometry Aircraft Design System (EGADS) [37, 38] application programming interface.

Details of the viscous goal-based metric construction process are provided by Park et al. [39]. The metric closely follows the method of Alauzet and Frazza [7, 14]. It is implemented with Hessian reconstruction of the conservative states with explicit weights instead of a direct primal or dual residual evaluation.

IV. Numerical results

For all the GGNS-T1/EPIC numerical results presented in this section, primal solutions are converged to machine-zero accuracy on each grid. More specifically, machine-zero is 9 to 13 orders of convergence for the local dual-volume scaled discrete residuals, and, respectively 7 to 10 orders of convergence for the discrete residuals without dual-volume scaling. The Spalart-Allmaras (SA) [40] RANS computations include the turbulence equation residual in the global L_2 discrete residual norm. For the adjoint solution, the linear system residual was required to reduce 10 orders of magnitude. The FUN3D-SFE primal and adjoint solutions are also converged to machine-zero accuracy. Previous studies in Refs. [41], [26], [42], and [43] show consistent results from GGNS-T1/EPIC and FUN3D-SFE/*refine* for the same metric formulation or mesh. Thus, with high level of confidence, the uncertainty in the code-to-code comparisons below is dominated by grid convergence due to differences in metric construction and mesh mechanics.

All the GGNS-T1/EPIC results use the following solver/adaptation strategy. First, a set of target grid complexities was defined, which was naturally case dependent). For each target complexity value, the mesh is adapted multiple times at the same complexity with multiplicity varying from 5 to 20 depending on the case and grid size. Having finished a prescribed number of mesh adaptation cycles at a complexity target, the algorithm will move to the next complexity, by doubling the complexity. *refine* increases complexity by a small fixed fraction between meshes without a pause for adaptation at a constant-complexity target. The algorithm starts on the first solver cycle with a coarse mesh containing low aspect ratio elements (appropriate for inviscid simulation) and freestream initial flow conditions. For calculations on the adapted meshes, the interpolated solution is used from the previous adapted grid.

A. Laminar Delta Wing

A delta wing is simulated with 3D laminar flow to introduce and evaluate the local anisotropic error estimates and metrics described in the previous sections. The 3D laminar delta wing case was described by Wang et al. [44] and was used in the first three International Workshops on High-Order CFD Methods (HIOCFD). This case was denoted BTC3 in the ADIGMA (Adaptive Higher-order Variational Methods for Aerodynamic Applications in Industry) project [45, 46] where adaptive results are detailed in Hartmann et al. [47]. Adaptive results were presented by Leicht and Hartmann [48], and Yano [49] published grid adapted results with the MOESS metric. Galbraith et al. [26] published results with MOESS, multiscale, and goal-based metrics. The geometry was modified from the original description in Klaij, van der Vegt, and van der Ven [50] based on the experiment of Riley and Lawson [51] to match the HIOCFD and ADIGMA cases.

The dimensions of the delta wing are provided in Table 1. The domain is constructed from planar facets, which eliminates the need for curved boundary recovery from the adaptive grid mechanics and curved elements for higher-order solution representations. The freestream conditions are 0.3 Mach, 4,000 Reynolds number based on the root chord length, and 12.5° angle of attack. The viscosity is assumed to be constant to match the workshop case. Isothermal no slip boundary conditions are set to freestream temperature. The reference (or freestream) temperature is not specified, because it is not required due to the constant viscosity assumption. The Prandtl number is 0.72.

Table 1 Reference geometry for the delta wing.

Root Chord	1.0
Span	0.267949223
Thickness	0.0244161374
Reference Area	0.133974596

Drag coefficient trajectories are shown in Fig. 2 for the Boeing General Geometry Navier-Stokes / T1 (GGNS-T1) solver with EPIC grid adaptation, which uses various local error indicators and metrics described in section II. The

spacing h is estimated as $h = \text{DOF}^{-1/3}$, where DOF is the number of vertices for these meshes. Drag output with $\epsilon^{(1)}$ (Eq. (6)) and $\epsilon^{(2)}$ (Eq. (10)) combined shows better accuracy on coarse meshes (large h) than $\epsilon^{(1)}$ alone, i.e. including adjoint anisotropy in the metric improves performance. Simplifying the error estimate to use the scalar $V^T \cdot U$ (Eq. (20)) when constructing the metric field for the Navier-Stokes equation has a similar convergence for coarse meshes as $\epsilon^{(1)}$, but has higher drag variation on fine meshes.

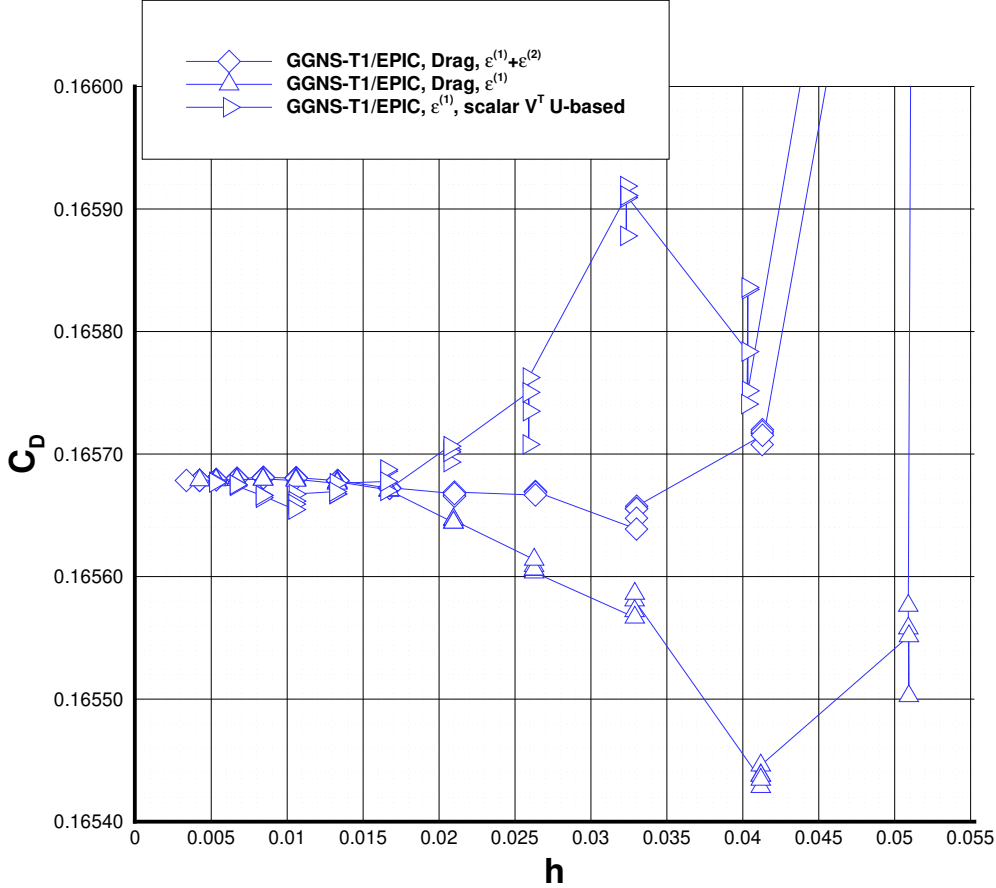


Fig. 2 Delta wing drag coefficient, GGNS-T1/EPIC error estimates.

The drag coefficient trajectories are shown in Fig. 3 for multiple approaches to verify the drag value of GGNS-T1 with MIT SANS-MOESS (also using EPIC for grid adaptation) and NASA FUN3D-SFE/*refine*. The SANS-MOESS results are reproduced from Ref. [26], which provides a complete description of SANS and MOESS. The goal-based SFE/*refine* runs used the implementation of the $\epsilon^{(1)}$ -type error estimate from the paper [7], where this “multicomponent” modification of the error estimate was originally proposed. GGNS-T1/EPIC with a multiscale error estimate based on the L_4 norm of the interpolation error of the Mach number is included to confirm asymptotic force values and show that goal-based metrics approach fine-mesh values on coarser meshes than the multiscale metric. The GGNS-T1 and FUN3D-SFE solvers use SUPG discretization. SANS uses a VMSSD discretization by Huang et al. [52] denoted VMSSDBR2.

Both of the GGNS-T1/EPIC drag error indicators are as accurate (if not occasionally better) than the $p = 1$ VMSSD MOESS results for this laminar test case. The entropy adjoint error indicator, described in the section II, leads to less accurate drag prediction on the coarse grids, which is comparable to the L_4 multiscale Mach interpolation error estimate.

The red delta symbols in Fig. 3 show GGNS-T1 with $p = 2$ solution and the same $p = 1$ error indicators, which by design are based for optimizing second-order error. This test of a $p = 2$ solution with a $p = 1$ error estimate significantly deteriorated accuracy as compared to the corresponding MOESS results and the GGNS-T1 $p = 1$ solution with a $p = 1$

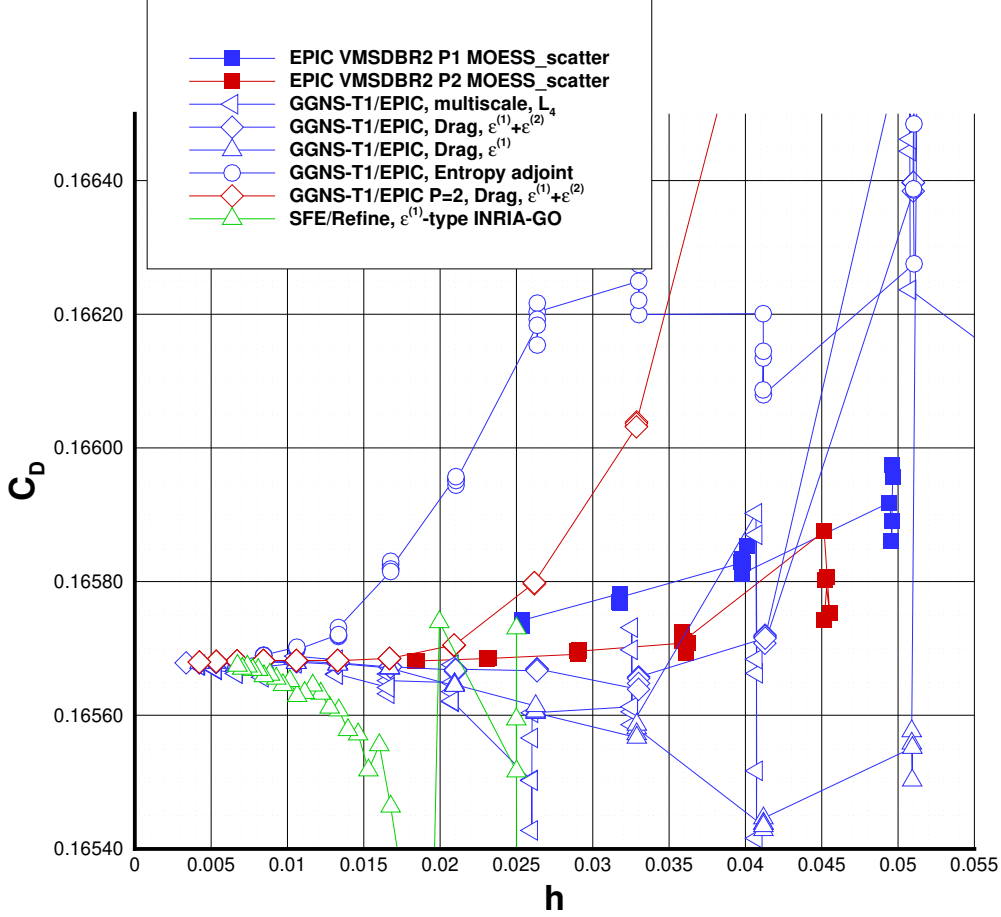


Fig. 3 Laminar delta wing drag coefficient.

error estimate. This observation supports the fact that at the infinite resolution grid limit $p = 2$ discretization has faster asymptotic convergence rate, as illustrated below with actual error convergence plots. Piecewise planar geometry is a special case that enables high-order discretization without introducing curvilinear grids.

The zoomed-in drag convergence plot shown in Fig. 4 illustrates the details of grid convergence behavior in the range of medium to fine grids. The finest grids run with GGNS-T1 for this test case had the number of DOFs as large as 25M nodes ($h = 0.0034$). The amount of the output functional variation for a fixed number of DOFs is higher for the multiscale error indicator. The entropy adjoint error indicator has fixed-DOFs variation between drag and multiscale error indicators. The goal-based error estimates led to much less variation at a fixed DOF. Extrapolating drag to infinite grid resolution ($h = 0$) for all cases (across multiple solvers) agreed very well.

In Fig. 5, the finest grid $p = 2$ result ($C_D = 0.1656787$) is used as a reference to compute a drag error and show error convergence on a log-log scale. Zero crossovers in the GGNS-T1 trajectories complicate the observation of convergence orders (slopes). The VMSD/MOESS $p = 2$ result demonstrates the expected (at least $2p$) convergence rate. The GGNS-T1 results also suggest higher than $p + 1$ convergence rates, but are more difficult to infer due to nonmonotonic convergence behavior. While VMSD is provably adjoint consistent, a proof for formal adjoint consistency is not available for the SUPG discretization used in GGNS-T1 and FUN3D-SFE. The inferred grid-converged values from the three presented solvers agree very well.

In Fig. 6, the convergence of lift coefficient C_L is shown. For this plot, we have extended the dataset by adding the GGNS-T1 results for the C_L used as the output functional. Observations of lift convergence are similar to those made for the drag convergence, where inferred grid-converged values from the three presented solvers agree very well.

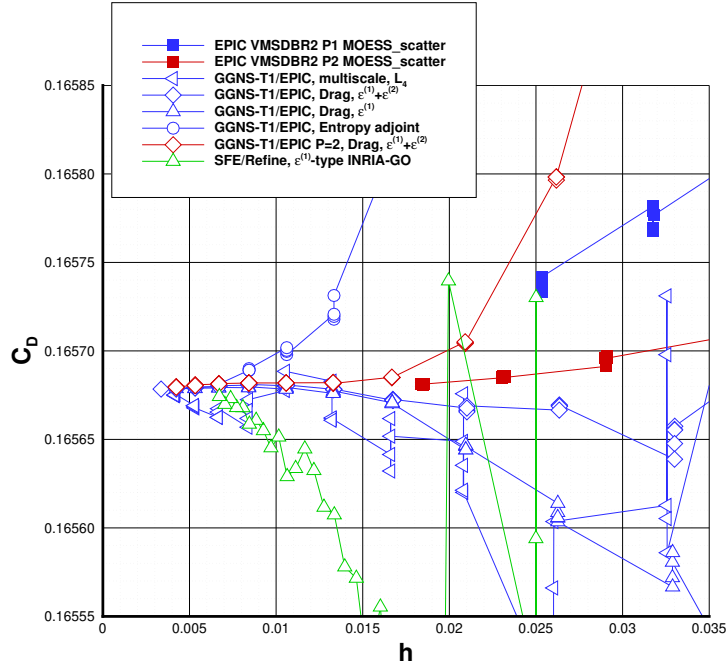


Fig. 4 Laminar delta wing drag coefficient, medium to fine grid range.

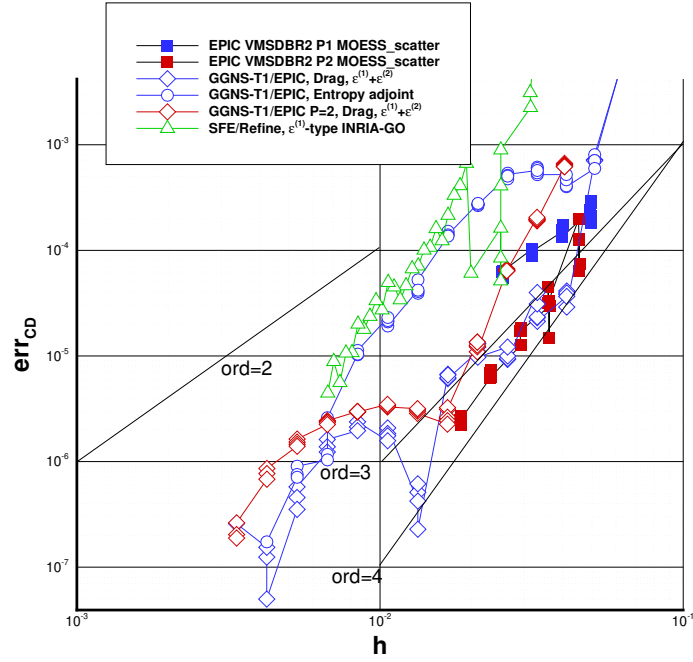


Fig. 5 Error vs. mesh size for the drag coefficient.

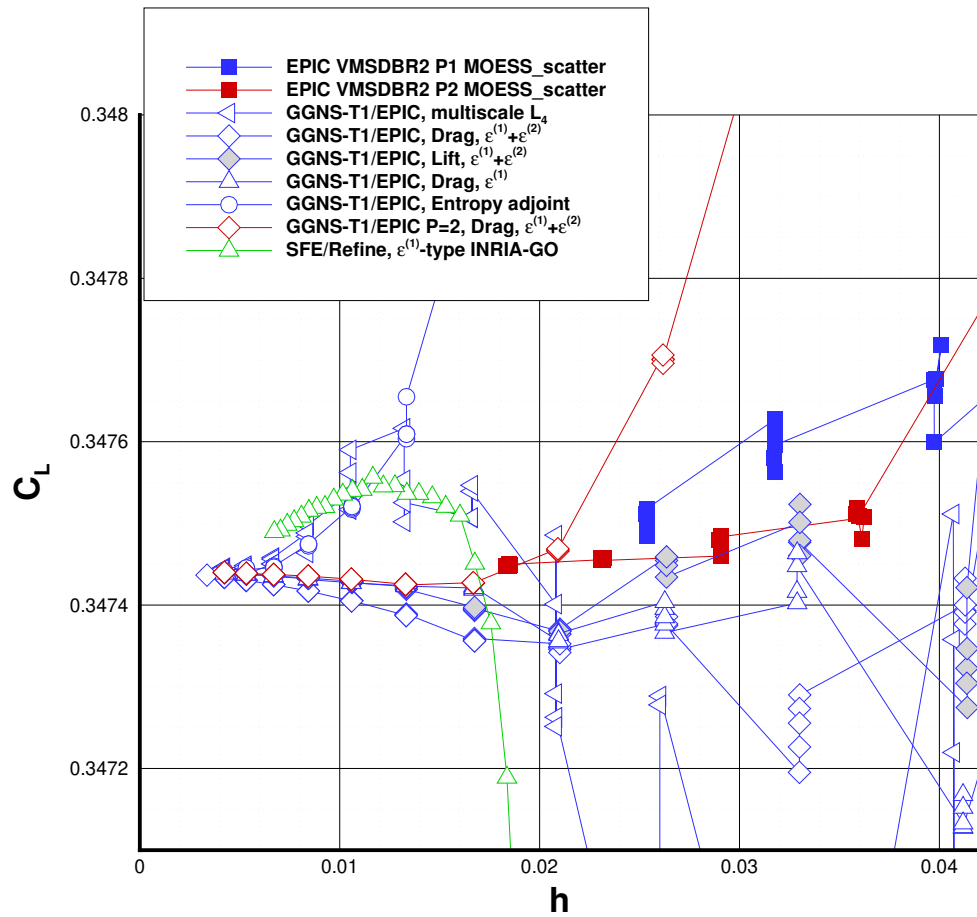


Fig. 6 Delta wing lift coefficient.

B. Turbulent Hemisphere Cylinder at $\alpha = 19^\circ$

The turbulent Hemisphere Cylinder test case at 19 degrees angle of attack is documented^f on the Turbulence Modeling Resource (TMR) website [53, 54]. The case was run, as prescribed by the verification test requirements, fully turbulent with SA-neg-noft2 model. Grid convergence for drag coefficient is plotted in Fig. 7 and grid convergence for lift coefficient is plotted in Fig. 8. All the solution adaptive results are with GGNS-T1 and EPIC and are compared against results on consistently-refined tetrahedral meshes [55] obtained with the FUN3D-SFE [56] and SANS results from Ref. [52] using a combination of the VMSD discretization with MOESS error estimation and EPIC adaptation. While an improvement is observed with the adaptive grids, there are no obvious winners among the different explicit error estimates for driving adaptation with GGNS-T1 unlike the laminar test case above. The MOESS-VMSD result definitely stands out in this case as it shows almost fourth order output convergence with $p = 1$ elements. The entropy adjoint result appears more accurate on the coarse grids. The multiscale convergence trajectory has a crossover when approaching the exact limit and, consistent with the observations for the laminar test case, shows more variation with mesh adaptation at a fixed DOF target.

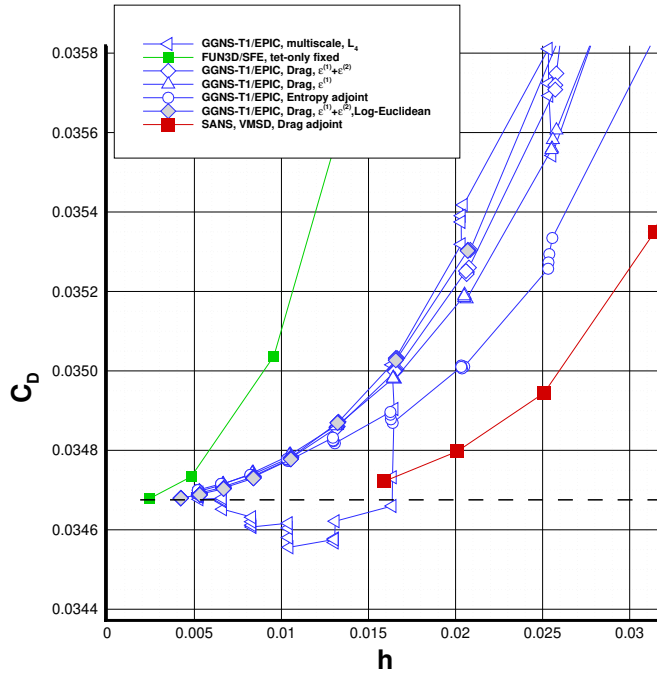


Fig. 7 Hemisphere cylinder, $\alpha = 19^\circ$, drag coefficient.

For this test case, the log-Euclidean averaging procedure is used to combine both the component metrics and the two error indicators. This is different from all the other GGNS-T1 results presented in this paper, where the arithmetic mean averaging is used. There is very little difference between the force convergence histories for these metric assembly methods. GGNS-T1 includes a turbulence equation in the local error estimates $\epsilon^{(1)}$ and $\epsilon^{(2)}$, but does not account for all of the terms related to the turbulent source linearization in Eq. (7). Implementation of these source terms in the local error estimate holds the possibility of improved goal-based accuracy.

Drag error is evaluated with respect to a very fine grid of 40M DOFs. Figure 9 shows drag error on hemisphere cylinder as a function of h . As in the laminar delta wing case, an order higher than $p + 1 = 2$ is observed for the functional output. The error does not have a zero crossing and the rate is easier to observe than the laminar delta wing error rate.

^fhttps://turbmodels.larc.nasa.gov/hc3dnumericpart2_val_sa.html [accessed 1 November 2021].

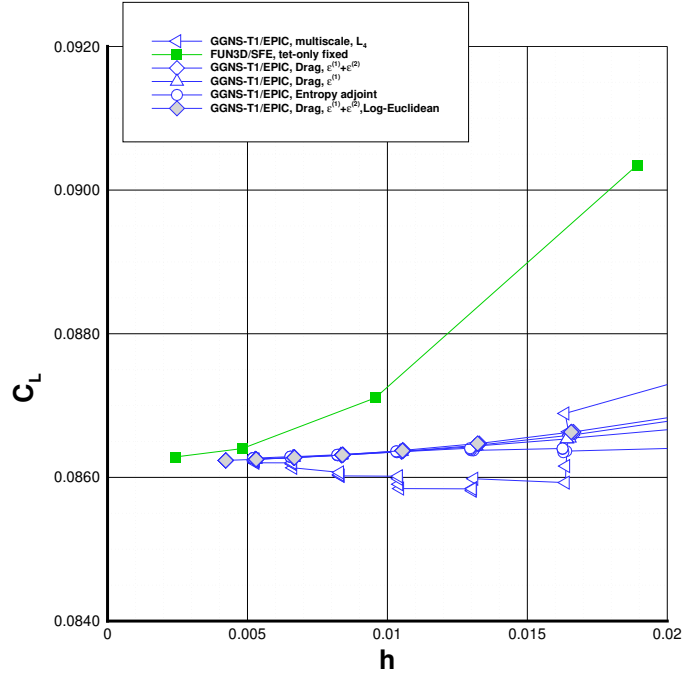


Fig. 8 Hemisphere cylinder, $\alpha = 19^\circ$, lift coefficient.

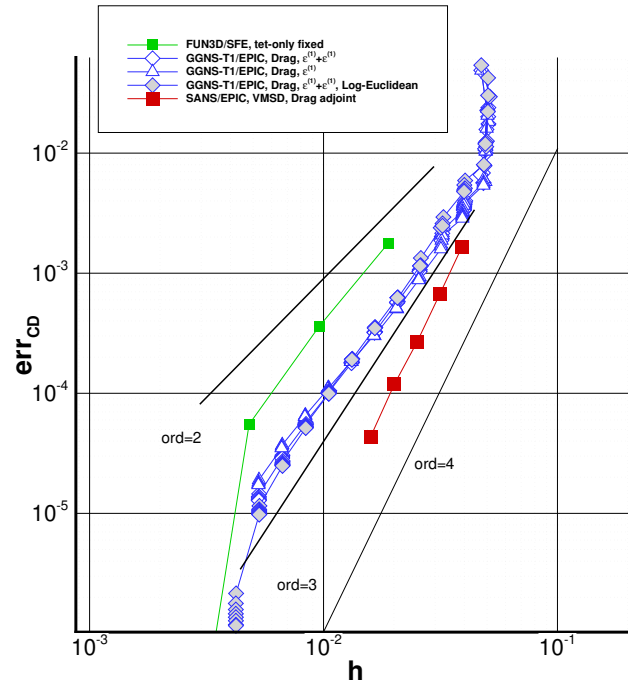


Fig. 9 Hemisphere cylinder, $\alpha = 19^\circ$, drag error.

C. HLPW-4 Nominal Test Case

To demonstrate the production capability of the introduced error estimates and the related error indicators, convergence for the “nominal” HLPW-4 grid convergence test case [57, 58]. The flow conditions for the free air test case are Mach $M = 0.2$, Reynolds number $Re = 5.49 \times 10^6$ based on mean aerodynamic chord of 275.8 inches, angle of attack $\alpha = 7.05^\circ$, and inboard/outboard flap deflections $40^\circ/37^\circ$.

Figure 10 shows grid convergence trajectories for the $\epsilon^{(1)} + \epsilon^{(2)}$ drag goal-based adjoint, multiscale based on the L_4 interpolation error of the Mach number, and entropy-adjoint $\epsilon^{(1)}$ error estimate. The adaptive runs use the Boeing GGNS-T1 solver and EPIC adaptation codes exclusively for this case. These adaptive trajectories are plotted with the fixed-grid sequence results obtained on tetrahedral grids provided by the HLPW-4 Committee (extra-coarse, coarse, and medium levels). A single mesh is shown for the $p = 2$ high-order solution obtained on the curvilinear Pointwise coarse grid with quadratic elements.

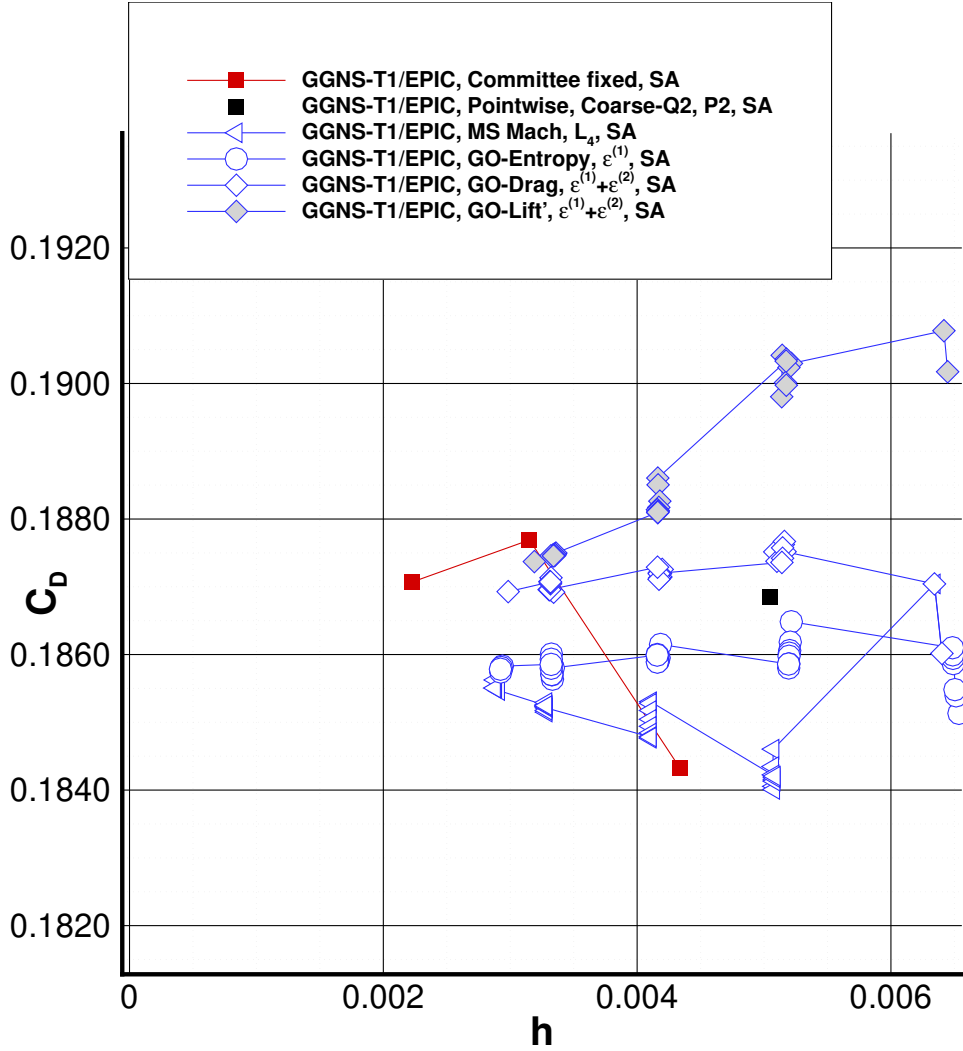


Fig. 10 HLPW-4 nominal case, $\alpha = 7.05^\circ$, drag coefficient.

Grid convergence for this case is more challenging to analyze than the previous cases on simpler geometries. This geometry has closely-coupled fuselage, wing, slat, flap, nacelle, and pylon elements which give rise to complex flow features like corner flows, vortices, and smooth body separation. The numerical data for the finest adaptive and fixed grid do not suggest conclusive statements about grid convergence. The drag convergence for the drag goal-based error estimate shows the “flattest” grid convergence trajectory, suggesting that results at about 7M grid nodes ($h \approx 0.005$)

may be suitable for increments (if not for the absolute values) of drag. Asymptotic grid convergence trends are more elusive for lift and pitching moment coefficients for the finest fixed and solution-adaptive grids, see Figs. 11 and 12.

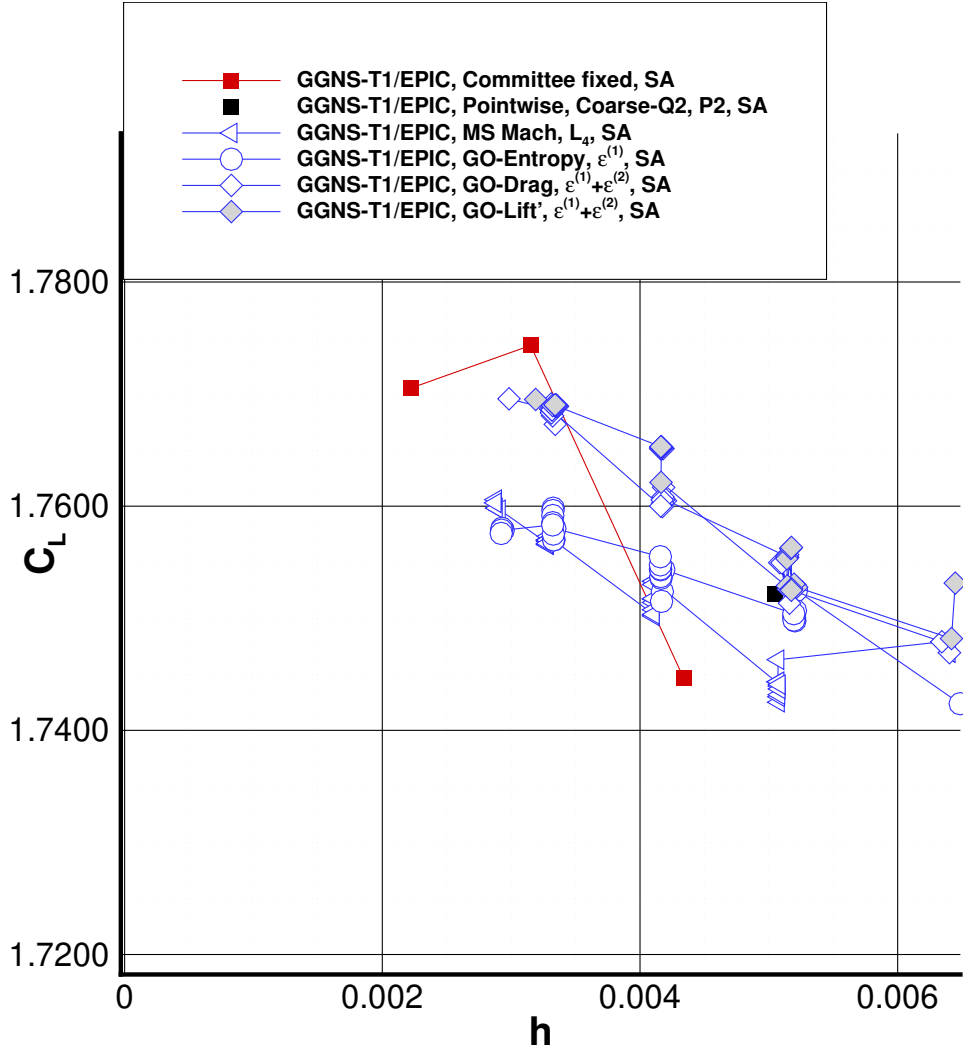


Fig. 11 HLPW-4 nominal case, $\alpha = 7.05^\circ$, lift coefficient.

Grid convergence studies for this test case are complicated by the presence of multiple solution branches, see Ref. [27] for examples on a simplified high-lift configuration. Understanding the implications of multiple solutions is important context for these HLPW-4 results. However, a detailed investigation of multiple solutions for this test case deserves an extensive study that is outside the scope of this paper. Mean numerical solutions with discretely distinct flow patterns (zones of flow separation in these cases) persist for the whole duration of the adaptation process. These different branches depend on the choice of error estimate and other parameters driving adaptation, which suggests the existence of multiple (though, isolated) solutions to the RANS PDEs.

In the attempt to avoid this sort of complication in the grid convergence studies for the considered test case, the multiscale trajectory was started from the goal-based drag mesh of 7.5M nodes ($h = 0.0051$), which corresponds to the 12th drag adapted mesh. The goal-based entropy adjoint trajectory begins from a coarse mesh with freestream initial conditions to form an independent trajectory. The goal-based trajectory shows better accuracy for the drag coefficient on the coarse grid levels than fixed-grid results. Asymptotic convergence is harder to deduce from the three forces and moment values than the mesh-adapted trajectories.

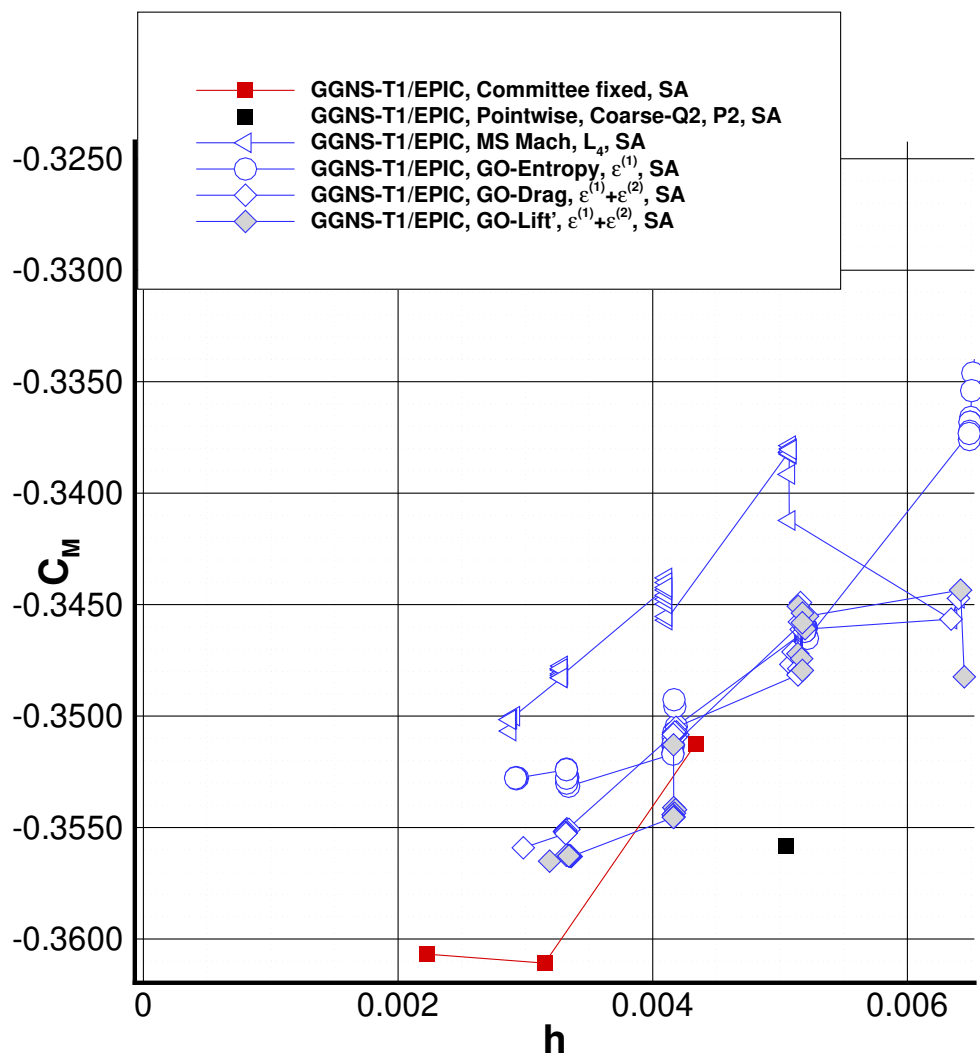


Fig. 12 HLPW-4 nominal case, $\alpha = 7.05^\circ$, pitching moment.

V. Conclusions

The error estimates, associated local error indicators, and anisotropic metrics originally proposed in Ref. [8] and their generalizations in Ref. [7] and Ref. [14] have been unified. Analysis is performed in the context of continuous stabilized finite element discretizations. This unification leads to further generalizations of the error estimates, most notably, the extensions giving rise to metrics incorporating adjoint anisotropy in addition to the primal anisotropy proposed by earlier work.

The implementation of the introduced error indicators naturally leads to cell-based error-indicators, as opposed to more conventional nodal storage schemes for the metric. The Boeing GGNS/T1 finite element RANS solver implementation coupled with the Boeing EPIC code is the focus of this paper. Comparisons of variations in the metric are presented to illustrate the relative benefit of the adjoint anisotropy terms. Comparisons are made with low variation between independent methods (e.g., multiscale metric, SANS/MOESS, FUN3D-SFE/*refine*) to initiate verification activities.

Through demonstration in three test cases that increase in geometric detail and flow complexity, the proposed explicit goal-based metric is comparable in accuracy with more complex approaches for the goal-based metric-based grid adaptation, i.e. the MIT developed MOESS approach. Numerical experiments for the 4th High Lift Prediction Workshop nominal test case demonstrate production capability of the introduced goal-based metrics when applied in the context of grid adaptive RANS solvers for the flows relevant to the aerospace industry.

Numerical evidence of convergence rates higher than the $p + 1$ convergence order is shown, where $p + 1$ convergence is expected for the functional outputs arising from the SUPG discretization on uniformly refined meshes. A formal proof of the adjoint consistency is not available to SUPG, but adjoint consistency issues did not appear to impede adaptive-mesh convergence. The application of $p = 1$ aware error estimation was not sufficient for the grid adaptation of $p = 2$ schemes and may lead to a significantly deteriorated accuracy of the high-order schemes on coarse grids.

There are multiple opportunities to further develop and extend the ideas of this paper. These could include, but are not limited to, taking into account stabilization terms in the error estimates and extending to other types of finite-element approximations (most importantly of the discontinuous Galerkin family). The estimates first presented by Alauzet and Frazza [7] have provided significant benefit to finite-volume solvers including INRIA Wolf, Boeing BCFD, and NASA FUN3D-FV. This paper is important foundational work to support the future research of optimal (accurate, efficient, and robust) adaptation strategies when using both multiscale and goal-based metrics in production-strength RANS applications relevant to the aerospace industry and the CFD Vision 2030 Study [1].

Acknowledgments

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