

Anisotropic Goal-Based Mesh Adaptation

Metric Clarification and Development

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Overview

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- Explicit anisotropic error estimates, error indicators and metrics
 - Derivation for the continuous stabilized FE methods
 - Original variants
 - Extensions. Accounting for the adjoint anisotropy
- Implementation details
- Numerical examples
 - Laminar delta wing
 - Turbulent hemisphere cylinder
 - HLPW-4 nominal case

Equations

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- Steady-state NS/RANS equations in $\Omega \subset \mathbb{R}^d$

$$U_{,t} + (F_i[U])_{,x_i} - (G_i[U, \nabla U])_{,x_i} - S[U, \nabla U] = 0, \quad U_{,t} = 0$$

- Conservation variables U , inviscid fluxes $F_i[U]$, viscous fluxes $G_i[U, \nabla U]$, $G_i = K_{ij}[U]U_{,x_j}$, $i, j = \overline{1, d}$ and source terms $S[U, \nabla U]$

Discretization

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- Finite-element unstructured grid in the domain Ω : $\bar{\Omega} = \cup \bar{K}$, $K \in \mathcal{T}_h$
- Globally continuous piecewise polynomial spaces $\mathcal{U}^{(h)}$ and $\mathcal{V}^{(h)}$
- FE projectional discretization: find $U^{(h)} \in \mathcal{U}^{(h)}$

$$b^{(h)}(W^{(h)}; U^{(h)}) = 0, \quad \forall W^{(h)} \text{ in a test space } \mathcal{V}^{(h)},$$

where $b^{(h)}$ is stabilized form linear in the 1st argument.
Boundary conditions incorporated into the form or as constraints on functions from $\mathcal{U}^{(h)}$, $\mathcal{V}^{(h)}$

- Functional of interest (output) $\phi(U)$

Error estimate derivation: step I

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- Leading error term: nonstabilized (Galerkin) part $b(\cdot; \cdot)$ of the $b^{(h)}(\cdot; \cdot)$ form:

$$b(W; U^{(h)}) \equiv \int_{\Omega} W_{,x_i}^T \left(-F_i[U^{(h)}] + G_i[U^{(h)}, \nabla U^{(h)}] \right) dx - \int_{\Omega} W^T S[U^{(h)}, \nabla U^{(h)}] dx$$

- Ignore BC-related terms (and errors)

Error estimate derivation: DWR

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- Apply standard dual weighted residual approach^(v)

$$\begin{aligned}\epsilon &= b'(Z - Z^{(h)}, U - U^{(h)}) = b'(\tilde{U}^{(h)}; Z - Z^{(h)}, U - U^{(h)}) \\ &= \int_{\Omega} \left(Z - Z^{(h)} \right)_{,x_i}^T \left(-A_i[\tilde{U}^{(h)}] \left(U - U^{(h)} \right) + K_{ij}[\tilde{U}^{(h)}] \left(U_{,x_j} - U_{,x_j}^{(h)} \right) \right) dx \\ &\quad - \int_{\Omega} \left(Z - Z^{(h)} \right)^T \left(S_{,U}[\tilde{U}^{(h)}, \nabla \tilde{U}^{(h)}] \left(U - U^{(h)} \right) + S_{,\nabla_i U}[\tilde{U}^{(h)}, \nabla \tilde{U}^{(h)}] \left(U_{,x_i} - U_{,x_i}^{(h)} \right) \right) dx\end{aligned}$$

$b'(\tilde{U}^{(h)}; \cdot, \cdot)$ is the linearization of the Galerkin $b(\cdot; \cdot)$ form taken at some state $\tilde{U}^{(h)}$. Z and $Z^{(h)}$ are exact and approximate adjoint solutions corresponding to the chosen functional $\phi(U)$

- ^(v) Roland Becker and Rolf Rannacher, Arie Iserles ed., *An Optimal Control Approach to a posteriori Error Estimation in Finite Element Methods*, Acta Numerica, **10**, Cambridge University Press, pp. 1–102, may 2001
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Error estimate – “Scalar” primal

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- Integration by parts, stripping derivatives from the primal solution $U^{(h)}$
- Localization on K followed by Cauchy-Schwarz and Hölder leads to the following “scalar” form^(vi):

$$\epsilon_K^{(1)} \leq \left\| \left| \mathcal{R}^*[\hat{Z}^{(h)}] \right| \right\|_{L_\infty(K)} \left\| |U - U^{(h)}| \right\|_{L_1(K)}$$

with $\mathcal{R}^*[\cdot]$ — adjoint PDE operator

- Uses discrete norm for the $\mathbb{R}^M$ -valued $U - U^{(h)}$ and $\mathcal{R}^*$
- Leads to a single Riemannian metric field for adaptation

^(vi) Todd Michal, Dmitry S. Kamenetskiy, Dave Marcum, Frédéric Alauzet, Loïc Frazza and Adrien Loseille, *Comparing Anisotropic Error Estimates for ONERA M6 Wing RANS Simulations*, section D. “Output-based metric in GGNS”, AIAA Paper 2018–920

Error estimate – “Multi-component” primal

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- Sharper estimate: (vii) (viii)

$$\epsilon_K^{(1)} \leq \sum_{m=1}^M \|\mathcal{R}_m^*\|_{L_\infty(K)} \|U_m - U_m^{(h)}\|_{L_1(K)}$$

- Automatically dimensionally consistent
- Need to combine multiple “component-wise” metrics into one for grid adaptation

- (vii) Frédéric Alauzet and Loïc Frazza, *3D RANS Anisotropic Mesh Adaptation on the High-Lift Version of NASA’s Common Research Model (HL-CRM)*, AIAA Paper, 2019–2947,
- (viii) Frédéric Alauzet and Loïc Frazza, *Feature-Based and Goal-Oriented Anisotropic Mesh Adaptation for RANS Applications in Aeronautics and Aerospace*, Journal of Computational Physics, **439**, aug, 2021
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Multi-component — dual form

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- Similar argument for the dual error estimate leads to:

$$\epsilon_K^{(2)} \leq \sum_{m=1}^M \|\mathcal{R}_m[\hat{U}^{(h)}]\|_{L_\infty(K)} \|Z_m - Z_m^{(h)}\|_{L_1(K)}$$

- $\mathcal{R}_m[\hat{U}^{(h)}]$ – m -th component of the primal PDE (residual) evaluated at $\hat{U}^{(h)}$

Converting to anisotropic metrics

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- Isotropic weights $r_m^* = \|\mathcal{R}_m^*\|_{L_\infty(K)}$ and $r_m = \|\mathcal{R}_m\|_{L_\infty(K)}$
- Interpolation error norms $\|U_m - U_m^{(h)}\|_{L_1(K)}$ are replaced by their computable counterparts: $\|\hat{U}_m^{(h)} - U_m^{(h)}\|_{L_1(K)}$ and are expressed in terms of the Hessians of $\hat{U}_m^{(h)}$
 - this introduces anisotropy into the metric
- Similarly for the adjoint solution error $\|Z_m - Z_m^{(h)}\|_{L_1(K)}$ in $\epsilon_K^{(2)}$
 - $\epsilon_K^{(1)}$ captures anisotropy in the primal, while $\epsilon_K^{(2)}$ – in the adjoint solution

Converting to anisotropic metrics (contd.)

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- For multi-component individual the weighted component Hessians are combined into one $\mathcal{H}^*$ or $\mathcal{H}$
 - in the simplest case: arithmetic mean of the matrix absolute values
- Mesh/metric duality approach leads to (L_p interpolation error with $p = 1$)(ix):

$$\mathcal{M}^* = \det(\mathcal{H}^*)^{-\frac{1}{2+d}} \mathcal{H}^* \quad \text{and} \quad \mathcal{M} = \det(\mathcal{H})^{-\frac{1}{2+d}} \mathcal{H}.$$

(ix) *Adrien Loseille and Frédéric Alauzet, Continuous Mesh Framework Part I: Well-Posed Continuous Interpolation Error, SIAM Journal on Numerical Analysis, 49, 1, 2011, pp. 38–60*

Solvers and grid adaptation codes: Boeing

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- GGNS-T1
 - SUPG discretization, baseline $p = 1$ finite elements
 - Pseudo-transient continuation to steady-state; Newton's method (Newton-Krylov-Schwarz)
 - linesearch; preconditioned GMRES for linear solver
- EPIC^(x)
 - Repeated application of edge break, edge collapse, element reconnection and vertex movement operations to modify a mesh to match a given anisotropic metric tensor field
 - The metric field on the adapted mesh is continuously interpolated from the initial metric field

^(x) Todd Michal and Joshua Krakos, *Anisotropic Mesh Adaptation Through Edge Primitive Operations*, AIAA Paper, 2012–159,

Solvers and grid adaptation codes: NASA and MIT

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- NASA: SFE + *refine*
- MIT: SANS-VMSD + MOESS/EPIC
- Details in the paper

Implementation of the error estimates (T1/EPIC)

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- Nodal gradient patch reconstruction (Zienkiewicz-Zhu) followed by linear interpolation in the element interiors
 - accounts for the interface derivatives jumps in the integration by parts scheme: $\hat{U}^{(h)}$ or $\hat{Z}^{(h)}$ in the weights
- Reconstructed nodal gradients used to recover quadratic mid-edge corrections leading to $p = 2$ globally continuous reconstructions: $\hat{U}^{(h)}$ or $\hat{Z}^{(h)}$ on the anisotropic interpolation error estimates
- Hessians evaluated from the quadratic reconstructions in the element interior
- Multiple weighted Hessian fields combined into one metric by averaging

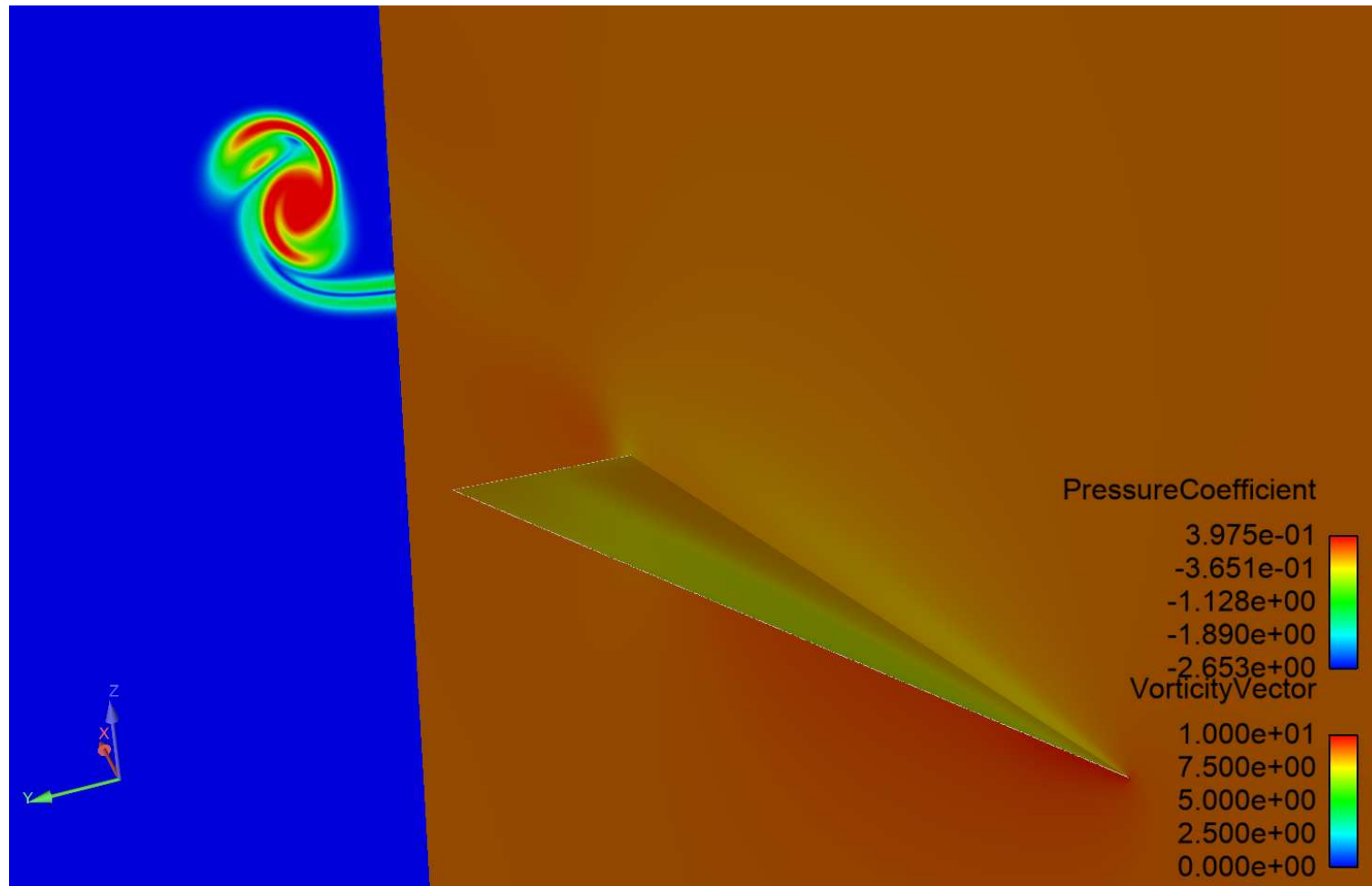
T1/EPIC adaptation strategy

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- For a sequence of a priori prescribed grid complexities:
 - for each complexity perform multiple solver/error estimation/grid adaptation cycles
 - in the numerical examples below, from 10 to 20 iterations on a given complexity performed
- Other details:
 - machine-zero converged T1 solution on most of the grids
 - 10 orders (relative) converged adjoint
 - SA-neg-noft2 model for the turbulent runs

Laminar Delta Wing

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Laminar Delta Wing

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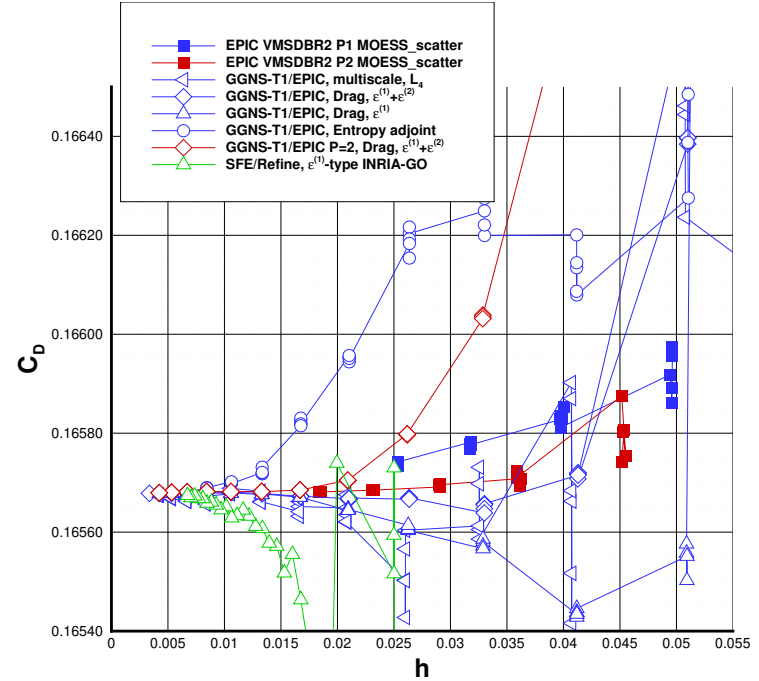
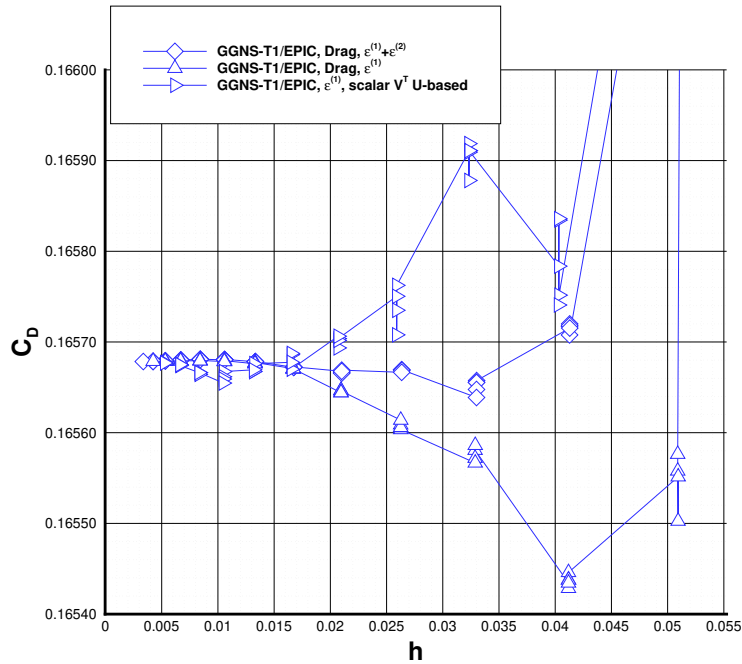
● (xi)

(xi) Z. J. Wang and Krzysztof Fidkowski and Rémi Abgrall and Francesco Bassi and Doru Caraeni and Andrew Cary and Herman Deconinck and Ralf Hartmann and Koen Hillewaert and H. T. Huynh and Norbert Kroll and Georg May and Per-Olof Persson and Bram van Leer and Miguel Visbal, *High-Order CFD Methods: Current Status and Perspective*, International Journal for Numerical Methods in Fluids, **72**, 8, july 2013, pp. 811–845,

Laminar delta wing

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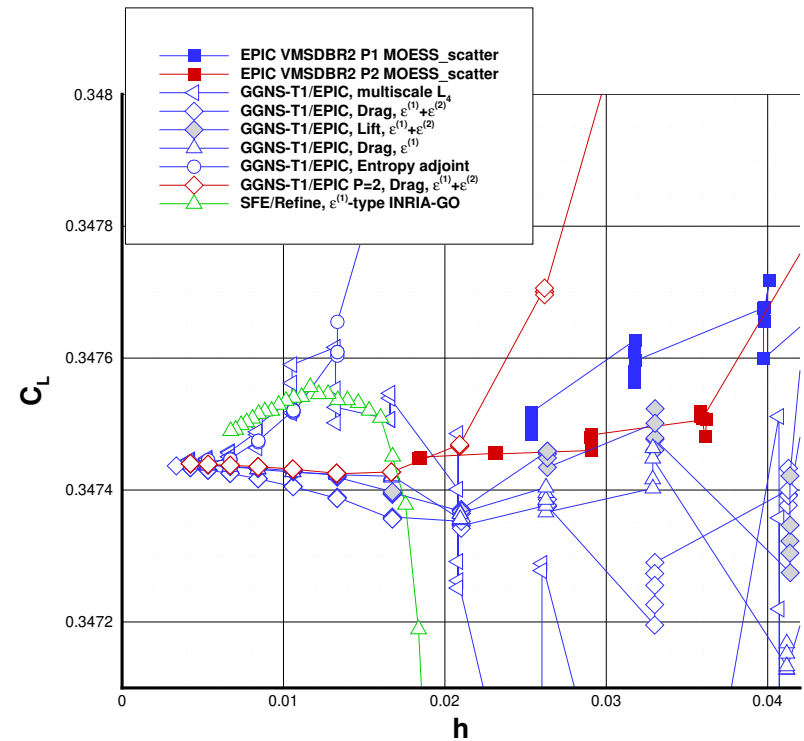
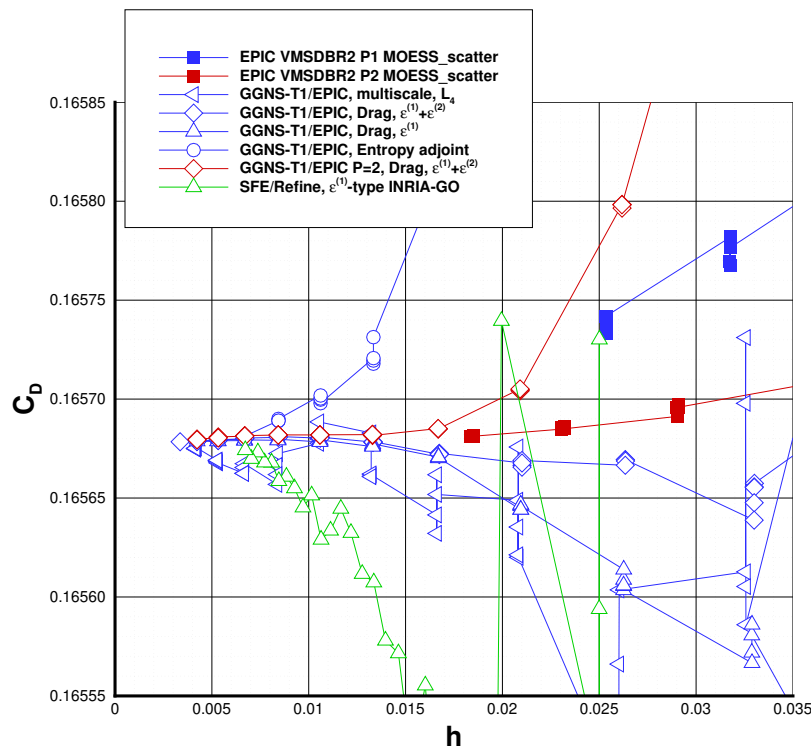
- Consistent $h \rightarrow 0$ result across the codes and grid sequences. $h = \text{nDOF}^{-1/3}$
- The blended $\epsilon_K^{(1)} + \epsilon_K^{(2)}$ with T1/EPIC most accurate in the coarse grid range
- GO more accurate than feature-based and entropy adjoint



Laminar delta wing (contd.)

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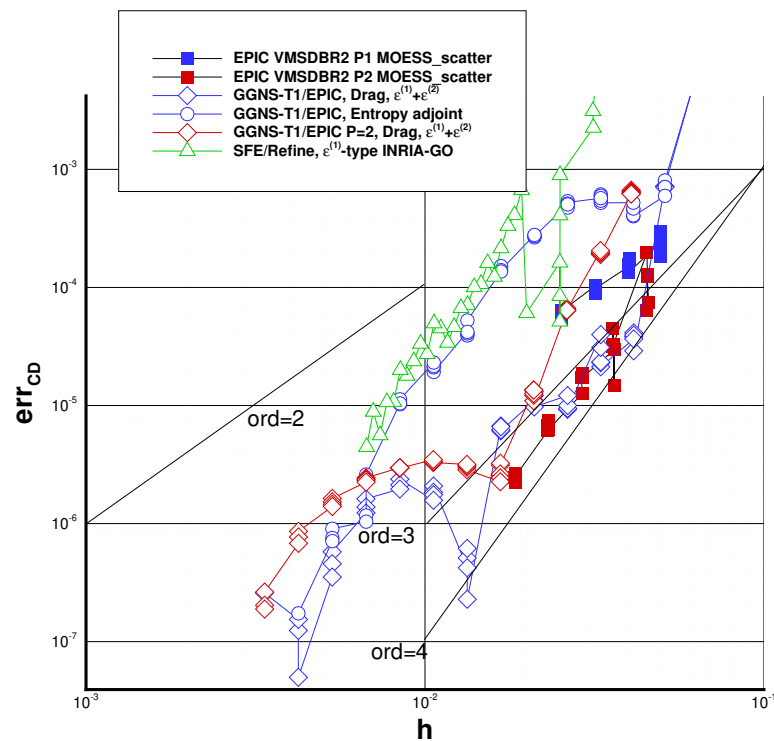
- Lift adjoint more accurate for C_L than drag adjoint
- High order $p = 2$ with second order estimate less accurate than $p = 1$ for coarse grids



Laminar delta wing (contd.)

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- Error wrt. the “accurate” HO, fine grid result
- Higher than second “observed” order for the output convergence for $p = 1$ SUPG



Turbulent hemisphere cylinder $\alpha = 19^\circ$

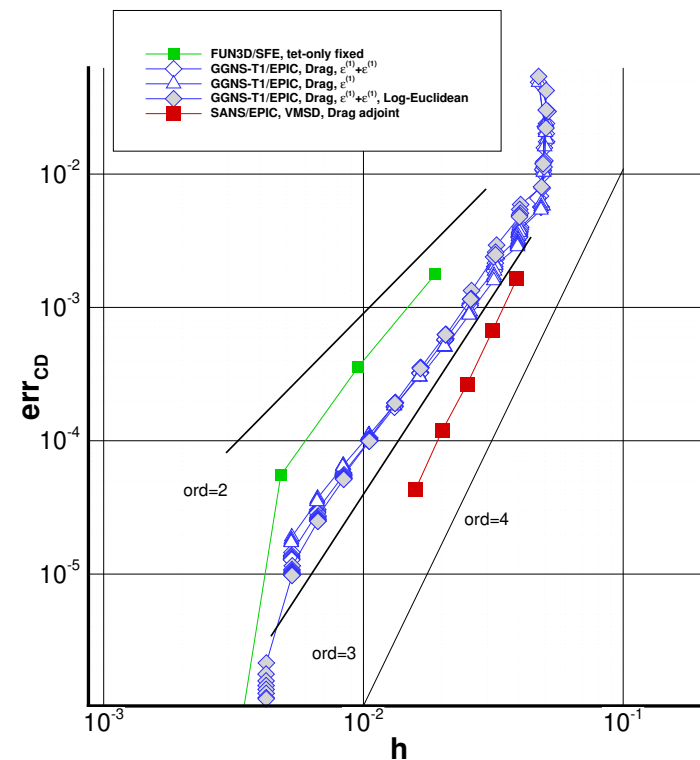
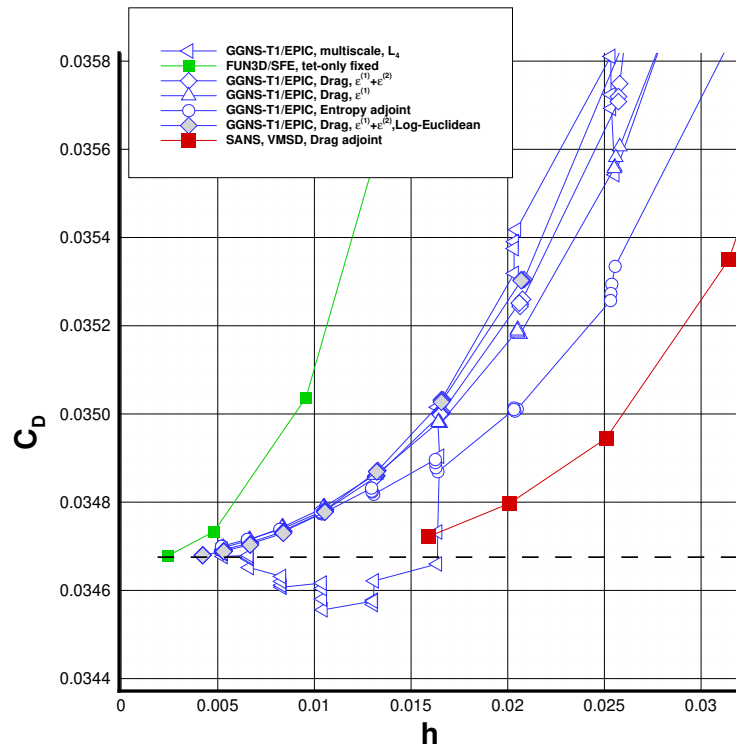
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https://turbmodels.larc.nasa.gov/hc3dnumericspart2_val.html

Turbulent hemisphere cylinder $\alpha = 19^\circ$

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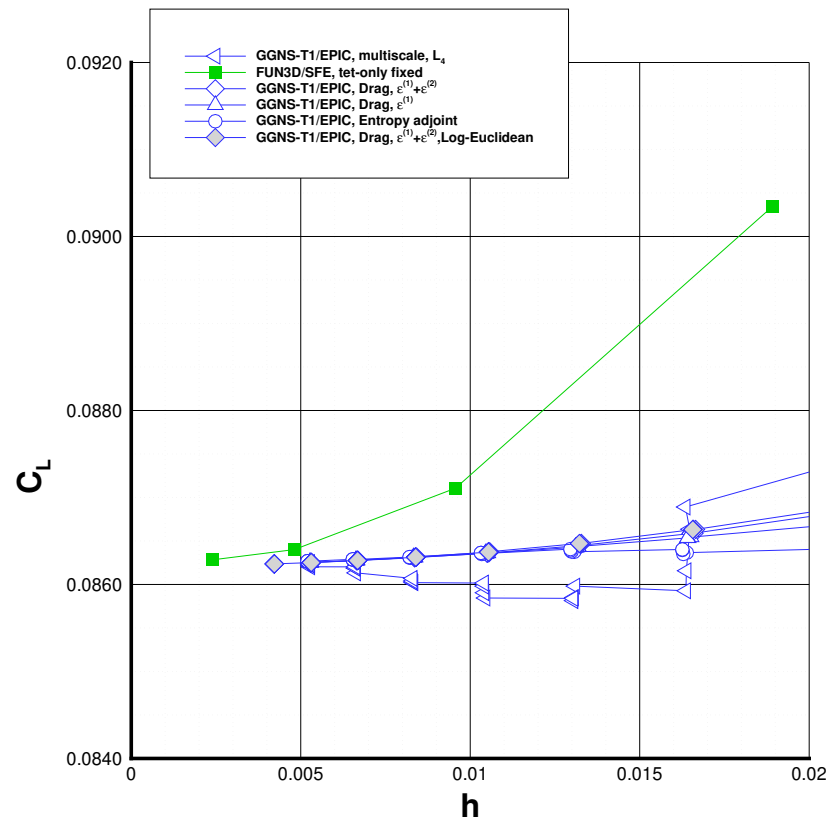
- All adaptive more accurate than SFE fixed grid
- Entropy-adjoint most accurate among T1/EPIC GO easimates
- SANS/VMSD/MOESS shows 4th-order (!) convergence for $p = 1$



Turbulent hemisphere cylinder $\alpha = 19^\circ$ (contd.)

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- All solution-adaptive significantly more accurate than fixed grids for C_L



HLPW-4 nominal case

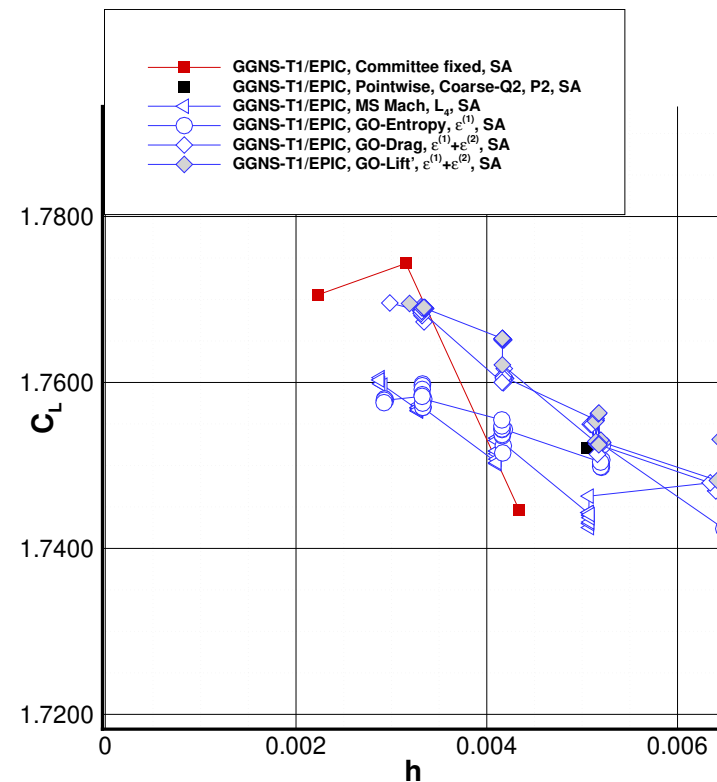
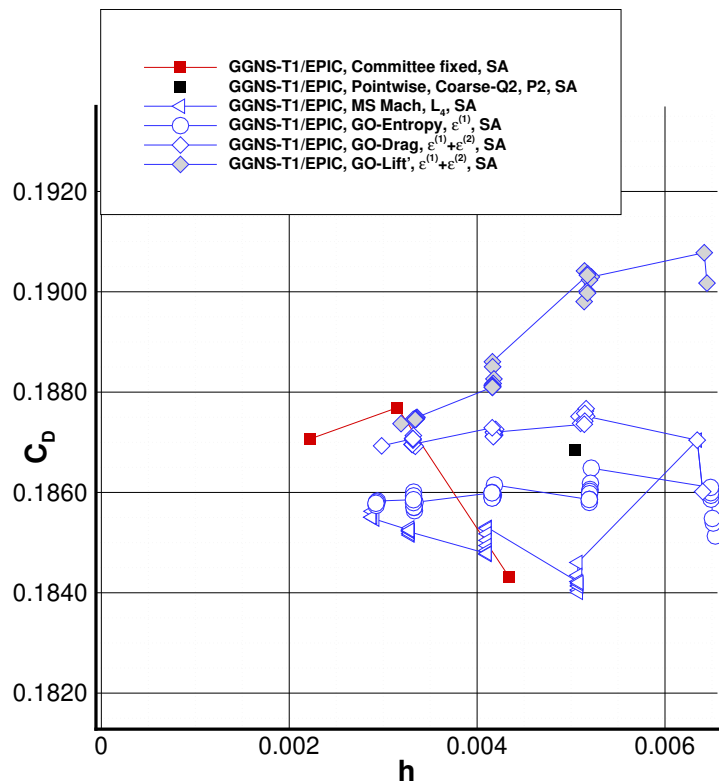
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https://hiliftpw.larc.nasa.gov/Workshop4/OfficialTestCases-HiLiftPW-4-2021_v15.pdf

HLPW-4 nominal case

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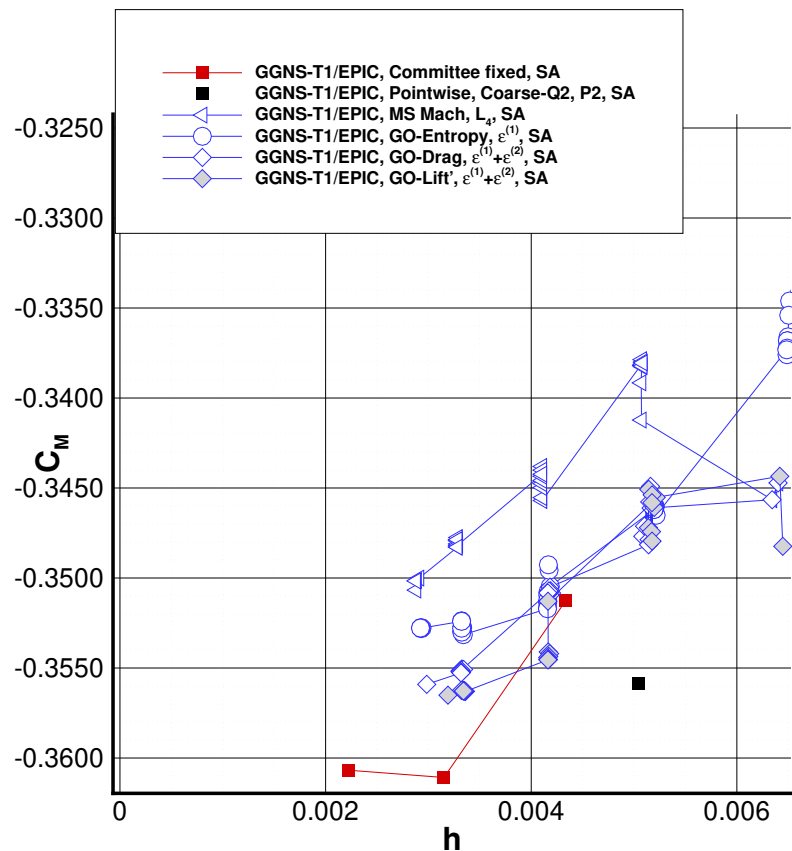
- Grid convergence and consistency at $h \rightarrow 0$ challenging due to the existence of solution *branches*
- Convergence plots for GO (except for C_L) “flatter” than for MS



HLPW-4 nominal case (contd.)

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- Fixed grid convergence (T1, HLPW-4 Committee grids)
inconclusive



Conclusions

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- Original approaches by Boeing and INRIA for the explicit GO viscous anisotropic error estimates for RANS have been unified in the context of the continuous stabilized FE discretizations
- Based on the proposed unified treatment extensions and generalizations are devised, in particular:
 - dual form of the error estimate taking into account anisotropy of the adjoint solution
 - systematic way of including turbulence equations into the error estimate
 - dimensionally consistent way of combining estimates (metric fields) for individual components into a single metric suggested

Conclusions

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- Numerical testing and verification of the newly proposed error estimates performed across a variety of solvers for a set of test cases of varying complexity

Using entropy variables for the dimensionally consistent scalar error estimate $\epsilon^{(1)}$

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- With the entropy variables V and positive definite matrix $\tilde{A}_0 = U_{,V}$ one gets using generalized discrete Cauchy-Schwarz:

$$\begin{aligned}\epsilon_K^{(1)} &\leq \int_K \left| \left(\mathcal{R}^* [\hat{Z}^{(h)}] \right)^T (U - U^{(h)}) \right| dx \\ &\leq \int_K \left(\tilde{A}_0 \mathcal{R}^*, \mathcal{R}^* \right) \left(\tilde{A}_0^{-1} (U - U^{(h)}), U - U^{(h)} \right) dx \\ &\leq \left\| |\mathcal{R}^*| \tilde{A}_0 \right\|_{L_\infty(K)} \left\| |U - U^{(h)}|_{\tilde{A}_0^{-1}} \right\|_{L_1(K)}\end{aligned}$$

- This is dimensionally consistent and suggests using $\rho s = V^T U$ as a single scalar field for the primal solution

Solvers and grid adaptation codes: NASA

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- FUN3D-SFE^(xii)
- *refine*^(xiii)
- Original $\epsilon^{(1)}$ -type goal-oriented “viscous” error estimate by F. Alauzet and L. Frazza

- (xii) W. Kyle Anderson, James C Newman and Steve L Karman, *Stabilized Finite Elements in FUN3D*, Journal of Aircraft, **55**, 2, pp. 696–714, mar 2018
- (xiii) Michael Andrew Park, *Anisotropic Output-Based Adaptation with Tetrahedral Cut Cells for Compressible Flows*, PhD Thesis, Massachusetts Institute of Technology, sep 2008, doi 1721.1/46363
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Solvers and grid adaptation codes: MIT

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- SANS-VMSD^(xiv)
- MOESS^(xv)
 - discretization-dependent local error sampling model
 - global constrained optimization problem to generate metric field
- EPIC – for grid generation

(xiv) Arthur C. Huang, Hugh A. Carson, Steven R. Allmaras, Marshall C. Galbraith, David L. Darmofal and Dmitry S. Kamenetskiy, *A Variational Multiscale Method with Discontinuous Subscales for Output-Based Adaptation of Aerodynamic Flows*, AIAA Paper 2020–1563

(xv) Masayuki Yano and David L. Darmofal, *An Optimization-Based Framework for Anisotropic Simplex Mesh Adaptation*, Journal of Computational Physics, 231, number = 22, pp. 7626–7649, sep 2012

T1/EPIC implementation

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- Regularizations for the isotropic weights L_∞ norms obtained by averaging over each cell
- Residual-weighted averages of the component Hessians converted to the L_p , $p = 1$ metrics using the mesh/metric duality formalism
- Cell-based metrics converted to the node based one internally by the EPIC adaptation code
 - Riemannian metric manipulation algebra